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Automatic Dirichlet Series

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Dirichlet series whose coefficients are generated by finite automata define meromorphic functions on the whole complex plane. As consequences, a new proof of Cobham's theorem on the existence of logarithmic frequencies of symbols in automatic sequences is given, and certain infinite products are explicitly computed. © 2000 Academic Press

Key Words: automatic sequences; Dirichlet series; logarithmic frequency.

1. INTRODUCTION

Automatic sequences have many properties ranging from number theory to harmonic analysis, from theoretical computer science to physics. See for example [11], [2], or [5]. An intuitive definition is that, given an integer $d \geqslant 2$, a sequence $(u_n)_{n \geqslant 0}$ with values in a finite set is d-automatic if its nth term can be computed by a finite-state machine using the base d expansion of the integer n. (A precise definition is given below.)

In his seminal paper on automatic sequences [10], Cobham proves that, given an automatic sequence $(u_n)_{n \geqslant 0}$, the set of integers such that u_n takes a given value, always has a logarithmic density. It may happen that this set *also* has a natural density, which then must be a rational number [10].

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A typical example is the sequence defined by $u_n = 1$ if n begins with a 1 in base 3, and $u_n = 0$ otherwise. This sequence is 3-automatic. The set of n's for which $u_n = 1$ does not have a natural density, but its logarithmic density exists and is equal to log 2/log 3. The proof of Cobham consists in showing that the sequence whose limit gives the existence of the logarithmic density, is bounded respectively from below and from above by two sequences converging towards a common limit.

In this paper we first study the Dirichlet series associated with automatic sequences: we prove that they possess a meromorphic continuation to the whole complex plane, and that the poles—if any—must lie on a finite number of left half-lattices. This result was announced in [1] but details were never written down; it is a generalization of the case of the Thue–Morse sequence with values ± 1, that was addressed in [3], and for which the Dirichlet series can be continued to an entire function in the complex plane. We then obtain as a consequence of the properties of automatic Dirichlet series, a new proof of the existence of logarithmic densities for automatic sequences, and an expression of these densities in terms of a numerical convergent series. The hint at this point is a result of analytic number theory stating that logarithmic density exists if and only if analytic density exists, and they are then equal. Finally we give applications to computing infinite products, in the spirit of [3] (see also [4] and [6] for real analysis methods).

2. DEFINITIONS. FIRST PROPERTIES

We first give a definition of d-automatic sequences.

DEFINITION 1. Let $d \geqslant 2$ be an integer. A sequence $(u_n)_{n \geqslant 0}$ with values in the set Σ is called d-automatic if and only if its d-kernel $\mathcal{N}_d(u)$ is finite, where the d-kernel of the sequence $(u_n)_{n \geqslant 0}$ is the set of subsequences defined by

$$\mathcal{N}_d(u) = \{n \mapsto u_{d^k n + a};\ k \geqslant 0,\ 0 \leqslant a \leqslant d^k - 1\}.$$

Remark 1. A d-automatic sequence necessarily takes finitely many values (take a large k, and look at the values u_a for $0 \leqslant a \leqslant d^k - 1$). Hence we can assume that the set Σ is finite.

The following characterizations of d-automatic sequences are easy consequences of the definition.

THEOREM 1. *Let $d \geqslant 2$ be an integer and $(u_n)_{n \geqslant 0}$ a sequence with values in Σ. Then, the following properties are equivalent.*

(i) *The sequence* $(u_n)_{n \geqslant 0}$ *is d-automatic.*

(ii) *There exist an integer* $t \geqslant 1$ *and a set of* t *sequences* $\mathcal{N}' = \{(u_n^{(1)})_{n \geqslant 0}, ..., (u_n^{(t)})_{n \geqslant 0}\}$, *such that*

— *the sequence* $(u_n^{(1)})_{n \geqslant 0}$ *is equal to the sequence* $(u_n)_{n \geqslant 0}$,

— *the set* $\mathcal{N}'$ *is closed under the maps* $(v_n)_{n \geqslant 0} \mapsto (v_{dn+i})_{n \geqslant 0}$, *for* $0 \leqslant i \leqslant d-1$.

(iii) *There exist an integer* $t \geqslant 1$ *and a sequence* $(U_n)_{n \geqslant 0}$ *with values in* Σ^t, *that we denote as a column vector. There exist d matrices of size* $t \times t$, *say* $A_0, A_1, ..., A_{d-1}$, *with the property that each row of each* A_i *has exactly one entry equal to* 1, *and the other* $t-1$ *entries equal to* 0, *such that*

— *the first component of the vector* $(U_n)_{n \geqslant 0}$ *is the sequence* $(u_n)_{n \geqslant 0}$;

— *for each* $i = 0, 1, ..., d-1$, *and for all* $n \geqslant 0$, *the equality* $U_{dn+i} = A_i U_n$ *holds.*

We recall now the definitions of natural, logarithmic and analytic densities.

Definition 2. Let E be a subset of the integers. We say that the set E has a *natural* density d if the limit

$$d = \lim_{x \to \infty} \frac{1}{x} \# \{n \leqslant x;\ n \in E\}$$

exists.

We say that the set E has a *logarithmic* density δ if the limit

$$\delta = \lim_{x \to \infty} \frac{1}{\log x} \sum_{n \leqslant x;\, n \in E} \frac{1}{n}$$

exists.

We say that the set E has an *analytic* density δ' if the limit

$$\delta' = \lim_{s \to 1+} (s-1) \sum_{n \geqslant 1,\, n \in E} \frac{1}{n^s}$$

exists.

Remark 2. It can be proved that if a set has a natural density, it also has a logarithmic density, and both densities are equal. The converse is not true as indicated in the introduction. The case of analytic density is addressed in the following proposition.

THEOREM 2. *A set has an analytic density if and only if it has a logarithmic density. Both densities are then equal.*

This result is given for example in [15, p. 96].

DEFINITION 3. Let $(u_n)_{n \geqslant 0}$ be a sequence with values in a set Σ, and let $x \in \Sigma$. We say that x occurs in the sequence $(u_n)_{n \geqslant 0}$ with *frequency* f (resp. *logarithmic frequency* f) if the set $\{n \in \mathbb{N} \backslash \{0\};\ u_n = x\}$ has a natural density (resp. has a logarithmic density) that is equal to f.

3. DIRICHLET SERIES OF AUTOMATIC SEQUENCES

With a sequence $(u_n)_{n \geqslant 0}$ taking its values in $\mathbb{C}$ we associate two Dirichlet series given by

$$\sum_{n=0}^{\infty} \frac{u_n}{(n+1)^s} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{u_n}{n^s}.$$

If the sequence $(u_n)_{n \geqslant 0}$ takes only finitely many values (this is in particular the case if $(u_n)_{n \geqslant 0}$ is automatic) both series converge for $\mathcal{R}s > 1$. The main theorem of this section was announced in [1]. It reads as follows.

THEOREM 3. *Let $d \geqslant 2$ be an integer and let $(u_n)_{n \geqslant 0}$ be a d-automatic sequence with values in $\mathbb{C}$. Then there exist an integer $t \geqslant 1$ and d matrices of size $t \times t$ (the matrices $A_0, A_1, ..., A_{d-1}$ defined in Theorem* 1(iii)), *such that the Dirichlet series*

$$\sum_{n=0}^{\infty} \frac{u_n}{(n+1)^s} \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{u_n}{n^s}$$

are the first components of Dirichlet vectors (i.e., vectors of Dirichlet series) $F(s)$ and $G(s)$, such that

- *F satisfies the infinite functional equation*

$$\left(I - d^{-s} \sum_{j=0}^{d-1} A_j\right) F(s) = \sum_{j=0}^{d-1} A_j \sum_{k=1}^{\infty} \binom{s+k-1}{k} (d-j-1)^k \frac{F(s+k)}{d^{s+k}}$$

G satisfies the infinite functional equation

$$\left(I - d^{-s} \sum_{j=0}^{d-1} A_j\right) G(s) = \left(\sum_{j=1}^{d-1} j^{-s} A_j\right) U_0 + \sum_{j=0}^{d-1} A_j \sum_{k=1}^{\infty} (-1)^k \binom{s+k-1}{k} j^k \frac{G(s+k)}{d^{s+k}}$$

- *F and G have meromorphic continuations to the whole complex plane, whose poles (if any) are located on a finite number of left semi-lattices.*

Proof. We give only the proof for F, since the proof for G follows the same lines. Let $A_0, A_1, ..., A_{d-1}$ be the matrices, and $(U_n)_{n \geqslant 0}$ be the sequence of vectors given in Theorem 1(iii). Define a Dirichlet vector $F(s)$ for $\mathscr{R}s \geqslant 1$ by

$$F(s) = \sum_{n=0}^{\infty} \frac{U_n}{(n+1)^s}.$$

We can write, using Theorem 1(iii),

$$\begin{aligned} F(s) &= \sum_{j=0}^{d-1} \sum_{n=0}^{\infty} \frac{U_{dn+j}}{(dn+j+1)^s} = \sum_{j=0}^{d-1} \sum_{n=0}^{\infty} \frac{A_j U_n}{(dn+j+1)^s} \\ &= \sum_{j=0}^{d-2} \sum_{n=0}^{\infty} \frac{A_j U_n}{(dn+j+1)^s} + \sum_{n=0}^{\infty} \frac{A_{d-1} U_n}{(dn+d)^s}. \end{aligned}$$

Hence, denoting by I the $t \times t$ unit matrix,

$$\begin{aligned} (I - d^{-s} A_{d-1})\, F(s) &= \sum_{j=0}^{d-2} A_j \sum_{n=0}^{\infty} \frac{U_n}{(dn+j+1)^s} \\ &= \sum_{j=0}^{d-2} A_j \sum_{n=0}^{\infty} d^{-s} (n+1)^{-s} \left(1 - \frac{d-j-1}{(n+1)\, d}\right)^{-s} U_n \\ &= \sum_{j=0}^{d-2} A_j \sum_{n=0}^{\infty} d^{-s} (n+1)^{-s}\, U_n \\ &\quad \times \sum_{k=0}^{\infty} \binom{s+k-1}{k} \frac{(d-j-1)^k}{(n+1)^k d^k}. \end{aligned} \tag{1}$$

This gives

$$(I - d^{-s} A_{d-1})\, F(s) = \sum_{j=0}^{d-2} A_j \sum_{k=0}^{\infty} \binom{s+k-1}{k} (d-j-1)^k \frac{F(s+k)}{d^{s+k}},$$

hence

$$
\begin{aligned}
&(I-d^{-s}(A_0+A_1+\cdots+A_{d-1}))\,F(s)\\
&\quad=\sum_{j=0}^{d-2} A_j \sum_{k=1}^{\infty} \binom{s+k-1}{k} (d-j-1)^k\,\frac{F(s+k)}{d^{s+k}}. \qquad (2)
\end{aligned}
$$

Let $\mathscr{A}=d^{-1}(\sum_{0\leqslant j\leqslant d-1} A_j)$, and let $\mathscr{M}(X)$ be the transpose of the comatrix of $(\mathscr{A}-XI)$, so that

$$\mathscr{M}(X)(\mathscr{A}-XI)=(\mathscr{A}-XI)\,\mathscr{M}(X)=\det(\mathscr{A}-XI)\,I.$$

Then, by multiplying (2) by $\mathscr{M}(d^{s-1})$, we get

$$
\begin{aligned}
&\det(\mathscr{A}-d^{s-1}I)\,F(s)\\
&\quad=-\mathscr{M}(d^{s-1})\sum_{j=0}^{d-2} A_j \sum_{k=1}^{\infty} \binom{s+k-1}{k} (d-j-1)^k\,\frac{F(s+k)}{d^{k+1}}. \qquad (3)
\end{aligned}
$$

Note that for any fixed $s\in\mathbb{C}$, the Dirichlet vector $F(s+k)$ is bounded for large k's. The righthand side of the above infinite functional equation (3) converges for $\mathscr{R}s>0$. Hence this gives a meromorphic continuation of $F(s)$ for $0<\mathscr{R}s\leqslant 1$ with poles (if any) at points $s\in\mathbb{C}$ such that d^{s-1} is an eigenvalue of the matrix $\mathscr{A}$. Now, if $-1<\mathscr{R}s\leqslant 0$, the righthand side converges, with the possible exception of those s for which d^s is an eigenvalue of the matrix $\mathscr{A}$. This gives a meromorphic continuation of F for $\mathscr{R}s>-1$, with poles (if any) for s such that either d^{s-1} or d^s is an eigenvalue of the matrix $\mathscr{A}$. Iterating this process shows that F has a meromorphic continuation to the whole complex plane with poles (if any) located at the points

$$s=\frac{\log\lambda}{\log d}+\frac{2ik\pi}{\log d}-\ell+1,$$

where λ is any eigenvalue of the matrix $\mathscr{A}$, $k\in\mathbb{Z}$, $\ell\in\mathbb{N}$, and log is a branch of the complex logarithm.

Remark 9. The previous proof also gives an "explicit" meromorphic continuation of F to the half-plane $\mathscr{R}s>0$. Namely, the first equality in (1) above reads

$$(I-d^{-s}A_{d-1})\,F(s)=\sum_{j=0}^{d-2} A_j \sum_{n=0}^{\infty} \frac{U_n}{(dn+j+1)^s}$$

hence

$$(I - d^{-s}(A_0 + A_1 + \cdots + A_{d-1}))\, F(s) = \sum_{j=0}^{d-2} A_j \sum_{n=0}^{\infty} U_n \left(\frac{1}{(dn+j+1)^s} - \frac{1}{(dn+d)^s} \right) \tag{4}$$

whose righthand side clearly converges for $\mathcal{R}s > 0$.

Remark 4. The Dirichlet series of *d*-regular sequences (see [7]) can be studied along similar lines: they converge for $\mathcal{R}s$ large enough (since the growth of *d*-regular sequences is at most polynomial [7]), and they can be analytically continued to meromorphic functions, by means of an infinite functional equation. For another interesting similar class of Dirichlet series, namely Dirichlet series of (completely) *d*-multiplicative sequences, see [12].

4. DENSITIES OF AUTOMATIC SEQUENCES

In his 1972 paper [10], Cobham proves the following two results.

THEOREM 4 (Cobham). *Let $d \geqslant 2$ be an integer. Let $(u_n)_{n \geqslant 0}$ be a d-automatic sequence with values in the set Σ. Then*

- *The logarithmic frequency of a in $(u_n)_{n \geqslant 0}$ exists for every $a \in \Sigma$;*
- *The frequency of a in $(u_n)_{n \geqslant 0}$ may not exist, but if it exists it must be a rational number.*

We give here another proof of the first part of this theorem. We first state two lemmas. The first one is classical, see for example [14, p. 40].

LEMMA 1. *Let B be a $t \times t$ matrix over a commutative field. Let $p_B(X) = \det(B - XI)$ be its characteristic polynomial and let $\pi_B(X)$ be its monic minimal polynomial. Let $\Delta(X)$ be the monic* gcd *of the entries of* (*the transpose of*) *the comatrix of the matrix $(B - XI)$. Then*

$$p_B(X) = (-1)^t\, \pi_B(X)\, \Delta(X).$$

The second lemma deals with *stochastic* matrices, i.e., matrices whose entries are nonnegative real numbers and for which every row sums up to 1. The first part of the lemma can be found for example in [13, Theorem 1.3, p. 45], the second part is an easy consequence of the first part.

LEMMA 2. *Let B be a $t \times t$ stochastic matrix. Then the elementary divisors of the matrix $(B - XI)$ of the form $(X-1)^k$ with $k \geqslant 1$ are all of degree 1, i.e., of the form $(X-1)$. In particular 1 is a simple root of the* minimal *polynomial of B.*

Remark 5. Note that if the stochastic matrix B is *irreducible*, i.e., if for each pair of indices (i, j), there exists a k such that the (i, j)-entry of the matrix B^k is positive (this is in particular the case if the matrix B is *primitive*, i.e., if there exists an integer ℓ such that all entries of B^ℓ are positive), then 1 is a simple root of the *characteristic* polynomial of B, see for example [13, Theorem 4.3, p. 14].

We now prove a theorem on the behaviour for $s \to 1_+$ of automatic Dirichlet series.

THEOREM 5. *Let $d \geqslant 2$ be an integer. Let $(u_n)_{n \geqslant 0}$ be a d-automatic sequence with values in $\mathbb{C}$. Let t be the integer, let $(U_n)_{n \geqslant 0}$ be the t-dimensional vector sequence, and let $A_0, A_1, ..., A_{d-1}$ be the $t \times t$ matrices, defined for the sequence $(u_n)_{n \geqslant 0}$ in Theorem* 1(iii). *Let $\mathscr{A} = d^{-1}(A_0 + A_1 + \cdots + A_{d-1})$. Let $\mathscr{M}$ be the transpose of the comatrix of the matrix $\mathscr{A} - XI$, and let $\Delta(X)$ be the monic* gcd *of the entries of $\mathscr{M}$. Let $F(s) = \sum_{n=0}^{\infty} \frac{U_n}{(n+1)^s}$. Then $\lim_{s \to 1_+} (s-1) F(s)$ exists and*

$$\lim_{s \to 1_+} (s-1) F(s) = \frac{(-1)^{t+1}}{\pi'_{\mathscr{A}}(1) \log d} \left(\frac{\mathscr{M}}{\Delta}(1) \right) \times \sum_{j=0}^{d-2} A_j \sum_{n=0}^{\infty} \frac{(d-j-1)}{(dn+j+1)(dn+d)} U_n.$$

Proof. Multiplying the last equality (actually valid for $\mathscr{R}s > 0$) of Remark 3 by $\frac{\mathscr{M}(d^{s-1})}{\Delta(d^{s-1})}$, and using Lemma 1, we obtain

$$(-1)^t d^{1-s} \pi_{\mathscr{A}}(d^{s-1}) F(s) = -\frac{\mathscr{M}(d^{s-1})}{\Delta(d^{s-1})} \times \sum_{j=0}^{d-2} A_j \sum_{n=0}^{\infty} U_n \left(\frac{1}{(dn+j+1)^s} - \frac{1}{(dn+d)^s} \right).$$

Hence

$$(s-1) F(s) = d^{s-1} (-1)^{t+1} \frac{s-1}{\pi_{\mathscr{A}}(d^{s-1})} \frac{\mathscr{M}(d^{s-1})}{\Delta(d^{s-1})} \times \sum_{j=0}^{d-2} A_j \sum_{n=0}^{\infty} U_n \left(\frac{1}{(dn+j+1)^s} - \frac{1}{(dn+d)^s} \right).$$

Since $\frac{\mathscr{M}(X)}{\Delta(X)}$ is a matrix with polynomial coefficients, the limit when s goes to 1 of $\frac{\mathscr{M}(d^{s-1})}{\Delta(d^{s-1})}$ exists, and is equal to $\frac{\mathscr{M}}{\Delta}(1)$. The limit when s goes to 1 of $\frac{s-1}{\pi_{\mathscr{A}}(d^{s-1})}$ is equal to $\frac{1}{\pi'_{\mathscr{A}}(1) \log d}$ since 1 is a simple root of $\pi_{\mathscr{A}}(X)$ (Lemma 2). Thus the limit when s goes to 1 of $(s-1) F(s)$ exists and

$$\lim_{s \to 1} (s-1) F(s) = \frac{(-1)^{t+1}}{\pi'_{\mathscr{A}}(1) \log d} \left(\frac{\mathscr{M}}{\Delta} (1) \right) \times \sum_{j=0}^{d-2} A_j \sum_{n=0}^{\infty} \frac{(d-j-1)}{(dn+j+1)(dn+d)} U_n.$$

We are now ready to state our theorem on frequencies.

THEOREM 6. *Let $d \geqslant 2$ be an integer. Let $(u_n)_{n \geqslant 0}$ be a d-automatic sequence with values in a set Σ. Let $\alpha \in \Sigma$. Define the map θ on Σ by $\theta(\alpha) = 1$, and $\theta(\beta) = 0$ for all $\beta \neq \alpha$. Let t be the integer, let $(U_n)_{n \geqslant 0}$ be the t-dimensional vector sequence, and let $A_0, A_1, ..., A_{d-1}$ be the $t \times t$ matrices, defined for the sequence $(u_n)_{n \geqslant 0}$ in Theorem* 1(iii). *Let $\mathscr{A} = d^{-1}(A_0 + A_1 + \cdots + A_{d-1})$. Let $\mathscr{M}$ be the transpose of the comatrix of the matrix $\mathscr{A} - XI$, and let $\Delta(X)$ be the gcd of the entries of $\mathscr{M}$. Let $(\theta(U_n))_n$ be the t-dimensional vector sequence whose components are the images by θ of the components of $(U_n)_n$. Then the logarithmic frequency of α in the sequence $(u_n)_{n \geqslant 0}$ exists, and it is equal to the first component of*

$$\frac{(-1)^{t+1}}{\pi'_{\mathscr{A}}(1) \log d} \left(\frac{\mathscr{M}}{\Delta} (1) \right) \sum_{j=0}^{d-2} A_j \sum_{n=0}^{\infty} \frac{(d-j-1)}{(dn+j+1)(dn+d)} \theta(U_n).$$

Proof. The sequence $(\theta(u_n))_n$ is d-automatic. Let t, $(U_n)_n$, and A_0, $A_1, ..., A_{d-1}$ be defined as above. It is easily checked that for all $j \in \{0, 1, ..., d-1\}$, we have

$$\theta(U_{dn+j}) = A_j \theta(U_n)$$

(remember that each row of A_j contains exactly one 1, and that all other entries in this row are equal to 0).

Using Theorem 6 above, we easily conclude by noting that the logarithmic frequency of α in the sequence $(u_n)_{n \geqslant 0}$ is equal to the logarithmic frequency of α in the sequence $(u_n)_{n \geqslant 1}$, and by applying Theorem 2.

5. INFINITE PRODUCTS

The reader can find in [3] applications of Dirichlet series associated with the Thue–Morse sequence to computing generalizations of the classical product

$$\prod_{n \geqslant 0} \left(\frac{2n+1}{2n+2} \right)^{\varepsilon_n} = \frac{\sqrt{2}}{2} \tag{5}$$

where $(\varepsilon_n)_{n\geqslant 0}$ is the Thue–Morse sequence on the alphabet $\{\pm 1\}$ (i.e., $\varepsilon_n = (-1)^{s_2(n)}$ where $s_2(n)$ is the sum of the binary digits of n). The proof of such formulas relies on functional equations linking the sums of the Dirichlet series associated with the Thue–Morse sequence.

We are going to give another derivation of these functional equations which applies to more general situations. But before doing so, we do not resist the temptation to give the simple proof of Equation (5) due to the first author (1987, unpublished; this was only written without reference in [8]).

5.1. *A simple approach*

Let P and Q be the infinite products defined by

$$P = \prod_{n=0}^{\infty} \left(\frac{2n+1}{2n+2}\right)^{\varepsilon_n}, \qquad Q = \prod_{n=1}^{\infty} \left(\frac{2n}{2n+1}\right)^{\varepsilon_n}.$$

Then

$$PQ = \frac{1}{2} \prod_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{\varepsilon_n} = \frac{1}{2} \prod_{n=1}^{\infty} \left(\frac{2n}{2n+1}\right)^{\varepsilon_{2n}} \prod_{n=0}^{\infty} \left(\frac{2n+1}{2n+2}\right)^{\varepsilon_{2n+1}}.$$

Of course all products are convergent, due to Abel's theorem and to the fact that the summatory function of the Thue–Morse sequence with values ± 1 is bounded. Now, since $\varepsilon_{2n} = \varepsilon_n$, and $\varepsilon_{2n+1} = -\varepsilon_n$, we get

$$PQ = \frac{1}{2} \prod_{n=1}^{\infty} \left(\frac{2n}{2n+1}\right)^{\varepsilon_n} \left(\prod_{n=0}^{\infty} \left(\frac{2n+1}{2n+2}\right)^{\varepsilon_n}\right)^{-1} = \frac{1}{2} QP^{-1}.$$

Since $Q \neq 0$, this gives $P^2 = 1/2$, hence the result (P is positive).

Remark 6. Note that other products could be studied using the same trick. We give two examples.

- Let f be a map from $\mathbb{N}$ to $\mathbb{R}^+$ such that there exists a real number a such that $f(2n) = af(n)$ for each n. Let

$$P' = \prod_{n=0}^{\infty} \left(\frac{f(2n+1)}{f(2n+2)}\right)^{\varepsilon_n}, \quad Q' = \prod_{n=1}^{\infty} \left(\frac{f(2n)}{f(2n+1)}\right)^{\varepsilon_n}.$$

Then, provided the infinite products converge, we have

$$P' = \frac{1}{\sqrt{af(1)}}.$$

Note that the condition $f(2n) = af(n)$ is satisfied by completely multiplicative functions (i.e., functions such that $f(mn) = f(m)\,f(n)$ for all m, n). But if a completely multiplicative function is non-decreasing and unbounded, it must be of the form n^α. (Even more is known, see [9].) Note that, if $f(n) = n^\alpha$, then the infinite products P' and Q' are respectively equal to P^α and Q^α.

- Define three maps u, v, w from $\mathbb{Z}/3\mathbb{Z}$ to $\mathbb{Z}$ by

n	0	1	2
$u(n)$	1	1	-2
$v(n)$	1	-2	1
$w(n)$	-2	1	1

Let $s_3(n)$ be the sum of digits of the base-3 expansion of the integer n. Then

$$\prod_{n=0}^{\infty} (3n+1)^{u(s_3(n))}\,(3n+2)^{v(s_3(n))}\,(3n+3)^{w(s_3(n))} = \frac{1}{3}.$$

This is proved by defining $\tau(n) = 3\{\frac{s_3(n)}{3}\} - 1$ (where $\{x\}$ is the fractional part of the real number x). Let

$$P'' = \prod_{n=1}^{\infty} \left(\frac{3n}{3n+1}\right)^{\tau(n)}, \qquad Q'' = \prod_{n=0}^{\infty} \left(\frac{3n+1}{3n+2}\right)^{\tau(n)}, \qquad R'' = \prod_{n=0}^{\infty} \left(\frac{3n+2}{3n+3}\right)^{\tau(n)}.$$

Then

$$P''Q''R'' = \left(\frac{1}{3}\right)^{\tau(0)} \prod_{n=1}^{\infty} \left(\frac{3n}{3n+3}\right)^{\tau(n)} = 3\prod_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{\tau(n)}.$$

Decomposing this last product according to n mod 3, and simplifying by P'' gives the result.

5.2. *An approach through integrals*

Now, let us consider a polynomial $P(X) = \sum_{j=0}^{k} a_j X^j$ such that $P(0) = 1$ and $P(1) = 0$. We are also given an integer $d \geqslant 2$.

Let us consider the function H defined in the unit disk by $H(z) = \prod_{\ell \geqslant 0} P(z^{d^\ell})$. Then H satisfies the functional equation $H(z) = P(z)\,H(z^d)$. This function as well as all its derivatives vanish at 1. It has an expansion in power series:

$$H(z) = \sum_{j \geqslant 0} \gamma_j z^j.$$

Such a sequence (γ_j) need not be automatic. Nevertheless this is the case when $d > k$ and when the coefficients a_j generate a finite multiplicative sub-monoid of $\mathbb{C}^*$. This is also the case in some other situations described below. The Thue-Morse sequence can be defined in this way (with $d=2$ and $P(X)=1-X$).

If $\mathcal{R}s>1$ and $\mathcal{R}z>0$, define $c(s,z)=\sum_{n\geqslant 0}\gamma_n(n+z)^{-s}$ and $G(s)=\sum_{n\geqslant 1}\gamma_n n^{-s}$.

The formula $\Gamma(s)\,w^{-s}=\int_0^{+\infty} t^{s-1}\mathrm{e}^{-tw}\,dt$, valid for $\mathcal{R}s>0$, implies the following equality

$$\Gamma(s)\,c(s,z)=\int_0^{+\infty} t^{s-1}\mathrm{e}^{-tz}H(\mathrm{e}^{-t})\,dt \tag{6}$$

which gives, for all z, an analytic continuation of $s\mapsto c(s,z)$ to an entire function.

The change of variable $t\to t/d$ in Formula (6) leads to

$$\begin{aligned}\Gamma(s)\,c(s,z)&=d^{-s}\int_0^{+\infty} t^{s-1}\mathrm{e}^{-tz/d}P(\mathrm{e}^{-t/d})\,H(\mathrm{e}^{-t})\,dt\\&=\Gamma(s)\,d^{-s}\sum_{j=0}^{k}a_jc\left(s,\frac{z+j}{d}\right).\end{aligned} \tag{7}$$

In the same way we get

$$\begin{aligned}\Gamma(s)\,G(s)&=\int_0^{+\infty} t^{s-1}(H(\mathrm{e}^{-t})-1)\,dt\\&=\kappa^s\int_0^{+\infty} t^{s-1}(H(\mathrm{e}^{-\kappa t})-1)\,dt\end{aligned}$$

and

$$(1-\kappa^{-s})\,\Gamma(s)\,G(s)=\int_0^{+\infty} t^{s-1}(H(\mathrm{e}^{-t})-H(\mathrm{e}^{-\kappa t}))\,dt, \tag{8}$$

where κ is any positive number. As a consequence, the function G itself has an analytic continuation to an entire function.

If we take $\kappa=d$ in the last formula, then

$$\begin{aligned}(1-d^{-s})\,\Gamma(s)\,G(s)&=\int_0^{+\infty} t^{s-1}(P(\mathrm{e}^{-t})-1)\,H(\mathrm{e}^{-td})\,dt\\&=d^{-s}\sum_{j=1}^{k}a_j\int_0^{+\infty} t^{s-1}\mathrm{e}^{-jt/d}H(\mathrm{e}^{-t})\,dt\\&=d^{-s}\Gamma(s)\sum_{j=1}^{k}a_jc(s,j/d).\end{aligned} \tag{9}$$

The negative integers are simple zeros of the functions G and $s \mapsto c(s, z)$. Moreover, we have $c(0, z) = 0$ and $G(0) = -1$. As a consequence of Formula (9) we have the relation

$$\sum_{j=1}^{k} a_j \frac{\partial c}{\partial s}(0, j/d) = -\log d.$$

This formula formally reads

$$\prod_{j=1}^{k} \prod_{n \geqslant 0} \left(\frac{dn+j}{d}\right)^{a_j \gamma_n} = d$$

Actually when $d > k$ this product converges provided its terms be suitably grouped (look at the summatory function of γ_n, remembering that $P(1) = 0$). In this case it can also be (formally) written

$$\prod_{j=1}^{k} \prod_{n \geqslant 0} (dn+j)^{a_j \gamma_n} = d.$$

When the polynomial P has no root of modulus less than 1, we consider the expansion $\frac{1}{P(X)} = \sum_{j \geqslant 0} \beta_j X^j$. Then Formula (6) reads

$$\begin{aligned}
\Gamma(s)\, c(s, z) &= d^s \int_0^{+\infty} t^{s-1} \mathrm{e}^{-dtz} H(\mathrm{e}^{-dt})\, dt \\
&= d^s \sum_{j \geqslant 0} \beta_j \int_0^{+\infty} t^{s-1} \mathrm{e}^{-t(dz+j)} H(\mathrm{e}^{-t})\, dt \\
&= \Gamma(s)\, d^s \sum_{j \geqslant 0} \beta_j c(s, dz+j).
\end{aligned}$$

5.3. *Examples*

We give some examples of the integral approach studied above.

Let α be a $(k+1)$-st root of 1 (other than 1), and let $P = \sum_{j=0}^{k} \alpha^j X^j$. Since we have $P(X) - 1 = \alpha X P(X) - X^{k+1}$, the first equality in Formula (9) yields the identity

$$(1 - d^{-s})\, G(s) = \alpha\, c(s, 1) - d^{-s} c\left(s, \frac{k+1}{d}\right),$$

from which we obtain

$$\sum_{n \geqslant 0} \gamma_n \left(\alpha \log(n+1) - \log\left(n + \frac{k+1}{d}\right)\right) = \log d. \tag{10}$$

(This is a formal relation, but again its terms can be grouped to ensure convergence when $d>k$. This will be also the case in the examples that follow.)

Let us assume now that $d>k$. The sequence (γ_j) is then d-automatic. It is easy to see that γ_n equals 0 if the base d expansion of n contains at least one digit larger than k. Otherwise, we have $\gamma_n=\alpha^{s_d(n)}$, where $s_d(n)$ stands for the sum of digits in the base d expansion of n. It will prove convenient to define $\sigma_d(n)$ to be $s_d(n)$ modulo d and to consider, if $2\leqslant b\leqslant d$, the map $\varphi_{b,d}$ which transforms $\sum_{j\geqslant 0} n_j\, b^j$ (where $0\leqslant n_j<b$) into $\sum_{j\geqslant 0} n_j\, d^j$.

Now, we can state several instances of Formula (10). If $d=2$, hence $k=1$, we get Formula (5). More generally, if $d>k=1$, we obtain

$$\prod_{n\geqslant 0}\left[(\varphi_{2,d}(n)+1)\left(\varphi_{2,d}(n)+\frac{2}{d}\right)\right]^{(-1)^{s_2(n)}}=\frac{1}{d}. \tag{11}$$

Let us take $k=2$, $d=3$. Then $\gamma_n=e^{2i\pi s_3(n)/3}$. By taking real and imaginary parts in (10), we get

$$\prod_{n\geqslant 0}(n+1)^{\sigma_3(n)-1}=3^{2/3} \qquad \text{and} \qquad \prod_{n\geqslant 0}(n+1)^{v(\sigma_3(n))}=3^{2/\sqrt{3}}$$

(where v has been defined in Remark 6) as well as a formula analogous to (11). In the same way, we obtain for $k=3$, $d=4$, $\alpha=i$,

$$\prod_{n\geqslant 0}(n+1)^{[\sigma_4(n)/2]-1/2}=2 \qquad \text{and} \qquad \prod_{n\geqslant 0}(n+1)^{|[(\sigma_4(n)-1)/2]|-1/2}=2.$$

We end our paper by showing that the condition $d>k$ is not necessary to get automaticity.

First take $k=d=2$ and $\alpha=e^{2i\pi/3}$. Then, the coefficients γ_n are determined by the following rules: $\gamma_0=1$, $\gamma_{2n+1}=\alpha\,\gamma_n$ for $n\geqslant 0$, and $\gamma_{2n}=\gamma_n+\alpha^2\gamma_{n-1}$. If v_n stands for the vector $\binom{\gamma_{n-1}}{\gamma_n}$, these recursion relations can be written $v_{2n}=\left(\begin{smallmatrix}\alpha & 0\\ \alpha^2 & 1\end{smallmatrix}\right) v_n$ and $v_{2n+1}=\left(\begin{smallmatrix}\alpha^2 & 1\\ 0 & \alpha\end{smallmatrix}\right) v_n$. If we denote by a and b the matrices $\left(\begin{smallmatrix}\alpha & 0\\ \alpha^2 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\alpha^2 & 1\\ 0 & \alpha\end{smallmatrix}\right)$, we have $a^3=b^3=1$ and $(ab)^2=(ba)^2=-\alpha$. These relations imply that a and b generate a finite group (of order 72). It easily follows that the sequence (γ_n) is 2-automatic.

Now take $k=d=3$ and $P(X)=1+iX-X^2-iX^3$. Let v_n stand for the vector whose components are γ_{n-1}, γ_n, and γ_{n+1}. Then we obtain $v_{3n}=a\,v_n$, $v_{3n+1}=b\,v_n$, and $v_{3n+2}=c\,v_n$, where

$$a=\begin{pmatrix}-1 & 0 & 0\\ -i & 1 & 0\\ 0 & i & 0\end{pmatrix}, \quad b=\begin{pmatrix}-i & 1 & 0\\ 0 & i & 0\\ 0 & -1 & 0\end{pmatrix}, \quad \text{and} \quad c=\begin{pmatrix}0 & i & 0\\ 0 & -1 & 0\\ 0 & -i & 1\end{pmatrix}.$$

But, it can be checked that a, b, and c generate a finite monoid (of order 110). It follows that the sequence (γ_n) is automatic.

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