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HYBRID BOUNDS FOR RANKIN-SELBERG L-FUNCTIONS

FEI HOU AND MENG ZHANG

ABSTRACT. Let M be a square-free integer and P be a prime such that $(P, M) = 1$. We prove a new hybrid bound for $L(\frac{1}{2}, f \otimes g)$ where f is a primitive holomorphic cusp form of level M and g a primitive (either holomorphic or Maass) cusp form of level P satisfying $P \sim M^\eta$ with $0 < \eta < 2/15$. Particularly in the range $\beta < \eta < (2 - 32\beta)/15$ with $\beta = 11/4875$ we present a strengthened level aspect hybrid subconvexity bound for $L(\frac{1}{2}, f \otimes g)$ relative to the current bounds obtained by Holowinsky-Munshi [11] and Ye [27].

1. INTRODUCTION

The subconvexity problem is concerned with the magnitude of an L-function on the critical line $s = 1/2$. Let π be an irreducible cuspidal representation of GL_2. Given an L-function $L(s, \pi)$ with analytic conductor denoted by $Q(s, \pi)$, one looks for a bound of the form

$$L\Big(\frac{1}{2}, \pi\Big) \ll Q\left(\frac{1}{2}, \pi\right)^{1/4-\theta}$$

for some constant $\theta > 0$. In the literature many authors have been established the exponent θ in a variety of settings. We refer the reader to [19, Chapter 4] and [17] for the backgrounds and relevant heuristics.

Several authors have made great strides in recent years to establish the level aspect hybrid bounds for twisted L-functions attached to modular forms. Michel-Ramakrishan [21] and later generalized by Feigon-Whitehouse [7] showed that there are "stable" formulas for the average central L-value of the Rankin-Selberg convolutions of some holomorphic forms of fixed even weight and large level against a fixed imaginary quadratic theta series, which as a consequence yield some hybrid subconvexity. To purely focus on subconvexity rather than stable averages as in the works [21, 7], Blomer and Harcos [2] obtained subconvexity for L-functions attached to twists $f \otimes \chi$ of a primitive cusp form f of level M and a primitive character modulo D, where M and D are any given positive integers. Particularly if one supposes M and D are co-prime varying at different rates, say $D \sim M^\eta$ for some $\eta > 0$, they may show that

$$L\Big(\frac{1}{2}, f \otimes \chi\Big) \ll Q^{1/4+\varepsilon} \min\left\{Q^{-\frac{\eta-2}{8(1+2\eta)}}, Q^{-\frac{\eta}{8(1+2\eta)}} + Q^{-\frac{1-\eta}{4(1+2\eta)}}\right\}$$

for $0 < \eta < 1$ or $\eta > 2$, where $f \in \mathcal{B}_\kappa^*(M)$ or $\mathcal{B}_\lambda^*(M)$ (see §2 for definitions) and $Q = MD^2$ is the size of the (arithmetic) conductor $Q(f \otimes \chi)$ of the L-function $L(s, f \otimes \chi)$ (see [14, Chapter 7]).

Assume M is square-free. Let N be a positive square-free integer co-prime with M satisfying $N \sim M^\eta$ for some $\eta > 0$. Given a primitive cusp form g of level N, one can verify that the (arithmetic) conductor $\mathcal{Q} := Q(f \otimes g)$ of the Rankin-Selberg L-function $L(s, f \otimes g)$ is $(NM)^2$ (see for instance [9]). Holowinsky and Munshi [11] showed that

$$L\Big(\frac{1}{2}, f \otimes g\Big) \ll \mathcal{Q}^{1/4+\varepsilon}\Big(\mathcal{Q}^{-\frac{\eta}{2(1+\eta)}} + \mathcal{Q}^{-\frac{2-21\eta}{64(1+\eta)}}\Big) \tag{1.1}$$

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for $0 < \eta < 2/21$, where $f \in \mathcal{B}_\kappa^*(M)$ and $g \in \mathcal{B}_k^*(N)$, N being restricted to be a prime. Further assuming each L-value $L(1/2, f \otimes g)$ to be non-negative, Holowinsky and Templier [13] obtained

$$L\Big(\frac{1}{2}, f \otimes g\Big) \ll \mathcal{Q}^{1/4+\varepsilon}\Big(\mathcal{Q}^{-\frac{\eta}{4(1+\eta)}} + \mathcal{Q}^{-\frac{1-\eta}{4(1+\eta)}}\Big)$$

for $0 < \eta < 1$, where $f \in \mathcal{B}_\kappa^*(M)$ and $g \in \mathcal{B}_k^*(N, \phi)$ with ϕ being an even Dirichlet character of conductor N. Recently Ye [27] obtained

$$L\Big(\frac{1}{2}, f \otimes g\Big) \ll \mathcal{Q}^{1/4+\varepsilon}\Big(\mathcal{Q}^{-\frac{\eta}{2(1+\eta)}} + \mathcal{Q}^{-\frac{\beta}{2(1+\eta)}} + \mathcal{Q}^{-\frac{3\beta+1-\eta}{6(1+\eta)}}\Big) \tag{1.2}$$

for $0 < \eta < 1 + 3\beta$, where $f \in \mathcal{B}_\kappa^*(M)$, $g \in \mathcal{B}_k^*(N)$ or $\mathcal{B}_\lambda^*(N)$ and $\beta = 11/4875$. In this paper we are able to prove:

Theorem 1.1. *Let M be a positive square-free integer and P be a prime such that $(P, M) = 1$. Let $\eta = \frac{\log P}{\log M}$. Let $k \geq 2$ be an even integer. Set $\mathcal{Q} = (PM)^2$. Then for two newforms $f \in \mathcal{B}_\kappa^*(M)$ and $g \in \mathcal{B}_k^*(P)$ (or $\mathcal{B}_\lambda^*(P)$) we have*

$$L\Big(\frac{1}{2}, f \otimes g\Big) \ll \mathcal{Q}^{1/4+\varepsilon}\left(\mathcal{Q}^{-\frac{\eta}{2(1+\eta)}} + \mathcal{Q}^{-\frac{2-15\eta}{64(1+\eta)}}\right). \tag{1.3}$$

Compared to (1.1) and (1.2) in situations where the level $N = P$, the estimate (1.3) is better than the former, and the latter whenever $\beta < \eta < (2-32\beta)/15$; while for $\eta \leq \beta$ the quantities in the parenthesises of (1.1)-(1.3) have the same order of $\mathcal{Q}^{-\frac{\eta}{2(1+\eta)}}$.

Our approach to achieve (1.3) is to study the average of the second moment of $L(1/2, f \otimes g)$ over a family of forms. The technique to treat the second moment usually involves invoking an approximate functional equation, the Petersson trace formula and the Voronoï summation formula.

Theorem 1.2. *Let $M, P, \mathcal{Q}$ be as in Theorem 1.1. Let h be a smooth function, compactly supported on $[1/2, 5/2]$ with bounded derivatives. Then for $\mathcal{Q}^{1/2-\delta} \leq X \leq \mathcal{Q}^{1/2+\varepsilon}$ and any newform $g \in \mathcal{B}_k^*(P)$ (or $\mathcal{B}_\lambda^*(P)$) we have*

$$\sum_{f \in \mathcal{B}_\kappa(M)} \omega_f^{-1} \left|\sum_n \psi_f(n)\psi_g(n)h(n/X)\right|^2 \ll_{\varepsilon,\delta} XPQ^\varepsilon \left(\frac{\mathcal{Q}^{2\delta}}{P} + \frac{1}{Q^\delta} + \mathcal{Q}^{\frac{5}{4}\delta}\frac{P^{\frac{15}{8}}}{M^{\frac{1}{4}}} + \mathcal{Q}^{3\delta}\frac{P^{\frac{5}{4}}}{M^{\frac{1}{4}}}\right), \tag{1.4}$$

where $\varepsilon, \delta > 0$ are arbitrary, $\psi_f(n)$ ($\psi_g(n)$ resp.) denotes the n-th Fourier coefficient of the form f (g resp.) and the spectral weights are given as $\omega_f := \frac{(4\pi)^{\kappa-1}}{\Gamma(\kappa-1)}\langle f, f\rangle$.

Our main result (1.3) is an immediate consequence of Theorem 1.2 by using the fact $\omega_f \ll_\kappa M$ (see [16]).

Remark 1.3. *The second moment method behaves like a harmonic detection process. One may seek the harmonic extraction by introducing the amplifier $\sum_{l \leq L} \alpha_l \psi_f(l)$ for some real sequence $\{\alpha_l : l \leq L\}$ in the sum of (1.4). However the choice of L is closely related to the levels of the forms (see* [12, Section 2] *for comparison) which in turn imposes an extra constraint on the parameter η in Theorem 1.1. In this paper we do not exploit the amplified second moment method.*

In the analytic theory of automorphic L-functions one often encounters sums of the form

$$\sum_{am \pm bn = l} \lambda_f(m)\lambda_g(n)F(m, n)$$

for some nice function F, for instance, smooth and compactly supported, where f, g are two primitive cusp forms and a, b, l are positive integers. In history these sums have been studied extensively (see for instance the works [5, 25, 18, 8, 23, 11, 1, 3, 26]). Non-trivial bounds of them often have deep implications, e.g. subconvexity and equidistribution (QUE). In this paper we establish the following bound for the shifted convolution sum of a special type, explicitly determining the dependence on the levels of the forms, which does not follow easily from any of the works in the current literature.

Theorem 1.4. *Suppose that l be a non-zero integer and $X, Y \geq 1$. Let $F(x,y)$ be a smooth function supported on $[1/2, 5/2] \times [1/2, 5/2]$ with partial derivatives satisfying*

$$x^i y^j \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial y^j} F\left(\frac{x}{X}, \frac{y}{Y}\right) \ll Z Z_x^i Z_y^j$$

for some $Z > 0$ and $Z_x, Z_y \geq 1$. Let $k \geq 2$ be an even integer. For any newforms $g_1, g_2 \in \mathcal{B}_k^*(P)$ (or $\mathcal{B}_\lambda^*(P)$) we define

$$\mathcal{S}_{X,Y}(l) = \sum \sum_{m=nP+l} \psi_{g_1}(n) \psi_{g_2}(m) F\left(\frac{n}{X}, \frac{m}{Y}\right).$$

Then we have

$$\mathcal{S}_{X,Y}(l) \ll (XYP)^\varepsilon P^{1/4} Z \sqrt{Z_x Z_y} \max\{XP, Y\}^{3/4} \max\{Z_x, Z_y\}^{5/4}. \tag{1.5}$$

2. Preliminaries

2.1. **Automorphic forms.** Let $k \geq 2$ be an even integer and $N > 0$ be an integer. Let χ be an even Dirichlet character of conductor N. We denote by $\mathcal{S}_k(N, \chi)$ the vector space of holomorphic cusp forms on $\Gamma_0(N)$ with nebentypus χ and weight k. For any $f \in \mathcal{S}_k(N, \chi)$ one has a Fourier expansion

$$f(z) = \sum_{n \geq 1} \psi_f(n) n^{\frac{k-1}{2}} e(nz)$$

for $\mathrm{Im} z > 0$. Analogously we denote by $\mathcal{S}_\lambda(N, \chi)$ the vector space of Maass forms on $\Gamma_0(N)$ with nebentypus χ, weight 0 and eigenvalue $\lambda = 1/4 + r^2 > 1/4$ (so that $r \in \mathbb{R}$). Then for any $f \in \mathcal{S}_\lambda(N, \chi)$ one has a Fourier expansion

$$f(z) = 2\sqrt{|y|} \sum_{n \neq 0} \psi_f(n) K_{ir}(2\pi |ny|) e(nx),$$

where $z = x + iy$ and K_{ir} denotes the K-Bessel function.

$\mathcal{S}_k(N)$ and $\mathcal{S}_\lambda(N)$ are finite dimensional Hilbert spaces which can be equipped with the Petersson inner products

$$< f_1, f_2 >= \int_{\Gamma_0(N)\backslash\mathbb{H}} f_1(z) \overline{f_2(z)} y^{k-2} \mathrm{d}x\mathrm{d}y$$

and

$$< f_1, f_2 >= \int_{\Gamma_0(N)\backslash\mathbb{H}} f_1(z) \overline{f_2(z)} \frac{\mathrm{d}x\mathrm{d}y}{y^2},$$

respectively. We recall the Hecke operators $\{T_n\}$ with $(n, N) = 1$ which satisfy the multiplicativity relation

$$T_n T_m = \sum_{d|(n,m)} \chi(d) T_{\frac{nm}{d^2}}. \tag{2.6}$$

The adjoint of T_n with respect to the Petersson inner products is $T_n^* = \overline{\chi(n)} T_n$, hence T_n is normal. One can find an orthogonal basis $\mathcal{B}_k(N, \chi)$ ($\mathcal{B}_\lambda(N, \chi)$ resp.) of $\mathcal{S}_k(N, \chi)$ ($\mathcal{S}_\lambda(N, \chi)$ resp.) consisting of common eigenfunctions of all the Hecke operators T_n with $(n, N) = 1$. For each $f \in \mathcal{B}_k(N, \chi)$ or $\mathcal{B}_\lambda(N, \chi)$, denote by $\lambda_f(n)$ the n-th Hecke eigenvalue which satisfies

$$T_n f(z) = \lambda_f(n) f(z)$$

for all $(n, N) = 1$. From (2.6) one has

$$\psi_f(m) \lambda_f(n) = \sum_{d|(n,m)} \chi(d) \psi_f\left(\frac{mn}{d^2}\right)$$

for any $m, n > 1$ with $(n, N) = 1$. In particular $\psi_f(1)\lambda_f(n) = \psi(n)$ if $(n, N) = 1$. Therefore

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(n,m)} \chi(d)\lambda_f\left(\frac{mn}{d^2}\right) \tag{2.7}$$

if $(mn, N) = 1$.

The Hecke eigenbasis $\mathcal{B}_k(N,\chi)$ ($\mathcal{B}_\lambda(N,\chi)$ resp.) also contains a subset of newforms $\mathcal{B}_k^*(N,\chi)$ ($\mathcal{B}_\lambda^*(N,\chi)$ resp.), those forms which are simultaneous eigenfunctions of all the Hecke operators T_n for any $n \geq 1$ and normalized to have first Fourier coefficient $\psi_f(1) = 1$. The elements of $\mathcal{B}_k^*(N,\chi)$ and $\mathcal{B}_\lambda^*(N,\chi)$ are usually called primitive forms.

When the nebentypus is trivial we remove it from the notations. Our primitive form f of level N is an element of $\mathcal{B}_k^*(N)$ or $\mathcal{B}_\lambda^*(N)$. For f such a primitive form the Hecke relations (2.7) hold for all integers $n, m > 1$ and it is also known (see [16]) that

$$|\lambda_f(p)| = p^{-1/2} \tag{2.8}$$

for any $p|N$.

We will need the following general Voronoï-type summation formula which is Theorem A.4 [18].

Lemma 2.1. *Let $k \geq 2$ be an even integer and $N > 0$ be an integer. Let $g \in B_k^*(N)$ (or $B_\lambda^*(N)$) be a newform. For $(a,q) = 1$ set $N_2 := N/(N,q)$. If $h \in \mathbb{C}^\infty(\mathbb{R}^{\times,+})$ is a Schwartz class function vanishing in a neighborhood of zero, then there exists a complex number η of modulus one, which depends on a, q and g, and a newform $g^\star \in B_k^*(N)$ (or $B_\lambda^*(N)$) such that*

$$\begin{aligned}\sum_n \lambda_g(n) e\left(\frac{an}{q}\right) h(n) = &\frac{2\pi\eta}{q\sqrt{N_2}} \sum_n \lambda_{g^\star}(n) e\left(-\frac{\overline{aN_2}n}{q}\right) \int_0^\infty h(\xi) J_g\left(\frac{4\pi\sqrt{n\xi}}{q\sqrt{N_2}}\right) \mathrm{d}\xi \\ &+ \frac{2\pi\eta}{q\sqrt{N_2}} \sum_n \lambda_{g^\star}(n) e\left(\frac{\overline{aN_2}n}{q}\right) \int_0^\infty h(\xi) K_g\left(\frac{4\pi\sqrt{n\xi}}{q\sqrt{N_2}}\right) \mathrm{d}\xi.\end{aligned}$$

In this formula,

- *if g is holomorphic of weight k then*

$$J_g(x) = 2\pi i^k J_{k-1}(x), \quad K_g(x) = 0;$$

- *if g is a Maass form with eigenvalue $\lambda = 1/4 + r^2$ then*

$$J_g(x) = \frac{-\pi}{\sin(\pi i r)}(J_{2ir}(x) - J_{-2ir}(x)), \quad K_g(x) = 4\cosh(\pi r) K_{2ir}(x).$$

2.2. **Rankin-Selberg convolutions of forms with co-prime levels.** Let N and M be two positive square-free co-prime integers, and let k and κ be two fixed positive even integers. Let χ and ϕ be Dirichlet characters modulo N and M, respectively. Given two forms $f \in \mathcal{B}_\kappa^*(M,\phi)$, $g \in \mathcal{B}_k^*(N,\chi)$ or $\mathcal{B}_\lambda^*(N,\chi)$ we consider the Rankin-Selberg convolution L-function

$$L(s, f \otimes g) = L^{(NM)}(\chi\phi, 2s) \sum_{n \geq 1} \frac{\lambda_f(n)\lambda_g(n)}{n^s},$$

where $L^{(NM)}(\chi\phi, 2s)$ denotes the partial Dirichlet L-function with the local factors at primes dividing NM removed. It admits an analytic continuation to all of $\mathbb{C}$ and a functional equation of the form

$$\Lambda(s, f \otimes g) = \left(\frac{NM}{4\pi^2}\right)^s L_\infty(s, f \otimes g) L(s, f \otimes g) = \epsilon(f \otimes g)\Lambda(1-s, f \otimes g),$$

where according to [18]

$$L_\infty(s, f\otimes g) = \Gamma\left(s+\frac{|k-\kappa|}{2}\right)\Gamma\left(s+\frac{k+\kappa}{2}-1\right) \quad \text{for } g \text{ holomorphic},$$
$$L_\infty(s, f\otimes g) = \Gamma\left(s+\frac{\kappa+2ir-1}{2}\right)\Gamma\left(s+\frac{\kappa-2ir-1}{2}\right) \quad \text{for } g \text{ a Maass form},$$

and the epsilon factor

$$\epsilon(f\otimes g) = \begin{cases}\chi(-M)\phi(N)\eta_f^2(M)\eta_g^2(N) & \text{if } g \text{ is holomorphic and } k\geq\kappa,\\ \chi(M)\phi(-N)\eta_f^2(M)\eta_g^2(N) & \text{otherwise.}\end{cases}$$

Here $\eta_f(M)$, $\eta_g(N)$ are the pseudo-eigenvalues of f, g for the Atkin-Lehner-Li operators W_M, W_N.

2.3. **Bessel functions.** We recall some properties of Bessel functions which can be found for instance in [9, Section 7].

Lemma 2.2. *Let $k\geq 2$ be an even integer and $N>0$ be an integer. Let $g\in B_k^*(N)$ (or $B_\lambda^*(N)$) be a newform. There exist smooth functions $F_g^{\pm}(x)$ such that*

$$x^j(F_g^{\pm})^{(j)}(x) \ll_{j,\nu_g} \frac{1}{(1+x)^{1/2}}$$

for all $j\in\mathbb{N}_0$ with $\nu_g=\pm 2ir$ or $k-1$, and

$$J_g(x) = F^+(x)e^{ix} + F^-(x)e^{-ix}.$$

Furthermore

$$K_g(x) \ll_\varepsilon \begin{cases}(1+|r|)^\varepsilon, & 0<x\leq 1+\pi|r|;\\ e^{-x}x^{-1/2}, & x>1+\pi|r|.\end{cases}$$

We will need the following two lemmas which have the flavours of [11, Lemma 2.1-2.3].

Lemma 2.3. *Let $k\geq 2$ be an even integer and $N>0$ be an integer. Let $g\in B_k^*(N)$ (or $B_\lambda^*(N)$) be a newform. For any $a,b,x,y>0$ define*

$$I(x,y) = \int_0^\infty h(\xi)J_g(4\pi a\sqrt{x\xi})J_{\kappa-1}(4\pi b\sqrt{y\xi})\mathrm{d}\xi$$

and

$$I'(x,y) = \int_0^\infty h(\xi)K_g(4\pi a\sqrt{x\xi})J_{\kappa-1}(4\pi b\sqrt{y\xi})\mathrm{d}\xi,$$

where h is a smooth function compactly supported on $[1/2,5/2]$ with bounded derivatives. We have

$$I(x,y) \ll_{j,\nu_g} |a\sqrt{x}-b\sqrt{y}|^{-j}, \quad I'(x,y) \ll_{j,\nu_g} \min\left\{e^{-2\pi a\sqrt{x}}, (b\sqrt{y})^{-j}\right\} \tag{2.9}$$

for any $j\geq 0$. Moreover for any non-negative integers i and j

$$\frac{\partial^i}{\partial x^i}\frac{\partial^j}{\partial y^j}I(x,y), \frac{\partial^i}{\partial x^i}\frac{\partial^j}{\partial y^j}I'(x,y) \ll_{i,j,\nu_g} \mathcal{E} \tag{2.10}$$

with

$$\mathcal{E} := \frac{1}{(1+a\sqrt{x})^{1/2}(1+b\sqrt{y})^{1/2}}(1+a\sqrt{x})^i(1+b\sqrt{y})^j.$$

Proof. A change of variables, $\xi = w^2$, gives

$$I(x,y) = 2\int_0^\infty h(w^2) w J_g(4\pi a\sqrt{x}w) J_{\kappa-1}(4\pi b\sqrt{y}w)\mathrm{d}w.$$

Therefore we see from Lemma 2.2 that $I(x,y)$ may be written as the sum of four similar terms, one of them being

$$\int_0^\infty e(2w(a\sqrt{x} - b\sqrt{y}))h(w^2)wF^+(2\pi a\sqrt{x}w)F^-(2\pi b\sqrt{x}w)\mathrm{d}w. \tag{2.11}$$

Repeated integration by parts gives the required bound of $I(x,y)$, as shown in (2.9). Repeating the procedure one may write $I'(x,y)$ as the sum of two similar terms, one of them being

$$\int_0^\infty e(2w(b\sqrt{y}))h(w^2)wK_g(2\pi a\sqrt{x}w)F^+(2\pi b\sqrt{x}w)\mathrm{d}w \tag{2.12}$$

which would lead to the bound of $I'(x,y)$ in (2.9). Differentiating (2.11), (2.12) together with Lemma 2.2 gives (2.10). □

Lemma 2.4. *Let $k \geq 2$ be an even integer and $N > 0$ be an integer. Let $g \in B_k^*(N)$ (or $B_\lambda^*(N)$) be a newform. Let P, q be positive integers. Take $Q > 1$ and $X, Y \geq 1$. For any $a, b > 0$ define*

$$I_{J,J}(a,b) = \int_0^\infty \int_0^\infty F\left(\frac{x}{X}, \frac{y}{Y}\right) h\left(\frac{q}{Q}, \frac{xP + l - y}{PQ^2}\right) J_g(4\pi a\sqrt{x})J_g(4\pi b\sqrt{y})\mathrm{d}x\mathrm{d}y, \tag{2.13}$$

and $I_{J,K}(a,b)$ to be the integral $I_{J,J}(a,b)$ with the second Bessel function J_g replaced by K_g in (2.13). Similarly we define $I_{K,J}(a,b)$ and $I_{K,K}(a,b)$. Here $h\left(\frac{q}{Q}, \frac{xP+l-y}{PQ^2}\right)$ is the function defined in Lemma 2.5 and F is a smooth function supported on $[1/2, 5/2] \times [1/2, 5/2]$ with partial derivatives satisfying

$$x^i y^j \frac{\partial^i}{\partial x^i}\frac{\partial^j}{\partial y^j} F\left(\frac{x}{X}, \frac{y}{Y}\right) \ll Z Z_x^i Z_y^j$$

for some $Z > 0$ and $Z_x, Z_y \geq 1$. Denote by $I_{*,*}$ any element belonging to $\{I_{J,J}, I_{J,K}, I_{K,J}, I_{K,K}\}$. Then for any non-negative integers i and j we have

$$I_{*,*}(a,b) \ll_{i,j,\nu_g} \mathcal{E}_0 \tag{2.14}$$

with

$$\mathcal{E}_0 := \frac{Q}{q}\frac{ZXY}{(1+a\sqrt{X})^{1/2}(1+b\sqrt{Y})^{1/2}}\left[\frac{1}{a\sqrt{X}}\left\{Z_x + \frac{X}{qQ}\right\}\right]^i \left[\frac{1}{b\sqrt{Y}}\left\{Z_y + \frac{Y}{PqQ}\right\}\right]^j.$$

Moreover

$$I_{*,*}(a,b) \ll_{\varepsilon,\nu_g} \mathcal{E}_1 \tag{2.15}$$

with

$$\mathcal{E}_1 := \frac{Z\sqrt{XY}Q^\varepsilon}{ab(1+a\sqrt{X})^{1/2}(1+b\sqrt{Y})^{1/2}}\frac{Q}{q}\min\left\{Z_x b\sqrt{Y}, Z_y a\sqrt{X}\right\}.$$

Proof. We only consider $I_{J,J}$ since the proofs for $I_{J,K}, I_{K,J}, I_{K,K}$ are similar. We change variables and integrate by parts once with respect to x and apply Lemma 2.2 to obtain

$$I_{J,J}(a,b) \ll_{\nu_g} \frac{Q}{q}\frac{ZXY}{(1+a\sqrt{X})^{1/2}(1+b\sqrt{Y})^{1/2}}\left[\frac{1}{a\sqrt{X}}\{Z_x + XPI\}\right] \tag{2.16}$$

with

$$I := \int_{1/2}^{5/2}\int_{1/2}^{5/2}{}_{2|xXP+l-yY|>qQP} \frac{1}{|xXP+l-yY|}\mathrm{d}x\mathrm{d}y.$$

Notice that (2.14) holds for $i = 1$ and $j = 0$ by using the trivial bound $I \ll (qQP)^{-1}$. Repeated integration by parts would then establish the result for all i and j. To prove (2.15) we replace x by a new variable $w = xXP + l - yY$ to get

$$I \ll (XP)^{-1} \int_{1/2}^{5/2} \int_{qQP/2}^{(XP+Y+|l|)Q^{\varepsilon}} \frac{1}{w} \mathrm{d}w\mathrm{d}y \ll (XP)^{-1} Q^{\varepsilon}.$$

Repeating the argument, for y instead of x, gives (2.15). □

2.4. **δ-method.** We will now briefly recall a version of the circle method. The δ-symbol method was developed in [4, 5] as variant of the circle method. The main purpose is to express $\delta(n, 0)$, the Dirac symbol at 0 (restricted to the integers n in some given range: $|n| \le N$), in terms of 'harmonics' $e(\frac{an}{q})$ for some integers a, q satisfying $(a, q) = 1$ and $q \le Q$, with Q being any fixed positive real number. In order to be of practical use one expects the δ-symbol method should be capable of providing an expression for $\delta(n, 0)$ in terms of harmonics of a small moduli. However the modulus in the circle method cannot be less than $N^{1/2}$, which corresponds to using Dirichlet's approximation theorem to produce values $q \le Q$ (see [10]). In our paper we shall exploit a modified δ-method motivated by the 'conductor lowering mechanism' (see [22], [23] or [24]).

We introduce a version of circle method for the latter use. One can follow the expositions of Heath-Brown in [10] (see also [11, Lemma 2.7 & (2.7), (2.8)]).

Lemma 2.5. *For any $Q > 1$ there exist a positive c_Q and an infinitely differentiable function $h(x, y)$ defined on the set $(0, \infty) \times \mathbb{R}$ such that*

$$\delta(n, 0) = \frac{c_Q}{Q^2} \sum_{q=1}^{\infty} \sideset{}{^*}\sum_{a \bmod q} e\left(\frac{an}{q}\right) h\left(\frac{q}{Q}, \frac{n}{Q^2}\right). \tag{2.17}$$

The constant c_Q satisfies $c_Q = 1 + O(Q^{-A})$ for any $A > 0$. $h(x, y)$ is non-zero only for $x \le \max\{1, 2|y|\}$ and $h(x, y) \ll x^{-1}$ for all y. Moreover

$$x^i \frac{\partial}{\partial x^i} h(x, y) \ll_i x^{-1} \quad \textit{and} \quad \frac{\partial}{\partial y} h(x, y) = 0 \quad (x \le 1, \quad |y| \le x/2).$$

When $|y| > x/2$ we have

$$x^i y^j \frac{\partial}{\partial x^i y^j} h(x, y) \ll_{i,j} x^{-1}.$$

3. Proof of Theorem 1.4

First we write $\mathcal{S}_{X,Y}(l)$ as

$$\mathcal{S}_{X,Y}(l) = \sum_n \sum_m \lambda_{g_1}(n) \lambda_{g_2}(m) F\left(\frac{n}{X}, \frac{m}{Y}\right) \delta(nP + l - m, 0). \tag{3.18}$$

Notice that for any positive integer K we have

$$\delta(r, 0) = \mathcal{C}_{K,r} \delta(r/K, 0),$$

where $\mathcal{C}_{K,r}$ is equal to 1 or 0 according as $K|r$ or not. Thus an application of Lemma 2.5 gives a expression of $\delta(n, 0)$:

$$\delta(r, 0) = \frac{c_Q}{KQ^2} \sum_{q=1}^{\infty} \sideset{}{^*}\sum_{a \bmod q} \sum_{b \bmod K} e\left(\frac{ar}{qK}\right) e\left(\frac{br}{K}\right) h\left(\frac{q}{Q}, \frac{r}{KQ^2}\right).$$

Here Q is the parameter appearing in (2.17), which will be determined later. Substituting the expression above into (3.18) with $r = nP + l - m$ and $K = P$ we obtain

$$\begin{aligned}\mathcal{S}_{X,Y}(l) =& \frac{c_Q}{PQ^2} \sum_{q=1}^{\infty} \sideset{}{^*}\sum_{a \bmod q} \sum_{b \bmod P} e\left(\frac{l(a+qb)}{qP}\right) \sum_n \lambda_{g_1}(n) e\left(\frac{an}{q}\right) \\ & \times \sum_m \lambda_{g_2}(m) e\left(-\frac{m(a+bq)}{qP}\right) h\left(\frac{q}{Q}, \frac{nP+l-m}{PQ^2}\right) F\left(\frac{n}{X}, \frac{m}{Y}\right).\end{aligned} \tag{3.19}$$

It turns out that one needs to investigate the cancellations of the averages involving the Fourier coefficients and the harmonics. We proceed with the argument by considering two cases according to $(a + bq, P) = 1$ or not in order to take into account the "conductor" of the sum over m in (3.19). Meanwhile one sees that the "conductor" of the sum over n is descended to q. This is where we are getting help from the parameter $K = P$ (actually one can check the choice of K is optimal to make a saving).

3.1. **Case 1: $(a + bq, P) = 1$.** If we write $q \to qP^\alpha$ with $(q, P) = 1$ for some integer $\alpha \geq 0$ we have

$$\begin{aligned}\mathcal{S}_{X,Y}(l) =& \frac{c_Q}{PQ^2} \sum_{\substack{q \geq 1 \\ (q,P)=1}} \sideset{}{^*}\sum_{a \bmod qP^\alpha} \sum_{\substack{b \bmod P \\ (a+bq,P)=1}} e\left(\frac{l(a+qP^\alpha b)}{qP^{1+\alpha}}\right) \\ & \times \sum_n \lambda_{g_1}(n) e\left(\frac{an}{qP^\alpha}\right) \sum_m \lambda_{g_2}(m) e\left(-\frac{m(a+qP^\alpha b)}{qP^{1+\alpha}}\right) G(qP^\alpha, n, m),\end{aligned} \tag{3.20}$$

where $G(q, n, m) = h\left(\frac{q}{Q}, \frac{nP+l-m}{PQ^2}\right) F\left(\frac{n}{X}, \frac{m}{Y}\right)$. Now applying the Voronoï summation formula (Lemma 2.1) to sums over n and m respectively yields

$$\begin{aligned}\mathcal{S}_{X,Y}(l) =& \frac{c_Q}{P^{2(1+\alpha)}\sqrt{P_\alpha}Q^2} \sum_{\substack{q \geq 1 \\ (q,P)=1}} \frac{1}{q^2} \sum_n \sum_m \lambda_{g_1}(n)\lambda_{g_2}(m) \{S_\alpha(n, m, l, q) G_{J,J}(qP^\alpha, n, m) \\ & + S_\alpha(n, -m, l, q) G_{J,K}(qP^\alpha, n, m) + S_\alpha(-n, m, l, q) G_{K,J}(qP^\alpha, n, m) \\ & + S_\alpha(-n, -m, l, q) G_{K,K}(qP^\alpha, n, m)\},\end{aligned} \tag{3.21}$$

where

$$\begin{aligned} S_\alpha(n, m, l, q) &= \sideset{}{^*}\sum_{a \bmod qP^\alpha} \sum_{\substack{b \bmod P \\ (a+bq,P)=1}} e\left(\frac{l(a+qP^\alpha b)}{qP^{1+\alpha}} - \frac{n\overline{aP_\alpha}}{qP^\alpha} + \frac{m\overline{a+qP^\alpha b}}{qP^{1+\alpha}}\right), \\ G_{J,J}(q, n, m) &= \int_0^\infty \int_0^\infty G(q, x, y) J_g\left(\frac{4\pi\sqrt{xn}}{q\sqrt{P_\alpha}}\right) J_g\left(\frac{4\pi\sqrt{ym}}{qP}\right) \mathrm{d}x\mathrm{d}y \end{aligned} \tag{3.22}$$

with $P_\alpha = P/(P, P^\alpha)$, $G_{J,K}$, $G_{K,J}$ and $G_{K,K}$ are defined similarly as $G_{J,J}$.

Lemma 3.1. *Let $S_\alpha(n, m, l, q)$ be defined as in (3.22). Then we have*

$$S_0(n, m, l, q) \ll_\varepsilon ((m - n, q)^{1/2} + (m, P)^{1/2})(qP)^{1/2+\varepsilon}.$$

Here $\overline{P}$ means $P\overline{P} \equiv 1 \pmod q$. For $\alpha \geq 1$, $S_\alpha(n, m, l, q)$ is zero unless the equation $lx^2 \equiv m \bmod P$ satisfying $(x, P) = 1$ is solvable, and in which case

$$S_\alpha(n, m, l, q) \ll_\varepsilon (nP - m, qP^\alpha)^{1/2}(qP^\alpha)^{1/2+\varepsilon} P.$$

Proof. For $\alpha = 0$ we have

$$S_0(n, m, l, q) = \sideset{}{^*}\sum_{a \bmod q} \sum_{\substack{b \bmod P \\ (a+bq,P)=1}} e\left(\frac{l(a+qb)}{qP} - \frac{n\overline{aP}}{q} + \frac{m\overline{a+qb}}{qP}\right) = \sideset{}{^*}\sum_{\gamma \bmod qP} e\left(\frac{l\gamma}{qP} - \frac{n\overline{\gamma P}}{q} + \frac{m\overline{\gamma}}{qP}\right).$$

We write $\gamma = xP + yq$ with $x \bmod q, (x,q) = 1$ and $y \bmod P, (y,P) = 1$. The last sum would be reduced to $S((m-n)\overline{P^2}, l; q)S(m\overline{q^2}, l; P)$. Using the Weil bound for individual Kloosterman sums we have

$$S_0(n,m,l,q) \ll_\varepsilon ((m-n,q)^{1/2} + (m,P)^{1/2})(qP)^{1/2+\varepsilon}.$$

Now we consider the situation where $\alpha \geq 1$. We can write $S_\alpha(n,m,l,q)$ as

$$\begin{aligned} &\sideset{}{^*}\sum_{a \bmod qP^\alpha} \sum_{b \bmod P} e\left(\frac{l(a+qP^\alpha b)}{qP^{1+\alpha}} - \frac{n\overline{a}}{qP^\alpha} + \frac{m\overline{a}(1-qP^\alpha b\overline{a+qP^\alpha b})}{qP^{1+\alpha}}\right) \\ &= \mathbb{1}_{\{a: la^2 \equiv m \bmod P\}} P \sideset{}{^*}\sum_{a \bmod qP^\alpha} e\left(\frac{la}{qP^{1+\alpha}} - \frac{n\overline{a}}{qP^\alpha} + \frac{m\overline{a}}{qP^{1+\alpha}}\right), \end{aligned}$$

where $\mathbb{1}_A : a \to \{0,1\}$ is the characteristic function of a subset A – it is equal to 1 when $a \in A$, and zero otherwise. Let $x_0 \pmod P$ be the solution of the equation $lx^2 \equiv m \bmod P$ satisfying $(x,P) = 1$. Now if we write $a = x_0(1+Py)$ with $y \bmod qP^{\alpha-1}, (y, qP^{\alpha-1}) = 1$, then the last sum becomes

$$e\left(\frac{lx_0}{qP^{1+\alpha}} - \frac{n\overline{x_0}}{qP^\alpha} + \frac{m\overline{x_0}}{qP^{1+\alpha}}\right) \sideset{}{^*}\sum_{y \bmod qP^{\alpha-1}} e\left(\frac{lx_0 y}{qP^\alpha} + \frac{n\overline{x_0(1+Py)}y}{qP^{\alpha-1}} - \frac{m\overline{x_0(1+Py)}y}{qP^\alpha}\right)$$

which by completing method (see [15, Chapter 12]) is

$$\begin{aligned} &\ll \left(1 + \frac{1}{P}\right) \sum_{0 \leq k \leq qP^\alpha} \frac{1}{1+k} \sideset{}{^*}\sum_{y \bmod qP^\alpha} e\left(\frac{(lx_0+k)y}{qP^\alpha} + \frac{n\overline{x_0(1+Py)}y}{qP^{\alpha-1}} - \frac{m\overline{x_0(1+Py)}y}{qP^\alpha}\right) \\ &= \left(1 + \frac{1}{P}\right) \sum_{0 \leq k \leq qP^\alpha} \frac{1}{1+k} \sideset{}{^*}\sum_{y \bmod qP^\alpha} e\left(\frac{(lx_0+k)\overline{y}}{qP^\alpha} + \frac{n\overline{x_0(1+P\overline{y})\overline{y}}}{qP^{\alpha-1}} - \frac{m\overline{x_0(1+P\overline{y})\overline{y}}}{qP^\alpha}\right) \\ &= \left(1 + \frac{1}{P}\right) \sum_{0 \leq k \leq qP^\alpha} \frac{1}{1+k} \sideset{}{^*}\sum_{y \bmod qP^\alpha} e\left(\frac{(lx_0+k)\overline{y} + (nP-m)\overline{x_0}\cdot\overline{y+P}}{qP^\alpha}\right). \end{aligned}$$

We claim the second sum is $\ll_\varepsilon (nP-m, qP^\alpha)^{1/2}(qP^\alpha)^{1/2+\varepsilon}$ which gives the required bound of $S_\alpha(n,m,l,q)$, $\alpha \geq 1$. Typically one may consider the sum

$$T(a,b,c;d) = \sideset{}{^*}\sum_{y \bmod d} e\left(\frac{a\overline{y} + b\overline{y+c}}{d}\right)$$

for any $a,b,c,d \in \mathbb{N}$. Write $y = y_1 q + y_2 P^\alpha$ with $y_1 \bmod P^\alpha, (y_1, P) = 1$ and $y_2 \bmod q, (y_2, q) = 1$. We have

$$\overline{(y+P)} \equiv \overline{(P+y_1q)}q + \overline{(P+y_2P^\alpha)P^\alpha}P^\alpha \pmod{qP^\alpha},$$

where the first inverse is mod qP^α, the second mod P^α and the third mod q. Hence by the Chinese Remainder Theorem $T(a,b,P;qP^\alpha) = T(a\overline{P^{2\alpha}}, b\overline{P^{2\alpha}}, \overline{P^{\alpha-1}}; q)T(a\overline{q^2}, b\overline{q^2}, \overline{q}P; P^\alpha)$. Changing the variable $y \to \overline{y}$ and then $y \to y - P^{\alpha-1}$ enable us to reduce $T(a\overline{P^{2\alpha}}, b\overline{P^{2\alpha}}, \overline{P^{\alpha-1}}; q)$ to a Kloosterman sum, up to a multiplicative factor of modulus one. Using the Weil bound gives that $T(a\overline{P^{2\alpha}}, b\overline{P^{2\alpha}}, \overline{P^{\alpha-1}}; q) \ll_\varepsilon (b,q)q^{1/2+\varepsilon}$. For the second sum, by [6, Theorem 1], one has $T(a\overline{q^2}, b\overline{q^2}, \overline{q}P; P^\alpha) \ll (b, P^\alpha)P^{\alpha/2}$. □

Now we turn to the estimate of the integral in (3.22). By Lemma 2.4 we have the following:

Lemma 3.2. *Denote by $G_{*,*}$ any element belonging to $\{G_{J,J}, G_{J,K}, G_{K,J}, G_{K,K}\}$. Then $G_{*,*}(q,n,m)$ is negligible unless $n \ll \frac{q^2 P_\alpha}{X}\left(Z_x + \frac{X}{qQ}\right)^2 (XYP)^\varepsilon$, $m \ll \frac{q^2P^2}{Y}\left(Z_y + \frac{Y}{PqQ}\right)^2 (XYP)^\varepsilon$, and in which case*

$$G_{*,*}(q,n,m) \ll_{\nu_g,\varepsilon} \frac{Z\sqrt{XY}Q^\varepsilon}{\left(\frac{\sqrt{n}}{q\sqrt{P_\alpha}} \cdot \frac{\sqrt{m}}{qP}\right)\left(1 + \frac{\sqrt{nX}}{q\sqrt{P_\alpha}}\right)^{1/2}\left(1 + \frac{\sqrt{mY}}{qP}\right)^{1/2}} \frac{Q}{q} \min\left\{\frac{Z_x\sqrt{mY}}{qP}, \frac{Z_y\sqrt{nX}}{q\sqrt{P_\alpha}}\right\}.$$

Consequently appealing to Lemma 3.1 and 3.2 we are ready to prove the required bound of $\mathcal{S}_{X,Y}(l)$.

Proposition 3.3. *Under Case 1 we have*

$$\mathcal{S}_{X,Y}(l) \ll (XYP)^{\varepsilon} Z P^{1/4} \sqrt{Z_x Z_y} \max\{XP, Y\}^{3/4} \max\{Z_x, Z_y\}^{5/4}. \tag{3.23}$$

Proof. Denote by

$$T_1 = \frac{q^2 P}{X}\left(Z_x + \frac{X}{qQ}\right)^2 (XYP)^{\varepsilon}, \quad T_2 = \frac{q^2P^2}{Y}\left(Z_y + \frac{Y}{PqQ}\right)^2 (XYP)^{\varepsilon},$$
$$T_3 = \frac{(qP^{\alpha})^2}{X}\left(Z_x + \frac{X}{qQ}\right)^2 (XYP)^{\varepsilon}, \quad T_4 = \frac{q^2 P^{2(1+\alpha)}}{Y}\left(Z_y + \frac{Y}{PqQ}\right)^2 (XYP)^{\varepsilon}.$$

When the parameters are such that either $T_1 < 1$, $T_2 < 1$ or $T_3 < 1$, then one has arbitrary saving in these situations. Otherwise, for $\alpha \geq 1$ by Lemma 3.1 and 3.2

$$\begin{aligned}
\mathcal{S}_{X,Y}(l) \ll Q^{\varepsilon}\Bigg(& \frac{1}{P^{5/2}Q}\sum_{q\leq Q}\frac{1}{q^3}\sum_{n\sim T_1}\sum_{m\sim T_2}\frac{Z(XY)^{1/4}(q^2P^{3/2})^{3/2}}{(nm)^{3/4}}\cdot\left((m-n,q)^{1/2}+(m,P)^{1/2}\right) \\
& \times (qP)^{1/2+\varepsilon} + \frac{1}{P^{2(1+\alpha)}Q}\sum_{q\leq Q/P^{\alpha}}\frac{1}{q^3}\sum_{n\sim T_3}\sum_{m\sim T_4}\frac{Z(XY)^{1/4}(q^2P^{1+2\alpha})^{3/2}}{(nm)^{3/4}} \\
& \times (nP-m, qP^{\alpha})^{1/2}(qP^{\alpha})^{1/2+\varepsilon}P\Bigg)\min\left\{Z_x\left(Z_y+\frac{Y}{PqQ}\right), Z_y\left(Z_x+\frac{X}{qQ}\right)\right\} \\
\ll{} & (XYP)^{\varepsilon} Z\sqrt{Z_xZ_y}Q^{3/2+\varepsilon}P\left(Z_x+\frac{X}{Q^2}\right)\left(Z_y+\frac{Y}{PQ^2}\right) \\
\ll{} & (XYP)^{\varepsilon} Z\sqrt{Z_xZ_y}Q^{3/2+\varepsilon}P\left(\max\{Z_x,Z_y\}+\frac{\max\{XP,Y\}}{PQ^2}\right)^2.
\end{aligned}$$

Choosing $Q = \left(\frac{\max\{XP,Y\}}{P\max\{Z_x,Z_y\}}\right)^{1/2}$ we derive that

$$\mathcal{S}_{X,Y}(l) \ll (XYP)^{\varepsilon} Z P^{1/4} \sqrt{Z_x Z_y} \max\{XP, Y\}^{3/4} \max\{Z_x, Z_y\}^{5/4}.$$

□

3.2. **Case 2:** $P|(a+bq)$**.** It is easy to see that $bq \equiv -a \pmod{P}$. If $(q,P)=1$ we have

$$b \equiv -a\overline{q} \pmod{P}.$$

Hence $\mathcal{S}_{X,Y}(l)$ becomes

$$\frac{c_Q}{PQ^2}\sum_{q\geq 1}\;\sideset{}{^*}\sum_{a \bmod q} e\left(\frac{laq'}{q}\right)\sum_n \lambda_{g_1}(n)e\left(\frac{an}{q}\right)\sum_m \lambda_{g_2}(m)e\left(-\frac{maq'}{q}\right)G(q,n,m), \tag{3.24}$$

where $q' = (1-q\overline{q})/P$. One may regard the expression above without P in the multiplicative factor as (3.20) with $P=1$, up to a factor q' co-prime with q. Repeating the process as in Case 1 we thus conclude (3.24) is bounded by the estimate in (3.23). If $P|q$ we have $P|a$. This contradicts the fact $(a,q)=1$.

4. Proof of Theorem 1.2

In this section we shall follow the argument in [11] to prove Theorem 1.2. Let us put

$$S_g(X) = \sum_{f\in\mathcal{B}_{\kappa}(M)}\omega_f^{-1}\left|\sum_n \psi_f(n)\psi_g(n)h(n/X)\right|^2.$$

Following the steps of the reduction of the second moment in [11, Section 3 & 4] (when it comes to applying the Voronoï summation formula we use Lemma 2.1 instead), we deduce the following:

Lemma 4.1. *Let $\delta > 0$. For $\mathcal{Q}^{1/2-\delta} \le X \le \mathcal{Q}^{1/2+\varepsilon}$ and any newform $g \in \mathcal{B}_k^*(P)$ (or $\mathcal{B}_\lambda^*(P)$) we have*

$$S_g(X) \ll_{\varepsilon,\delta} \mathcal{Q}^{\varepsilon} \left(X + PX\mathcal{Q}^{-\delta} + \sum_{M \le D \le X\mathcal{Q}^{2\delta}} R_{g,D}(X) \right), \tag{4.25}$$

where $R_{g,D}(X)$ is defined as

$$R_{g,D}(X) = \sum_n \sum_m \lambda_g(n) h\left(\frac{n}{X}\right) \lambda_g(m) h\left(\frac{m}{X}\right) \sum_{\substack{d>0 \\ d\equiv 0 \bmod M}} \frac{S(n,m;d)}{c} J_{\kappa-1}\left(\frac{4\pi\sqrt{nm}}{d}\right) \eta_D(d)$$

with η_D being a smooth function supported on $[D/2, 5D/2]$. Moreover

$$R_{g,D}(X) \ll \sum_{LR=P} \frac{1}{\sqrt{L}} \sum_{\substack{d>0 \\ (d,L)=1 \\ d\equiv 0 \bmod RM}} \frac{\eta_D(d)}{d} \sum_{bc=d} \frac{1}{b} (|\Sigma_d(L;c)| + |\Sigma_d'(L;c)|), \tag{4.26}$$

where

$$\Sigma_d(L;c) = \sum_n \sum_{m \equiv nL \bmod c} \lambda_g(n) \lambda_{g^\star}(m) I_d(n,m) \tag{4.27}$$

with the function $I_d(n,m)$ being defined as

$$I_d(n,m) = h\left(\frac{n}{X}\right) \int_0^\infty h\left(\frac{\xi}{X}\right) J_{\kappa-1}\left(\frac{4\pi\sqrt{n\xi}}{d}\right) J_g\left(\frac{4\pi\sqrt{m\xi}}{d\sqrt{L}}\right) \mathrm{d}\xi,$$

and

$$\Sigma_d'(L;c) = \sum_n \sum_{m+nL \equiv 0 \bmod c} \lambda_g(n) \lambda_{g^\star}(m) I_d'(n,m) \tag{4.28}$$

with the function $I_d'(n,m)$ being defined as

$$I_d'(n,m) = h\left(\frac{n}{X}\right) \int_0^\infty h\left(\frac{\xi}{X}\right) J_{\kappa-1}\left(\frac{4\pi\sqrt{n\xi}}{d}\right) K_g\left(\frac{4\pi\sqrt{m\xi}}{d\sqrt{L}}\right) \mathrm{d}\xi.$$

In (4.27) and (4.28) $g^\star$ is a newform depending on g with the same level P.

As shown in Lemma 2.3 $I_d(n,m)$ and $I_d'(n,m)$ enjoy, at least for our needs, the same properties. They determine the main contributions to the sums (4.27) and (4.28) respectively, which occur at $n \sim X$ and $m = nL + O(dL(1+d/X)\mathcal{Q}^{\varepsilon})$. Moreover one sees that the estimate (2.10) can also be adapted to either of them. In this sense it therefore suffices to deal only with the sum $\Sigma_d(L;c)$ involving $I_d(n,m)$. The argument of $\Sigma_d'(L;c)$ follows similarly with that of $\Sigma_d(L;c)$.

In what follows we shall consider $\Sigma_d(L;c)$ from two situations: $L=1$ and $L=P$. For $L=1$ by Lemma 2.3 we have the estimate $I_d(n,m) \ll X \min\{(d/X)^{1/2}, 1\}$, whereby we obtain

$$\Sigma_d(1;c) \ll \mathcal{Q}^{\varepsilon} X \sum_{n\sim X} \sum_{\substack{m = n + O\left(\frac{d^2}{X}\mathcal{Q}^{\varepsilon}\right) \\ m\equiv n \bmod c}} \ll X^2 \Big(1 + \frac{d^2}{Xc}\Big) \mathcal{Q}^{\varepsilon}$$

which would contribute $R_{g,D}(X)$ an amount

$$\mathcal{Q}^{\varepsilon} X^2 \sum_{\substack{d>0 \\ d\equiv 0 \bmod PM}} \frac{\eta_D(d)}{d} \Big(1 + \frac{d}{X}\Big) \ll X\mathcal{Q}^{2\delta+\varepsilon}. \tag{4.29}$$

Next we consider the case $L = P$. First we treat the "zero shift", that is, the terms satisfying $m = nL = nP$ in (4.27). Repeating the procedure of dealing with the case $L = 1$ one sees that an application of the estimate $|\lambda_{g^*}(nP)| = |\lambda_{g^*}(n)\lambda_{g^*}(P)| = |\lambda_{g^*}(n)|P^{-1/2}$ (see (2.8)) together with the bound $I_d(n, nP) \ll X \min\{(d/X)^{1/2}, 1\}$ gives that

$$\Sigma_d(P; c) \ll \frac{X^2}{\sqrt{P}}$$

so that the zero shift contribution to $R_{g,D}(X)$ is

$$\ll \frac{\mathcal{Q}^\varepsilon X^2}{P} \sum_{\substack{d>0 \\ (d,P)=1 \\ d\equiv 0 \bmod M}} \frac{\eta_D(d)}{d} \ll X\mathcal{Q}^\varepsilon. \tag{4.30}$$

We are now left with the non-zero shifts $m \neq nP$ for $\Sigma_d(P; c)$ which may be rewritten as

$$\sum_{0\neq|r|\ll \frac{dP}{c}\left(1+\frac{d}{X}\right)\mathcal{Q}^\varepsilon} \sum_{m=nP+cr} \lambda_g(n)\lambda_{g^*}(m)I_d(n, m). \tag{4.31}$$

For this sum we shall estimate "trivially" by taking the absolute value of each summand in (4.31) in the first step. We shall resort to Theorem 1.5 if the trivial estimate is not enough to achieve subconvexity. First we write

$$I_d(n, m) = X \sum_Y F\left(\frac{n}{X}, \frac{m}{Y}\right) \tag{4.32}$$

by taking a smooth partition of unity, where Y runs over values 2^v with $v = -1, 0, 1, 2, \dots$ such that $m = nP + c$ is soluble when $m \sim Y$ and $F(x, y)$ is a smooth function supported on $[1/2, 5/2] \times [1/2, 5/2]$. By Lemma 2.3 one can show that

$$x^i y^j \frac{\partial^i}{\partial x^i}\frac{\partial^j}{\partial y^j} F\Big(\frac{x}{X}, \frac{y}{Y}\Big) \ll \frac{1}{\left(1+\frac{X}{d}\right)^{1/2}\left(1+\frac{\sqrt{XY}}{d\sqrt{P}}\right)^{1/2}} \left(1+\frac{X}{d}\right)^i \left(1+\frac{\sqrt{XY}}{d\sqrt{P}}\right)^j \tag{4.33}$$

for any non-negative integers i and j. Therefore it follows that the sum in (4.31) is

$$\ll X^2 \sum_Y \frac{1}{\left(1+\frac{X}{d}\right)^{1/2}\left(1+\frac{\sqrt{XY}}{d\sqrt{P}}\right)^{1/2}} \frac{dP}{c}\left(1+\frac{d}{X}\right)\mathcal{Q}^\varepsilon \tag{4.34}$$

which is bounded by $X^2 dP\mathcal{Q}^\varepsilon/c$. This will contribute $R_{g,D}(X)$ an amount

$$\frac{1}{\sqrt{P}} \sum_{\substack{d>0 \\ (d,P)=1 \\ d\equiv 0 \bmod M}} \frac{\eta_D(d)}{d} X^2 P\mathcal{Q}^\varepsilon \ll XP^{3/2}\mathcal{Q}^\varepsilon. \tag{4.35}$$

If d is small compared with X, $d \ll X\mathcal{Q}^{-\delta}$ say, one sees that $m = nP + cr$ is soluble only when $Y \sim XP$. Thus (4.34) satisfies the stronger bound

$$\left(1+\frac{d}{X}\right)\frac{Xd^2PQ^\varepsilon}{c}$$

which will lead to a contribution of the magnitude

$$\ll X\sqrt{P}Q^\varepsilon \sum_{\substack{d>0 \\ (d,P)=1 \\ d\equiv 0 \bmod M}} \eta_D(d)\left(1+\frac{d}{X}\right) \ll XP\mathcal{Q}^\varepsilon\left(\frac{D}{\sqrt{P}M}\right). \tag{4.36}$$

As refereed in §1 one has $\omega_f \ll_\kappa M$, whence the bound $S_g(X) \ll_\varepsilon XP\mathcal{Q}^\varepsilon$ for $X \le \mathcal{Q}^{1/2+\varepsilon}$ would lead to the convexity bound for any individual $L(f \otimes g, 1/2)$. Observe that if $D \ge \sqrt{P}M$ the bound in (4.36) cannot beat the convexity bound barrier. We now apply Theorem 1.4 to seek another bound as in Lemma 4.2, which would be better than $XP\mathcal{Q}^\varepsilon$ if the level P is chosen properly.

Lemma 4.2. *Let $\delta > 0$. If $\sqrt{P}M\mathcal{Q}^{-\delta} \le D \le X\mathcal{Q}^{2\delta}$ we have*

$$R_{g,D}(X) \ll XP\mathcal{Q}^\varepsilon \left(\frac{P^{15/8}\mathcal{Q}^{5\delta/4}}{M^{1/4}} + \frac{P^{5/4}\mathcal{Q}^{3\delta}}{M^{1/4}} \right). \tag{4.37}$$

Proof. Let us put

$$\mathcal{S}_{X,Y}(cr) = \sum_{m=nP+cr} \lambda_g(n)\lambda_{g^\star}(m) F\left(\frac{n}{X}, \frac{m}{Y}\right)$$

with $F(x,y)$ being as in (4.32). Upon noting (4.33), an application of Theorem 1.4 gives that

$$\mathcal{S}_{X,Y}(cr) \ll (XYP)^\varepsilon P^{1/4} Z\sqrt{Z_x Z_y} \max\{XP, Y\}^{3/4} \max\{Z_x, Z_y\}^{5/4},$$

where

$$Z = \frac{1}{\left(1+\frac{X}{d}\right)^{1/2}\left(1+\frac{\sqrt{XY}}{d\sqrt{P}}\right)^{1/2}}, \quad Z_x = \left(1+\frac{X}{d}\right) \quad \text{and} \quad Z_y = \left(1+\frac{\sqrt{XY}}{d\sqrt{P}}\right).$$

Hence the contribution of the non-zero shifts to $R_{g,D}(X)$ is bounded by

$$XP^{3/4}\mathcal{Q}^\varepsilon \sum_{\substack{d>0\\(d,P)=1\\ d\equiv 0 \bmod M}} \frac{\eta_D(d)}{d}\left(1+\frac{d}{X}\right)\sum_Y (XYP)^\varepsilon Z\sqrt{Z_xZ_y}\max\{XP,Y\}^{3/4}\max\{Z_x,Z_y\}^{5/4}. \tag{4.38}$$

If $\sqrt{P}M\mathcal{Q}^{-\delta} \le D < X$ one has that $Y \sim XP$, whence (4.38) reduces to

$$XP^{3/4}\cdot(XP)^{3/4}\mathcal{Q}^\varepsilon \sum_{\substack{d>0\\(d,P)=1\\ d\equiv 0 \bmod M}} \frac{\eta_D(d)}{d}\left(1+\frac{X}{d}\right)^{5/4} \ll \mathcal{Q}^{5\delta/4+\varepsilon}(XP)\cdot\frac{P^{15/8}}{M^{1/4}}.$$

If $X \le D \le X\mathcal{Q}^{2\delta}$ one has that $Y \ll d^2P\mathcal{Q}^\varepsilon/X$, whence (4.38) reduces to

$$\begin{aligned} &XP^{3/4}\mathcal{Q}^\varepsilon \sum_{\substack{d>0\\(d,P)=1\\ d\equiv 0 \bmod M}} \frac{\eta_D(d)}{d} \max_{Y \ll \frac{d^2P\mathcal{Q}^\varepsilon}{X}} \left\{ Y^{3/4}\left(1+\frac{\sqrt{XY}}{d\sqrt{P}}\right)^{9/4}\right\} \\ &\ll XP^{3/4}\mathcal{Q}^\varepsilon\cdot\left(\frac{d^2P}{X}\right)^{3/4} \sum_{\substack{d>0\\(d,P)=1\\ d\equiv 0 \bmod M}} \frac{\eta_D(d)}{d} \ll \mathcal{Q}^{3\delta+\varepsilon}(XP)\cdot\frac{P^{5/4}}{M^{1/4}}. \end{aligned}$$

Combining (4.36) and (4.37) with (4.29) and (4.30) and inserting these bounds into (4.25) completes the proof of Theorem 1.2. □

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