Journal of Number Theory **94**, 136–176 (2002)
doi:10.1006/jnth.2001.2733, available online at http://www.idealibrary.com on IDEAL®

Measures of Simultaneous Approximation for Quasi-Periods of Abelian Varieties

Pierre Grinspan

Institut de Mathématiques, Théorie des Nombres, Université de Paris VI, Paris, France
E-mail: grinspan@math.jussieu.fr

Communicated by J.-B. Bost

Received November 20, 2000

We examine various extensions of a series of theorems proved by Chudnovsky in the 1980s on the algebraic independence (transcendence degree 2) of certain quantities involving integrals of the first and second kind on elliptic curves; these extensions include generalizations to abelian varieties of arbitrary dimensions, quantitative refinements in terms of measures of simultaneous approximation, as well as some attempt at unifying the aforementioned theorems. In the process we develop tools that might prove useful in other contexts, revolving around explicit "algebraic" theta functions on the one hand, and Eisenstein's theorem and G-functions on the other hand.

Key Words: transcendence; simultaneous approximation; algebraic independence; Chudnovsky; periods; second kind; abelian varieties; Weierstrass functions; Eisenstein's theorem; G-functions.

1. INTRODUCTION

This text has its source in several, both old and recent, results:

• First and foremost, a series of theorems of Chudnovsky from the 1970s [Chu84, Chap. 7] which, as they are central here and somewhat scattered in Chudnovsky's book, we presently recall:

THEOREM 1.1 (Chudnovsky's Theorems). *Let $\Lambda = \mathbf{Z}\omega + \mathbf{Z}\omega' \subset \mathbf{C}$ be a lattice with invariants g_2, g_3, Weierstrass functions $\wp$, ζ and quasi-periods η, η', all defined in the usual way (see [Sil94]). Each of the sets below contains at least two algebraically independent numbers: first*

(1) $\{g_2, g_3, \frac{\eta}{\omega}, \frac{\pi}{\omega}\}$,

(2) $\{g_2, g_3, \omega, \eta, \omega', \eta'\}$;

0022-314X/02 $35.00

if we assume $g_2, g_3 \in \bar{\mathbf{Q}}$*:*

(3) $\{\frac{\eta}{\omega}, \frac{\pi}{\omega}\}$,

(4) $\{\omega, \omega', \eta, \eta'\}$;

and if, still assuming $g_2, g_3 \in \bar{\mathbf{Q}}$*, we consider two complex numbers* u *and* u'*,* **Q***-linearly independent and such that* $\wp(u), \wp(u') \in \bar{\mathbf{Q}}$*:*

(5) $\{\frac{\eta}{\omega}, \zeta(u) - \frac{\eta}{\omega} u\}$,

(6) $\{u, u', \zeta(u), \zeta(u')\}$.

It can be noticed that the above assertions are related, regardless of their truthfulness, by the following logical implications :

$$\begin{array}{ccccc} (1) & \Rightarrow & (3) & \Leftarrow & (5) \\ \Downarrow & & \Downarrow & & \\ (2) & \Rightarrow & (4) & \Leftarrow & (6) \end{array}$$

• Second, a result announced in Chudnovsky's book [Chu84, Theorem 9, p. 9] which extends assertion (4) above to abelian varieties of arbitrary dimension; a complete proof was recently given in [Vas96].

• Third, Theorem 4.1 of [RW97], where a measure of simultaneous approximation is established which has Theorem 1.1(4) as a corollary.

• Fourth, a "trick" introduced by Chudnovsky in [Chu82] to prove a sharp measure of algebraic independence refining assertion (3) above; recently rediscovered in [Phi99, Bru99], it consists in relating elliptic and quasi-elliptic (like Weierstrass's ζ) functions to G-functions, allowing better arithmetic estimates in the transcendence proof and, ultimately, an optimal dependence of measures in the parameter controlling the height.

One important feature of our results, coming from point Three above, is the following:

DEFINITION 1.1. A **(simultaneous) approximation measure** for $(\theta_1, \ldots, \theta_n) \in \mathbf{C}^n$ is a function $\phi: \mathbf{N} \times \mathbf{R}_+ \to \mathbf{R}_+$ such that for some constant $C > 0$, for any $(\alpha_1, \ldots, \alpha_n) \in \bar{\mathbf{Q}}^n$ with $[\mathbf{Q}(\alpha_1, \ldots, \alpha_n) : \mathbf{Q}] \leqslant D$, $h(\alpha_i) \leqslant h$ (absolute logarithmic height as defined in [Wal92]) and $D, h \geqslant C$ one has

$$\log \max_i |\theta_i - \alpha_i| \geqslant -C\phi(D, h).$$

Before stating our main result, we recall a few more definitions. Let A be an abelian variety of dimension g defined over a subfield K of $\mathbf{C}$. A rational (hence meromorphic) differential on A is said to be of the second kind if it has no residues [GH78, p. 454]; the quotient space of second-kind by exact differentials has dimension $2g$ and will be denoted by $H^1_{DR}(A, K)$, or $H^1_{DR}(A)$ when we implicitly take $K = \mathbf{C}$ (the notation H^1_{DR} is justified by [FW84, p. 192]). On the other hand, $H_1(A, \mathbf{Z})$ will denote the usual first homology group of $A(\mathbf{C})$.

THEOREM 1.2. *Let A be an abelian variety defined over a field $K \subset \mathbf{C}$, $(\omega_1, \dots, \omega_{2g})$ representing a basis of $H^1_{DR}(A, K)$ and $u_1, \dots, u_r \in T_0A(\mathbf{C})$ (tangent space at the origin), $\mathbf{Q}$-linearly independent and such that $\exp_A(u_j) \in A(K)$. We let $\rho = \frac{r}{g}$.*

(1) *If $K \subset \bar{\mathbf{Q}}$, the set of $\int_0^{u_j} \omega_i$ $(1 \leqslant i \leqslant 2g, 1 \leqslant j \leqslant r)$ admits the following approximation measure:*

$$\phi_1(D, h) = D^{\frac{3}{2}+\frac{1}{\rho}}(\log D)^{-1/2}\,(D^{2/\rho}+h).$$

(2) *If all the u_j are periods (elements of the period lattice $\Lambda = \ker \exp_A$), the set made up by all $\int_0^{u_j} \omega_i$ $(1 \leqslant i \leqslant 2g, 1 \leqslant j \leqslant r)$ together with a generating system of K over $\mathbf{Q}$ admits the following approximation measure:*

$$\phi_2(D, h) = [D(h+\log D)]^{3/\rho}.$$

Using a theorem of Laurent and Roy [LR99, Théorème 1.2] we can deduce from assertions (1) and (2) (resp.) of the preceding theorem extensions in arbitrary dimension of assertions (6) and (2) (resp.) of Theorem 1.1:

COROLLARY 1.1. (1) *If $K \subset \bar{\mathbf{Q}}$ and $r = 2g$, the $\int_0^{u_j} \omega_i$ generate a field of transcendence degree at least 2.*

(2) *If $r \geqslant g+1$ and the u_j are periods, the field generated over K by the $\int_0^{u_j} \omega_i$ has transcendence degree (over $\mathbf{Q}$) at least 2.*

The text is arranged as follows. Section 2 briefly reviews embeddings of extensions of abelian varieties by powers of the additive group [FW84], and ends with a zero estimate, corollary of [Phi96], tailored to this particular type of algebraic groups. Section 3 contains addition, multiplication and differentiation formulae for the functions involved in these embeddings, much in the spirit and continuation of [MW93, Sect. 3]; there we also construct, starting from classical theta functions, "sigma functions" which are nothing but the "algebraic" theta functions whose existence is proved in [Bar70]. The next section (Section 4) develops, in the context of the algebraic groups described above, Chudnovsky's "G-trick" mentioned earlier (point Four); it is based on a quite general and effective version of Eisenstein's classical theorem stating that every algebraic power series is a G-function (see [PS76, VIII.3.3 and VIII.4.4]). In Section 5, we review briefly the very special but classical case $g=1$ of the general results contained in the previous two sections. In Section 6 we state in full detail and carry out the proof of our main result; and in Section 7, we discuss various results closely related to our main theorem, state some with summary indications of proofs, and examine which technical difficulties (coming from the insufficiency of known Schwarz/interpolation lemmas) arise in proving more.

2. EMBEDDINGS AND ZERO LEMMA

Here we will describe the type of embeddings we will be using for extensions, by powers of the additive group $\mathbf{G}_a$, of principally polarized abelian varieties.

Let A be such an abelian variety, $\Lambda \subset \mathbf{C}$ a lattice such that $A(\mathbf{C}) \simeq \mathbf{C}/\Lambda$. For $i=1\ldots g$ write $\partial_i = \partial/\partial z_i$, and for any derivation ∂, $\partial \log f = \frac{\partial f}{f}$. The following, elementary but fundamental, lemma provides us with an explicit basis for the quotient $H^1_{DR}(A, \mathbf{C})$ of the space of first-order meromorphic differentials of the second kind on A (i.e., without residues) by that of exact differentials. We refer the reader to [Lan82] for both the definition of a nondegenerate theta function (all of those considered in this text will be so) and the reduction process used in the proof, as they both are classical and will not be used in the rest of the text.

LEMMA 2.1. *For any nondegenerate theta function θ for the lattice Λ, the differential forms $dz_1, \ldots, dz_g$ (coordinates in $\mathbf{C}^g$) and*

$$d\partial_1 \log \theta, \ldots, d\partial_g \log \theta$$

make up a basis of $H^1_{DR}(A, \mathbf{C})$.

Proof. We actually show the nondegeneracy of the (quasi-)period matrix

$$\left(\int_0^{\lambda_j} \omega_i\right)_{1 \leqslant i, j \leqslant 2g},$$

where $(\lambda_1, \ldots, \lambda_{2g})$ is a basis of Λ and $\omega_i = dz_i$, $\omega_{g+i} = d\partial_i \log \theta$ for $i=1\ldots g$; this obviously implies the lemma. For the proof, by the same process as in [Lan82, pp. 93–94] (and without affecting the rank of our matrix), we reduce θ to a theta function with a particularly simple automorphy factor, *viz.* (for some Frobenius basis $(e_1, \ldots, e_g, v_1, \ldots, v_g)$ of Λ)

$$\theta(z+e_i) = \theta(z),$$

$$\theta(z+v_i) = \theta(z) \exp(c_i z_i + d_i)$$

with every $c_i \neq 0$; the (quasi-)period matrix then appears to be triangular and trivially nondegenerate. ∎

Assume now that the complex torus $\mathbf{C}^g/\Lambda$ is embedded into a projective space $\mathbf{P}_N$ using a "theta embedding" $\Theta = (\theta_0 : \cdots : \theta_N)$. The above lemma then allows us to associate to each $\omega \in H^1_{DR}(A, \mathbf{C})$ a derivation ∂ and a linear form L, both uniquely determined, such that in $H^1_{DR}(A)$ the equality $\omega = d\partial \log \theta_0 + dL$ takes place. We will denote by $\tilde{\Theta}_\omega$ the following application, which defines an embedding of (the complex locus of) the extension of A by $\mathbf{G}_a$ associated with ω [FW84, III.2]:

$$\begin{aligned} \mathbf{C}^g \times \mathbf{C} &\to \mathbf{P}_{2N+1}(\mathbf{C}) \\ (z; t) &\mapsto (\theta_0(z) : \cdots : \theta_N(z) : \\ &\qquad (\partial + L(z) + t)\, \theta_0(z) : \cdots : (\partial + L(z) + t)\, \theta_N(z)). \end{aligned}$$

In the following we will often write, just as we did here, $(\partial + L(z) + t)\, \theta_i(z)$ for $\partial\theta_i(z) + (L(z) + t)\, \theta_i(z)$ and, once ∂ and L have been fixed, $\tilde{\theta}_i(z; t) = (\partial + L(z) + t)\, \theta_i(z)$.

For $\underline{\omega} = (\omega_1, \ldots, \omega_l)$ a family of differentials of the second kind, linearly independent in $H^1_{DR}(A)$, associated in the above way to derivations $\partial_1, \ldots, \partial_l$ and linear forms $L_1, \ldots, L_l$, we define the application

$$\begin{aligned} \tilde{\Theta}_{\underline{\omega}} : \mathbf{C}^g \times \mathbf{C}^l &\to \mathbf{P}_{(l+1)(N+1)-1}(\mathbf{C}) \\ (z; t) &\mapsto (\theta_0(z) : \cdots : \theta_N(z) : \\ &\qquad (\partial_1 + t_1 + L_1(z))\, \theta_0(z) : \cdots : (\partial_l + t_l + L_l(z))\, \theta_N(z)) \end{aligned}$$

obtained by "concatenation" of the $\tilde{\Theta}_{\omega_i}$ $(i = 1 \ldots l)$; it defines an embedding of the extension G of A by $\mathbf{G}_a^l$ associated to $\underline{\omega}$. Note that, in particular, the subgroup $\mathbf{G}_a^l$ is naturally defined within G by a system of equations $\theta_i(0)\, X_j = \theta_j(0)\, X_i$ $(0 \leqslant i < j \leqslant N)$.

We can now state the zero lemma we will be using, a corollary of Théorème 9 from [Phi96]:

Proposition 2.1 (Zero Lemma). *For any $g \in \mathbf{N}^*$, $l \in \mathbf{N}$ there exists $c > 0$ with the following property. Let G be an algebraic group of dimension $d = g + l$ defined over a subfield K of $\mathbf{C}$, extension of an abelian variety A of dimension g by $\mathbf{G}_a^l$,*

$$0 \to \mathbf{G}_a^l \xrightarrow{\iota} G \xrightarrow{\pi} A \to 0$$

and embedded in a projective space $\mathbf{P}_M$ in the fashion described above. If $\mathscr{E}$ is a subset of $G(K)$ containing 0; if a homogeneous polynomial $P \in K[X_0, \ldots, X_M]$, with degree bounded by L_2, vanishes on $d\mathscr{E} - d\mathscr{E}$ (where $d\mathscr{E}$

denotes the sum of d terms $\mathscr{E}+\cdots+\mathscr{E}$, and the difference $d\mathscr{E}-d\mathscr{E}$ has similar meaning) to the order T along some subspace $\mathscr{V}$ of the tangent space T_G to G at the origin, without however vanishing identically on G; if, finally, its restriction to $\mathbf{G}_a^l \subset G$ (see above) can be written as a polynomial of degree bounded by L_1; then for some proper algebraic subgroup G' of G the inequality

$$N'T^{d'} \deg G' \leqslant cL_1^{l'} L_2^{a'}$$

holds, where N' is the cardinality of $(\mathscr{E}+G')/G'$, d' the dimension of $(\mathscr{V}+T_{G'})/T_{G'}$, l' that of $L/(L\cap G')$, and a' that of $A/\pi(G')$.

3. THETA AND SIGMA FUNCTIONS

Here we will recall a few properties of classical theta functions and construct from these some analogues in higher dimension of Weierstrass's sigma function from the theory of elliptic functions.

Let g be a non-zero integer, and fix an element τ of Siegel's upper half-plane $\mathscr{H}_g$ formed by all square complex matrices of size g, symmetrical and with definite-positive imaginary part. The theta function with characteristic $m=(m', m'') \in (\mathbf{R}^g)^2$ associated to τ is defined by

$$\theta_m(\tau, u) = \sum_{n \in \mathbf{Z}^g} \exp[i\pi((n+m')\, \tau^t(n+m')+2(n+m')^t\, (u+m''))];$$

most of the time we will omit its dependence in τ and simply write $\theta_m(u)$. We will, however, say that some object (e.g., polynomial) which depends *a priori* on τ is "locally independent" of τ if it is constant on each element of some open covering of $\mathscr{H}_g$.

If now m is restricted to $(\frac{1}{2}\,\mathbf{Z}^g)^2$, the classical relations

$$\theta_m(-u) = \theta_{-m}(u),$$
$$\theta_{m+n}(u) = \exp(2i\pi m'^t n'')\, \theta_m(u)(n=(n', n'') \in (\mathbf{Z}^g)^2)$$

imply in particular that θ_m is either even or odd, depending on whether $2m'^t m''$ is an integer or not; they also suggest that we deal only with m in some fixed system $\mathscr{Z}_2$ of representatives of $(\frac{1}{2}\,\mathbf{Z}^{2g})/\mathbf{Z}^{2g}$.

One of the fundamental properties of theta functions is the existence of "Riemann relations" (see [MW93, relation (3.1) and Lemma 3.2]). It is easily checked that for any homogeneous quadratic polynomial $Q \in \mathbf{C}[u]$, those relations are still satisfied by the family $(\sigma_m)_{m \in \mathscr{Z}_2}$ defined by

$$\sigma_m = \theta_m e^Q;$$

thus any property of the family $(\theta_m)_{m\in\mathscr{Z}_2}$ which follows from them is still valid for the $(\sigma_m)_{m\in\mathscr{Z}_2}$. For the following, we will fix such a family and write $\Theta=(\sigma_m)_{m\in\mathscr{Z}_2}$. First we can deduce from the above-mentioned results [MW93]:

LEMMA 3.1 (Riemann Relations). (1) *For any* $m, n, p, q\in\mathbf{R}^{2g}$ *and* $z, w\in\mathbf{C}^g$ *the following relation holds,*

$$\begin{aligned}\sigma_m(z+w)\,\sigma_n(z-w)\,\sigma_p(0)\,\sigma_q(0)\\ =2^{-g}\sum_{\beta\in\mathscr{Z}_2} c_\beta\sigma_{a+\beta}(z)\,\sigma_{b+\beta}(z)\,\sigma_{c+\beta}(w)\,\sigma_{d+\beta}(w),\end{aligned}$$

where $c_\beta=\pm1$ *depends on* β *and* m, *while*

$$(a, b, c, d)=\frac{1}{2}(m, n, p, q)\begin{pmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{pmatrix}.$$

(2) *Moreover, for any* $m, n\in\mathscr{Z}_2$ *there exist* $p\equiv m+\alpha$, $q\equiv n+\alpha$ $(\alpha\in\mathscr{Z}_2)$ *in* $\mathscr{Z}_2$ *such that* a, b, c, d *defined by the formula above are half-integers* $(a, b, c, d\in\frac{1}{2}\mathbf{Z}^{2g})$ *and that* $\sigma_p(0)\,\sigma_q(0)\neq 0$.

Let $N+1=4^g$; this is the number of elements in $\mathscr{Z}_2$. We define T, X, X_1, X_2, Y_1, Y_2 to be families of $(N+1)$ variables, all independent; furthermore $D\in\mathbf{N}^*$ will be an integer depending only on g and whose precise value will be of little importance to us; finally, we will call homogeneous of degree 0 in some set of variables, say T, any quotient of two homogeneous polynomials of the same degree in T.

We define a basis of derivations $(\partial_1, \ldots, \partial_g)$ on $\mathbf{C}^g$ as follows. We assume the elements of $\mathscr{Z}_2$ (and accordingly, functions θ_m and σ_m) to be numbered so that $\theta_0(0)\neq 0$ (thus θ_0, σ_0 are even) and that the jacobian matrix

$$P=\frac{1}{\theta_0(0)}\left(\frac{\partial\theta_i}{\partial u_j}(0)\right)_{1\leqslant i, j\leqslant g}$$

of $\sigma_i/\sigma_0=\theta_i/\theta_0$ $(i=1\ldots g)$ at the origin is invertible (thus θ_i and σ_i, $i=1\ldots g$, are odd). We then let

$$\left(\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_g}\right)=(\partial_1, \ldots, \partial_g)\,P;$$

we also introduce coordinates $z=(z_1, \ldots, z_g)$ in $\mathbf{C}^g$ so that $(dz_1, \ldots, dz_g)$ is dual to $(\partial_1, \ldots, \partial_g)$, and by a slight abuse of notation we will from now on write the σ_m as functions of z. As a last piece of notation we define, for any functions f and g, $[f,g]_i = g^2\partial_i(f/g) = g\partial_i f - f\partial_i g$ (in a manner similar to [Dav89]).

Using what was said above about the σ_m, we deduce from Lemmas 3.1 and 3.7 of [MW93]:

Proposition 3.1 (Masser–Wüstholz). *There exist finite families*

$$(D_{mni})_{\substack{m,n\in\mathscr{Z}_2\\ i=1,\dots,g}}, \quad (F^{\mu})_{\mu\in M},$$

with $F^{\mu}\in\mathbf{Q}[T,X]$ *bi-homogeneous of degree* $(D,2)$ *and* $D_{mni}\in\mathbf{Q}(T)[X]$ *of degree* $(0,2)$, *locally independent of* τ *and satisfying:*

(1) *The polynomials obtained by specializing the* F^{μ} *at* $T=\Theta(0)$ *provide a system of equations for the image of* Θ; *the family of their differentials at* $X=\Theta(0)$ *has rank* $N-g+1$.

(2) *For any* $m,n\in\mathscr{Z}_2$ *and* $i=1\dots g$ *we have the following equality (between entire functions on* $\mathbf{C}^g$*):*

$$[\sigma_m,\sigma_n]_i = D_{mni}(\Theta(0),\Theta).$$

The purpose of this section will be to establish, using only the Riemann relations, similar results for the functions $\tilde{\sigma}_M$ appearing in the embedding $\tilde{\Theta}=\tilde{\Theta}_\omega$ defined in Section 2, associated to a derivation ∂ and a linear form L.

3.1. *Addition*

We start from the Riemann relation stated in Lemma 3.1; fixing $(z-w)$ and differentiating (with ∂) with respect to $(z+w)$, we get

$$\begin{aligned}
&2^{g+1}\partial\sigma_m(z+w)\,\sigma_n(z-w)\,\sigma_p(0)\,\sigma_q(0)\\
&\quad=\sum_{\beta\in\mathscr{Z}_2} c_\beta[\partial\sigma_{a+\beta}(z)\,\sigma_{b+\beta}(z)\,\sigma_{c+\beta}(w)\,\sigma_{d+\beta}(w)\\
&\qquad+\sigma_{a+\beta}(z)\,\partial\sigma_{b+\beta}(z)\,\sigma_{c+\beta}(w)\,\sigma_{d+\beta}(w)\\
&\qquad+\sigma_{a+\beta}(z)\,\sigma_{b+\beta}(z)\,\partial\sigma_{c+\beta}(w)\,\sigma_{d+\beta}(w)\\
&\qquad+\sigma_{a+\beta}(z)\,\sigma_{b+\beta}(z)\,\sigma_{c+\beta}(w)\,\partial\sigma_{d+\beta}(w)];
\end{aligned}$$

it follows easily that for $t, u \in \mathbf{C}$,

$$\begin{aligned} 2^{g+1}\tilde{\sigma}_m(z+w; t+u)\, \sigma_n(z-w)\, \sigma_p(0)\, \sigma_q(0) &= \sum_{\beta \in \mathscr{Z}_2} c_\beta[\sigma_{b+\beta}(z)\, \tilde{\sigma}_{a+\beta}(z; t)(\sigma_{c+\beta}\sigma_{d+\beta})(w) \\ &\quad + \sigma_{a+\beta}(z)\, \tilde{\sigma}_{b+\beta}(z; t)(\sigma_{c+\beta}\sigma_{d+\beta})(w) \\ &\quad + (\sigma_{a+\beta}\sigma_{b+\beta})(z)\, \sigma_{d+\beta}(w)\, \tilde{\sigma}_{c+\beta}(w; u) \\ &\quad + (\sigma_{a+\beta}\sigma_{b+\beta})(z)\, \sigma_{c+\beta}(w)\, \tilde{\sigma}_{d+\beta}(w; u)]. \end{aligned}$$

Similarly to the proof of Lemma 3.3 in [MW93], the latter equality together with Lemma 3.1 yields

PROPOSITION 3.2 (Addition Formulae). *There exist finite families*

$$(A_m^\xi)_{m \in \mathscr{Z}_2,\, \xi \in \Xi}, \qquad (B_m^\xi)_{m \in \mathscr{Z}_2,\, \xi \in \Xi}$$

of elements of $\mathbf{Q}[T, X_1, X_2, Y_1, Y_2]$*, locally independent of* τ*, with the following properties:*

- *Each polynomial* A_m^ξ *is multi-homogeneous in* (T, X_1, X_2, Y_1, Y_2) *with degree* $(D, 2, 0, 2, 0)$*; each polynomial* B_m^ξ *is homogeneous in* T *with degree* D*, in* (X_1, X_2) *with degree* 2*, in* (Y_1, Y_2) *with degree* 2*, and in* (X_2, Y_2) *with degree* 1.
- *For any* $z, w \in \mathbf{C}^g$*,* $t, u \in \mathbf{C}$ *and* $\xi \in \Xi$*, the family* $((A_m^\xi)_{m \in \mathscr{Z}_2}, (B_m^\xi)_{m \in \mathscr{Z}_2})$ *at* $T = \Theta(0)$*,* $X_1 = (\sigma_i(z))_{0 \leqslant i \leqslant N}$*,* $X_2 = (\tilde{\sigma}_i(z; t))_{0 \leqslant i \leqslant N}$*,* $Y_1 = (\sigma_i(w))_{0 \leqslant i \leqslant N}$*,* $Y_2 = (\tilde{\sigma}_i(w; u))_{0 \leqslant i \leqslant N}$*, provides a system of projective coordinates for the point* $\tilde{\Theta}(z+w; t+u)$*, unless identically zero.*
- *For any* $z_0, w_0 \in \mathbf{C}^g$ *there is* $\xi \in \Xi$ *such that for all* $t, u \in \mathbf{C}$*,* $z \in \mathbf{C}^g$ *near* z_0 *and* $w \in \mathbf{C}^g$ *near* w_0*, the family above does not vanish identically.*

This proposition does not, however, stress the particularly simple form taken by the addition law whenever w belongs to the period lattice $\Lambda_\tau = 2\mathbf{Z}^g + \mathbf{Z}^g\tau$; the latter form can be found by a different method. First, differentiating with respect to w then evaluating at $w = \frac{\omega}{2}$ $(\omega \in \Lambda_\tau)$ the Riemann relation with $n = m$, $q = p$, we obtain

$$\begin{aligned} &\left[\partial\sigma_m\left(z+\frac{\omega}{2}\right)\sigma_m\left(z-\frac{\omega}{2}\right) - \sigma_m\left(z+\frac{\omega}{2}\right)\partial\sigma_m\left(z-\frac{\omega}{2}\right)\right]\sigma_p^2(0) \\ &\quad = 2^{1-g} \sum_{\beta \in \mathscr{Z}_2} c_\beta \sigma_{m+p+\beta}(z)\, \sigma_{m-p+\beta}(z)(\sigma_\beta \partial\sigma_\beta)\left(\frac{\omega}{2}\right). \end{aligned}$$

Subtracting $2(\partial\sigma_n/\sigma_n)(\frac{\omega}{2})$ times the original Riemann relation (still with $n=m$, $p=q$, $w=\frac{\omega}{2}$, and with $n\in\mathscr{Z}_2$ chosen so that $\sigma_n(\frac{\omega}{2})\neq 0$), we deduce

$$\begin{aligned}\partial\sigma_m\left(z+\frac{\omega}{2}\right)\sigma_m\left(z-\frac{\omega}{2}\right)-\sigma_m\left(z+\frac{\omega}{2}\right)\partial\sigma_m\left(z-\frac{\omega}{2}\right)\\ -2\,\frac{\partial\sigma_n}{\sigma_n}\left(\frac{\omega}{2}\right)\sigma_m\left(z+\frac{\omega}{2}\right)\sigma_m\left(z-\frac{\omega}{2}\right)\\ =\frac{2^{1-g}}{\sigma_p^2(0)}\sum_{\beta\in\mathscr{Z}_2}c_\beta\sigma_{m+p+\beta}(z)\,\sigma_{m-p+\beta}(z)\left(\frac{\sigma_\beta}{\sigma_n}\,\partial\,\frac{\sigma_\beta}{\sigma_n}\right)\left(\frac{\omega}{2}\right)\sigma_n^2\left(\frac{\omega}{2}\right).\end{aligned}$$

Now we notice that for any $\beta\in\mathscr{Z}_2$ the function $f=\sigma_\beta/\sigma_n$, being on the one hand periodic, on the other hand either even or odd, satisfies

$$f\partial f\left(-\frac{\omega}{2}\right)=f\partial f\left(\frac{\omega}{2}\right)=-f\partial f\left(-\frac{\omega}{2}\right);$$

this means that the preceding expression is in fact zero. Finally we let $x=z-\frac{\omega}{2}$ to get, for any $x\in\mathbf{C}^g$, $m\in\mathscr{Z}_2$ with $\sigma_m(x)\neq 0$ and $n\in\mathscr{Z}_2$ with $\sigma_n(\frac{\omega}{2})\neq 0$:

$$\frac{\partial\sigma_m}{\sigma_m}(x+\omega)-\frac{\partial\sigma_m}{\sigma_m}(x)=2\,\frac{\partial\sigma_n}{\sigma_n}\left(\frac{\omega}{2}\right).$$

Taking $x=0$ we find in particular, for any $p\in\mathscr{Z}_2$ such that $\sigma_p(\omega)\neq 0$ (so σ_p is even),

$$\frac{\partial\sigma_p}{\sigma_p}(\omega)=2\,\frac{\partial\sigma_n}{\sigma_n}\left(\frac{\omega}{2}\right).$$

We can now write

$$\partial\sigma_m(x+\omega)=\frac{\sigma_m(x+\omega)}{\sigma_m(x)}\left(\partial\sigma_m(x)+\sigma_m(x)\,\frac{\partial\sigma_p}{\sigma_p}(\omega)\right);$$

it follows that the vector

$$((\sigma_p(\omega)\,\sigma_m(x))_{m\in\mathscr{Z}_2},(\sigma_p(\omega)\,\partial\sigma_m(x)+\sigma_m(x)\,\partial\sigma_p(\omega))_{m\in\mathscr{Z}_2})$$

is equal to the quotient of

$$((\sigma_m(x+\omega))_{m\in\mathscr{Z}_2},(\partial\sigma_m(x+\omega))_{m\in\mathscr{Z}_2})$$

by the quantity

$$\frac{\sigma_m(x+\omega)}{\sigma_p(\omega)\,\sigma_m(x)},$$

independent of $m \in \mathscr{Z}_2$ (as long as $\sigma_m(x) \neq 0$) since $\omega \in \Lambda_\tau$; finally, adding the t variable we readily conclude:

Proposition 3.3. *For any $x \in \mathbf{C}^g$, $\omega \in \Lambda_\tau$, and $t, u \in \mathbf{C}$, the family*

$$((\sigma_p(\omega)\,\sigma_m(x))_{m \in \mathscr{Z}_2}, (\sigma_p(\omega)\,\tilde{\sigma}_m(x;t)+\sigma_m(x)\,\tilde{\sigma}_p(\omega;u))_{m \in \mathscr{Z}_2}),$$

with $p \in \mathscr{Z}_2$ such that $\sigma_p(0) \neq 0$, provides a system of projective coordinates for $\tilde{\Theta}(x+\omega, t+u)$.

3.2. *Multiplication*

In a way similar to [Rém00, Proposition 5.2], we now deduce from the addition law a multiplication law which, although not optimal (the right degree, according to Serre's appendix to [Wal87], is only n^2), will be enough for our purpose. Note that since all our functions are either even or odd, we really do not need to consider subtractions or multiplications by $a<0$.

Proposition 3.4 (Multiplication Formulae). *There is a constant $C>0$ (depending only on g) and families of polynomials*

$$(M^{\rho}_{ma})_{\substack{m \in \mathscr{Z}_2,\, a \in \mathbf{N}^* \\ \rho \in \mathbf{P}}}, (\tilde{M}^{\rho}_{ma})_{\substack{m \in \mathscr{Z}_2,\, a \in \mathbf{N}^* \\ \rho \in \mathbf{P}}},$$

locally independent of τ, with, for each a, $M^{\rho}_{ma}, \tilde{M}^{\rho}_{ma} \in \mathbf{Q}[T, X_1, X_2]$ homogeneous of degree $d(a) \leqslant 4a^2$ in (X_1, X_2), with degree $d'(a) \leqslant (4a^2/3)\,D$ in T, length at most $L(a)$ with $\log L(a) \leqslant C(4a^2/3)$, and finally with total degrees in variables X_2 bounded by 0 for M^{ρ}_{ma} and 1 for $\tilde{M}^{\rho}_{ma}$, such that:

- *For any $a \in \mathbf{N}^*$, $\rho \in \mathbf{P}$, $z \in \mathbf{C}^g$, and $t \in \mathbf{C}$, the family*

$$((M^{\rho}_{ma})_{m \in \mathscr{Z}_2}, (\tilde{M}^{\rho}_{ma})_{m \in \mathscr{Z}_2})$$

evaluated at $T=\Theta(0)$, $X_1=(\sigma_i(z))_{0 \leqslant i \leqslant N}$, $X_2=(\tilde{\sigma}_i(z;t))_{0 \leqslant i \leqslant N}$ provides a system of projective coordinates for $\tilde{\Theta}(az;at)$ unless identically zero.

- *For any $z_0 \in \mathbf{C}^g$, $a \in \mathbf{N}^*$ there exist $\rho \in \mathbf{P}$ such that for $z \in \mathbf{C}^g$ close to z_0 and any $t \in \mathbf{C}$, the preceding family does not vanish identically.*

Proof. Write C for the logarithm of the greatest length of polynomials A_m^ξ, B_m^ξ $(\xi \in \Xi)$ appearing in Proposition 3.2. We define functions d, d', and L by $d(1)=1$, $d'(1)=0$, $L(1)=0$ and, in accordance with Proposition 3.2,

$$d(2a)=4d(a), \qquad d(2a+1)=2d(a)+2d(a+1),$$

$$d'(2a)=D+4d'(a), \qquad d'(2a+1)=D+2d'(a)+2d'(a+1),$$

$$L(2a)=e^C L(a)^4, \qquad L(2a+1)=e^C L(a)^2 L(a+1)^2.$$

It follows by induction that $d'(a)=\frac{d(a)-1}{3}D$ and $\log L(a)=C\frac{d(a)-1}{3}$, hence we only have to bound $d(a)$. Now it is easily seen, on the one hand that for $a=2^k$ we have $d(a)=2^{2k}=a^2$, on the other hand that the function d is non-decreasing; this allows us to (very roughly) bound $d(a)$ by $d(2^k)$ for any a between 2^{k-1} and 2^k. Note that, when applied with care, the above formulae yield the value $d(a)=2^{k-1}(3a-2^k)$ for $2^{k-1}\leqslant a\leqslant 2^k$. ∎

On the other hand, we can deduce from Proposition 3.3 the following

COROLLARY 3.1. *For any $x\in\mathbf{C}^g$, $\omega\in\Lambda_\tau$, $t, u\in\mathbf{C}$ and $a\in\mathbf{N}^*$, the family*

$$((\sigma_p(\omega)\,\sigma_m(x))_{m\in\mathscr{Z}_2}, (\sigma_p(\omega)\,\tilde\sigma_m(x;t)+a\sigma_m(x)\,\tilde\sigma_p(\omega;u))_{m\in\mathscr{Z}_2}),$$

with $p\in\mathscr{Z}_2$ such that $\sigma_p(0)\neq 0$, provides a system of projective coordinates for $\tilde\Theta(x+a\omega, t+au)$.

3.3. *Differentiation*

We now turn to differentiating the $\tilde\sigma_i$; here the modification we made from the θ_i to the σ_i will, at last, play a (crucial) part, and so will the choice of the derivation $\partial=\sum_{j=1}^g(\partial z_j)\,\partial_j$ and linear form $L(z)=\sum_{i=1}^g(\partial_i L)\,z_i$ used in constructing the $\tilde\sigma_i$.

We start again from the addition formula for $\tilde\sigma_m$, which we summarize as

$$(\tilde A)\ \ 2^{g+1}\tilde\sigma_m(z+w;t+u)\,\sigma_n(z-w)\,\sigma_p(0)\,\sigma_q(0)$$

$$=\sum_{a,b,c,d\in\mathscr{Z}_2} c_{abcd}[\sigma_a(z)\,\tilde\sigma_b(z;t)\,\sigma_c(w)\,\sigma_d(w)+\sigma_a(z)\,\sigma_b(z)\,\tilde\sigma_c(w;u)\,\sigma_d(w)],$$

where (with some new notation) c_{abcd} is an integer depending on m, n, p, q, a, b, c, d. We differentiate it (using ∂_i, $1\leqslant i\leqslant g$) with respect to w, then let $u=w=0$ to obtain

$$2^{g+1}\sigma_p(0)\,\sigma_q(0)[\tilde\sigma_m,\sigma_n]_i$$

$$=\sum_{a,b,c,d\in\mathscr{Z}_2} c_{abcd}\sigma_a[(\sigma_d\partial_i\sigma_c+\sigma_c\partial_i\sigma_d)(0)\,\tilde\sigma_b$$

$$+(\sigma_d(\partial_i\partial+\partial_i L)\,\sigma_c+\partial\sigma_c\partial_i\sigma_d)(0)\,\sigma_b].$$

In order to get rid of first-order derivatives at 0, we note that (by Proposition 3.1(2)) every function σ_k ($k \in \mathscr{Z}_2$) and derivation ∂_j ($j=1\dots g$) satisfy, since σ_0 is even,

$$\sigma_0(0)\,\partial_j\sigma_k(0) = [\sigma_k, \sigma_0]_j\,(0) = D_{k0j}(\Theta(0), \Theta(0)).$$

Applying this to σ_c and σ_d, to ∂_i and ∂, we get

$$\partial_i\sigma_c(0) = \frac{1}{\sigma_0(0)}\,D_{c0i}(\Theta(0), \Theta(0)),$$

$$\partial_i\sigma_d(0) = \frac{1}{\sigma_0(0)}\,D_{d0i}(\Theta(0), \Theta(0))$$

while

$$\partial\sigma_c(0) = \sum_{j=1}^{g} \partial z_j \partial_j \sigma_c(0) = \sum_{j=1}^{g} \partial z_j\,\frac{1}{\sigma_0(0)}\,D_{c0j}(\Theta(0), \Theta(0));$$

all together this yields

$$\begin{aligned} &2^{g+1}\sigma_p(0)\,\sigma_q(0)\,\sigma_0^2(0)[\tilde{\sigma}_m, \sigma_n]_i \\ &\quad = \sum_{a,\,b \in \mathscr{Z}_2} [P_{abi}(\Theta(0), \Theta(0))\,\sigma_a\tilde{\sigma}_b + Q_{abi}(\Theta(0), \Theta(0), (\partial z_j)_{1 \leqslant j \leqslant g}, \partial_i L)\,\sigma_a\sigma_b] \\ &\qquad + \sum_{a,\,b,\,c,\,d \in \mathscr{Z}_2} c_{abcd}\sigma_0^2(0)\,\partial_i\partial\sigma_c(0)\,\sigma_d(0)\,\sigma_a\sigma_b, \end{aligned}$$

where P_{abi}, Q_{abi} are homogeneous (as rational fractions) of degree 0 in their first set of variables, (as polynomials) of degree 4 in their second one, and furthermore Q_{ab} is (polynomially) homogeneous of degree 1 in its last $(g+1)$ variables. Now all we have to do is get rid of the second-order derivatives $\partial_i\partial\sigma_c(0)$. To do this, we apply ∂ to the equality

$$[\sigma_c, \sigma_0]_i = D_{c0i}(\Theta(0), \Theta)$$

from Proposition 3.1(2) then specialize the result at 0; we thus get (remembering that σ_0 is even)

$$\sigma_0(0)\,\partial\partial_i\sigma_c(0) - \sigma_c(0)\,\partial\partial_i\sigma_0(0) = \partial[D_{c0i}(\Theta(0), \Theta)](0),$$

exhibiting on the right-hand side a linear combination, with coefficients linear in $(\partial z_j)_{1 \leqslant j \leqslant g}$, of terms $\sigma_k(0)\,\partial_j\sigma_l(0)$ to which, after mutliplying them with $\sigma_0(0)$, we can apply Proposition 3.1(2) again. This allows us to express

$$(\sigma_0^2\partial\partial_i\sigma_c - \sigma_0\sigma_c\partial\partial_i\sigma_0)(0)$$

as a bi-homogeneous polynomial of degree 3 in $\Theta(0)$ and 1 in $(\partial z_j)_{1\leqslant j\leqslant g}$, with coefficients homogeneous of degree 0 in $\mathbf{Q}(\Theta(0))$. Eventually,

$$2^{g+1}\sigma_p(0)\,\sigma_q(0)\,\sigma_0^2(0)[\tilde\sigma_m,\sigma_n]_i=\sum_{a,b\in\mathscr{Z}_2}P_{abi}(\Theta(0),\Theta(0))\,\sigma_a\tilde\sigma_b$$
$$+\sum_{a,b\in\mathscr{Z}_2}\sigma_a\sigma_b[R_{abi}(\Theta(0),\Theta(0),(\partial z_j)_{1\leqslant j\leqslant g},\partial_i L)$$
$$+\sigma_0(0)\,\partial\partial_i\sigma_0(0)\,S_{abi}(\Theta(0),\Theta(0))],$$

where P_{abi}, R_{abi} are homogeneous of degree 0 in their first set of variables and 4 in their second one, R_{abi} is also linear (homogeneous of degree 1) in $((\partial z_j)_{1\leqslant j\leqslant g},\partial_i L)$, and $S_{abi}\in\mathbf{Q}(T)[X]$ has bi-degree $(0,2)$. We can conclude:

Proposition 3.5. *Assume the polynomial Q used in defining $(\sigma_m)_{m\in\mathscr{Z}_2}$ is chosen so that all second-order derivatives of σ_0 vanish at 0. Then there exists a family $(E_{mni})_{\substack{m,n\in\mathscr{Z}_2\\ i=1,\ldots,g}}$, locally independent ($\partial$ being fixed) of τ, of elements of*

$$\mathbf{Q}(T)[X_1,X_2,Z_1,\ldots,Z_g,Z']$$

homogeneous of degree 0 in T, 2 in (X_1,X_2) and 1 in $(X_2,Z_1,\ldots,Z_g,Z')$, such that for any $m,n\in\mathscr{Z}_2$ and $i=1,\ldots,g$ we have

$$[\tilde\sigma_m,\sigma_n]_i=E_{mni}(\Theta(0))((\sigma_i)_{0\leqslant i\leqslant N},(\tilde\sigma_i)_{0\leqslant i\leqslant N},\partial z_1,\ldots,\partial z_g,\partial_i L).$$

Finally, whenever $m=n$ we expect the terms $\tilde\sigma_i$ to vanish from the expression of $[\tilde\sigma_m,\sigma_m]_i$; to check this we go back to Eq. $(\tilde A)$ above (with $n=m$) and subtract the "symmetrical" equation

$$2^{g+1}\tilde\sigma_m(z-w;t-u)\,\sigma_m(z+w)\,\sigma_p(0)\,\sigma_q(0)$$
$$=\sum_{a,b,c,d\in\mathscr{Z}_2}c_{abcd}[\sigma_a(z)\,\tilde\sigma_b(z;t)\,\sigma_c(w)\,\sigma_d(w)-\sigma_a(z)\,\sigma_b(z)\,\tilde\sigma_c(w;u)\,\sigma_d(w)]$$

obtained by differentiating Riemann's relation with respect to $(z-w)$ instead of $(z+w)$; in this way we find an expression for

$$2^g(\tilde\sigma_m(z+w;t+u)\,\sigma_m(z-w)-\tilde\sigma_m(z-w;t-u)\,\sigma_m(z+w))\,\sigma_p\sigma_q(0)$$

without any $\tilde\sigma_b(z)$. Differentiating again as above (using derivation ∂_i) with respect to w before letting $u=w=0$, we get

$$2^{g+1}[\tilde\sigma_m,\sigma_m]_i\sigma_p\sigma_q(0)$$
$$=\sum_{a,b,c,d\in\mathscr{Z}_2}c_{abcd}\sigma_a(z)\,\sigma_b(z)[(\partial_i L+\partial_i\partial)\,\sigma_c(0)\,\sigma_d(0)+\partial\sigma_c(0)\,\partial_i\sigma_d(0)];$$

clearing out derivatives at 0, as we did above, yields:

COROLLARY 3.2. *Under the same hypotheses, if $m=n$, the variables X_2 do not appear in polynomials E_{mni}.*

Throughout this paper, the functions σ_i will be assumed to be normalized as above, i.e., **so that all second-order derivatives of σ_0 vanish at 0.**

3.4. *Conclusion*

The preceding results allow us to specify the "normalized" embeddings Θ and $\tilde{\Theta}_{\underline{\omega}}$ (notation of Section 2) we will use throughout the text:

PROPOSITION 3.6. *For any lattice $\Lambda \in \mathbf{C}^g$, there exists a family $\Theta = (\sigma_m)_{m\in\mathscr{Z}_2}$ of theta functions for Λ and a choice of coordinates $(z_1, ..., z_g)$ in $\mathbf{C}^g$, such that for all K-linear combinations $\partial^{(1)}, ..., \partial^{(l)}$ of $\partial_1 = \partial/\partial z_1, ..., \partial_g = \partial/\partial z_g$ and linear forms $L_1, ..., L_l \in \mathrm{Vect}_K(z_1, ..., z_g)$ with $\mathbf{C} \supset K \supset \mathbf{Q}((\sigma_1/\sigma_0)(0), ..., (\sigma_N/\sigma_0)(0))$, the embedding $\tilde{\Theta}_{\underline{\omega}}$ associated in section 2 to $\omega_1 = d\partial^{(1)} \log \sigma_0 + dL_1$, ..., $\omega_l = d\partial^{(l)} \log \sigma_0 + dL_l$, as well as derivations $\partial_1, ..., \partial_g$, are defined over K; moreover the family*

$$\left(dz_1, ..., dz_g, d\frac{\partial \log \sigma_0}{\partial z_1}, ..., d\frac{\partial \log \sigma_0}{\partial z_g}\right)$$

provides a basis for the space $H^1_{DR}(A, K)$ of K-rational classes of differentials of the second kind on A.

Remark 3.1. The function σ_0 constructed here is essentially ϑ_X attached to $X = \mathrm{Div}\theta_0$ in [Bar70].

4. EISENSTEIN'S THEOREM AND CONSEQUENCES

4.1. *A Variant of Eisenstein's Theorem*

The following result is an effective version (cf. [DvdP92; HS00, Sect. E.9]) of an extension (cf. [MW93, Lemma 5.3]) of a classical theorem of Eisenstein [PS76, VIII.3.3 and VIII.4.4], in the simple case where the implicit function theorem would apply.

PROPOSITION 4.1. *Let g and n be non-zero integers, $K = \mathbf{Q}(\theta_1, ..., \theta_m) = \mathbf{Q}(\theta)$ a field of finite type and $\mathcal{O} = \mathbf{Z}[\theta]$ its "integer ring," $X = (X_1, ..., X_g)$ and $Y = (Y_1, ..., Y_n)$ two families of independent variables, $(F_d)_{d\in\mathbf{N}}$ a family of n-tuples with coefficients in $\mathcal{O}[[X]][Y]$, F_d being homogeneous of degree d in Y, and $F = \sum_{d\in\mathbf{N}} F_d$. Letting, for each $d \in \mathbf{N}$, $F_d = (F_{d1}, ..., F_{dn})$ and*

$F_{dj}=\sum_{k\in\mathbf{N}} F_{djk}$ with $F_{djk}\in\mathbf{Z}[\theta, X, Y]$ homogeneous of degree k in X, we assume that we have, for every $(d, k, j)\in\mathbf{N}^2\times\{1, ..., n\}$, $\deg_\theta F_{djk}\leqslant d_0 d+d_1 k$ and $L(F_{djk})\leqslant L_0^d L_1^k$. We assume moreover that $F(0,0)=0$, and that the determinant $\Delta\in\mathcal{O}[[X]]$ of

$$F_1(X, Y)\in(\mathcal{O}[[X]]\,Y_1+\cdots+\mathcal{O}[[X]]\,Y_n)^n$$

in the basis $(Y_1, ..., Y_n)$ satisfies $\Delta(0)=\delta\neq 0$. Then the equation

$$(E)\colon F(X, y)=0$$

has a unique solution $y=(y_1, ..., y_n)\in(K[[X]])^n$ vanishing at 0, and the homogeneous polynomials $y_{jk}\in K[X]$ $(\deg_X y_{jk}=k)$ such that $y_j=\sum_{k\in\mathbf{N}^*} y_{jk}$ satisfy the following:

(1) *$z_{jk}=\delta^{2k-1}y_{jk}$ belongs to $\mathcal{O}[X]$;*

(2) *$z_{jk}\in\mathbf{Z}[\theta, X]$ has degree in θ bounded by $[(2k-1)\,n-1]\,d_0+kd_1$, and length at most $L_0^{(2k-1)\,n-1}c_k L_1^k$, where $w=\sum_{k\geqslant 1} c_k T^k$ is such that*

$$2w+\frac{1}{(1-T)^{n-1}}=\frac{1}{(1-T)^n(1-w)}$$

and the sequence of integers $(c_k)_{k\geqslant 1}$ grows at most geometrically: $c_k\leqslant C^k$ for some constant $C=C(n)>0$.

Remark 4.1. The fact that F is not necessarily a polynomial already appeared in [PS76, VIII, No. 153].

Proof. (1) The proof relies on a rewriting of (E) as

$$F_1(X, y)=-F_0(X)-\sum_{d\geqslant 2} F_d(X, y)$$

or, denoting by $F_1(X)$ the matrix associated to F_1, whose determinant is Δ:

$$F_1(X)\,y=-F_0(X)-\sum_{d\geqslant 2} F_d(X, y).$$

Guided by the shape of the desired property we let $\tilde{y}=\frac{y}{\delta}$ and $\tilde{X}=\frac{X}{\delta^2}$; we now have to show that $\tilde{y}\in(\mathcal{O}[[\tilde{X}]])^n$. We use the formula $F_1(X)^{-1}=(\det F_1(X))^{-1}\operatorname{com} F_1(X)$, where com is the comatrix, together with the usual formula for the reciprocal of a power series; we thus obtain

$$F_1(X)^{-1}=\frac{1}{\delta}\operatorname{com} F_1(X)\sum_{r\in\mathbf{N}}\left(1-\frac{\Delta(X)}{\delta}\right)^r$$

and finally the following equation for $\tilde{y}$:

$$(\tilde{E})\ \tilde{y} = -\mathrm{com}\ F_1(X) \sum_{r\in\mathbb{N}} \left(1 - \frac{\Delta(X)}{\delta}\right)^r \left[\frac{F_0(X)}{\delta^2} + \sum_{d\geqslant 2} \delta^{d-2} F_d(X, \tilde{y})\right].$$

Now we only have to notice that $\Delta(X) \in \delta\mathcal{O}[[\tilde{X}]]$, $F_d(X, Y) \in \mathcal{O}[[X]][Y] \subset \mathcal{O}[[\tilde{X}]][Y]$ $(d \geqslant 1)$ and (since it has no constant term) $F_0(X) \in \delta^2\mathcal{O}[[\tilde{X}]]$ to conclude from $(\tilde{E})$, by induction on k, that indeed all the coefficients z_{jk} belong to $\mathcal{O}$.

(2) We assume that for any k' strictly less than k we have (for all $j = 1 \ldots n$) $\deg_\theta z_{jk'} \leqslant [(2k'-1)\,n-1]\,d_0 + k'd_1$, and estimate $\deg_\theta z_{jk}$ using formula $(\tilde{E})$. Imagining the latter fully expanded, we focus on the coefficient in the product of terms $\tilde{X}^a$ from $\mathrm{com}\,F_1$, $\tilde{X}^{l_1} \ldots \tilde{X}^{l_r}$ from $(1-\frac{\Delta}{\delta})^r$, $\tilde{X}^b Y_{i_1} \ldots Y_{i_d}$ from F_d and $\tilde{X}^{k_m}$ in each $\tilde{y}_{i_m}$ $(m = 1 \ldots d)$, with

$$k = |a| + |l_1| + \cdots + |l_r| + |b| + |k_1| + \cdots + |k_d|$$

(where $|x|$ denotes, for any tuple x, the sum of the absolute values of its components). The total degree in θ of this product is bounded by

$$(n-1)\,d_0 + |a|\,d_1 + 2\,|a| \deg_\theta \delta + \sum_{m=1}^{r} [nd_0 + |l_m|\,d_1 + (2\,|l_m| - 1) \deg_\theta \delta]$$

$$+ dd_0 + |b|\,d_1 + 2\,|b| \deg_\theta \delta + (d-2) \deg_\theta \delta + \sum_{m=1}^{d} [(2\,|k_m| - 1)\,n - 1]\,d_0$$

$$+ \sum_{m=1}^{d} |k_m|\,d_1$$

or simply, since $\deg_\theta \delta \leqslant nd_0$ and $k = |a| + |l_1| + \cdots + |l_r| + |b| + |k_1| + \cdots + |k_d|$, by $(2kn - n - 1)\,d_0 + kd_1$ as announced.

What remains now is to compute the length of z_{jk}. To do this, we think in terms of "majorization"

$$\sum a_k X^k \ll \sum b_k X^k \Leftrightarrow (\forall k,\ |a_k| \leqslant b_k)$$

and notice the following. If the length were a non-archimedean quantity like, say, $\exp(\deg_\theta)$, then the same method that allowed us to bound the degree of z_{jk} by $[(2k-1)\,n-1]\,d_0 + kd_1$ would here lead to the bound $L_0^{(2k-1)\,n-1} L_1^k$ for its length; now the problem reduces to computing what additional factor stems from the (cumulated) number of terms adding up at each step of the induction defining the z_{jk}. This leads to a similar but simpler induction, defining a sequence of integers c_k which can be conveniently, if somewhat artificially, dealt with by introducing the series $w = \sum_{k \in \mathbf{N}^*} c_k T^k$ satisfying

$$w = \left(\frac{1}{1-T}\right)^{n-1} \cdot \sum_{r \in \mathbf{N}} \left[\left(\frac{1}{1-T}\right)^n - 1\right]^r \cdot \left[\left(\frac{1}{1-T}\right)\left(1 + \sum_{d \geqslant 2} w^d\right) - 1\right]$$

or equivalently

$$2w + \left(\frac{1}{1-T}\right)^{n-1} = \left(\frac{1}{1-T}\right)^n \frac{1}{1-w}.$$

This formal power series is algebraic, hence [Rui93, p. 106] has a non-zero convergence radius; this implies the existence of $C = C(n)$ such that $|c_k| \leqslant C^k$. ∎

We give separately, as we will make no use of this estimate, an explicit value for the constant $C(n)$ appearing above:

LEMMA 4.1. *For the constant $C = C(n)$ above, we can take $C(n) = 60n$.*

Proof. We use, as in [Ahl66, Sect. 8.2.2], a corollary of the residue formula which states that w, defined implicitly by an equation $f(w, T) = 0$, can be expressed by an integral $w = (1/2i\pi) \int_{C_\varepsilon} z(\partial_1 f / f)(z, T)\, dz$ along a circle C_ε of radius ε small enough to separate the point $w_0 = 0$ from the other roots of $f(w_i, 0) = 0$. Applying this to

$$f(z, T) = 2z + \left(\frac{1}{1-T}\right)^{n-1} - \frac{1}{1-z}\left(\frac{1}{1-T}\right)^n$$

with $\varepsilon = \frac{1}{3}$, we get

$$w = \frac{1}{2i\pi} \int_{|z| = \frac{1}{3}} \frac{2z - \dfrac{z}{(1-z)^2}\left(\dfrac{1}{1-T}\right)^n}{2z + \left(\dfrac{1}{1-T}\right)^{n-1} - \dfrac{1}{1-z}\left(\dfrac{1}{1-T}\right)^n}\, dz.$$

Using once more the "majorization" method, we then find

$$w \ll \frac{1}{3} \frac{\dfrac{2}{3} + \dfrac{3}{4}\left(\dfrac{1}{1-T}\right)^n}{\dfrac{1}{6} - 3\left[\left(\dfrac{1}{1-T}\right)^n - 1\right]} \ll \frac{1/2}{1/6} \frac{\left(\dfrac{1}{1-T}\right)^n}{1 - 18\left[\left(\dfrac{1}{1-T}\right)^n - 1\right]}.$$

Then, since $1/(1-T)^n \ll 1/(1-nT)$,

$$w \ll \frac{3}{1-19nT} \ll 3 + \sum_{k \in \mathbf{N}^*} (60nT)^k$$

whence, since we also know that w has no constant term, the desired result. ▮

The following (classical) two lemmas will allow us to estimate "common denominators" for certain power series expansions obtained by integration:

LEMMA 4.2. *There exists an absolute constant $c > 0$ such that, for any $n \in \mathbf{N}^*$, $\operatorname{lcm}(1, 2, ..., n) \leqslant e^{cn}$.*

Proof.

$$\log \operatorname{lcm}(1, 2, ..., n) = \sum_{\substack{p \leqslant n \\ p \text{ prime}}} (\log p) \max_{k \leqslant n} v_p(k) \leqslant \sum_{\substack{p \leqslant n \\ p \text{ prime}}} \log p \frac{\log n}{\log p} = \pi(n) \log n$$

where $\pi(n) = \operatorname{card}\{p \text{ prime} \leqslant n\}$. A weak form of the prime number theorem now suffices to prove $\pi(n) \log n = O(n)$ and the lemma. ▮

COROLLARY 4.1. *For any $r \in \mathbf{N}^*$ the integer*

$$d_n(r) = \operatorname{lcm}\{n_1 ... n_{r'} \mid r' \leqslant r, (\forall i)\, n_i \neq 0, n_1 + \cdots + n_{r'} \leqslant n\}$$

is bounded by $(er)^{cn}$, c being the constant from the previous lemma.

Proof. Without loss of generality we can assume $n_1, ..., n_{r'}$ to be in non-increasing order, whence $in_i \leqslant n$ or $n_i \leqslant \frac{n}{i}$ $(1 \leqslant i \leqslant r')$; the l.c.m. we want is thus bounded by

$$\prod_{i=1}^{r} \operatorname{lcm}\left(1, ..., \left[\frac{n}{i}\right]\right) \leqslant \prod_{i=1}^{r} e^{cn/i} \leqslant e^{cn(1+\log r)},$$

which concludes the proof. ▮

4.2. *Application to Quasi-Abelian Functions*

We now deduce from Proposition 4.1 some results concerning the various functions introduced in Section 3.

For $i = 0 ... N$ we let $f_i = \sigma_i / \sigma_0$, then $\underline{f} = (f_1, ..., f_N)$. We consider the linear system from Proposition 3.1(1), which we dehomogenize with respect to variable X_0, defining $G^\mu(Z_1, ..., Z_N) = F^\mu(\underline{f}(0); 1, Z_1, ..., Z_N)$. The statement made there regarding differentials of the F^μ (which is nothing but the characteristic property of an embedding) implies that the family $(dG^{\mu}(\underline{f}(0)))_{\mu \in M}$ has rank $N-g$. Thus we can find $M' \subset M$ with cardinality $N-g$ corresponding to equations whose differentials at $\underline{f}(0)$ are independent. We can then apply Proposition 4.1, taking $n = N-g$ and $\theta = \underline{f}(0)$, the parameters d_0, d_1, L_0, and L_1 being equal to some constants depending on g; finally $X_i = Z_i - f_i(0)$ $(i = 1 \ldots g)$, assuming (just like in Section 3) $\mathrm{Jac}_0(f_i)_{i=1\ldots g} \neq 0$, and the vector y is then made up, if we write $\tilde{f}_i = f_i - f_i(0)$, by expansions in powers of $\tilde{f}_1, \ldots, \tilde{f}_g$ of the functions $\tilde{f}_{g+1}, \ldots, \tilde{f}_N$ near 0. From all this, we deduce the following (in this and the next few statements, C_i $(i \in \mathbf{N})$ is a constant depending only on g):

Corollary 4.2. *In a neighbourhood of* 0, *the functions* $f_j = \sigma_j / \sigma_0$ $(j = g+1, \ldots, N)$ *can be expanded as*

$$f_j = \sum_{k \in \mathbf{N}^g} f_{jk} \tilde{f}^k,$$

where the $f_{jk} \in K = \mathbf{Q}(f_1(0), \ldots, f_N(0))$ *satisfy*

(1) *there exists* $\delta \in \mathcal{O} = \mathbf{Z}[f_1(0), \ldots, f_N(0)]$, *with degree and length bounded by* C_0, *such that for all* j, k *we have* $\delta^{|k|} f_{jk} \in \mathcal{O}$;

(2) *the degree and logarithm of the length of* $\delta^{|k|} f_{jk} \in \mathcal{O}$ *are bounded by (resp.)* $(1+|k|)\, C_0$ *and* $C_0^{|k|}$;

(3) *the polynomials (in* $\mathbf{Z}[Z_1, \ldots, Z_N]$*) giving the expressions of* δ *and* $\delta^{|k|} f_{jk}$ *are locally independent of* $\tau \in \mathscr{H}_g$.

Let us introduce the

Definition 4.1. We say a function f holomorphic near the origin in $\mathbf{C}^g$ is a G-**function of type** (δ, C, C', r) $(\delta \in \mathcal{O},\ r \in \mathbf{N},\ C, C' > 0)$ if it can be written $f = \sum_{k \in \mathbf{N}^g} f_k \tilde{f}^k$ with, for any $k \in \mathbf{N}^g$:

(1) $\delta^{|k|} d_{|k|}(r)\, f_k \in \mathcal{O}$, the sequence $(d_n(r))_{n \in \mathbf{N}^*}$ being that of Lemma 4.1 (with the convention that $d_n(0) = 1$);

(2) the degree and length of $\delta^{|k|} f_k$ are bounded by (resp.) $C' + |k|\, C$ and $C' C^{|k|}$.

So, for example, f_j $(1 \leqslant j \leqslant N)$ has type $(\delta, C_0, C_0, 0)$.

Now, define a new set of derivations $(\bar{\partial}_1, \ldots, \bar{\partial}_g)$ on $K(A)$ by

$$(\partial_1, \ldots, \partial_g) = (\bar{\partial}_1, \ldots, \bar{\partial}_g)\, J,$$

where

$$J = (\partial_j f_i)_{1 \leqslant i, j \leqslant g};$$

note that the only difference between ∂_i and $\bar{\partial}_i$ is that the matrix linking the latter to the dz_i is not the jacobian matrix of the f_i evaluated at 0, but indeed their jacobian matrix as a function. What makes the $\bar{\partial}_i$ interesting for us is that they differentiate, by construction, "with respect to the $\tilde{f}_i$" ($i = 1 \dots g$): that is, if a function f expands as

$$f = \sum_{k \in \mathbf{N}^g} f_k \tilde{f}^k$$

near the origin, then for any $i = 1, \dots, g$ we have

$$\bar{\partial}_i f = \sum_{k \in \mathbf{N}^g} f_k k_i \frac{\tilde{f}^k}{\tilde{f}_i}.$$

By integrating the $\tilde{f}$-expansions of derivatives $\bar{\partial}_i f$ ($i = 1 \dots g$) we can therefore deduce, up to its constant term, that of f. Furthermore, the matrix J has (by Proposition 3.1(2)) entries polynomial in the f_i; its reciprocal is a matrix with similar shape divided by the determinant ($\det J$), which again is a polynomial in the f_i ($i = 1 \dots N$) and which, by definition of $\bar{\partial}_j$, has value 1 at the origin; since $\frac{1}{1-u} = \sum_{n \in \mathbf{N}} u^n$ this entails that $\frac{1}{\det J}$ is a G-function of type $(\delta, C_1, C_1, 0)$, and so are the entries of J^{-1}. Thus, if the derivatives $\partial_i f$ of a function f all have type $(\delta, C, C, 0)$ then the $\bar{\partial}_j f$ have type $(\delta, CC_2, CC_2, 0)$ and integrating them, as suggested above, yields the g-expansion of f:

Lemma 4.3. *If all derivatives $\partial_i f$ of a function f with $f(0) = 0$ are G-functions of type $(\delta, C, C, 0)$ then f has type $(\delta, CC_2, 0, 1)$.*

Remark 4.2. Actually, integration yields slightly more information than this as the denominator of f_k appears to divide $\delta^{|k|} \gcd(k_1, \dots, k_g)$.

We can now apply this to the coordinate functions $z_1, \dots, z_g$ of $\mathbf{C}^g$ whose differentials form the dual basis of $(\partial_1, \dots, \partial_g)$.

Corollary 4.3. *The coordinate functions $u_1, \dots, u_g$ of $\mathbf{C}^g$ defined by $\partial_i u_j = \delta_{ij}$ are G-functions of type $(\delta, C_3, 0, 1)$.*

Now we turn to $\partial_i \sigma_0 / \sigma_0$. According to Corollary 3.2, applied with $\partial = \partial_i$ and $L = 0$, each $\partial_i(\tilde{\sigma}_0/\sigma_0) = \partial_i(\partial\sigma_0/\sigma_0)$ can be written as a homogeneous quadratic polynomial, with coefficients in K, in the functions f_j ($j = 1 \dots N$); therefore we can apply the above lemma to get

Corollary 4.4. *Each function $\partial_i \sigma_0 / \sigma_0$ $(i = 1 \dots g)$ is a G-function of type $(\delta, C_3, 0, 1)$.*

Eventually, we will have to deal with monomials in the f_i, u_j, and $\partial_j\sigma_0/\sigma_0$. Noticing (the sequence $d_n(r)$ was introduced just for this) that a product of G-functions of types (δ, C, C', e_1), ..., (δ, C, C', e_r) is a G-function of type $(\delta, C_4C, (C_4C')^r, e_1+\cdots+e_r)$, we get:

COROLLARY 4.5. *Every monomial of degree L_2 in the f_i, u_j, $\partial_j\sigma_0/\sigma_0$, and L_1 in the u_j and $\partial_j\sigma_0/\sigma_0$, is a G-function of type $(\delta, C_5, C_5^{L_2}, L_1)$.*

5. THE ELLIPTIC CASE

In this section, focusing on the case $g=1$, we relate the functions constructed in Section 3, and their properties exhibited in Section 4, to some classical facts from elliptic function theory [Law89, Cha85].

When $g=1$, the four classical theta functions are

$$\theta_3(\tau, z) = \theta_{0,0}(\tau, z) = \sum_{n\in\mathbf{Z}} \exp\left[i\pi(n^2\tau + 2nz)\right],$$

$$\theta_4(\tau, z) = \theta_{0,\frac{1}{2}}(\tau, z) = \sum_{n\in\mathbf{Z}} \exp\left[i\pi(n^2 + 2n(z+\tfrac{1}{2}))\right],$$

$$\theta_2(\tau, z) = \theta_{\frac{1}{2},0}(\tau, z) = \sum_{n\in\mathbf{Z}} \exp\left[i\pi((n+\tfrac{1}{2})^2 + 2z(n+\tfrac{1}{2}))\right],$$

$$\theta_1(\tau, z) = \theta_{-\frac{1}{2},\frac{1}{2}}(\tau, z) = -\sum_{n\in\mathbf{Z}} \exp\left[i\pi((n+\tfrac{1}{2})^2 + 2(n+\tfrac{1}{2})(z+\tfrac{1}{2}))\right]$$

whose dependence on τ will be mostly "forgotten" by writing $\theta_1(z)$, etc., once the parameter $\tau \in \mathscr{H}$ (upper half-plane) has been fixed. Their link with Weierstrass functions for the lattice $\Lambda_\tau = \mathbf{Z} + \mathbf{Z}\tau$ is based on the relation

$$\sigma_\tau(z) = \frac{1}{\theta_1'(0)} \exp\left(-\frac{\theta_1'''(0)}{6\theta_1'(0)} z^2\right) \theta_1(z),$$

where σ_τ denotes the Weierstrass sigma function for Λ_τ. The function σ_τ satisfies $\sigma_\tau'(0) = 1$, $\sigma_\tau'''(0) = \sigma_\tau''(0) = \sigma_\tau(0) = 0$; it is, roughly speaking, the function obtained from θ_1 by the "normalization" process described in Section 3, with a slight difference since here, for an *odd* theta function, it is the third derivative that is equated to zero. Next, letting $-\theta_1'''(0)/3\theta_1'(0) = \eta_\tau$ we find

$$(\log\theta_1)'(z) = \zeta_\tau(z) - \eta_\tau z,$$

$$(\log\theta_1)''(z) = -\wp_\tau(z) - \eta_\tau,$$

where $\zeta_\tau = (\log \sigma_\tau)'$ and $\wp_\tau = -(\log \sigma_\tau)''$; the notation η_τ traditionally reserved for the quasi-period $2\zeta_\tau(\frac{1}{2}) = \zeta_\tau(z+1) - \zeta_\tau(z)$ is perfectly justified here since the 1-periodicity of $(\theta_1)^2$ implies that of $(\log \theta_1)'(z) = \zeta_\tau(z) - \eta_\tau z$.

For any $\omega \in \mathbf{C}^*$ we introduce the Weierstrass sigma function for the lattice $\Lambda = \omega\Lambda_\tau$ by letting $\sigma(z) = \omega\sigma_\tau(\frac{z}{\omega})$; then we let

$$\sigma_i(z) = \exp\left(\frac{\eta_\tau}{2}\left(\frac{z}{\omega}\right)^2\right)\frac{\theta_{i+1}\left(\dfrac{z}{\omega}\right)}{\theta_{i+1}(0)}$$

for $i = 1, 2, 3$; the four functions σ, σ_1, σ_2, σ_3 define an embedding of $\mathbf{C}/\Lambda$ into $\mathbf{P}_3(\mathbf{C})$.

The construction of "algebraic" derivations ∂_i made in section 3 shows that $\frac{d}{dz}$ is one if we let $\omega = (\theta_1'/\theta_3)(0)$. However, any ω whose ratio with the latter belongs to $K = \mathbf{Q}((\theta_2/\theta_3)(0), (\theta_4/\theta_3)(0))$ will be just as good, and Jacobi's relation $\theta_1'(0) = \pi(\theta_2\theta_3\theta_4)(0)$ shows that it is, in particular, the case for $\omega = \pi\theta_3^2(0)$; it is the latter normalization that leads to the classical Jacobi functions $\mathrm{sn} = \sigma/\sigma_3$, $\mathrm{cn} = \sigma_1/\sigma_3$, $\mathrm{dn} = \sigma_2/\sigma_3$.

If we now let $\zeta^*(z) = (\sigma_3'/\sigma_3)(z)$ and $\eta = \eta_\tau/\omega = \zeta^*(\omega)$, then Jacobi's Z function can be defined by

$$\mathrm{Z}(z) = \frac{\sigma_3'}{\sigma_3}(z) - \frac{\eta}{\omega}z;$$

it satisfies in particular [Law89, 3.6] $\mathrm{Z}(z+\omega) = \mathrm{Z}(z)$ and $\mathrm{Z}(\omega') = -\frac{2i\pi}{\omega}$.

As for assertions made in Section 4.2, here the formula $\mathrm{sn}' = \mathrm{cn}.\mathrm{dn}$, together with $\mathrm{cn} = \sqrt{1-\mathrm{sn}^2}$ and $\mathrm{dn} = \sqrt{1-\lambda\mathrm{sn}^2}$, first yields

$$z = \int \frac{d(\mathrm{sn})}{\sqrt{(1-\mathrm{sn}^2)(1-\lambda\mathrm{sn}^2)}};$$

then from the system [Cha85, VII.4]

$$\mathrm{sn}' = \mathrm{cn}.\mathrm{dn}, \qquad \mathrm{cn}' = -\mathrm{sn}.\mathrm{dn}, \qquad \mathrm{dn}' = -\lambda\mathrm{sn}.\mathrm{cn}$$

(where, as usual, $\lambda = k^2 = (\theta_2/\theta_3)(0)^4$), differentiating and evaluating at 0 we can deduce

$$\sigma_1''(0) = \frac{\lambda-2}{3}, \qquad \sigma_2''(0) = \frac{1-2\lambda}{3}, \qquad \sigma_3''(0) = \frac{1+\lambda}{3};$$

this, together with the "Riemann" identity

$$\sigma_3(x+y)\,\sigma_3(x-y) = \sigma_3^2(x)\,\sigma_3^2(y) + \lambda\sigma^2(x)\,\sigma^2(y),$$

allows us to find, using the method described in Section 3.3,

$$(\log \sigma_3)' = \int \left(\frac{1+\lambda}{3} + \lambda \mathrm{sn}^2 \right) \frac{d(\mathrm{sn})}{\sqrt{(1-\mathrm{sn}^2)(1-\lambda \mathrm{sn}^2)}}.$$

6. STATEMENT AND PROOF OF THE MAIN RESULT

6.1. *Statement*

Let A be an abelian variety of dimension g, which for simplicity (replacing if necessary our variety with another one isogenous to one of its factors) we assume to be principally polarized and simple, associated through the relation $A(\mathbf{C}) \simeq \mathbf{C}^g/(\mathbf{Z}^g + \mathbf{Z}^g\tau)$ to an element τ of the Siegel upper half-plane $\mathscr{H}_g$ formed by all square complex matrices of size g whose imaginary part is positive definite; we also assume, as we can without loss of generality, that τ is in the fundamental domain defined in [Igu72] (V.4). We define as in section 3 an embedding $\Theta = (\sigma_0, \ldots, \sigma_N)$ (depending on τ), with kernel $\Lambda = \mathbf{Z}^g + \mathbf{Z}^g\tau$, of $\mathbf{C}^g/\Lambda$ into $\mathbf{P}_N(\mathbf{C})$, and let

$$P = \frac{1}{\sigma_0(0)} \left(\frac{\partial \sigma_i}{\partial z_j}(0) \right)_{1 \leqslant i, j \leqslant g}$$

$$(\partial_1, \ldots, \partial_g) = \left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_g} \right) P^{-1},$$

$$\mathrm{Z}_1(z) = (\partial_1 \log \sigma_0(z), \ldots, \partial_g \log \sigma_0(z)),$$

$$\mathrm{Z}_2(z) = (z_1, \ldots, z_g)^t\, P$$

(with ^{t} denoting transposition) and $\mathrm{Z} = (\mathrm{Z}_1; \mathrm{Z}_2) = (\zeta_1, \ldots, \zeta_{2g})$, meromorphic function from $\mathbf{C}^g$ to $\mathbf{C}^{2g}$. We will consider the algebraic group G, extension of A by $\mathbf{G}_a^{2g}$ associated with $\underline{\omega} = d\mathrm{Z}$; it is embedded into the projective space $\mathbf{P}_{(2g+1)(N+1)-1}(\mathbf{C})$ by means of $\tilde{\Theta} = \tilde{\Theta}_{\underline{\omega}}$ as described in Section 2.

Let $u_1, \ldots, u_p \in \mathbf{C}^g$, $\kappa = \frac{p}{g}$. In the following, $\|.\|$ denotes the supremum norm in $\mathbf{C}^n$ (for arbitrary n) and for $x, y \in \mathbf{P}_N(\mathbf{C})$, we let

$$\|x - y\| = \frac{\max_{0 \leqslant i, j \leqslant N} |x_i y_j - x_j y_i|}{\max_{0 \leqslant i \leqslant N} |x_i| \max_{0 \leqslant i \leqslant N} |y_i|};$$

finally, for any n-tuple $k = (k_1, \ldots, k_n)$ we let $|k| = |k_1| + \cdots + |k_n|$.

PROPOSITION 6.1. *There exists positive constants C_0 (depending on g and p) and C_1 (depending on g, p, and an upper bound for $\|\Im m\tau\|$) with the following properties:*

(1) *If A is simple and $\Theta(0)=J\in\mathbf{P}_N(\bar{\mathbf{Q}})$, if $D, h_0, h_1\geqslant C_0$ and the u_j satisfy*

- $(\forall j)\ \|u_j\|\leqslant C_1$,
- $(\forall j)\ \Theta(u_j)=\xi_j\in\mathbf{P}_N(\bar{\mathbf{Q}})$,
- *the following hypothesis (H) holds:*

$$(H)\quad \forall q=(q_1,\ldots,q_p)\in\mathbf{Z}^p\setminus\{0\},\ \|q\|\leqslant C_1[D(h_0)+\log D)]^{1/\kappa}\Rightarrow q.u\neq 0;$$

if, moreover, $\max(h(J),\max_{1\leqslant j\leqslant p}(h(\xi_1),\ldots,h(\xi_p)))\leqslant h_0$ *and the*

$$\alpha_j\in\mathbf{P}_{(2g+1)(N+1)-1}(\bar{\mathbf{Q}})\qquad(1\leqslant j\leqslant p)$$

are such that

$$\max_{1\leqslant j\leqslant p} h(\alpha_j)\leqslant h_1,$$

$$[\mathbf{Q}(J,\xi,\alpha):\mathbf{Q}]\leqslant D$$

(where $\xi=(\xi_1,\ldots,\xi_p)$, $\alpha=(\alpha_1,\ldots,\alpha_p)$), then we have

$$\max_{1\leqslant j\leqslant p}\|\alpha_j-\tilde{\Theta}(u_j)\|\geqslant\exp(-C_1\phi_1(D,h_0,h_1))$$

(here and later the second parameter of $\tilde{\Theta}$ will be omitted when it is 0) with

$$\phi(D,h_0,h_1)=\frac{D^{\frac{3}{2}+\frac{1}{\kappa}}(h_0+\log D)^{\frac{1}{\kappa}}h_0^{\frac{1}{2}}}{(\log(Dh_0))^{\frac{1}{2}+\frac{1}{\kappa}}}\left[\left(\frac{D(h_0+\log D)}{\log(Dh_0)}\right)^{\frac{2}{\kappa}}h_0+h_1\right].$$

(2) *If, furthermore, all the u_j belong to the lattice Λ, then ϕ can be improved to*

$$\phi(D,h_0,h_1)=D^{\frac{1}{\kappa}}(h_0+\log D)^{\frac{1}{\kappa}}(D(h_1+\log D)+D^{\frac{2}{\kappa}}(h_0+\log D)^{\frac{2}{\kappa}}).$$

Assertion (1) of Theorem 1.2 follows immediately from assertion (1) above, together with Proposition 3.6. As for assertion (2) of Theorem 1.2, it will follow from the

LEMMA 6.1. *Assertions* (1) *and* (2) *above remain valid if we*

- *replace hypothesis (H) with*

$$(H')\quad\forall S\in\mathbf{N}^*,\ \forall q\in\mathbf{Z}^p\setminus\{0\},\ \|q\|\leqslant S\Rightarrow\log\|q.u\|\geqslant -C_1S^3,$$

- *do not assume* $J, \xi_j \in \mathbf{P}_N(\bar{\mathbf{Q}})$ *to be equal to* $\Theta(0)$, *resp.* $\Theta(u_j)$ *anymore, AND*
- *in the conclusion, replace* $\max_{1 \leqslant j \leqslant p} \|\alpha_j - \tilde{\Theta}(u_j)\|$ *with*

$$\max(\|J - \Theta(0)\|, \max_{1 \leqslant j \leqslant p} \|\alpha_j - \tilde{\Theta}(u_j)\|, \max_{1 \leqslant j \leqslant p} (\|\xi_j - \Theta(u_j)\|)).$$

Proof. We can assume without loss of generality that J belongs to the algebraic variety spanned by all points $\Theta(0) \in \mathbf{P}_N(\mathbf{C})$ as A varies, together with τ and Θ; then, in view of [Igu72, V.4, Corollary of Theorem 4], the implicit function theorem allows us to find $\tau' \in \mathscr{H}_g$ such that (exceptionally indexing Θ with the modulus τ') $\Theta_{\tau'}(0) = J$, with τ' "close" to τ in the sense that $\log \|\tau' - \tau\|$ is at most an absolute constant times $\log \|J - \Theta(0)\|$. Note that it follows, e.g., from [Sas83] that τ' must correspond, just like τ, to a simple abelian variety, say A'. Now we construct, using the implicit function theorem again, points u_j' close to the u_j in the above sense (so that $\tilde{\Theta}_{\tau'}(u')$ is still close to α) such that $\Theta_{\tau'}(u_j') = \xi_j$; it remains to deduce property (H) for the u_j' from (H') for the u_j. Indeed, if $\|q\| \leqslant S = C_1[D(h_0) + \log D)]^{1/\kappa}$ and $q.u' = 0$ then $\log \|q.u\|$ is at most a constant times $\log \max_{1 \leqslant j \leqslant p}(\|\xi_j - \Theta(u_j)\|)$; but this, because S^3 is smaller than (a constant times) $\phi(D, h_0, h_1)$, would contradict hypothesis (H'). ∎

Remark 6.1. Using the same lemma, one can obtain a result where J and the $\Theta(u_j)$ are not assumed to be algebraic, nor are the u_j assumed to be in Λ, but then an *ad hoc*, "technical" assumption must be included in the hypotheses in order to ensure (H') above.

6.2. *Parameters*

We let $K_0 = \mathbf{Q}(J)$, $K_1 = K_0(\xi)$, $K = K_1(\alpha)$ and, in order to treat both cases simultaneously yet smoothly, we introduce a parameter ρ equal to 2 in case (1), 0 in case (2). Arguing by contradiction, we suppose found $\alpha \in (\mathbf{P}^{(2g+1)(N+1)-1}(\bar{\mathbf{Q}}))^p$ with $\max_{1 \leqslant j \leqslant p} h(\alpha_j) \leqslant h_1$, $[\mathbf{Q}(J, \xi, \alpha) : \mathbf{Q}] \leqslant D$ and $\max_{1 \leqslant j \leqslant p} \|\alpha_j - \tilde{\Theta}(u_j)\| < e^{-V}$ where, depending on the case,

$$(1) \quad V = c_0^{20} \frac{D^{\frac{3}{2}+\frac{1}{\kappa}}(h_0 + \log D)^{\frac{1}{\kappa}} h_0^{\frac{1}{2}}}{(\log(Dh_0))^{\frac{1}{2}+\frac{1}{\kappa}}} \left[\left(\frac{D(h_0 + \log D)}{\log(Dh_0)} \right)^{\frac{2}{\kappa}} h_0 + h_1 \right],$$

$$(2) \quad V = c_0^{20} D^{\frac{1}{\kappa}}(h_0 + \log D)^{\frac{1}{\kappa}} (D(h_1 + \log D) + D^{\frac{2}{\kappa}}(h_0 + \log D)^{\frac{2}{\kappa}}),$$

c_0 being a constant "sufficiently large" and depending on g, p and an upper bound for $\|\Im m\tau\|$ as stated in the proposition.

We can also assume without loss of generality that the first $N+1$ coordinates of each α_j are nothing but those of ξ_j, and that α is also an element of G, i.e., the relations

$$z_l \sigma_i(u_j) = \frac{\sigma_i}{\sigma_0}(u_j)(z_l \sigma_0)(u_j),$$

$$\sigma_0(u_j)\,\partial_l \sigma_i(u_j) = \sigma_i(u_j)\,\partial_l \sigma_0(u_j) + [\sigma_i, \sigma_0]_l\,(u_j) \qquad (l = 1 \ldots g,\ i = 0 \ldots N)$$

between entries of $\tilde{\Theta}(u_j)$ still hold for their counterparts in α_j; as a consequence (which will be used in Section 6.4), the field $K_0(\alpha)$ is generated over K_1 by the quantities

$$\frac{\alpha_{j,\,iN}}{\alpha_{j,\,0}} \qquad (j = 1 \ldots p,\ i = 1 \ldots 2g)$$

or equivalently, for any choice of indices i_j ($j = 1 \ldots p$), by the

$$\frac{\alpha_{j,\,kN+i_j}}{\alpha_{j,\,i_j}} \qquad (j = 1 \ldots p,\ k = 1 \ldots 2g)$$

provided these quantities are well-defined, i.e., have non-zero denominators.

Letting $\hbar_0 = h_0 + \log D$, $\hbar_1 = h_1 + \log D$ we define the following parameters, depending on which case we are considering:

$$(1) \qquad T = \left[c_0^{2+\frac{15}{\kappa}} D^{\frac{1}{2}+\frac{1}{\kappa}} \hbar_0^{\frac{1}{\kappa}-1} h_0^{1/2} (\log(Dh_0))^{-\frac{1}{2}-\frac{1}{\kappa}} \left(\left(\frac{D\hbar_0}{\log(Dh_0)} \right)^{\frac{2}{\kappa}} h_0 + h_1 \right) \right],$$

$$S = \left[c_0^{5/\kappa} \left(\frac{D\hbar_0}{\log(Dh_0)} \right)^{1/\kappa} \right],$$

$$E = Dh_0,$$

$$L_1 = \left[c_0^{2+\frac{5}{\kappa}} \left(\frac{D\hbar_0}{\log(Dh_0)} \right)^{1/\kappa} \left(\frac{Dh_0}{\log(Dh_0)} \right)^{1/2} \right],$$

$$L_2 = \left[c_0^{2+\frac{5}{\kappa}} D^{\frac{1}{2}-\frac{1}{\kappa}} \hbar_0^{-1/\kappa} h_0^{-1/2} (\log(Dh_0))^{-1/2} \left(\left(\frac{D\hbar_0}{\log(Dh_0)} \right)^{\frac{2}{\kappa}} h_0 + h_1 \right) \right];$$

$$(2) \qquad T = [c_0^{2+\frac{15}{\kappa}} (D\hbar_0)^{\frac{1}{\kappa}-1}\,(D\hbar_1 + (D\hbar_0)^{2/\kappa})],$$

$$S = [c_0^{5/\kappa} (D\hbar_0)^{1/\kappa}],$$

$$E = 1,$$

$$L_1 = [c_0^{2+\frac{5}{\kappa}} (D\hbar_0)^{1/\kappa}],$$

$$L_2 = [c_0^{2+\frac{5}{\kappa}} (D\hbar_0)^{-1/\kappa}\,(D\hbar_1 + (D\hbar_0)^{2/\kappa})].$$

Note that the modulus $\tau \in \mathscr{H}_g$ (Siegel upper half-plane) of our abelian varieties will be scarcely mentioned explicitly throughout the proof; the unspecified dependence of constants in our results on bounds for $\|\Im m\tau\|$ essentially comes from the basic fact that any continuous function is bounded on the compact subset ($\|\Im m\tau\| \leqslant C$) of the fundamental domain for $\mathscr{H}_g$.

6.3. *The "Baker–Coates–Anderson–Chudnovsky Trick"*

It is known (see [Dav91, Théorème 3.1]) that for any $z \in \mathbf{C}^g$ at least one of the σ_i ($i = 0 \ldots N$) has modulus bounded from below at z by $\exp(-c_0 \|z\|^2)$; from now on for $z = t.u = t_1 u_1 + \cdots + t_p u_p$ ($t \in \mathbf{Z}^p$) we will denote by i_t an index such that σ_{i_t} is such a function.

In the following lemma, $X, Y_1, \ldots, Y_p, Z_1, \ldots, Z_p, T$ are families of $(N+1)$ variables, all independent. The functions $\sigma_0, \ldots, \sigma_g$ being still as in section 3, we let $f_i = \sigma_i / \sigma_0$ for $i = 1, \ldots, g$.

LEMMA 6.2. *There exists polynomials $D \in \mathbf{Z}[X]$ (homogeneous), $D_t \in \mathbf{Z}[X, Y_1, \ldots, Y_p, T]$ (homogeneous in each of their $(p+2)$ sets of $(N+1)$ variables) for $(t_1, \ldots, t_p) \in \mathbf{Z}^p$ and, for any $r \in \mathbf{N}^{2g}$ with $|r| \leqslant L_1$ and $s \in \mathbf{N}^g$ with $|s| \leqslant L_2$, $Q_{rstn} \in \mathbf{Q}[X, Y_1, Z_1, \ldots, Y_p, Z_p]$ (homogeneous in X and in each pair (Y_j, Z_j)) with the following properties. For $z \in \mathbf{C}^g$ in a neighborhood of* 0,

$$D_t\left(J, \xi, \frac{\Theta}{\sigma_0}(z)\right)^{L_1+L_2} \left[\left(\frac{\sigma_0}{\sigma_{i_t}}\right)^{L_1+L_2} \left(\frac{\sigma_1}{\sigma_0}\right)^{s_1} \cdots \left(\frac{\sigma_g}{\sigma_0}\right)^{s_g} \mathbf{Z}^r\right](t.u+z)$$

$$= \sum_{n \in \mathbf{N}^g} Q_{rstn}(J, \tilde{\Theta}(u)) \left(\frac{f}{D(J)}\right)^n (z)$$

with $D_t(J, \xi, \frac{\Theta}{\sigma_0}(0)) \neq 0$. Moreover the partial degrees and logarithmic length of D_t are bounded by $c_0\, |t|^\rho$, those of D by some constant depending only on g, and

$$\deg_X Q_{rstn} = |n| \deg_X D + (L_1 + L_2) \deg_X D_t,$$

$$\deg_{(Y_j, Z_j)} Q_{rstn} = (L_1 + L_2) \deg_{Y_j} D_t,$$

$$\deg_{(Z_1, \ldots, Z_p)} Q_{rstn} \leqslant |r|$$

and

$$L(Q_{rstn}) \leqslant (1 + |t|)^{|r|}\, c_0^{|n| + (L_1 + L_2)\, |t|^\rho}.$$

Finally, for any $n \neq 0$ *we have* $d_n(|r|)\, Q_{rstn} \in \mathbf{Z}[X, Y, T]$, $d_n(|r|)$ *being the quantity introduced in Corollary* 4.1.

Proof. We treat only case (1), the other one being treated in a similar but easier way using Proposition 3.3 and Corollary 3.1 instead of their "non-periodic" equivalents. First Proposition 3.2 allows us to write, for each function σ_i $(i = 0 \ldots g)$ and each index $j = 1 \ldots g$,

$$\frac{\sigma_i}{\sigma_{i_t}}(t.u+z) = \frac{A_i(\Theta(t.u);\Theta(z))}{A_{i_t}(\Theta(t.u);\Theta(z))},$$

$$\left(\frac{\sigma_0}{\sigma_{i_t}}\zeta_j\right)(t.u+z) = \frac{B_j(\tilde{\Theta}(t.u);\tilde{\Theta}(z))}{A_{i_t}(\Theta(t.u);\Theta(z))}$$

where the parameters $\Theta(0)$ and ξ, being both fixed, are omitted; thus A_i, B_j here have coefficients in K_0. Then, in order to express the value of $\tilde{\Theta}$ at the point $t.u = t_1 u_1 + \cdots + t_r u_r$ we apply Proposition 3.4, together with the addition formula again, which now tells us that the family $\tilde{\Theta}(t_k u_k)$ $(k = 1 \ldots p)$ is proportional to some

$$(M_{0t_k}(\Theta), \ldots, M_{Nt_k}(\Theta), \tilde{M}_{0t_k}(\tilde{\Theta}), \ldots, \tilde{M}_{Nt_k}(\tilde{\Theta}))(u_k).$$

The quantities $(\sigma_i/\sigma_{i_t})(t.u+z)$, $((\sigma_0/\sigma_{i_t})\,\zeta_j)(t.u+z)$ now appear as rational functions in the values of $\tilde{\Theta}$ at points u_k on the one hand, z on the other hand, with numerators and denominators homogeneous of degree at most $c_0\,|t|^\rho$ $(\rho = 2$ in case (1)) in the former, and 2 in the latter. Dehomogenizing with respect to $\sigma_0(z)$ allows us to express $(\sigma_i/\sigma_{i_t})(t.u+z)$ and $((\sigma_0/\sigma_{i_t})\,\zeta_j)(t.u+z)$ as rational functions, with coefficients in K_0, in the values of $\tilde{\Theta}$ at points u_k, and at z, of $(1/\sigma_0)\,\tilde{\Theta}$. The latter function, whose coordinates are the functions (σ_i/σ_0), $(\sigma_i/\sigma_0)\,z_j$ and

$$\frac{\partial_l \sigma_i}{\sigma_0} = \frac{\sigma_i}{\sigma_0}\frac{\partial_l \sigma_0}{\sigma_0} + \partial_l \frac{\sigma_i}{\sigma_0} \qquad (l = 1 \ldots g),$$

can be rewritten according to Proposition 3.1(2) using only the σ_i/σ_0 and ζ_j $(j = 1 \ldots 2g)$. Now we only have to substitute for the latter their respective g-expansions as given by Corollaries 4.3 and 4.4, then invoke Corollary 4.5 to conclude the proof. ∎

6.4. *Construction of an invertible matrix with algebraic entries*

We form the matrix $\mathscr{M}_0$ whose entries are the

$$d_n(L_1)\, Q_{rstn}(J, \alpha)$$

from Lemma 6.2, with lines indexed by $(3g)$-tuples (r, s) ($|r| < L_1$, $|s| < L_2$) and columns by (t, n) ($|t| < S$, $|n| < T$). Here and in the following, unless otherwise specified, any mention of a projective point $\Theta(t.u)$, resp. $\tilde{\Theta}(t.u)$, as an argument to a polynomial actually refers to the quotient $(\Theta/\sigma_{i_t})(t.u)$, resp. $(\tilde{\Theta}/\sigma_{i_t})(t.u)$, with i_t as in Lemma 6.2; similarly, in such a situation α_j will be understood to mean $\alpha_j/\alpha_{j, i_{e_j}}$ where $(e_1, \ldots, e_p)$ is the canonical basis of $\mathbf{Z}^p$.

We denote by I_0 and J_0 the sets indexing the lines, resp. columns of $\mathscr{M}_0$. The purpose of this section will be to prove the

LEMMA 6.3. *The matrix $\mathscr{M}_0$ has maximal rank, that is, $L_1^{2g} L_2^g$.*

This will allow us to extract from $\mathscr{M}_0$ a square non-degenerate submatrix $\mathscr{M}(\alpha)$ of size $L = L_1^{2g} L_2^g$, with

$$\mathscr{M} \in \mathscr{M}at_{L_1^{2g} L_2^g}(\mathbf{Z}[J][Y_1, Z_1, \ldots, Y_p, Z_p])$$

(Y_j, Z_j being as in Lemma 6.2); we will let J_1 be the subset of J_0 indexing its columns and $\Delta = \det \mathscr{M}$, so that by construction $\Delta_{ar} = \Delta(\alpha) \in \bar{\mathbf{Q}}^*$.

We now turn to the proof of the above lemma. Assume its conclusion to be false; then a non-trivial linear combination of its lines vanishes:

$$|n| < T,\ |t| < S \Rightarrow \sum_{\substack{|r| < L_1 \\ |s| < L_2}} \lambda_{rs} Q_{rstn}(J, \xi, \alpha) = 0.$$

The algebraic group G being the image of $\tilde{\Theta}$ as above, we define

$$P(X_0, \ldots, X_{(2g+1)(N+1)-1}) = \sum_{\substack{|r| < L_1 \\ |s| < L_2}} \lambda_{rs} X_0^{L_1 + L_2 - |r| - |s|} X_1^{s_1} \ldots X_g^{s_g} \prod_{i=1}^{2g} X_{i(N+1)}^{r_i}$$

and

$$\mathscr{V} = \mathrm{Vect}_{\mathbf{C}}\left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_g}\right)$$

in the tangent space at the origin of G identified in a natural way with $\mathbf{C}^g \times \mathbf{C}^{2g}$. Finally, the set $\mathscr{E}$ is that of points $\gamma_t \in G(K)$ ($|t| < S$), images through $\tilde{\Theta}$ of $t.(u; \varepsilon)$ where

$$\varepsilon_{jk} = \frac{\alpha_{j, kN + i_{e_j}}}{\alpha_{j, i_{e_j}}} - \partial_k \log \sigma_{i_{e_j}}(u_j) \qquad (k \leqslant g),$$

$$\varepsilon_{jk} = \frac{\alpha_{j, kN + i_{e_j}}}{\alpha_{j, i_{e_j}}} - u_{j, k-g} \qquad (g + 1 \leqslant k \leqslant 2g)$$

(so that in particular $\|\varepsilon\| \leqslant \max_{1 \leqslant j \leqslant p} \|\alpha_j - \tilde{\Theta}(u_j)\| \leqslant e^{-V}$); hypothesis (H) in the Proposition ensures that $t \neq t' \Rightarrow t.u \neq t'.u$.

It is easily seen, going over the proof of Lemma 6.2 (see also [Phi88, Sect. 3.1.3]) that the quantities $Q_{rstn}(J, \alpha)$ are nothing but the coefficients in the f-expansion of the expression from the left-hand side of Lemma 6.2 with each Z ($j = 1 \ldots 2g$) replaced with $\mathrm{Z} + t.\varepsilon$; it follows that the polynomial P vanishes along $\mathscr{V}$ with order at least T at each point of $\mathscr{E}$. This allows us to deduce from Proposition 2.1 the existence of a proper algebraic subgroup G' of G such that, with the notation from that Proposition,

$$N' T^{d'} \deg G' \leqslant c L_1^{l'} L_2^{a'}$$

However, since the variety A is assumed to be simple, there are only two kinds of such subgroups:

- $\mathbf{G}' \subset \mathbf{G}_a^{2g} \subset \mathbf{G}$.

Then we have, to begin with, $d' = g$. Furthermore, if two points γ_t and $\gamma_{t'}$ are congruent modulo G' then their projections in A are equal, which implies $(t - t').u \in \Lambda$. Now the important point is to notice that G' cannot contain more than $(\dim G')$ $\mathbf{Z}$-linearly independent elements of $\mathscr{E}$; this follows easily from the fact that for ε small enough (which we can assume here), any family of "perturbed" elements $\mathrm{Z}(t.u) + t.\varepsilon \in \mathbf{C}^{2g}$ (t being such that $t.u \in \Lambda$) has the same rank over $\mathbf{C}$ as the "unperturbed" family of the $\mathrm{Z}(t.u)$, which by Lemma 2.1 is just the rank of the $\mathbf{Z}$-module generated by the corresponding t's. From this we deduce

$$S^{p - \min(p, \dim G')} T^g \leqslant c_0 L_1^{2g - \dim G'} L_2^g$$

or equivalently

$$S^{\kappa - \min(\kappa, \frac{1}{g} \dim G')} T \leqslant c_0^{1/g} L_1^{2 - \frac{1}{g} \dim G'} L_2$$

which leads us to a contradiction for values 0 and $\min(\kappa, 2)$, hence also for all others, of the ratio $\frac{1}{g} \dim G'$—this is due to the inequalities

$$c_0^{1/g} L_1^2 L_2 < T S^\kappa,$$
$$S \leqslant L_1.$$

- **G' contains G_0, universal extension of A by $\mathbf{G}_a^g$.**

In this case G/G' is isomorphic to a quotient of $G/G_0 \simeq \mathbf{G}_a^g$ by a subgroup G^* of dimension strictly less than g; then d' is equal to $(g - \dim G^*)$ and we get

$$T^{g - \dim G^*} \leqslant N' T^{g - \dim G^*} \leqslant c_0 L_1^{g - \dim G^*},$$

incompatible once again with our choice of parameters.

6.5. *Application of a Schwarz Lemma*

Let $I \subset I_0$ and $J \subset J_0$ with the same cardinality m (in the following, I_0 and J_0 will sometimes be identified with $\{1, ..., m\}$). We define a function ϕ_{IJ} of one complex variable by

$$\phi_{IJ}(v) = \det(\partial_z^{(n)} \psi_i(vt.u))_{\substack{i=(r,s) \in I \\ j=(t,n) \in J}},$$

where $\partial_z = (\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_g})$ and for any $n \in \mathbf{N}^g$,

$$\partial_z^{(n)} = \frac{1}{n!} \partial_z^n = \frac{1}{n_1! \dots n_g!} \left(\frac{\partial}{\partial z_1}\right)^{n_1} \cdots \left(\frac{\partial}{\partial z_g}\right)^{n_g},$$

while

$$\psi_i = \psi_{rs} = \sigma_0^{L_1+L_2} \left(\frac{\sigma_1}{\sigma_0}\right)^{s_1} \cdots \left(\frac{\sigma_g}{\sigma_0}\right)^{s_g} \mathbf{Z}^r.$$

LEMMA 6.4. *The function ϕ_{IJ} vanishes at the origin with order at least*

$$\Omega_m = m\left(\frac{g^2}{(g+1)\, e} m^{1/g} - 2g - T\right).$$

Proof. The same reasoning as in [Wal97, Sect. 5(d), *Premier pas* and Lemme 5.2] leads to

$$\Omega_m + mT \geqslant A \binom{A+g-1}{g} - \sum_{a=0}^{A-1} \binom{a+g}{g}$$

$$= A \binom{A+g-1}{g} - \binom{A+g}{g+1} = \binom{A+g}{g} \left(\frac{A^2}{A+g} - \frac{A}{g+1}\right)$$

where the integer A is defined by

$$\binom{A+g-1}{g} \leqslant m < \binom{A+g}{g};$$

it follows that

$$\Omega_m + mT \geqslant m \frac{A(A-1)\, g}{(A+g)(g+1)}.$$

But then,

$$m \leqslant \binom{A+g}{g} \leqslant \left(\frac{e}{g}(A+g)\right)^g$$

implies $\frac{A+g}{g} \geqslant \frac{1}{e} m^{1/g}$ hence, writing $Y = \frac{1}{e} m^{1/g}$,

$$\Omega_m + mT \geqslant m \frac{g(Y-1)}{(g+1)\,Y} [gY - (g+1)] \geqslant m \left(\frac{g^2}{g+1} Y - 2g \right).$$

∎

We can now apply to ϕ_{IJ} the following, most elementary form of Schwarz lemma:

LEMMA 6.5. *Let $R \geqslant 1$, Ω a positive integer, ϕ analytical in the disk $D(0, R) \subset \mathbf{C}$ and vanishing with order at least Ω at the origin; then*

$$|\phi(1)| \leqslant R^{-\Omega}\, |\phi|_R,$$

where $|\phi|_R = \sup_{D(0,\, R)} |\phi|$.

A rough estimate (using Cauchy's formula for derivatives) of $|\phi_{IJ}|_R$ yields

$$|\phi_{IJ}|_R \leqslant m!(\max_i\ |\psi_i|_{c_0 SR+1})^m \leqslant e^{c_0 m(L_1+L_2)(SR)^2}$$

hence the lemma leads to

$$|\phi_{IJ}(\underline{t}.u)| \leqslant \exp(-\Omega_m \log R + c_0 m(L_1+L_2)(SR)^2).$$

Now assume $m \geqslant \frac{1}{2} L$. Then $\Omega_m \geqslant \frac{1}{12} L_1^2 L_2 m$; on the other hand $L_1 + L_2 \leqslant 2L_2$, and we can take

$$R = \frac{1}{c_0} \frac{L_1}{S}$$

to finally obtain the

LEMMA 6.6. *For any $I \subset I_0$ and $J \subset J_0$ with the same cardinality $m \geqslant \frac{L}{2}$, for $\underline{t} = (t_j)_{j=(t_j,\, n_j) \in J}$ we have*

$$|\phi_{IJ}(\underline{t}.u)| \leqslant \exp\left(-\frac{1}{c_0} (L^{1/g})\, m \log E \right)$$

(see above, in the list of parameters, the definition of E).

6.6. *Conclusion of the Proof*

We let, as in Section 4.2, $\bar{\partial}_1 = \partial/\partial f_1, \ldots, \bar{\partial}_g = \partial/\partial f_g$, corresponding to the local parameters $(f_i = \sigma_i/\sigma_0)_{1 \leqslant i \leqslant g}$ at the origin; then we define, as we did for ∂_z, $\bar{\partial}^{(n)} = \frac{1}{n!}\bar{\partial}^n$ for any $n \in \mathbf{N}^g$. We recall that by definition of the polynomials Q_{rsnt},

$$Q_{rsnt}(J, \tilde{\Theta}(u)) = D(J)^{|n|}\, \bar{\partial}^{(n)} \left[D_t \left(J, \xi, \frac{\Theta}{\sigma_0}(z) \right)^{L_1+L_2} (\sigma_{i_t}^{-(L_1+L_2)} \psi_i)(t.u+z) \right]_{z=0}$$

(where $i = (r, s)$ as before).

We use the following formula, easily read from power series substitution rules,

$$\bar{\partial}^{(n)} = \sum_{\substack{k \in \mathbf{N}^g \\ |k| \leqslant |n|}} \left[\sum_{\substack{i_1, \ldots, i_{|k|} \in \mathbf{N}^g \\ i_1 + \cdots + i_{|k|} = n}} \prod_{j=1}^{g} \left(\prod_{l=k_1+\cdots+k_{j-1}+1}^{k_1+\cdots+k_j} \bar{\partial}^{(i_l)} z_j \right) \right] \partial_z^{(k)},$$

together with the product differentiation formula (Leibniz rule), to find

$$\begin{aligned} Q_{rsnt}(J, \tilde{\Theta}(u)) = D(J)^{|n|} \sum_{\substack{k \in \mathbf{N}^g \\ |k| \leqslant |n|}} & \left[\sum_{\substack{i_1, \ldots, i_{|k|} \in \mathbf{N}^g \\ i_1 + \cdots + i_{|k|} = n}} \prod_{j=1}^{g} \left(\prod_{l=k_1+\cdots+k_{j-1}+1}^{k_1+\cdots+k_j} \bar{\partial}^{(i_l)} z_j(0) \right) \right] \\ & \times \sum_{l_1+l_2+l_3=k} \partial_z^{(l_1)} \left(D_t \left(J, \xi, \frac{\Theta}{\sigma_0}(z) \right)^{L_1+L_2} \right)(0) \\ & \times \partial_z^{(l_2)} (\sigma_{i_t}^{-(L_1+L_2)})(t.u)\, \partial_z^{(l_3)} \psi_i(t.u) \end{aligned}$$

expressing the left-hand side as a linear combination of the $\partial_z^{(k)} \psi_i(t.u)$ ($|k| \leqslant |n|$),

$$Q_{rsnt}(J, \tilde{\Theta}(u)) = \sum_{|k| \leqslant |n|} \lambda_{nkt} \partial_z^{(k)} \psi_i(t.u).$$

Thus, the columns of $\mathscr{M}(\tilde{\Theta}(u))$ can be written as linear combinations of those of

$$(\partial_z^{(n)} \psi_i(t.u))_{\substack{i=(r,s) \in I_0 \\ j=(t,n) \in J_0}}$$

with coefficients equal to the λ_{nkt} from the formula above.

To bound these λ_{nkt}, we bound the number of terms in their expression by a rough c_0^T, the modulus $|\sigma_{i_t}(t.u)^{-1}|$ by $e^{c_0^2 S^2}$, while by Cauchy's formula differentiation introduces at most a factor c_0^T; we thus find

$$|\lambda_{nkt}| \leqslant c_0^{3T} (e^{c_0^2 S^2} L(D_t)\, c_0^{\deg D_t})^{L_1+L_2} \leqslant c_0^{3T} \exp(2c_0^2(L_1+L_2)\, S^2).$$

Now let $\sigma \in \mathbf{N}^{(2g+1)(N+1)p}$ with length $|\sigma| = m' \leqslant L$. Writing $\mathscr{L}_i$ for the line of index i in $\mathscr{M}$, we use the standard formula for differentiating determinants to express $\partial^{(\sigma)}\Delta(\tilde{\Theta}(u))$ as a sum

$$\partial^{(\sigma)}\Delta(\tilde{\Theta}(u)) = \sum_{\sigma_1 + \cdots + \sigma_L = \sigma} \det\,(\partial^{(\sigma_i)}\mathscr{L}_i(\tilde{\Theta}(u)))_{i \in I_0}$$

of determinants where at least $(L-m')$ out of L lines have not been differentiated, i.e., are equal to the initial $\mathscr{L}_i$. Developing these determinants yields (according to the Laplace formula) a sum of minors of size $m = L-m'$ coming from $\mathscr{M}(\tilde{\Theta}(u))$, multiplied by their cofactors in

$$(\partial^{(\sigma_i)}\mathscr{L}_i(\tilde{\Theta}(u)))_{i \in I_0}.$$

According to the calculations made earlier, these minors of $\mathscr{M}(\tilde{\Theta}(u))$ can in turn be expressed as linear combinations of minors (with the same size m) of the shape $\phi_{IJ}(\underline{t}u)$; using Lemma 6.6 together with trivial estimates for the cofactors yields the following: for any $\sigma \in \mathbf{N}^{(2g+1)(N+1)p}$ with $|\sigma| \leqslant L$, letting $|\sigma| = m'$ and $m = L-m'$, we have

$$\begin{aligned}
|\partial^{(\sigma)}\Delta(\tilde{\Theta}(u))| &\leqslant c_0^L \max_{\sigma_1 + \cdots + \sigma_L = \sigma} |\det\,(\partial^{(\sigma_i)}\mathscr{L}_i(\tilde{\Theta}(u)))_{i \in I_0}| \\
&\leqslant c_0^L (\max_{|n| < T} d_n(L_1))^L \max_{\substack{I \subset I_0,\, J \subset J_1 \\ |I| = |J| = m}} |\det\,(Q_{rstn}(J, \tilde{\Theta}(u)))_{\substack{(r,s) \in I \\ (t,n) \in J}}| \\
&\quad \times \binom{L}{m} \max_{\sum_i |\sigma_i| \leqslant m'} |\det\,(\partial^{(\sigma_i)}Q_{rstn}(J, \tilde{\Theta}(u)))_{\substack{i = (r,s) \in I_0 \setminus I \\ j = (t,n) \in J_1 \setminus J}}| \\
&\leqslant c_0^L (\max_{n < T} d_n(L_1))^L\, T^{c_0 m} \max_{\substack{|n|,\, |k| < T \\ |t| < S}} |\lambda_{nkt}|^m \max_{\substack{I \subset I_0,\, J \subset J_0 \\ |I| = |J| = m}} |\phi_{IJ}(\underline{t}u)| \\
&\quad \times \frac{L!}{m!} \max_{(r,s,t,n)} (\deg Q_{rstn} L(Q_{rstn})\, c_0^{\deg Q_{rstn}})^{m'};
\end{aligned}$$

this brings us to

$$\begin{aligned}
&|\partial^{(\sigma)}\Delta(\tilde{\Theta}(u))| \\
&\quad \leqslant (c_0 L)^L\, L_1^{c_0 TL} (c_0^{4T} e^{2c_0^2 (L_1 + L_2) S^2})^m \\
&\qquad \times (e^{c_0^2 (T + (L_1 + L_2) S^2)})^{m'} \max_{\substack{I \subset I_0,\, J \subset J_0 \\ |I| = |J| = m}} |\phi_{IJ}(\underline{t}.u)| \\
&\quad \leqslant \exp[3c_0^2 L(T \log L_1 + (L_1 + L_2)\, S^2)] \exp\left(-\frac{1}{c_0}(L^{1/g})\, m \log E\right)
\end{aligned}$$

or, taking into account the fact that $c_0^4 T \log L_1 \leqslant L_1^2 L_2 \log E$ and

$$c_0^2 S \leqslant L_1 \leqslant L_2,$$

Lemma 6.7. *For any $\sigma \in \mathbf{N}^{(2g+1)(N+1)p}$ such that $m = L - |\sigma| \geqslant \frac{1}{2} L$, we have*

$$|\partial^{(\sigma)} \Delta(\tilde{\Theta}(u))| \leqslant \exp\left(-\frac{1}{2c_0}(L^{1/g})\, m \log E\right).$$

Now all we have to do is write the Taylor formula with order $M = [\frac{L}{2}]$,

$$\left|\Delta(\alpha) - \sum_{|\sigma| = m' \leqslant M} (\alpha - \tilde{\Theta}(u))^\sigma\, \partial^{(\sigma)} \Delta(\tilde{\Theta}(u))\right|$$

$$\leqslant \exp(-(M+1)\, V) \sum_{|\sigma| = M+1} \sup_{\|x - \tilde{\Theta}(u)\| \leqslant e^{-V}} |\partial^{(\sigma)} \Delta(x)|$$

whence, using again a trivial estimate for $|\partial^{(\sigma)} \Delta(x)|$,

$$|\Delta(\alpha)| \leqslant L^{c_0} \exp\left(-L \min\left(\frac{1}{2c_0} L^{1/g} \log E, V\right)\right)$$
$$+ L^{c_0} \exp(-(M+1)\, V)\, L!(e^{c_0^2(T + (L_1 + L_2)\, S^2)})^L$$

and finally

$$|\Delta(\alpha)| \leqslant \exp\left(-\frac{1}{2} L \min\left(\frac{1}{2c_0} L^{1/g} \log E, V\right)\right)$$
$$+ \exp\left[L\left(2c_0^2(T + (L_1 + L_2)\, S^2) - \frac{1}{2} V\right)\right].$$

Since

$$L^{1/g} \log E = L_1^2 L_2 \log E \leqslant c_0 V,$$

this entails

$$|\Delta(\alpha)| \leqslant \exp\left(-\frac{1}{5c_0} L^{1+\frac{1}{g}} \log E\right).$$

Now we can apply to $\Delta(\alpha)$, which is in fact the value at (J, α) of a polynomial $\tilde{\Delta} \in \mathbf{Z}[X, Y_1, Z_1, \dots, Y_p, Z_p]$, the classical Liouville inequality [Wal92]. Since

$$\begin{aligned}
\deg_T \tilde{\Delta} &\leqslant LL_1, \\
\deg_X \tilde{\Delta} &\leqslant c_0 L(T + (L_1 + L_2)\, S^\rho), \\
\deg_Y \tilde{\Delta} &\leqslant c_0 L(L_1 + L_2)\, S^\rho
\end{aligned}$$

and

$$L(\tilde{\Delta}) \leqslant (L_1^{c_0 T} S^{L_1} e^{c_0 (T + (L_1 + L_2)\, S^\rho)})^L,$$

that inequality turns out to be incompatible with the conditions

$$\begin{aligned}
c_0^3 D(L_1 + L_2)\, S^\rho h_0 &\leqslant L_1^2 L_2 \log E, \\
c_0^3 DT(h_0 + \log L_1) &\leqslant L_1^2 L_2 \log E, \\
c_0^3 DL_1(h_1 + \log S) &\leqslant L_1^2 L_2 \log E
\end{aligned}$$

satisfied by our parameters. Our proof by contradiction is therefore complete.

7. QUESTIONS AND (PARTIAL) ANSWERS

We conclude by raising, and beginning to answer, a few questions that may seem natural on comparing Corollary 1.1 to Theorem 1.1.

• **How about combining Chudnovsky's statements**, say (1) and (5)?

It is indeed tempting to conjecture that in general at least two numbers among $g_2, g_3, \frac{\eta}{a}\omega, \wp(u), \zeta(u) - \frac{\eta}{a}\omega u$ (notation from Theorem 1.1) are algebraically independent. As noted in Remark 6.1, a measure of simultaneous approximation for these numbers seems to require an additional, "technical" hypothesis; in any case, the measure obtained is far from one that would grant algebraic independence. The preceding conjecture, although only a corollary of the Main Conjecture of [Ber00], thus appears to be still out of reach through this "classical" method.

• **What about differential forms with vanishing periods**, such as $d\zeta - \frac{\eta}{\omega}\, dz$ in the elliptic case?

Their treatment requires that the basic Schwarz Lemma we used in the proof be replaced with a more sophisticated one, such as the following:

Lemma 7.1 [Wal93, Lemma 7.1; Gra99, Théorème n]. *Let $\mathscr{E}$ be a subset of $\mathbf{C}$ with cardinality S inside the ball of radius $r > 0$ centered at the origin, let $\Omega > 0$ and $R = 7r$, and let ϕ be an analytical in the poly-disc of radius R centered at 0 in $\mathbf{C}^n$, vanishing with order at least Ω at each point of $\mathscr{E}^n \subset \mathbf{C}^n$; then*

$$|\phi|_r \leqslant e^{-S\Omega} \, |\phi|_R.$$

Indeed, this particular lemma combined with the method of proof described above yields the following quantitative refinements of assertions (1) and (5) from Theorem 1.1:

Proposition 7.1. *Let E be an elliptic curve defined over a field $K \subset \mathbf{C}$ and $\omega \in H^1_{DR}(E, \mathbf{C})$, defined over $K' \supset K$, with a non-trivial zero period:*

$$\exists \lambda \in H_1(E, \mathbf{Z}), \, \lambda \neq 0, \int_\lambda \omega = 0.$$

Let $u \in T_0 E(\mathbf{C})$ (tangent space at the origin) be non-zero and such that $\exp_E(u) \in A(K)$.

(1) *If $K \subset \bar{\mathbf{Q}}$, the set made up by $\int_0^u \omega$ together with a generating system of K' over K admits the following approximation measure:*

$$\phi_1(D, h) = D^{7/4} (\log D)^{3/2} \, (h + \sqrt{D} \log D).$$

(2) *If u is a period (element of the period lattice $\Lambda = \mathrm{ker}\exp_E$) independent of λ, the set made up by $\int_0^u \omega$ together with a generating system of K' over $\mathbf{Q}$ admits the following approximation measure:*

$$\phi_2(D, h) = [D(h + \log D)]^{3/2}.$$

The desirable extensions to higher dimensions would require corresponding extensions of the preceding lemma to accommodate functions ϕ and sets $\mathscr{E}$ in a space $\mathbf{C}^g$ of arbitrary dimension (at least for an $\mathscr{E}$ contained in a fixed lattice of $\mathbf{C}^g$); such a lemma does not seem to be known at present.

Note, however, that according to a recent result of D. Roy [Roy00] it could be deduced from a "satisfactory" interpolation lemma for lattices in arbitrary dimension; this is not available in full generality, but has been proved by Masser (see [Mas78b, Theorem B; and Mas78a, Lemma 7]) for lattices (coming from abelian varieties) with complex multiplication. As a consequence, one can prove (still in much the same way as above) the

PROPOSITION 7.2. *Let A be an abelian variety of dimension g with complex multiplication, $\omega_1, ..., \omega_f$ be linearly independent in $H^1_{DR}(A, K)$ and such that, for some integer $q \leqslant 2g$,*

$$\forall i = 1...f, \forall j = q+1...2g, \qquad \eta_{ij} = \int_{\lambda_j} \omega_i = 0.$$

Then the set made up by all quasi-periods $\eta_{ij} = \int_{\lambda_j} \omega_i$ $(i = 1...f,\ j = 1...q)$ together with a generating system of K over $\mathbf{Q}$ admits as a measure of simultaneous approximation the function ϕ defined by

$$\phi(D, h) = CD(h + \log D)(D \log D)^{q/2f}.$$

This provides, in the case of complex multiplication only, a quantitative refinement of the following algebraic independence statement which turns out to be true in general:

PROPOSITION 7.3. *Let A be an abelian variety defined over $K \subset \mathbf{C}$, $(\lambda_1, ..., \lambda_{2g})$ a basis of $H_1(A, \mathbf{Z})$, $\omega_1, ..., \omega_f$ $(f \leqslant 2g)$ independent elements of $H^1_{DR}(A, K)$ sharing $p = 2g - q$ independent periods, and satisfying moreover $\int_0^{\lambda_j} \omega_i \in K$ for all $i \leqslant f$ and $j \leqslant 2g$. If $2f > q$, then* deg tr $K \geqslant 2$.

The latter result is shown by yet the same method but, being of a qualitative nature, instead of an interpolation lemma it requires only a Schwarz lemma (for a function having many zeroes, not just small values) which can be found, e.g., in [Wal87, Proposition 7.4.1].

Remark 7.1. In his most recent text [Roy01], Roy establishes one of the (so far unknown) interpolation lemmas mentioned above: he basically (up to constants) extends Masser's result from [Mas78a] to arbitrary, non-CM varieties. This allows the removal from Proposition 7.2 of the unnatural hypothesis of complex multiplication, as well as the inclusion among the quantities involved of a generating system (not necessarily algebraic anymore) of A over $\mathbf{Q}$, giving a measure of the same shape as Proposition 6.1(2):

$$\phi(D, h_0, h_1) = D(h_1 + h_0 + \log D)[D(h_0 + \log D)]^{\frac{q}{2f}}.$$

ACKNOWLEDGMENTS

I thank Michel Waldschmidt for his constant help, Daniel Bertrand for introducing me to "algebraic" theta functions, and Gaël Rémond for a couple of very helpful conversations on theta functions in general.

REFERENCES

[Ahl66] L. V. Ahlfors, "Complex Analysis: An Introduction of the Theory of Analytic Functions of One Complex Variable," 2nd ed., McGraw–Hill, New York, 1966.

[Bar70] I. Barsotti, Considerazioni sulle funzioni theta, *in* "Symposia Mathematica, Vol. III, INDAM, Rome, 1968/69," pp. 247–277, Academic Press, London, 1970.

[Ber00] D. Bertrand, $\Theta(\tau, z)$ and transcendence, *in* "Introduction to Algebraic Independence Theory" (Yu. Nesterenko and P. Philippon, Eds.), Chap. 1, 2000.

[Bru99] S. Bruiltet, D'une mesure d'approximation simultanée à une mesure d'irrationalité : Le cas de $\Gamma(1/4)$ et $\Gamma(1/3)$, *Acta Arithmetica*, to appear.

[Cha85] K. Chandrasekharan, "Elliptic Functions," Springer-Verlag, Berlin, 1985.

[Chu82] G. V. Chudnovsky, Measures of irrationality, transcendence and algebraic independence: Recent progress, *in* "Number Theory Days, Exeter, 1980," pp. 11–82, Cambridge Univ. Press, Cambridge, UK, 1982.

[Chu84] G. V. Chudnovsky, "Contributions to the theory of Transcendental Numbers," Amer. Math. Soc., Providence, RI, 1984.

[Dav89] S. David, Théorie de Baker pour des familles de groupes algébriques commutatifs, *in* "Fonctions thêta, formes modulaires et approximation diophantienne," Thèse de Doctorat, Chap. 2, Université de Paris VI, Février 1989.

[Dav91] S. David, Fonctions thêta et points de torsion des variétés abéliennes, *Compositio Math.* **78** (1991), 121–160.

[DvdP92] B. M. Dwork and A. J. van der Poorten, The Eisenstein constant, *Duke Math. J.* **65** (1992), 23–43.

[FW84] G. Faltings and G. Wüstholz, Einbettungen kommutativer algebraischer Gruppen und einige ihrer Eigenschaften, *J. Reine Angew. Math.* **354** (1984), 175–205.

[GH78] Ph. Griffiths and J. Harris, "Principles of Algebraic Geometry," Pure and Applied Mathematics, Wiley–Interscience, New York, 1978.

[Gra99] F. Gramain, Lemme de Schwarz pour des produits cartésiens, *in* "Annales Mathématiques Blaise Pascal, 2001," to appear.

[HS00] M. Hindry and J. H. Silverman, "Diophantine Geometry," Springer-Verlag, New York, 2000.

[Igu72] Jun-ichi Igusa, "Theta Functions," Die Grundlehren der Mathematischen Wissenschaften, Vol. 194, Springer-Verlag, New York, 1972.

[Lan82] S. Lang, "Introduction to Algebraic and Abelian Functions," 2nd ed., Springer-Verlag, New York, 1982.

[Law89] D. F. Lawden, "Elliptic Functions and Applications," Springer-Verlag, New York, 1989.

[LR99] M. Laurent and D. Roy, Sur l'approximation algébrique en degré transcendance un, *Ann. Inst. Fourier (Grenoble)* **49** (1999), 27–55.

[Mas78a] D. W. Masser, Diophantine approximation and lattices with complex multiplication, *Invent. Math.* **45** (1978), 61–82.

[Mas78b] D. W. Masser, Polynomial interpolation in several complex variables, *J. Approx. Theory* **24** (1978), 18–34.

[MW93] D. Masser and G. Wüstholz, Periods and minimal abelian subvarieties, *Ann. of Math.* **137** (1993), 407–458.

[Phi88] G. Philibert, Une mesure d'indépendance algébrique, *Ann. Inst. Fourier (Grenoble)* **38** (1988), 85–103.

[Phi96] P. Philippon, Nouveaux lemmes de zéros dans les groupes algébriques commutatifs, *Rocky Mountain J. Math.* **26** (1996), 1069–1088.

[Phi99] P. Philippon, Mesures d'approximation de valeurs de fonctions analytiques, *Acta Arith.* **88** (1999), 113–127.

[PS76] G. Pólya and G. Szegő, "Problems and Theorems in Analysis. Vol. II. Theory of Functions, Zeros, Polynomials, Determinants, Number Theory, Geometry," Grundlehren der Mathematischen Wissenschaften, Vol. 216, Springer-Verlag, New York, 1976.

[Rém00] G. Rémond, Inégalité de Vojta en dimension supérieure, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **29** (2000), 101–151.

[Roy00] D. Roy, Interpolation formulas and auxiliary functions, *J. Number Theory*, to appear.

[Roy01] D. Roy, Interpolation sur des perturbations d'ensembles produits, preprint, 2001.

[Rui93] J. M. Ruiz, "The Basic Theory of Power Series," Vieweg, Braunschweig, 1993.

[RW97] D. Roy and M. Waldschmidt, Simultaneous approximation and algebraic independence, *Ramanujan J.* **1** (1997) 379–430.

[Sas83] R. Sasaki, Modular forms vanishing at the reducible points of the Siegel upper-half space, *J. Reine Angew. Math.* **345** (1983), 111–121.

[Sil94] J. H. Silverman, "Advanced Topics in the Arithmetic of Elliptic Curves," Springer-Verlag, New York, 1994.

[Vas96] K. G. Vasil'ev, On the algebraic independence of the periods of Abelian integrals, *Mat. Zametki* **60** (1996), 681–691, 799.

[Wal87] M. Waldschmidt, Nombres transcendants et groupes algébriques, *Astérisque* (1987), 69–70, with appendices by Daniel Bertrand and Jean-Pierre Serre, and an English summary.

[Wal92] M. Waldschmidt, "Linear Independence of Logarithms of Algebraic Numbers," Institute of Mathematical Sciences, Madras, 1992, with an appendix by Michel Laurent.

[Wal93] M. Waldschmidt, Approximation diophantienne sur les groupes algébriques commutatifs, Sect. 7, Mai 1993.

[Wal97] M. Waldschmidt, Approximation diophantienne dans les groupes algébriques commutatifs. I. Une version effective du théorème du sous-groupe algébrique, *J. Reine Angew. Math.* **493** (1997), 61–113.