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Oscillations of the remainder term related to the Euler totient function ☆

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ABSTRACT

We split the remainder term in the asymptotic formula for the mean of the Euler phi function into two summands called the arithmetic and the analytic part respectively. We show that the arithmetic part can be studied with a mild use of the complex analytic tools, whereas the study of the analytic part heavily depends on the properties of the Riemann zeta function and on the distribution of its non-trivial zeros in particular.

1. Introduction

Let $\varphi(n)$ denote the classical Euler totient function and let

$$E(x)=\sum_{n\leqslant x}\varphi(n)-\frac{3}{\pi^2}x^2 \tag{1.1}$$

be the associated error term. Our aim is to study its oscillatory properties. According the classical result of Dirichlet, $E(x)\ll x^{1+\varepsilon}$ for every $\varepsilon>0$. This was improved by F. Mertens [13] to $E(x)\ll x\log x$, and then by A. Walfisz [16], who proved that $E(x)\ll x(\log x)^{2/3}(\log\log x)^{4/3}$. Walfisz' estimate remains the best result of this type up to date. Typically, $E(x)$ is proportional to x as proved by S.S. Pillai and S.D. Chowla [15]:

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$$\sum_{n\leqslant x} E(n) \sim \frac{3}{2\pi^2}x^2$$

and S.D. Chowla [1]:

$$\int_0^x E(t)^2\,dt \sim \frac{x^3}{6\pi^2}.$$

There are however large deviations from this mean. The best omega result for $E(x)$ belongs to H.L. Montgomery [14]:

$$E(x) = \Omega_\pm\big(x\sqrt{\log\log x}\big). \tag{1.2}$$

The smoothed error:

$$\tilde{E}(x) := \int_0^x E(t)\,\frac{dt}{t} = \sum_{n\leqslant x}\varphi(n)\log\frac{x}{n} - \frac{3}{2\pi^2}x^2 \tag{1.3}$$

was studied by the present authors in [12]. It was proved that for x tending to infinity we have

$$\tilde{E}(x) = \Omega_\pm\big(x^{\frac{1}{2}}\log\log\log x\big)$$

and moreover, assuming the Riemann Hypothesis, there exists a positive constant B such that

$$\tilde{E}(x) \ll x^{\frac{1}{2}}\exp\left(B\frac{\log x}{\log\log x}\right).$$

1.1. *Splitting the error term*

It turns out that it is convenient to split $E(x)$ into two summands. To this end let us consider the following Volterra integral equation of second type:

$$F(x) - \int_0^\infty K(x,t)F(t)\,dt = E(x) \quad (x \geqslant 1), \tag{1.4}$$

where $F(x)$ is the unknown function and the kernel $K(x,t)$ is defined as follows:

$$K(x,t) = \begin{cases} 1/t & \text{if } 0 < t \leqslant x, \\ 0 & \text{if } 1 \leqslant x < t. \end{cases}$$

According to the general theory, (1.4) has a uniquely determined solution given as an absolutely convergent Neumann series (see for instance [2, Chapter 3]). Let us be more explicit. We define the operator δ_1 on the linear space $\mathcal{X}$ of functions $g : (0,\infty) \to \mathbb{R}$ which are Lebesgue locally integrable and such that

$$\int_0^1 \big|g(t)\big||\log t|^N \frac{dt}{t} < \infty$$

for every integer $N \geqslant 1$, as follows:

$$\delta_1(g)(x) = \int_0^x \frac{g(t)}{t}\,dt \quad (x > 0,\ g \in \mathcal{X}). \tag{1.5}$$

Moreover, let δ_k denote the k-fold iteration of δ_1:

$$\delta_k = \delta_1 \circ \cdots \circ \delta_1 \quad (k \text{ times}) \tag{1.6}$$

and let

$$R_k(x) = \delta_k(E)(x), \tag{1.7}$$

where $E(x)$ is defined in (1.1). Then the Neumann series in question gives the following formal expansion:

$$F(x) = E(x) + \sum_{k=1}^{\infty} R_k(x). \tag{1.8}$$

Notice that δ_1 was used also for instance in [5,6,9,11] and most recently in [7] and [8] showing its usefulness in proving results on the distribution of values of various arithmetic error terms.

Our first aim is to solve (1.4) explicitly. This is the content of our first theorem. To formulate it let us put

$$f(x) = -\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left\{\frac{x}{n}\right\} \tag{1.9}$$

for every $x \geqslant 0$, where $\mu(n)$ denotes the Möbius function and $\{\theta\} = \theta - [\theta]$ is the fractional part of a real number θ.

Theorem 1.1. *The unique solution of* (1.4) *is* $F(x) = xf(x)$.

For $x \geqslant 0$ let us write

$$g(x) = \sum_{n=1}^{\infty} \mu(n) \left\{\frac{x}{n}\right\}^2. \tag{1.10}$$

Theorem 1.2. *For* $x \geqslant 1$ *we have*

$$E(x) = xf(x) + \frac{1}{2}g(x) + \frac{1}{2}.$$

Let us remark that Theorems 1.1 and 1.2 are equivalent. Our proof of the latter shows clearly how it follows from Theorem 1.1. Equally well one can proceed in the opposite direction, establishing Theorem 1.2 first.

According to Theorem 1.2, for $x \geqslant 1$ we can split $E(x)$ as follows:

$$E(x) = E^{\mathrm{AR}}(x) + E^{\mathrm{AN}}(x), \tag{1.11}$$

where

$$E^{\mathrm{AR}}(x) = xf(x) \quad \text{and} \quad E^{\mathrm{AN}}(x) = \frac{1}{2}g(x) + \frac{1}{2} \tag{1.12}$$

with $f(x)$ and $g(x)$ given by (1.9) and (1.10) respectively. Using Theorem 1.1, (1.4) and (1.8) it is easy to see that

$$E^{\mathrm{AN}}(x) = -\sum_{k=1}^{\infty} R_k(x) \tag{1.13}$$

and hence in particular

$$E^{\mathrm{AN}}(x) = -\int_0^x E^{\mathrm{AR}}(t)\,\frac{dt}{t}.$$

We see also that $E^{\mathrm{AN}}(x)$ is absolutely continuous. Note that $\tilde{E}(x)$ in (1.3) coincides up to the sign with the first term of the expansion (1.13). We make use of the two initial terms of the Neumann series (1.13) in the proof of Theorem 1.8 below.

We call $E^{\mathrm{AR}}(x)$ and $E^{\mathrm{AN}}(x)$ the *arithmetic* and the *analytic part* of $E(x)$ respectively. As we shall see, while $E^{\mathrm{AN}}(x)$ depends heavily on the distribution of non-trivial zeros of the Riemann zeta function, $E^{\mathrm{AR}}(x)$ does not. Our treatment of the latter is quite general, with $\mu(n)$ in (1.9) replaced by an arithmetic function $\alpha(n)$ with certain axiomatic properties; see the definition of the class $\mathcal{A}$ in Section 1.2. When $\alpha = \mu$, which is the case that applies to $E^{\mathrm{AR}}(x)$, axioms can be verified in an elementary way using the method of Erdös and Selberg for proving the Prime Number Theorem. In other cases, including those discussed in Section 1.3, the analysis can be done with a mild use of the zeros of the involved L-functions using the standard complex integration method and appealing to a de la Valleé–Poussin type zero-free region.

1.2. Analysis of the arithmetic part

To place the subject in a proper context we state a general theorem for a class of functions similar to $f(x)$ in (1.9). Let $\mathcal{A}$ denote the set of all arithmetic functions $\alpha(n)$ satisfying the following conditions:

(i) $\alpha(n)$ is real and multiplicative.
(ii) There exists a positive real number $\theta < 1$ such that

$$a(n) \ll n^{\theta}. \tag{1.14}$$

(iii) We have

$$\sum_{n=1}^{\infty} \frac{\alpha(n)}{n^2} \neq 0. \tag{1.15}$$

(iv) For every $N \geqslant 1$ we have

$$\sum_{n=1}^{N} \bigl|\alpha(n)\bigr| \ll N. \tag{1.16}$$

(v) The series

$$\sum_{n=1}^{\infty} \frac{\alpha(n)}{n}$$

converges.

(vi) There exists a positive real number η and a sequence of positive numbers $x_\nu \to \infty$ such that

$$\sum_{\substack{p \leqslant x_\nu \\ \alpha(p)<0 \\ p\equiv 3 \ (\mathrm{mod}\ 4)}} \frac{|\alpha(p)|}{p} \geqslant \eta \log\log x_\nu + O(1) \tag{1.17}$$

for all $\nu \geqslant 1$.

For such $\alpha(n)$ we write

$$f(x,\alpha) = \sum_{d=1}^{\infty} \frac{\alpha(d)}{d} s\left(\frac{x}{d}\right), \tag{1.18}$$

where $s(x)$ denotes the saw tooth function:

$$s(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Z}, \\ \frac{1}{2} - \{x\} & \text{otherwise.} \end{cases}$$

Recalling that $\sum_n \mu(n)/n = 0$ we have

$$f(x,\mu) = \frac{1}{2}\big(f(x-0) + f(x+0)\big),$$

where $f(x)$ is defined in (1.9). Hence for $x \notin \mathbb{Z}$ we have $f(x,\mu) = f(x)$.

Moreover, for every $x \geqslant 1$ let us put

$$R(x,\alpha) = \sup_{y \geqslant x} \left| \sum_{n>y} \frac{\alpha(n)}{n} \right|$$

and

$$R^*(x,\alpha) = \sqrt{R(\sqrt{x},\alpha)} + \frac{1}{x}. \tag{1.19}$$

It is evident that $R^*(x,\alpha)$ is positive, monotonic and according to (v) $R^*(x,\alpha) \to 0$ as $x \to \infty$.

Theorem 1.3. *Let $\alpha \in \mathcal{A}$. Then we have*

$$f(x,\alpha) = \Omega_\pm\left(\left(\log\log \frac{1}{R^*(x^{2/3},\alpha)}\right)^{\eta}\right)$$

as $x \to \infty$.

Taking the Möbius μ-function as $\alpha(n)$ we see that we can apply the above result with $\eta = 1/2$. Using de la Valleé–Poussin zero-free region and complex integration method one can easily show that

$$R^*(x,\alpha) \ll \exp(-c_0\sqrt{\log x}) \tag{1.20}$$

for certain positive constant c_0 and all $x \geqslant 1$. Consequently we have the following corollary which was implicitly proved in [14].

Corollary 1.4. *Let $f(x)$ be defined in* (1.9)*. Then*

$$f(x) = \Omega_\pm(\sqrt{\log\log x}) \quad \textit{and} \quad E^{\mathrm{AR}}(x) = \Omega_\pm(x\sqrt{\log\log x})$$

as $x \to \infty$.

Remark. Most probably estimate of type (1.20), possibly with $(\log x)^\beta$, $\beta < 1/2$, in place of $\sqrt{\log x}$, can be proved in an elementary way, and hence one can make our treatment of $E^{\mathrm{AR}}(x)$ completely independent of the theory of the Riemann zeta function.

1.3. Two examples

We show here that besides the Möbius function there are other interesting arithmetic functions belonging to the class $\mathcal{A}$. Let ϕ be a holomorphic newform of weight k and level q with the following normalized Fourier expansion at infinity:

$$\phi(z) = \sum_{n=1}^{\infty} \lambda_\phi(n) n^{\frac{k-1}{2}} e(nz) \quad (z \in \mathbb{H}).$$

We refer to [4] for basic definitions and results used in this section.

Proposition 1.5. *The function $n \mapsto \mu(n)|\lambda_\phi(n)|^2$ belongs to the class $\mathcal{A}$ and satisfies condition* (vi) *with $\eta = 1/2$. In particular, for the corresponding function f we have the following omega estimate*:

$$f\big(x, \mu|\lambda_\phi|^2\big) = \Omega_\pm(\sqrt{\log\log x})$$

as $x \to \infty$.

Condition (i) is obviously satisfied since ϕ is an eigenfunction of the Hecke algebra. Moreover, (ii) holds true with every positive θ according to the celebrated Deligne bound

$$\big|\lambda_\phi(n)\big| \leqslant d(n), \tag{1.21}$$

$d(n)$ being the classical divisor function. Other conditions from the definition of $\mathcal{A}$ can easily be verified using the known properties of the corresponding L-function

$$L(s,\phi) = \sum_{n=1}^{\infty} \frac{\lambda_\phi(n)}{n^s} \quad (\sigma > 1),$$

its twist by the primitive Dirichlet character (mod 4) $L(s, \phi \otimes \chi_4)$, and the Rankin–Selberg convolution $L(s, \phi \otimes \overline{\phi})$. Using Euler products it is easy to see that for $\sigma > 1$ we have

$$\sum_{n=1}^{\infty} \frac{\mu(n)|\lambda_\phi(n)|^2}{n^s} = \frac{h(s)}{L(s, \phi \otimes \bar{\phi})}, \tag{1.22}$$

where $h(s)$ is holomorphic and non-vanishing for $\sigma > 1/2$. Putting $s = 2$ we check condition (iii), whereas (iv) follows immediately from the properties of the Rankin–Selberg convolution which imply

$$\sum_{n=1}^{\infty} \left|\lambda_\phi(n)\right|^2 \sim c_0 x$$

for a certain positive $c_0 = c_0(\phi)$ as $x \to \infty$. Applying the usual complex integration technique to (1.22) we obtain

$$\sum_{n \leqslant x} \mu(n) \left|\lambda_\phi(n)\right|^2 \ll x e^{-c_1 \sqrt{\log x}}$$

for a certain positive $c_1 = c_1(\phi)$ as $x \to \infty$. This implies (v) and shows also that

$$R^*\left(x, \mu|\lambda_\phi|^2\right) \ll e^{-c_2 \sqrt{\log x}}$$

for a certain positive $c_2 = c_2(\phi)$ as $x \to \infty$. Finally, using the Rankin–Selberg convolution once again, we can prove that

$$\sum_{p \leqslant x} \frac{|\lambda_\phi(p)|^2}{p} = \log\log x + O(1)$$

and

$$\sum_{p \leqslant x} \frac{|\lambda_\phi(p)|^2 \chi_4(p)}{p} = O(1)$$

as $x \to \infty$. This implies

$$\sum_{\substack{p \leqslant x \\ p \equiv 3 \ (\mathrm{mod}\ 4)}} \frac{|\lambda_\phi(p)|^2}{p} = \frac{1}{2} \log\log x + O(1) \tag{1.23}$$

as $x \to \infty$ and gives (vi) with $\eta = 1/2$, as required. Proposition 1.5 therefore follows.

Let us now assume that coefficients $\lambda_\phi(n)$ are real. This happens for instance when ϕ is associated to an elliptic curve defined over $\mathbb{Q}$.

Proposition 1.6. *Let ϕ be as above and have real Fourier coefficients. Then $\lambda_\phi(n)$ belongs to the class $\mathcal{A}$ and satisfies condition* (vi) *with $\eta = 1/8$. In particular, we have*

$$f(x, \lambda_\phi) = \Omega_\pm\left((\log\log x)^{1/8}\right)$$

as $x \to \infty$.

One verifies axioms (i)–(v) along similar lines as before. Using (1.23) and the fact that $|\lambda_\phi(p)| \leqslant 2$ (see (1.21)), we have

$$\sum_{\substack{p \leqslant x \\ p \equiv 3 \ (\mathrm{mod}\ 4)}} \frac{|\lambda_\phi(p)|}{p} \geqslant \frac{1}{4} \log\log x + O(1).$$

Since

$$\sum_{\substack{p \leqslant x \\ p \equiv 3 \ (\mathrm{mod}\ 4)}} \frac{\lambda_\phi(p)}{p} = O(1),$$

we easily deduce that

$$\sum_{\substack{p \leqslant x \\ p \equiv 3 \ (\mathrm{mod}\ 4) \\ \lambda_\phi(p) < 0}} \frac{|\lambda_\phi(p)|}{p} \geqslant \frac{1}{8} \log\log x + O(1),$$

and condition (vi) follows with $\eta = 1/8$ as required.

Let us remark that the exponent $1/8$ in the last proposition can in many cases be improved. The authors hope to address this problem in a future paper.

1.4. Analysis of the analytic part

The properties of $E^{\mathrm{AN}}(x)$ directly depend on the distribution of non-trivial zeros of the Riemann zeta function. First of all let us observe that

$$E^{\mathrm{AN}}(x) \ll x \exp\left(-A \frac{(\log x)^{3/5}}{(\log\log x)^{1/5}} \right) \tag{1.24}$$

for a suitable positive constant A and $x \to \infty$. Indeed, (1.24) follows from (5.6) below after a suitable modification of the path of integration and by appealing to the classical zero-free region of Korobov and Vinogradov. Details are similar to those in the proof of Theorem 12.2, [3], and can be omitted. Let us remark that (1.24) together with Theorem 1.2 and Corollary 1.4 imply Montgomery's estimate (1.2).

We can restate the Riemann Hypothesis in terms of $E^{\mathrm{AN}}(x)$ as follows.

Theorem 1.7. *The following statements are equivalent.*

(1) *The Riemann Hypothesis is true.*
(2) *There exists a positive constant A such that for $x \geqslant e^e$ we have*

$$E^{\mathrm{AN}}(x) \ll x^{\frac{1}{2}} \exp\left(A \frac{\log x}{\log\log x} \right). \tag{1.25}$$

(3) *For every $\varepsilon > 0$ and $x \geqslant 1$ we have*

$$E^{\mathrm{AN}}(x) \ll_\varepsilon x^{\frac{1}{2}+\varepsilon}.$$

The following result gives an unconditional lower estimate for $E^{\mathrm{AN}}(x)$.

Theorem 1.8. *For x tending to infinity we have*

$$E^{\mathrm{AN}}(x) = \Omega_{\pm}\bigl(x^{\frac{1}{2}} \log\log\log x\bigr).$$

2. Proof of Theorems 1.1 and 1.2

Consider the subsidiary function defined for $x \geqslant 0$ by the formula

$$R(x) = E(x) - xf(x). \tag{2.1}$$

We will prove Theorem 1.1 by showing that $R(x) = \int_0^x f(t)\,dt$ or all $x > 0$.

Let us observe that $R(x)$ is continuous. Indeed, for $x = 0$ or for positive x which are not integers it is evident. If $x = N \in \mathbb{N}$ then

$$R(N+0) - R(N-0) = \varphi(N) - N\bigl(f(N+0) - f(N-0)\bigr).$$

Since

$$\left\{\frac{N+0}{n}\right\} - \left\{\frac{N-0}{n}\right\} = \begin{cases} 0 & \text{if } n \nmid N, \\ -1 & \text{if } n \mid N, \end{cases}$$

we have

$$f(N+0) - f(N-0) = \sum_{n|N} \frac{\mu(n)}{n} = \prod_{p|N}\left(1 - \frac{1}{p}\right) = \frac{\varphi(N)}{N}.$$

Therefore $R(N+0) - R(N-0) = 0$, and continuity of $R(x)$ follows. Let now x be positive and not an integer. Taking derivative of both sides of (2.1) we find

$$R'(x) = -\frac{1}{\zeta(2)}x - f(x) - xf'(x).$$

For $x \notin \mathbb{N}$, $x > 0$, we have $\{x/n\}' = 1/n$, and hence $f'(x) = -1/\zeta(2)$. Consequently, we obtain

$$R'(x) = -f(x) \quad (x > 0,\ x \notin \mathbb{N}).$$

Hence

$$R(x) = -\int_0^x f(t)\,dt + R(0). \tag{2.2}$$

So far we proved (2.2) for $x > 0$, $x \notin \mathbb{N}$, but since both sides are continuous, equality holds for all positive x. Theorem 1.1 follows now from (2.1) and (2.2) since $R(0) = 0$.

Let us prove Theorem 1.2 now. By Theorem 1.1 it is enough to show that for $x \geqslant 1$ we have

$$\int_0^x f(t)\,dt = -\frac{1}{2}g(x) - \frac{1}{2}. \tag{2.3}$$

This can be done as follows. For $x > 0$ we have

$$\int_0^x \{t\}\,dt = \frac{1}{2}\{x\}^2 + \frac{1}{2}[x]$$

and hence recalling (1.10) we obtain

$$\int_0^x f(t)\,dt = -\sum_{n=1}^{\infty} \mu(n) \int_0^{x/n} \{t\}\,dt$$
$$= -\frac{1}{2}g(x) - \frac{1}{2}\sum_{n=1}^{\infty} \mu(n)\left[\frac{x}{n}\right].$$

But for $x \geqslant 1$ the last sum equals

$$\sum\sum_{nm \leqslant x} \mu(n) = \sum_{n \leqslant x}\sum_{d|n} \mu(d) = 1,$$

and (2.3) follows. The proof is complete.

3. Some subsidiary estimates needed for the proof of Theorem 1.3

Lemma 3.1. *Let $\alpha \in \mathcal{A}$. For $y \geqslant \max(xR^*(x,\alpha), \sqrt{x})$ we have*

$$f(x,\alpha) = \sum_{d \leqslant y} \frac{\alpha(d)}{d} s\left(\frac{x}{d}\right) + O\big(R^*(x,\alpha)\big).$$

Proof. For $y \geqslant x$ we have by partial summation

$$\sum_{n > y} \frac{\alpha(d)}{d^2} \ll \frac{1}{x} R(x,\alpha)$$

and hence

$$f(x,\alpha) = \sum_{d \leqslant y} \frac{\alpha(d)}{d} s\left(\frac{x}{d}\right) + \sum_{d > y} \frac{\alpha(d)}{d}\left(\frac{1}{2} - \frac{x}{d}\right)$$
$$= \sum_{d \leqslant y} \frac{\alpha(d)}{d} s\left(\frac{x}{d}\right) + O\big(R(x,\alpha)\big),$$

which is more than enough since $R(x,\alpha) \ll R^*(x,\alpha)$.

Suppose now that

$$\max\big(xR^*(x,\alpha), \sqrt{x}\big) \leqslant y < x. \tag{3.1}$$

Using the first part of the proof we write

$$f(x,\alpha)=\sum_{d\leqslant y}\frac{\alpha(d)}{d}s\left(\frac{x}{d}\right)+\sum_{y<d\ll x}\frac{\alpha(d)}{d}s\left(\frac{x}{d}\right)+O\big(R(x,\alpha)\big)$$

and split the range $y<d\ll x$ into $\ll x/y$ subranges $(x/k)\leqslant d<x/(k-1)$. In every such subrange, $s(x/d)$ is monotonic, and hence by partial summation

$$\sum_{(x/k)\leqslant d<x/(k-1)}\frac{\alpha(d)}{d}s\left(\frac{x}{d}\right)\ll R\left(\frac{x}{k},\alpha\right)\ll R(y,\alpha).$$

Since $R(y,\alpha)$ is monotonic as a function of y, and $y\geqslant\sqrt{x}$ by (3.1), the total contribution of these terms is

$$\ll\frac{x}{y}R(\sqrt{x},\alpha).$$

Since by (3.1) and (1.19) we have $y\geqslant xR^*(x,\alpha)\geqslant x\sqrt{R(\sqrt{x},\alpha)}$, the last expression is

$$\leqslant\sqrt{R(\sqrt{x},\alpha)}\leqslant R^*(x,\alpha).$$

This completes the proof. □

Lemma 3.2. *Let b and $r>0$ be relatively prime integers, and let β be a real number. Then for any positive N,*

$$\sum_{n=1}^{N}s\left(n\frac{b}{r}+\beta\right)=\frac{N}{r}s(r\beta)+O(r).$$

Proof. This is Lemma 3 in [14]. Note that there is a misprint in (13) of [14], and "$\ll$" there should read as "$=$". □

Lemma 3.3. *Let $\alpha\in\mathcal{A}$. For sufficiently large square-free integer*

$$q\leqslant\frac{1}{2}\min\big(R^*(N,\alpha)^{-1},\sqrt{N}\big)$$

and every real number $0<a<q$ we have

$$\sum_{n=1}^{N}f(nq+a,\alpha)=c(q,a,\alpha)N+O(N),$$

where

$$c(q,a,\alpha)=\left(\sum_{e|q}\frac{s(a/e)}{e}\sum_{f_1|e^\infty}\frac{\alpha(ef_1)}{f_1^2}\right)\sum_{(f_2,q)=1}\frac{\alpha(f_2)}{f_2^2}. \tag{3.2}$$

The notation $f_1\mid e^\infty$ means that all the prime divisors of f_1 divide e.

Proof. We apply Lemma 3.1 with $y = N$ and $x = nq + a$. Since $q \leqslant \sqrt{N}/2$ we have $\sqrt{nq+a} \leqslant \sqrt{(N+1)q} \leqslant N$. If $nq+a \leqslant N$ then obviously $(nq+a)R^*(nq+a,\alpha) \leqslant N$ for sufficiently large q. Otherwise, since $q \leqslant R^*(N,\alpha)^{-1}/2$, and using monotonicity of $R^*(x,\alpha)$, we have $(nq+a)R^*(nq+a,\alpha) \leqslant N$. Hence in all cases

$$N \geqslant \max\bigl((nq+a)R^*(nq+a,\alpha), \sqrt{nq+a}\bigr)$$

if q is large enough, so that we can apply Lemma 3.1. Consequently, using in addition Lemma 3.2 with $b = q/(q,d)$ and $\beta = a/d$ we obtain

$$\begin{aligned}\sum_{n=1}^{N} f(nq+a,\alpha) &= \sum_{n=1}^{N}\sum_{d=1}^{N} \frac{\alpha(d)}{d} s\left(\frac{nq+a}{d}\right) + O(N)\\ &= \sum_{d=1}^{N}\sum_{n=1}^{N} \frac{\alpha(d)}{d} s\left(\frac{nq+a}{d}\right) + O(N)\\ &= N\sum_{d=1}^{N} \frac{\alpha(d)(q,d)}{d^2} s\left(\frac{a}{(q,d)}\right) + O(N).\end{aligned}$$

Putting in the last sum $d = ef_1f_2$, where $e = (q,d)$, $f_1 \mid e^{\infty}$, $(f_2,q) = 1$ and using multiplicativity of α we arrive at (3.2) which ends the proof. $\square$

Lemma 3.4. *Let $\alpha \in \mathcal{A}$. Then there exists a positive constant $c_0 = c_0(\alpha)$ such that for every integer q we have*

$$\left|\sum_{(n,q)=1} \frac{\alpha(n)}{n^2}\right| \geqslant c_0.$$

Proof. Let

$$\sum_{n=1}^{\infty} \frac{\alpha(n)}{n^2} = \prod_p \sum_{l=0}^{\infty} \frac{\alpha(p^l)}{p^{2l}} = \prod_p A_p,$$

say. By (1.15) we have that $A_p \neq 0$ for all primes p. Using (1.14) it is easy to see that there exist $\theta' = \theta'(\alpha) > 1$ and $p_0 = p_0(\theta')$ such that

$$\sum_{l=1}^{\infty} \frac{|\alpha(p^l)|}{p^{2l}} \leqslant \frac{1}{p^{\theta'}}$$

for $p > p_0$. Then

$$\prod_{\substack{p>p_0\\ p\nmid q}} A_p \geqslant \prod_p \left(1 - \frac{1}{p^{\theta'}}\right) = \frac{1}{\zeta(\theta')}.$$

Moreover, let $p_1, \dots, p_r$ denote all primes $\leqslant p_0$ and let

$$b_0 := \min\Bigl(\min_{1\leqslant j\leqslant r} |A_{p_j}|, 1\Bigr).$$

Then

$$\left|\sum_{(n,q)=1}\frac{\alpha(n)}{n^2}\right| = \prod_{\substack{1\leqslant j\leqslant r\\ p_j\nmid q}}|A_{p_j}|\prod_{\substack{p>p_0\\ p\nmid q}}A_p \geqslant \frac{b_0^r}{\zeta(\theta')}$$

and the lemma follows. □

4. Proof of Theorem 1.3

Let x_ν be as in the definition of the class $\mathcal{A}$, see Section 1. Using (1.14) it is easy to see that there exist positive constants $\theta' = \theta'(\alpha)$ and $c_1 = c_1(\alpha)$ such that

$$\sum_{l=1}^{\infty}\frac{|\alpha(p^{l+1})|}{p^{2l}} \leqslant \frac{1}{p^{2\theta'}} \tag{4.1}$$

for all primes $p \geqslant c_1$. We have obviously

$$\sum_{|\alpha(p)|\leqslant p^{-\theta'}}\frac{|\alpha(p)|}{p} \ll 1$$

and hence according to (1.17) we have

$$\sum_{\substack{p\leqslant x_\nu\\ \alpha(p)<-p^{-\theta'}\\ p\equiv 3 \ (\mathrm{mod}\ 4)}}\frac{|\alpha(p)|}{p} \geqslant \eta\log\log x_\nu + O(1). \tag{4.2}$$

Let

$$q = q_\nu := \prod_{\substack{c_2(\nu)\leqslant p\leqslant x_\nu\\ \alpha(p)<-p^{-\theta'}\\ p\equiv 3 \ (\mathrm{mod}\ 4)}} p, \tag{4.3}$$

where $c_2(\nu) \geqslant c_1$ is chosen in such a way that $q \equiv 1 \pmod 4$ and $c_2(\nu) \ll 1$ as $\nu \to \infty$. For $p \mid q$ we have using (4.1) and inequality $|\alpha(p)| > p^{-\theta'}$

$$\begin{aligned}\sum_{l=0}^{\infty}\frac{\alpha(p^{l+1})}{p^{2l}} &= \alpha(p) + \vartheta_p\sum_{l=1}^{\infty}\frac{|\alpha(p^{l+1})|}{p^{2l}}\\ &= \alpha(p)\bigl(1+\vartheta'_p p^{-\theta'}\bigr)\end{aligned}$$

for certain $|\vartheta_p|, |\vartheta'_p| \leqslant 1$. Hence for every $e \mid q$ we have

$$\sum_{f_1|e^\infty}\frac{\alpha(ef_1)}{f_1^2}=\prod_{p|e}\sum_{l=0}^{\infty}\frac{\alpha(p^{l+1})}{p^{2l}}$$
$$=\alpha(e)\prod_{p|e}\bigl(1+\vartheta'_p p^{-\theta'}\bigr). \tag{4.4}$$

Let now suppose that ν is sufficiently large and put

$$N:=\min\left\{k\geqslant 1:\ \frac{1}{2}\min\bigl(R^*(k,\alpha)^{-1},\sqrt{k}\bigr)\geqslant q\right\},$$
$$a=q/4, \tag{4.5}$$

and

$$\varepsilon=\operatorname{sgn}\sum_{(n,q)=1}\frac{\alpha(n)}{n^2}.$$

Observe that for large ν, ε is independent of ν. This follows from the fact that factors A_p in the proof of Lemma 3.4 are positive for sufficiently large p's. By Lemma 3.3 we have

$$\sum_{n=1}^{N}\varepsilon f(nq+a,\alpha)=\varepsilon c(q,a,\alpha)N+O(N). \tag{4.6}$$

Observe that for $e\mid q$, the value $s(a/e)$ is positive (and equals $1/4$) if and only if $e\equiv 1 \pmod 4$, which happens exactly when e has an even number of prime factors. Hence

$$\operatorname{sgn}s(a/e)=\operatorname{sgn}\alpha(e),$$

and thus using in addition Lemma 3.4 and (4.4) we have

$$\varepsilon c(q,a,\alpha)=\bigl|c(q,a,\alpha)\bigr|\gg\sum_{e|q}\frac{|\alpha(e)|}{e}\prod_{p|e}\bigl(1+\vartheta'_p p^{-\theta'}\bigr)$$
$$=\prod_{p|q}\left(1+\frac{|\alpha(p)|}{p}\bigl(1+\vartheta'_p p^{-\theta'}\bigr)\right)\gg\exp\left(\sum_{p|q}\frac{|\alpha(p)|}{p}\right).$$

Using (4.2) and (4.3) this is $\gg(\log x_\nu)^\eta$. Now using $q=\exp(O(x_\nu))$ and recalling (4.5) we obtain

$$\log x_\nu\gg\log\log q\gg\min\left(\log\log\frac{1}{R^*(N-1,\alpha)},\log\log(N-1)\right)$$
$$=\log\log\frac{1}{R^*(N-1,\alpha)}.$$

The last equality follows from the inequality $R^*(N-1,\alpha)\geqslant 1/(N-1)$ which holds according to (1.19).

Gathering the above estimates and using

$$nq+a\leqslant(N+1)q\leqslant(N-1)^{3/2}$$

we obtain using (4.6)

$$\max_{1\leqslant x\leqslant (N-1)^{3/2}} \bigl(\varepsilon f(x,\alpha)\bigr) \gg \left(\log\log \frac{1}{R^*(N-1,\alpha)}\right)^{\eta} + O(1)$$

which gives

$$\varepsilon f(x,\alpha) = \Omega_+\left(\left(\log\log \frac{1}{R^*(x^{2/3},\alpha)}\right)^{\eta}\right).$$

Choosing $a = 3q/4$ we obtain Ω_--estimate and the result follows.

5. Proof of Theorem 1.8

It is convenient to derive the assertion from the results proved in [12].

Lemma 5.1. *With the notation* (1.7) *we have*

$$R_1(x) + R_2(x) = \Omega_\pm(\sqrt{x}\log\log\log x)$$

as $x \to \infty$.

Proof. Analogous result for $R_1(x)$ in the place of $R_1(x) + R_2(x)$ was established in [12]. The present lemma follows by repeating all steps in the proof of Theorem 1.1 in [12]. The required modifications are straightforward and shall not be described here in details. □

Lemma 5.2. *For* $\sigma > 2$ *we have*

$$\int_1^\infty E^{\mathrm{AN}}(x)x^{-s-1}\,dx = \frac{3}{\pi^2}\frac{1}{s-2} + \frac{\zeta(s-1)}{s(1-s)\zeta(s)}. \tag{5.1}$$

Proof. According to (1.11) we have $E^{\mathrm{AN}}(x) = E(x) - xf(x)$. We insert this into the integral in (5.1) and use the following identities which are easy to verify by a straightforward computations

$$\int_1^\infty E(x)x^{-s-1}\,dx = \frac{\zeta(s-1)}{s\zeta(s)} - \frac{3}{\pi^2}\frac{1}{s-2}$$

and

$$\int_1^\infty f(x)x^{-s}\,dx = \frac{\zeta(s-1)}{(s-1)\zeta(s)} - \frac{6}{\pi^2}\frac{1}{s-2}.$$

The lemma then follows. □

Lemma 5.3. *Suppose that a measurable, locally bounded function $h : [1, \infty) \to \mathbb{R}$ satisfies $h(x) = O(x^A)$, and $h(x) \leqslant Bx^a \log\log x$ or $h(x) \geqslant -Bx^a \log\log x$ for certain positive a, A and B and all large x. Moreover, let its Mellin transform $F(s) = \int_1^\infty h(x)x^{-s-1}\,dx$ be holomorphic on the interval $[a, A]$. Then $F(s)$ is holomorphic for $\sigma > a$ and*

$$F(s) \ll \frac{1}{\sigma - a} \log\left(\frac{1}{\sigma - a}\right)$$

uniformly for $a < \sigma < a + (1/2)$.

Proof. This is Corollary 3.2 in [10] or Lemma 2.1 in [12]. □

Lemma 5.4. *Suppose that*

$$\exists_{x_0, C_0 > 0}\ \forall_{x \geqslant x_0} \quad E^{\mathrm{AN}}(x) \leqslant C_0 x^{\frac{1}{2}} \log\log x \tag{5.2}$$

or

$$\exists_{x_0, C_0 > 0}\ \forall_{x \geqslant x_0} \quad E^{\mathrm{AN}}(x) \geqslant -C_0 x^{\frac{1}{2}} \log\log x. \tag{5.3}$$

Then the Riemann Hypothesis is true, all non-trivial zeros of the Riemann zeta function are simple, and denoting by $\rho = (1/2) + i\gamma$ a generic non-trivial zero we have

$$\frac{\zeta(\rho - 1)}{\zeta'(\rho)} \ll \gamma^2 \log|\gamma|. \tag{5.4}$$

Proof. We apply Lemmas 5.2 and 5.3 to $h(x) = E^{\mathrm{AN}}(x)$, $a = 1/2$, $A = 2$ and conclude that $\zeta(s-1)/\zeta(s)$ is holomorphic for $\sigma > (1/2)$. Moreover, we obtain that

$$\frac{\zeta(s-1)}{\zeta(s)} \ll \frac{|s|^2}{\sigma - (1/2)} \log \frac{1}{\sigma - (1/2)} \tag{5.5}$$

for $(1/2) < \sigma < 1$. Consequently, Riemann Hypothesis holds true and zeros on the critical line are simple. Estimate (5.4) follows from (5.5) as it was shown in the proof of Lemma 2.3 in [12]. □

Now we can conclude the proof of Theorem 1.8. We can assume that (5.2) and (5.3) are true since otherwise there is nothing left to be proved. Taking the inverse Mellin transform in (5.1) we obtain after some trivial computations

$$E^{\mathrm{AN}}(x) = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\zeta(s-1)}{\zeta(s)} \frac{x^s}{s(1-s)}\,ds, \tag{5.6}$$

where the path of integration $\mathcal{L}$ consists of the half-line $[3 - i\infty, 3 - 2i]$, the semi-circle $s = 3 + 2e^{i\theta}$, $3\pi/2 > \theta > \pi/2$ and the half-line $[3 + 2i, 3 + i\infty]$. Writing

$$\frac{1}{s(1-s)} = -\frac{1}{s^2} - \frac{1}{s^3} + \frac{1}{s^3(1-s)}$$

we split the last integral into three parts

$$-R_1(x) - R_2(x) + \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\zeta(s-1)}{\zeta(s)}\frac{x^s}{s^3(1-s)}\,ds = -R_1(x) - R_2(x) + I,$$

say. Shifting the line of integration to the left we have

$$I = \sum_{\rho} \frac{\zeta(\rho-1)}{\rho^3(1-\rho)\zeta'(\rho)}x^{\rho} + \frac{1}{2\pi i}\int_{(1/4)-i\infty}^{(1/4)+i\infty} \frac{\zeta(s-1)}{\zeta(s)}\frac{x^s}{s^3(1-s)}\,ds.$$

According to Lemma 5.4, the sum over zeros is

$$\ll x^{1/2}\sum_{\rho}\frac{\log|\gamma|}{\gamma^2} \ll x^{1/2},$$

whereas the integral is

$$x^{1/4}\int_{(1/4)-i\infty}^{(1/4)+i\infty} |s|^{-3+\varepsilon}\,|ds| \ll x^{1/4},$$

and hence

$$E^{\mathrm{AN}}(x) = -R_1(x) - R_2(x) + O\big(x^{1/2}\big).$$

The assertion now follows from Lemma 5.1 and Theorem 1.2.

6. Proof of Theorem 1.7

Let us assume that the Riemann Hypothesis is true. Then shifting the line of integration in (5.6) we obtain

$$E^{\mathrm{AN}}(x) = \frac{1}{2\pi i}\left(\int_{(5/2)-ix^2}^{(1/2)+\eta-ix^2} + \int_{(1/2)+\eta-ix^2}^{(1/2)+\eta+ix^2} + \int_{(1/2)+\eta+ix^2}^{(5/2)+ix^2}\right)\frac{\zeta(s-1)}{\zeta(s)}\frac{x^s}{s(1-s)}\,ds + O\big(x^{1/2}\big),$$

where $\eta = 1/\log(2\log x)$. We estimate these integrals exactly in the same way as in the proof of Theorem 1.2 in [12] getting (1.25). Hence (1) implies (2). Moreover, (3) follows from (2) in a trivial way. To conclude the proof let us remark that if $E^{\mathrm{AN}}(x) \ll x^{(1/2)+\varepsilon}$ then the integral in (5.1) defines for $\Re(s) > 1/2$ a holomorphic function and hence $\zeta(s)$ has no zeros in this half-plane. We see therefore that (3) implies (1) and the result follows.

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