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From quadratic polynomials and continued fractions to modular forms

Paloma Bengoechea

Department of Mathematics, University of York, York, YO10 5DD, United Kingdom

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ABSTRACT

We study certain real functions defined in a very simple way by Zagier as sums of powers of quadratic polynomials with integer coefficients. These functions give the even parts of the period polynomials of the modular forms which are the coefficients in the Fourier expansion of the kernel function for the Shimura–Shintani correspondence. We give three different representations of these sums in terms of a finite set of polynomials coming from reduction of binary quadratic forms and in terms of the infinite set of transformations occurring in a continued fraction algorithm of the real variable. We deduce the exponential convergence of the sums, which was conjectured by Zagier as well as one of the three representations.

1. Introduction

In [14], D. Zagier studied a class of functions which are defined as sums over certain quadratic polynomials with integer coefficients, and discovered that these functions have many surprising properties. That is, for $x \in \mathbb{R}$, Zagier considered sums over all quadratic functions $Q(X) = aX^2 + bX + c$ with integer coefficients and fixed discriminant that

E-mail address: paloma.bengoechea@york.ac.uk.

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are negative at infinity (which for convenience we write as $Q(\infty) < 0$) and positive at x. For example, Zagier found that:

Theorem 1.1. *Let D be a positive non-square discriminant. Then the sum*

$$A_D(x) := \sum_{\substack{\operatorname{discr}(Q)=D \\ Q(\infty)<0<Q(x)}} Q(x) \tag{1}$$

converges for all $x \in \mathbb{R}$ and has a constant value α_D independent of x.

For example, one finds $A_5(x) = 2$, and more generally $\alpha_D = -5L(-1, \chi_D)$, where $L(s, \chi_D)$ is the Dirichlet L-series of the character $\chi_D(n) = (D/n)$ (Kronecker symbol). We denote

$$\mathcal{Q} = \left\{aX^2 + bX + c \;\middle|\; a, b, c \in \mathbb{Z},\, b^2 - 4ac > 0,\, b^2 - 4ac \text{ not a square}\right\},$$

and for any positive non-square discriminant D,

$$\mathcal{Q}_D = \left\{aX^2 + bX + c \in \mathcal{Q} \;\middle|\; b^2 - 4ac = D\right\},$$
$$\mathcal{Q}_D\langle x\rangle = \left\{Q \in \mathcal{Q}_D \;\middle|\; Q(\infty) < 0 < Q(x)\right\}.$$

The sum (1) is the special case $k = 2$ of the function

$$A_{k,D}(x) := \sum_{Q\in\mathcal{Q}_D\langle x\rangle} Q(x)^{k-1} \quad (k \geq 2). \tag{2}$$

Theorem 1.1 is still true for $k = 4$ with α_D now replaced by $L(-3, \chi_D)$, but this fails for even $k \geq 6$ because of the existence of cusp forms of weight $2k$ on the full modular group. More explicitly, for even $k \geq 6$, the function $A_{k,D}(x)$ is a linear combination of a constant function and the functions $\sum_{n\geq 1} n^{1-2k} a_f(n) \cos(2\pi nx)$, where f runs over the normalized Hecke eigenforms in $S_{2k}(\mathrm{PSL}(2,\mathbb{Z}))$ and $a_f(n)$ denotes the n-th Fourier coefficient of f. The function $A_{k,D}(x)$ arose from studying the modular form of weight $2k$ which is the D-th coefficient in the Fourier expansion of the kernel function for the Shimura and Shintani lifts between half-integral and integral weight cusp forms [8,9]. The function $A_{k,D}(x)$ is (modulo a constant multiple of $X^{2k-2} - 1$) the even part of its Eichler integral on $\mathbb{R}$ and so gives the even part of its period polynomial (see [6,7,14] or [2] for a recent proof).

The convergence of $A_{k,D}(x)$ is not immediate at all. As Zagier observed, one has $Q(x) = O(1/a_Q)$ for all the Q occurring in (2) and one sees easily that there are only $O(1)$ quadratic functions Q for each a-value. Therefore the series (2) converges at most like $\sum_{a>0} a^{1-k}$ if $k \geq 4$. In the case $k = 2$, however, this argument fails and Zagier could only deduce the convergence of (1) from the fact that the function is finite and constant ($= -5L(-1, \chi_D)$) when x is rational. In fact, if for some value of x the sum diverged, then, since all summands are positive, there would be finitely many quadratic functions Q whose sum at x already exceeded $-5L(-1,\chi_D)$, and the sum of their values at a sufficiently nearby rational number would also exceed $-5L(-1,\chi_D)$.

Following this proof, the sum might converge extremely slowly. However, experiments carried out for the value $x = 1/\pi$ suggested that the series (1) (and hence also the series (2)) converge extremely rapidly. More precisely, for $x = 1/\pi$ and $D = 5$, Zagier found experimentally that the elements of $\mathcal{Q}_D\langle x\rangle$ belong to the union of two lists, each having values that tend to 0 exponentially quickly. Each quadratic function Q in each list is obtained from the preceding one by applying a fairly simple element of $\mathrm{PGL}(2,\mathbb{Z})$ giving the positive continued fraction of x. The two lists start with the opposite of the simple forms coming from the reduction theory of binary quadratic forms with fixed discriminant 5: $[1,-1,-1]$ and $[1,1,-1]$.

The following tables give the first five functions Q, and the corresponding values of $Q(1/\pi)$ in each list. The first five functions produced by the first list appear in the sum $A_5(x)$. This is not the case for the functions in the second list, for which we have chosen to include only the ones appearing in the sum.

Q	$Q(1/\pi)$
$[-1,1,1]$	1.216989
$[-11,7,-1]$	0.113636
$[-541,345,-55]$	0.00215
$[-117\,731,74\,951,-11\,929]$	0.000008
$[-133\,351,84\,893,-13\,511]$	0.000008
Sum:	1.332791

Q	$Q(1/\pi)$
$[-1,-1,1]$	0.580369
$[-5,5,-1]$	0.084943
$[-409,259,-41]$	0.001896
$[-5\,959\,340\,757\,998\,441,3\,793\,834\,156\,817\,819,-603\,807\,459\,328\,429]$	6.856501 E-17
$[-7\,755\,390\,254\,828\,071,493\,723\,477\,865\,040,-785\,785\,320\,227\,431]$	1.568047 E-18
Sum:	0.667208

Zagier conjectured that all the functions in one list and some in the second one occur in $A_5(x)$, but he found no criterion to decide which ones. He suggested there is a similar situation in the general case.

We prove Zagier's conjectures, giving a criterion for the second list, and establish a similar result for the general case, obtaining the exponential convergence for $A_{k,D}(x)$ and a direct proof for the convergence in the case $k=2$. Marie Jameson addresses the same sort of questions in [5] in very different terms. We also prove other descriptions of $A_{k,D}(x)$ analogous to the one conjectured by Zagier: first we replace the simple forms with the reduced forms and the usual algorithm of positive continued fraction with a different one (with sign + as well). Later we keep the simple forms but we consider the (usual) negative continued fraction. We give correspondences between all these situations. This is done in the third section.

In the fourth section we prove an unexpected result: the values at x of the functions in the lists that do not appear in the sum $A_{k,D}(x)$ cancel each other out. Thus the sum over all the functions in the lists is equal to $A_{k,D}(x)$. A similar phenomenon holds for the description with the negative continued fraction. This is a consequence of the more general Corollary 4.2 which gives the even part of the Eichler integral on $x \in \mathbb{R}$ of a cusp form for $\mathrm{PSL}(2,\mathbb{Z})$ in terms of the even part of its period polynomial and the continued fraction of x.

2. Reduction theories

In this section we give the main connections between reduction theory of binary quadratic forms with positive non-square discriminant for $\mathrm{PSL}(2,\mathbb{Z})$ and the continued fractions of a real number. We also recall some simple properties of positive and negative continued fractions and give the nonstandard algorithm that we use in the next sections.

We denote

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

The matrices ε, σ, T generate the group $\Gamma = \mathrm{PGL}(2,\mathbb{Z})$ and S, T generate the group $\Gamma_1 = \mathrm{PSL}(2,\mathbb{Z})$. Clearly

$$\varepsilon^2 = \sigma^2 = S^2 = \varepsilon\sigma S = 1$$

so $\{1, \varepsilon, \sigma, S\}$ form a Klein 4-group. We write $\hat{\Gamma}$ to mean Γ or Γ_1. The group $\hat{\Gamma}$ acts on the projective line by fractional linear transformations:

$$\gamma(x) := \frac{rx+s}{tx+u} \quad \left(\gamma = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \hat{\Gamma}\right),$$

and on the set of binary quadratic forms by

$$Q|\gamma(X,Y) = Q(rX+sY, tX+uY). \tag{3}$$

For a positive non-square fixed discriminant D, a *reduction theory* of binary quadratic forms with discriminant D for the action of the group Γ_1 consists in giving a finite system $\mathcal{R}$ of such forms (called a system of reduced forms) and an algorithm such that:

(i) each form with discriminant D is Γ_1-equivalent to some element of $\mathcal{R}$ by applying the algorithm a finite number of times;

(ii) the image by the algorithm of an element of $\mathcal{R}$ still belongs to $\mathcal{R}$. In other words, the elements of $\mathcal{R}$ form cycles such that two elements of $\mathcal{R}$ are Γ_1-equivalent if and only if they belong to the same cycle. In particular, the number of cycles is the number of Γ_1-equivalence classes for D.

Usually (and in all cases we consider) the reduction algorithm for binary quadratic forms $Q = [a, b, c]$ $(:= aX^2 + bXY + cY^2)$ is obtained by applying a reduction algorithm for real numbers (usually some sort of continued fraction algorithm) to one of the roots of $Q(X, 1)$ or $Q(X, -1)$. We denote by

$$w_Q = \frac{-b - \sqrt{D}}{2a}, \qquad w'_Q = \frac{-b + \sqrt{D}}{2a}$$

the two roots of $Q(X, 1)$ ($\sqrt{D}$ denotes the positive square root).

We use two different sets $\mathcal{R}$ of forms $[a, b, c]$:

$$a > 0, \qquad c > 0, \qquad b > a + c$$

which we call *reduced* (see [13] for details), and

$$a > 0 > c$$

which we call *simple*.

The bijection $[a, b, c] \mapsto [-c, -b, -a]$ exchanges the simple forms with positive and negative values of $a + b + c$ (the value 0 cannot occur for non-square D). The bijection

$$\begin{gathered} \text{reduced} \to \text{simple with } a + b + c > 0 \\ [a, b, c] \mapsto [a, b - 2a, c - b + a] \end{gathered}$$

proves that there are exactly half as many reduced forms as simple forms. We denote by Red and Sim the sets of quadratic polynomials that correspond to the reduced and simple forms respectively

$$\begin{aligned} \mathrm{Red} &= \bigl\{Q(X) \in \mathcal{Q} \bigm| Y^2 Q(X/Y) \text{ is reduced}\bigr\}, \\ \mathrm{Sim} &= \bigl\{Q(X) \in \mathcal{Q} \bigm| Y^2 Q(X/Y) \text{ is simple}\bigr\}. \end{aligned}$$

We can translate the inequalities for the coefficients of simple or reduced forms into inequalities for the corresponding quadratic irrationalities:

$$\begin{aligned} Q(X, Y) \text{ reduced} \quad &\Longleftrightarrow \quad w_Q < -1 < w'_Q < 0, \\ Q(X, Y) \text{ simple} \quad &\Longleftrightarrow \quad w_Q < 0 < w'_Q. \end{aligned}$$

The positive and negative continued fraction of a real number x, denoted by

$$x = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\ddots}}} \qquad (n_i \in \mathbb{Z},\ n_i \geq 1\ \forall i \geq 1),$$

$$x = m_0 - \cfrac{1}{m_1 - \cfrac{1}{m_2 - \cfrac{1}{\ddots}}} \quad (m_i \in \mathbb{Z},\, m_i \geq 2\ \forall i \geq 1),$$

are produced by the algorithms

$$x_0 = x, \qquad n_i = \lfloor x_i \rfloor, \qquad x_{i+1} = \frac{1}{x_i - n_i} = \varepsilon T^{-n_i}(x_i) \quad (i \geq 0), \tag{4}$$

$$x_0 = x, \qquad m_i = \lceil x_i \rceil, \qquad x_{i+1} = \frac{1}{m_i - x_i} = ST^{-m_i}(x_i) \quad (i \geq 0), \tag{5}$$

where $\lfloor x \rfloor$ and $\lceil x \rceil$ are respectively the floor and ceiling parts. The positive and negative continued fractions of w_Q are periodic for a quadratic form Q. Moreover, the negative continued fraction of $-w_Q$ is purely periodic if and only if Q is reduced. Thus the cycle of reduced forms that are equivalent to a given form Q corresponds to the cycle of real quadratic irrationalities x_i given by the algorithm (5) with $x_0 = -w_Q$.

In a similar way, a quadratic form Q is simple if and only if $-w_Q$ is purely periodic for the algorithm (see [3])

$$x_0 = x, \qquad x_{i+1} = \begin{cases} x_i + 1 = T(x_i) & \text{if } x_i \leq 0, \\ \frac{x_i}{1-x_i} = T^{-1}ST^{-1}(x_i) & \text{if } 0 < x_i < 1, \\ x_i - 1 = T^{-1}(x_i) & \text{if } x_i \geq 1. \end{cases}$$

This gives an expansion in negative continued fraction that is slower than the usual one.

The cycle of simple forms which are equivalent to a given form Q corresponds to the cycle of real quadratic irrationalities x_i given by the algorithm above with $x_0 = -w_Q$.

Clearly each x_i in (4) is the image of x by a matrix $\gamma_i = \gamma_{i,x} \in \Gamma$, given explicitly by

$$\gamma_0 := \mathrm{Id}, \qquad \gamma_i := \begin{pmatrix} 0 & 1 \\ 1 & -n_{i-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & -n_0 \end{pmatrix} \quad (i \geq 1)$$

and recursively by

$$\gamma_0 = \mathrm{Id}, \qquad \gamma_{i+1} = \varepsilon T^{-n_i} \gamma_i \quad (i \geq 0).$$

We denote

$$\Gamma(x) := \{\gamma_1, \gamma_2, \gamma_3, \ldots\} \subset \Gamma.$$

There is an explicit description of $\Gamma(x)$ in terms of the convergents of x. The i-th convergent of x is denoted by $\frac{p_i}{q_i} = [n_0, \ldots, n_i]$. The integers p_i and q_i satisfy the recurrence

$$p_{-2} = 0, \qquad p_{-1} = 1, \qquad p_i = n_i p_{i-1} + p_{i-2} \quad (i \geq 0),$$

$$q_{-2} = 1, \qquad q_{-1} = 0, \qquad q_i = n_i q_{i-1} + q_{i-2} \quad (i \geq 0),$$

the equation

$$p_{i+1}q_i - p_i q_{i+1} = (-1)^i$$

and the inequalities

$$q_i \geq q_{i-1} \geq 0 \quad (i \geq 0), \qquad q_i > q_{i-1} > 0 \quad (i \geq 2), \tag{6}$$

$$|p_i| \geq |p_{i-1}| \quad (i \geq 2), \qquad |p_i| > |p_{i-1}| \quad (i \geq 4). \tag{7}$$

The numbers δ_i $(i \geq -1)$ defined by

$$\delta_i = (-1)^i (p_{i-1} - q_{i-1}x)$$

satisfy the recurrence

$$\delta_{-1} = x, \qquad \delta_0 = 1, \qquad \delta_{i+1} = -n_i\delta_i + \delta_{i-1} \quad \text{with } n_i = \left\lfloor \frac{\delta_{i-1}}{\delta_i} \right\rfloor$$

and the inequalities $1 = \delta_0 > \delta_1 > \ldots \geq 0$. If x is rational, then $x_i = p_i/q_i$ for some i and the recurrence stops with $\delta_{i+1} = 0$; if x is irrational, the δ_i are all positive and converge to 0 with exponential rapidity. With these notations, one has

$$\gamma_i^{-1} = \begin{pmatrix} p_{i-1} & p_{i-2} \\ q_{i-1} & q_{i-2} \end{pmatrix}, \qquad \gamma_i \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} \delta_{i-1} \\ \delta_i \end{pmatrix}.$$

Now we consider the slower version of the algorithm of reduction (4)

$$x_0 = x, \qquad x_{i+1} = \begin{cases} x_i + 1 = T(x_i) & \text{if } x_i \leq 0, \\ \frac{1}{x_i} - 1 = T^{-1}\varepsilon(x_i) & \text{if } 0 < x_i \leq 1, \\ x_i - 1 = T^{-1}(x_i) & \text{if } x_i > 1, \end{cases} \tag{8}$$

such that with this algorithm the expansion of x in continued fraction is

$$x = \underbrace{\pm 1 \pm \cdots \pm 1}_{|n_0|} + \cfrac{1}{\underbrace{1 + \cdots + 1}_{n_1} + \cfrac{1}{\underbrace{1 + \cdots + 1}_{n_2} + \cfrac{1}{\ddots}}}$$

where $n_0, n_1, n_2, \ldots,$ are given in (4) and each $\pm$ equals the sign of n_0.

Each x_i in algorithm (8) is the image of x by a matrix $\gamma_i' = \gamma_{i,x}' \in \Gamma$ given recursively by

$$\gamma_0' = \mathrm{Id}, \qquad \gamma_{i+1}' = \begin{cases} T\gamma_i' & \text{if } x_i \leq 0, \\ T^{-1}\varepsilon\gamma_i' & \text{if } 0 < x_i \leq 1, \\ T^{-1}\gamma_i' & \text{if } x_i > 1. \end{cases}$$

We denote

$$\Gamma(x)' := \{\gamma'_{|n_0|+1}, \gamma'_{|n_0|+2}, \gamma'_{|n_0|+3}, \dots\} \subset \Gamma$$

so in $\Gamma(x)'$ we are missing the first several γ'_i which arise from the n_0 term. We note that

$$\Gamma(x)' = \{T^{-k}\gamma_i,\ 1 \le k \le n_i\}_{i\ge 1}.$$

The following two propositions, whose proofs are given in [1], describe the sets $\Gamma(x)$ and $\Gamma(x)'$ for any non-integer x as subsets of elements of Γ defined by certain simple linear inequalities:

Proposition 2.1. *For any non-integer x, the set $\Gamma(x)$ equals $W - (W_1 \cup W_2)$, where*

$$W = \{\gamma \in \Gamma \mid -1 \le \gamma(\infty) \le 0,\, \gamma(x) > 1\},$$

$$W_1 = \{\gamma \in W \mid \gamma(\infty) = 0,\, \det(\gamma) = 1\} = \left\{\begin{pmatrix} 0 & -1 \\ 1 & -1-n_0 \end{pmatrix}\right\},$$

$$W_2 = \{\gamma \in W \mid \gamma(\infty) = -1,\, \det(\gamma) = -1\} = \begin{cases} \{\left(\begin{smallmatrix} -1 & 1+n_0 \\ 1 & -n_0 \end{smallmatrix}\right)\} & \textit{if } n_1 \ge 2, \\ \emptyset & \textit{if } n_1 = 1. \end{cases}$$

Proposition 2.2. *For any non-integer x, the set $\Gamma(x)'$ equals $W' - W'_1$, where*

$$W' = \{\gamma \in \Gamma \mid \gamma(\infty) \le -1,\, \gamma(x) > 0\}$$

and

$$W'_1 = \{\gamma \in W' \mid \gamma(\infty) = -1,\, \det(\gamma) = 1\} = \left\{\begin{pmatrix} 1 & -n_0 \\ -1 & n_0+1 \end{pmatrix}\right\}.$$

Remark 2.3. If $x \in \mathbb{Z}$, then $\Gamma(x) = W - \left(\begin{smallmatrix} -1 & 1+x \\ 1 & -x \end{smallmatrix}\right)$ and $\Gamma(x)' = W'$.

Each x_i in (5) is the image of x by a matrix $\tilde{\gamma}_i = \tilde{\gamma}_{i,x} \in \Gamma_1$ defined by

$$\tilde{\gamma}_0 = \mathrm{Id}, \qquad \tilde{\gamma}_{i+1} = ST^{-m_i}\tilde{\gamma}_i = \begin{pmatrix} -\tilde{q}_{i-1} & \tilde{p}_{i-1} \\ -\tilde{q}_i & \tilde{p}_i \end{pmatrix} \quad (i \ge 0), \tag{9}$$

where $\frac{\tilde{p}_i}{\tilde{q}_i}$ is the i-th negative convergent of x. The set of matrices from (9) will be denoted by

$$\Gamma_1(x) := \{\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3, \dots\} \subset \Gamma_1.$$

The integers $\tilde{p}_i$ and $\tilde{q}_i$ satisfy the recurrence

$$\tilde{p}_{-2} = 0, \qquad \tilde{p}_{-1} = 1, \qquad \tilde{p}_i = m_i \tilde{p}_{i-1} - \tilde{p}_{i-2} \quad (i \geq 0),$$
$$\tilde{q}_{-2} = -1, \qquad \tilde{q}_{-1} = 0, \qquad \tilde{q}_i = m_i \tilde{q}_{i-1} - \tilde{q}_{i-2} \quad (i \geq 0),$$

the equation

$$\tilde{p}_i \tilde{q}_{i+1} - \tilde{p}_{i+1} \tilde{q}_i = 1,$$

and the inequalities

$$\tilde{q}_i \geq \tilde{q}_{i-1} \geq 0 \quad (i \geq 0), \qquad \tilde{q}_i > \tilde{q}_{i-1} > 0 \quad (i \geq 1), \tag{10}$$
$$|\tilde{p}_i| \geq |\tilde{p}_{i-1}| \quad (i \geq 1). \tag{11}$$

In a similar way to the positive continued fraction, the numbers $\tilde{\delta}_i$ $(i \geq -1)$ defined by

$$\tilde{\delta}_i = \tilde{p}_{i-1} - \tilde{q}_{i-1} x$$

satisfy the recurrence

$$\tilde{\delta}_{-1} = x, \qquad \tilde{\delta}_0 = 1, \qquad \tilde{\delta}_{i+1} = m_i \tilde{\delta}_i - \tilde{\delta}_{i-1} \quad \text{with } m_i = \left\lceil \frac{\tilde{\delta}_{i-1}}{\tilde{\delta}_i} \right\rceil$$

and the inequalities $1 = \tilde{\delta}_0 > \tilde{\delta}_1 > \ldots \geq 0$. If x is rational, then $x_i = \tilde{p}_i / \tilde{q}_i$ for some i and the recurrence stops with $\tilde{\delta}_{i+1} = 0$; if x is irrational, the $\tilde{\delta}_i$ are all positive and converge to 0 with exponential rapidity.

3. Combining reduction theories: proofs of the conjectures

For $d \in 2\mathbb{N}$, the group $\hat{\Gamma}$ acts on homogeneous polynomials of degree d by (3) or, equivalently, on the space of polynomials of degree $\leq d$ in one variable by

$$(P|_{-d}\gamma)(x) := (tx + u)^d P\left(\frac{rx + s}{tx + u}\right) \quad \left(\gamma = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \hat{\Gamma}\right).$$

We write

$$\mathcal{F}_d = \left\{ P \in \mathbb{Z}[X]_{\leq d} \;\middle|\; P \text{ has exactly 2 real roots, both irrational} \right\}.$$

Given $P \in \mathcal{F}_d$, we denote by w_P and w'_P its two real roots such that

$$\operatorname{sign}\bigl(P(\infty)\bigr) \cdot w_P < \operatorname{sign}\bigl(P(\infty)\bigr) \cdot w'_P.$$

If $\mathcal{A}$ is a $\hat{\Gamma}$-equivalence class in $\mathcal{F}_d$, we define

$$\begin{aligned}\mathcal{A}^{\mathrm{Red}} &= \{P \in \mathcal{A} \mid P(\infty) > 0,\, P(-1) < 0 < P(0)\} = \{P \in \mathcal{A} \mid w_P < -1 < w'_P < 0\},\\ \mathcal{A}^{\mathrm{Sim}} &= \{P \in \mathcal{A} \mid P(0) < 0 < P(\infty)\} = \{P \in \mathcal{A} \mid w_P < 0 < w'_P\},\\ \mathcal{A}\langle x\rangle &= \{P \in \mathcal{A} \mid P(\infty) < 0 < P(x)\}.\end{aligned}$$

In the special case $d = 2$, we have $\mathcal{F}_2 = \mathcal{Q}$ and

$$\mathcal{A}^{\mathrm{Red}} = \mathcal{A} \cap \mathrm{Red}, \qquad \mathcal{A}^{\mathrm{Sim}} = \mathcal{A} \cap \mathrm{Sim}\,.$$

In this case, $\mathcal{A}^{\mathrm{Red}}$ and $\mathcal{A}^{\mathrm{Sim}}$ are both finite. If $\mathcal{A}$ is a $\hat{\Gamma}$-equivalence class in $\mathcal{Q}$, we define

$$A_{k,\mathcal{A}}(x) = \sum_{Q \in \mathcal{A}\langle x\rangle} Q(x)^{k-1} \quad (x \in \mathbb{R},\, k \ge 2).$$

Then the sum $A_{k,D}$ defined in (2) is given by

$$A_{k,D}(x) = \sum_{\mathcal{A} \in \mathcal{Q}_D/\hat{\Gamma}} A_{k,\mathcal{A}}(x). \tag{12}$$

The two theorems below are stated in the general case $\mathcal{F}_d$ with even $d \ge 2$.

Theorem 3.1. *For $d \in 2\mathbb{N}$, $\mathcal{A}$ a Γ-equivalence class in $\mathcal{F}_d$ and $x \in \mathbb{R}$, the following bijection holds*

$$\begin{aligned}\left\{\begin{array}{c}(P,\gamma) \in \mathcal{A}^{\mathrm{Sim}} \times \Gamma(x) : \\ P\big(\gamma(\infty)\big) < 0 < P\big(\lfloor \gamma(x) \rfloor\big)\end{array}\right\} &\xrightarrow{\cong} \mathcal{A}\langle x\rangle\\ (P,\gamma) &\mapsto P|\gamma.\end{aligned}$$

Proof. We only have to check that $P(\lfloor \gamma(x)\rfloor) > 0$ implies $P(\gamma(x)) > 0$ to prove that the map is well defined. This follows from $P(\lfloor \gamma(x)\rfloor) > 0$ and $\lfloor \gamma(x)\rfloor > 0$, which imply $\lfloor \gamma(x)\rfloor > w'_P$, so $\gamma(x) > w'_P$, and thus $P(\gamma(x)) > 0$.

We now prove that the map is a bijection. Let $P \in \mathcal{A}$ satisfy $P(\infty) < 0 < P(x)$. For $j \gg 0$, the convergents $\frac{p_j}{q_j}$ of x belong to (w'_P, w_P), because $x \in (w'_P, w_P)$. Moreover, if two consecutive convergents $\frac{p_j}{q_j}$ and $\frac{p_{j+1}}{q_{j+1}}$ belong to (w'_P, w_P), so do all the later convergents.

If $\lfloor x\rfloor \notin (w'_P, w_P)$, we define i to be the unique positive integer such that $\frac{p_{i-1}}{q_{i-1}} \notin (w'_P, w_P)$ but $\frac{p_i}{q_i}, \frac{p_{i+1}}{q_{i+1}} \in (w'_P, w_P)$. If $\lfloor x\rfloor \in (w'_P, w_P)$, we set $i = 0$. Since $P(\infty) < 0$, in both cases we have

$$P\left(\frac{p_{i-1}}{q_{i-1}}\right) < 0 < P\left(\frac{p_i}{q_i}\right), \qquad P\left(\frac{p_{i+1}}{q_{i+1}}\right) > 0. \tag{13}$$

Put

$$\gamma = \begin{pmatrix} q_{i-1} & -p_{i-1} \\ -q_i & p_i \end{pmatrix},$$

and $R = P|\gamma^{-1}$. We have

$$R(0) = q_{i-1}^d P\left(\frac{p_{i-1}}{q_{i-1}}\right) < 0, \qquad R(\infty) = q_i^d P\left(\frac{p_i}{q_i}\right) > 0.$$

The inequality $P(\frac{p_{i+1}}{q_{i+1}}) > 0$ is equivalent to the condition $R(\lfloor\gamma(x)\rfloor) > 0$ because $q_{i+1}^d P(\frac{p_{i+1}}{q_{i+1}}) = R(n_{i+1})$ and $n_{i+1} = \lfloor\gamma(x)\rfloor$. The condition $P(\infty) < 0$ is equivalent to the condition $R(\gamma(\infty)) < 0$. Thus (R, γ) belongs to the left hand set of the map in Theorem 3.1.

The uniqueness of the preimage (R, γ) follows from the equivalence between the condition $R(\lfloor\gamma(x)\rfloor) > 0$ and the inequality $P(\frac{p_{i+1}}{q_{i+1}}) > 0$, together with the uniqueness of i satisfying (13). $\square$

Note that we did not use the fact that $\Gamma(x)$ comes from the continued fraction of x to prove that the map above is well defined, but rather the description given by Proposition 2.1. The argument for the bijectivity is in fact a "local" phenomenon: we did not need the whole continued fraction of x, but only three consecutive convergents. One could certainly prove Theorem 3.1 without using continued fractions and using the description for $\Gamma(x)$ with linear inequalities, but the proof given here seemed to the author to be simple and attractive.

Remark 3.2. If we replace $\lfloor\gamma(x)\rfloor$ by $\gamma(x)$ in the above definition, then the map

$$\begin{array}{rcl} \left\{ \begin{array}{c} (P, \gamma) \in \mathcal{A}^{\mathrm{Sim}} \times \Gamma(x) : \\ P\big(\gamma(\infty)\big) < 0 < P\big(\gamma(x)\big) \end{array} \right\} & \longrightarrow & \mathcal{A}\langle x\rangle \\ (P, \gamma) & \mapsto & P|\gamma \end{array}$$

is still surjective but in general not injective.

Theorem 3.3. *For $d \in 2\mathbb{N}$, $\mathcal{A}$ a Γ-equivalence class in $\mathcal{F}_d$ and $x \in \mathbb{R}$, the following bijection holds*

$$\begin{array}{rcl} \big\{(P, \gamma) \in \mathcal{A}^{\mathrm{Red}} \times \Gamma(x)' : P\big(\gamma(\infty)\big) < 0\big\} & \xrightarrow{\cong} & \mathcal{A}\langle x\rangle \\ (P, \gamma) & \mapsto & P|\gamma. \end{array}$$

Proof. Note that the above map can be expressed as the composition of the map from Theorem 3.1 together with the map ψ given by

$$\big\{(P,\gamma)\in\mathcal{A}^{\mathrm{Red}}\times\Gamma(x)':P\big(\gamma(\infty)\big)<0\big\}\xrightarrow{\psi}\left\{\begin{array}{c}(P,\gamma)\in\mathcal{A}^{\mathrm{Sim}}\times\Gamma(x):\\ P\big(\gamma(\infty)\big)<0<P\big(\lfloor\gamma(x)\rfloor\big)\end{array}\right\}$$
$$\big(P,T^{-k}\gamma_i\big)\ \mapsto\ \big(P|T^{-k},\gamma_i\big).$$

Thus by Theorem 3.1, we need only to show that ψ is well-defined and bijective. To prove that ψ is well-defined we only have to check, given $(P,\gamma)=\psi(R,\tilde\gamma)$ with $\tilde\gamma=T^{-k}\gamma$, two conditions: $P\in\mathcal{A}^{\mathrm{Sim}}$ and $P(\lfloor\gamma(x)\rfloor)>0$. We have

$$w_P=w_R+k,\qquad w'_P=w'_R+k.$$

From $w'_R>-1$ and $k\geq 1$, we deduce that $w'_P>0$. The inequality $R(\tilde\gamma(\infty))<0$ and the equality $\tilde\gamma(\infty)=-\frac{q_{i-2}}{q_{i-1}}-k$ imply $-w_R>\frac{q_{i-2}}{q_{i-1}}+k$, where each term is positive, so $k<-w_R$, and thus $w_P<0$. Hence $P\in\mathcal{A}^{\mathrm{Sim}}$. The condition $P(\lfloor\gamma(x)\rfloor)>0$ follows from

$$\lfloor\gamma(x)\rfloor=\left\lfloor\frac{\delta_{i-1}}{\delta_i}\right\rfloor=n_i\geq k=w'_P-w'_R>w'_P.$$

We consider the map φ

$$\left\{\begin{array}{c}(P,\gamma)\in\mathcal{A}^{\mathrm{Sim}}\times\Gamma(x):\\ P\big(\gamma(\infty)\big)<0<P\big(\lfloor\gamma(x)\rfloor\big)\end{array}\right\}\xrightarrow{\varphi}\left\{\begin{array}{c}(P,\gamma)\in\mathcal{A}^{\mathrm{Red}}\times\Gamma(x)':\\ P\big(\gamma(\infty)\big)<0\end{array}\right\}$$
$$(P,\gamma_i)\ \mapsto\ \big(P|T^{\lfloor w'_P\rfloor+1},T^{-\lfloor w'_P\rfloor-1}\gamma_i\big).$$

To prove that φ is well defined we check, given $(R,\tilde\gamma)=\varphi(P,\gamma_i)$, two conditions: $R\in\mathcal{A}^{\mathrm{Red}}$ and $\lfloor w'_P\rfloor+1\leq n_i$. The condition $R\in\mathcal{A}^{\mathrm{Red}}$ is immediate: from the equalities

$$w_R=w_P-\lfloor w'_P\rfloor-1,\qquad w'_R=w'_P-\lfloor w'_P\rfloor-1$$

and the inequalities $w_P<0<w'_P$, we deduce $w_R<-1<w'_R<0$.

The condition $\lfloor w'_P\rfloor+1\leq n_i$ follows from $\lfloor\gamma_i(x)\rfloor=n_i$ and $P(\lfloor\gamma_i(x)\rfloor)>0$.

Finally φ is the inverse of ψ. Indeed, it is clear that $\psi\circ\varphi$ is the identity, and we deduce the same statement for $\varphi\circ\psi$ from the equation below for $(P,\gamma_i)=\psi(R,T^{-k}\gamma_i)$:

$$-k+\lfloor w'_P\rfloor+1=-k+\lfloor w'_R+k\rfloor+1=\lfloor w'_R\rfloor+1=0.\qquad\square$$

Corollary 3.4. *Let x be a real number, $k\geq 2$ an integer and $\mathcal{A}$ a Γ-equivalence class in $\mathcal{Q}$. Then the following equalities hold*

$$A_{k,\mathcal{A}}(x)=\sum_{Q\in\mathcal{A}^{\mathrm{Sim}}}\ \sum_{\substack{\gamma\in\Gamma(x)\\ Q(\lfloor\gamma(x)\rfloor)>0\\ Q(\gamma(\infty))<0}}(Q|\gamma)(x)^{k-1}=\sum_{Q\in\mathcal{A}^{\mathrm{Red}}}\ \sum_{\substack{\gamma\in\Gamma(x)'\\ Q(\gamma(\infty))<0}}(Q|\gamma)(x)^{k-1}.$$

Corollary 3.5. *For $x\in\mathbb{R}$, the sum $A_{k,D}(x)$ has exponential convergence.*

Proof. The function $A_{k,D}(x)$ is the sum of the sums that appear in each side of Corollary 3.4 over all the Γ-equivalence classes in $\mathcal{Q}_D$. We can prove its exponential convergence looking at the sum on $\mathcal{A}^{\mathrm{Sim}}$ or on $\mathcal{A}^{\mathrm{Red}}$. Let us look at the sum on $\mathcal{A}^{\mathrm{Sim}}$. On the one hand, the set of polynomials that belong to Sim with fixed positive discriminant is finite. On the other hand, for an element $Q(X) = aX^2 + bX + c$ of Sim, $Q|\gamma_i(x) = a\delta_{i-1}^2 + b\delta_{i-1}\delta_i + c\delta_i^2$. Now the exponential convergence of the series $\delta_i = |p_{i-1} - q_{i-1}x|$ to 0 proves the corollary. $\square$

Theorem 3.6. *For $d \in 2\mathbb{N}$, $\mathcal{B}$ a Γ_1-equivalence class in $\mathcal{F}_d$ and $x \in \mathbb{R}$, the following bijection holds*

$$\begin{aligned} \left\{ \begin{array}{c} (P,\gamma) \in \mathcal{B}^{\mathrm{Sim}} \times \Gamma_1(x) : \\ P\big(\gamma(\infty)\big) < 0 < P\big(\gamma(x)\big) \end{array} \right\} &\xrightarrow{\cong} \mathcal{B}\langle x\rangle \\ (P,\gamma) &\mapsto P|\gamma. \end{aligned}$$

Proof. The map is well defined because of its definition. Let $P \in \mathcal{B}$ satisfy $P(\infty) < 0 < P(x)$. For $j \gg 0$, the convergents $\frac{\tilde{p}_j}{\tilde{q}_j}$ of x belong to (w'_P, w_P), because $x \in (w'_P, w_P)$. Moreover, if one convergent belongs to (w'_P, w_P), so do all the later convergents.

If $\lceil x \rceil \notin (w'_P, w_P)$, we define i to be the unique positive integer such that $\frac{\tilde{p}_{i-1}}{\tilde{q}_{i-1}} \notin (w'_P, w_P)$ but $\frac{\tilde{p}_i}{\tilde{q}_i} \in (w'_P, w_P)$. If $\lceil x \rceil \in (w'_P, w_P)$, we set $i = 0$. Since $P(\infty) < 0$, in both cases we have

$$P\left(\frac{\tilde{p}_{i-1}}{\tilde{q}_{i-1}}\right) < 0 < P\left(\frac{\tilde{p}_i}{\tilde{q}_i}\right). \tag{14}$$

Put

$$\gamma = \begin{pmatrix} -\tilde{q}_{i-1} & \tilde{p}_{i-1} \\ -\tilde{q}_i & \tilde{p}_i \end{pmatrix},$$

and $R = P|\gamma^{-1}$. We have

$$R(0) = \tilde{q}_{i-1}^d P\left(\frac{\tilde{p}_{i-1}}{\tilde{q}_{i-1}}\right) < 0, \qquad R(\infty) = \tilde{q}_i^d P\left(\frac{\tilde{p}_i}{\tilde{q}_i}\right) > 0,$$

thus (R, γ) belongs to the left hand set of the map in Theorem 3.6. The uniqueness of the preimage (R, γ) follows from the uniqueness of i satisfying (14). $\square$

Corollary 3.7. *Let x be a real number, $k \geq 2$ an integer and $\mathcal{B}$ a Γ_1-equivalence class in $\mathcal{Q}$. Then*

$$\sum_{Q \in \mathcal{B}\langle x\rangle} Q(x)^{k-1} = \sum_{Q \in \mathcal{B}^{\mathrm{Sim}}} \sum_{\substack{\gamma \in \Gamma_1(x) \\ Q(\gamma(x))>0 \\ Q(\gamma(\infty))<0}} (Q|\gamma)(x)^{k-1}.$$

4. Continued fractions and modular forms

There is a canonical choice for the Eichler integral F of a cusp form $f(\tau) = \sum_{n=1}^{\infty} a_n q^n$ $(q = e^{2\pi i\tau})$ of weight $2k$ $(k \geq 1)$ for Γ_1:

$$F(\tau) = \int_{\tau}^{i\infty} f(z)(\tau - z)^{2k-2}\, dz \doteq \sum_{n=1}^{\infty} \frac{a_n}{n^{2k-1}} q^n \quad (\tau \in \mathcal{H})$$

where the symbol $\doteq$ denotes equality with constant. The sum in the second equality defining $F(\tau)$ converges also for $\tau \in \mathbb{R}$, so we can expand the definition domain of F to $\mathcal{H} \cup \mathbb{R}$. The image $F|_{2-2k}(1-\gamma)$ belongs to the space $\mathbb{C}[X]_{\leq 2k-2}$ because of Bol's identity between the $(k-1)$st derivative of $F|_{2-2k}\gamma$ and the image by $|_{2k}\gamma$ of the $(k-1)$st derivative of F. Moreover, the map

$$\begin{aligned} \Gamma_1 &\longrightarrow \mathbb{C}[X]_{\leq 2k-2} \\ \gamma &\mapsto F|(1-\gamma) \end{aligned}$$

is a parabolic 1-cocyle. Since Γ_1 is generated by T and S, the 1-cocyle above is determined by $T \mapsto 0$ and

$$S \mapsto r_f(X) = F|(1-S) = \int_0^{i\infty} f(z)(X - z)^{2k-2}\, dz.$$

The polynomial $r_f(X)$ is called *period polynomial* of the cusp form f. More generally, a map $\Gamma_1 \to \mathbb{C}[X]_{\leq 2k-2}$ which sends T to 0 and S to a complex polynomial $P(X)$ is a parabolic 1-cocyle if and only if P satisfies (see [15])

$$P|(1+S) = 0, \qquad P\big|\big(1 + U + U^2\big) = 0. \tag{15}$$

We can easily check that if $P = A|(1-S)$, with $A(x)$ a periodic real function, then P satisfies (15).

The space W_{2k} of polynomials in $\mathbb{C}[X]_{\leq 2k}$ satisfying (15) splits into the subspaces of even and odd polynomials W_{2k}^{+} and W_{2k}^{-}. Thus $r_f = r_f^{+} + r_f^{-}$ with $r_f^{+}(X) \in W_{2k-2}^{+}$ and $r_f^{-}(X) \in W_{2k-2}^{-}$; such polynomials give rise to the isomorphisms

$$\begin{aligned} r^{-} : S_{2k}(\Gamma_1) &\longrightarrow W_{2k-2}^{-}, \\ f &\mapsto r_f^{-}(X) \end{aligned} \qquad \begin{aligned} r^{+} : S_{2k}(\Gamma_1) &\longrightarrow W_{2k-2}^{+}/\big\langle X^{2k-2} - 1\big\rangle \\ f &\mapsto r_f^{+}(X), \end{aligned}$$

where $S_{2k}(\Gamma_1)$ is the space of cusp forms of weight $2k$ for Γ_1 $(k \geq 1)$.

In fact we can find the even and odd parts of r_f from the even and odd parts of the Eichler integral of f on $\mathbb{R}$:

$$F^+(x)=\sum_{n=1}^{\infty}\frac{a_n}{n^{2k-1}}\cos(2\pi nx),\qquad F^-(x)=\sum_{n=1}^{\infty}\frac{a_n}{n^{2k-1}}\sin(2\pi nx),$$
$$r_f^+=F^+|(1-S),\qquad r_f^-=F^-|(1-S).$$

Given $P(X)\in\mathbb{C}[X]_{\le 2k}$, we denote

$$P^{\Gamma}(x)=\sum_{\gamma\in\Gamma(x)}(P|\gamma)(x),\qquad P^{\Gamma_1}(x)=\sum_{\gamma\in\Gamma_1(x)}(P|\gamma)(x)\quad(x\in\mathbb{R}).$$

We will see that, when $P(X)\in W_{2k}^+$, the function $P^{\Gamma}(x)$ is the even part of the Eichler integral on $\mathbb{R}$ of a cusp form whose even part of the period polynomial is (modulo $X^{2k}-1$) the polynomial $-P(X)$. As a consequence, we obtain that we can drop the conditions $Q(\gamma(\infty))<0<Q(\lfloor\gamma(x)\rfloor)$ in Corollary 3.4 and $Q(\gamma(\infty))<0<Q(\gamma(x))$ in Corollary 3.7.

For $k\ge 1$ and $P(X)\in\mathbb{C}[X]_{\le 2k}$, the theorem below gives the differences between P^{Γ} and its image by each generator of Γ in terms of P, $P|(1+\varepsilon)$ and $P|(1+U-SU^2)$ (the last two vanish when P belongs to W_{2k}^+).

Theorem 4.1. *For $k\ge 1$ and $P(X)\in\mathbb{C}[X]_{\le 2k}$, we have*

(i) $P^{\Gamma}|(1-T)=0$.

(ii) If we denote $P_1=P|(1+\varepsilon)$ and $P_2=P|(1+U-SU^2)\varepsilon$, then $P^{\Gamma}|(1-\sigma)$ is equal to

$$\sum_{n\in 2\mathbb{Z}}\big(\chi_{(\frac{n}{2},\frac{n+1}{2}]}\big(-P_1|T^{n/2+1}\sigma+P_2|T^{-n/2}\big)+\chi_{(\frac{n+1}{2},\frac{n+2}{2})}\big(P_1|T^{-n/2}-P_2|T^{n/2+1}\sigma\big)\big).$$

(iii) If $x>0$ and $x\ne 1$, then

$$\big(P^{\Gamma}\big|(1-\varepsilon)\big)(x)=\chi_{(0,1)}\big(P\big|(1+\varepsilon)\big)(x)-P(x).$$

Proof. Statement (i) follows from the calculation

$$\begin{aligned}P^{\Gamma}(x+1)&=\sum_{\gamma\in\Gamma(x+1)}(P|\gamma)(x+1)\\&=\sum_{\gamma\in\Gamma(x)}\big(P\big|\gamma T^{-1}\big)(x+1)\quad\text{because }\Gamma(x+1)=\Gamma(x)T^{-1}\\&=P^{\Gamma}(x).\end{aligned}$$

If $x\in\mathbb{Z}$, then the matrices $\left(\begin{smallmatrix}0&1\\1&-x\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&1\\1&x\end{smallmatrix}\right)$ are the only elements in $\Gamma(x)$ and $\Gamma(-x)$ respectively, so $P^{\Gamma}(x)=P(\infty)=(P^{\Gamma}|\sigma)(x)$.

Let $x=[n_0,n_1,\ldots]$ be a non-integer number and $-x=[-n_0-1,n_1',\ldots]$ its opposite. The inequality $0<x-n_0\le\frac{1}{2}$ is equivalent to $n_1\ge 2$. It is also equivalent to $\frac{1}{2}\le -x+n_0+1<1$, so to $n_1'=1$. By Proposition 2.1, $P^{\Gamma}(-x)$ is equal to

$$\sum_{\substack{-1\le\gamma(\infty)\le 0\\ \gamma(-x)>1}} (P|\gamma)(-x) - \big(P\big|\varepsilon T^{-n_0}\big)(x) - \begin{cases} (P|T^{-1}\varepsilon T^{n_0+1}\sigma)(x) & \text{if } n_1 = 1\\ \emptyset & \text{if } n_1 \ge 2.\end{cases}$$

Since

$$\sum_{\substack{-1\le\gamma(\infty)\le 0\\ \gamma(-x)>1}} (P|\gamma)(-x) = \sum_{\substack{-1\le\gamma\sigma(\infty)\le 0\\ \gamma\sigma(x)>1}} (P|\gamma\sigma)(x) = \sum_{\substack{-1\le\gamma(\infty)\le 0\\ \gamma(x)>1}} (P|\gamma)(x),$$

by Proposition 2.1 again, we have

$$\begin{aligned} P^{\Gamma}|(1-\sigma) &= P|\varepsilon T^{-n_0} - P|\varepsilon T^{n_0+1}\sigma + \begin{cases} -P|T^{-1}\varepsilon T^{-n_0} & \text{if } n_1\ge 2\\ P|T^{-1}\varepsilon T^{n_0+1}\sigma & \text{if } n_1 = 1\end{cases}\\ &= \begin{cases} -P|(1+\varepsilon)T^{n_0+1}\sigma + P|(1+U-SU^2)\varepsilon T^{-n_0} & \text{if } n_1\ge 2\\ P|(1+\varepsilon)T^{-n_0} - P|(1+U-SU^2)\varepsilon T^{n_0+1}\sigma & \text{if } n_1 = 1\end{cases}\end{aligned}$$

because $T^{-n_0} = U\varepsilon T^{n_0+1}\sigma$, $T^{n_0+1}\sigma = U\varepsilon T^{-n_0}$ and $SU^2 = T^{-1}$. Thus statement (ii) is proved.

Suppose $x > 1$. For $i \ge 1$, the i-th term of the real series $(x_i)_{i\ge 0}$ defined in (4) which gives the continued fraction of $1/x$ is equal to the $(i-1)$-th term of the series which gives the continued fraction of x. So

$$\Gamma(1/x) = \Gamma(x)\varepsilon \cup \left\{\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\right\}.$$

From that we get

$$\begin{aligned} x^{2k}P^{\Gamma}(1/x) &= x^{2k}\sum_{\gamma\in\Gamma(x)} (P|\gamma\varepsilon)(1/x) + x^{2k}(P|\varepsilon)(1/x)\\ &= P^{\Gamma}(x) + P(x).\end{aligned}$$

Now suppose $0 < x < 1$ and denote $y = 1/x$. We have

$$\begin{aligned} x^{2k}P^{\Gamma}(1/x) &= \frac{1}{y^{2k}}P^{\Gamma}(y)\\ &= P^{\Gamma}(1/y) - \frac{1}{y^{2k}}P(y) \quad \text{(by the previous case)}\\ &= P^{\Gamma}(x) - x^{2k}P(1/x). \qquad \square\end{aligned}$$

Corollary 4.2.

(i) For $k \ge 1$ and $P(X) \in W_{2k}^{+}$, the function $P^{\Gamma}(x)$ is even, periodic and satisfies $P^{\Gamma}|(1-S) = -P$.

(ii) Let f be a cusp form of weight $2k$ for Γ_1 and $P(X)$ be the even part of its period polynomial. The even part of the Eichler integral of f on $\mathbb{R}$ is (modulo a constant multiple of $X^{2k-2}-1$) the function $(-P)^\Gamma(x)$.

Proof. (i) Let $P(X)$ be an element of W_{2k}^+. By the statement (i) of Theorem 4.1, the function $P^\Gamma(x)$ is periodic. Since $1+U-SU^2 = 1+U+U^2-(1+S)U^2$ and $P(X)$ is even (so $P|S = P|\varepsilon$), the statement (ii) implies that $P^\Gamma(x)$ is even.

The statement (iii) gives us that $P^\Gamma|(1-S)(x) = -P(x)$ for $x \neq 0, \pm 1$. If $x = 0$, then $(P^\Gamma|(1-S))(0) = P(\infty) = -P(0)$ because $P|(1+S) = 0$. If $x = 1$, then $(P^\Gamma|(1-S))(1) = 0 = P(1)$ again because $P|(1+S) = 0$ and $P|(1-\sigma) = 0$.

(ii) Let f be a cusp form of weight $2k$ for Γ_1 and $P(X)$ be the even part of its period polynomial. Then $P(X)$ belongs to W_{2k-2}^+, so $(-P)^\Gamma$ is periodic and satisfies $(-P)^\Gamma|(1-S) = P$ by the statement (i) of the corollary. Hence $(-P)^\Gamma(x)$ is (modulo $X^{2k-2}-1$) the even part of the Eichler integral of f for $x \in \mathbb{R}$. $\square$

Given an even integer $k \geq 2$ and a Γ-equivalence class $\mathcal{A}$ of $\mathcal{Q}$, we define the polynomial

$$P_{k,\mathcal{A}}(X) = \sum_{Q \in \mathcal{A}^{\mathrm{Sim}}} Q(X)^{k-1}.$$

Kohnen and Zagier proved in [7] that, for every positive non-square discriminant D, the polynomial

$$P_{k,D}(X) = \sum_{\mathcal{A} \in \mathcal{Q}_D/\Gamma} P_{k,\mathcal{A}}(X)$$

is (up to multiplication by a constant and modulo a multiple of $X^{2k-2}-1$) the even part of the period polynomial of the cusp form of weight $2k$ for the modular group Γ_1

$$f_{k,D}(z) = \sum_{\substack{(a,b,c)\in\mathbb{Z}^3 \\ b^2-4ac=D}} \frac{1}{(az^2+bz+c)^k} \quad (z \in \mathcal{H},\ k \geq 2 \text{ even}).$$

This function arose in [11], in the case where D is a fundamental discriminant, by considering the restriction to the diagonal $z_1 = z_2$ of a family of Hilbert modular forms $w_m(z_1, z_2)$ $(m = 0, 1, 2, \ldots)$ of weight k for the Hilbert modular group $\mathrm{SL}_2(\mathcal{O})$, where $\mathcal{O}$ was the ring of integers of the real quadratic field with discriminant D. The functions $w_m(z_1, z_2)$ are the Fourier coefficients of the kernel function for the Doi–Naganuma correspondence between elliptic modular forms and Hilbert modular forms. They are well defined for all positive discriminants D and so is $f_{k,D}(z)$, except that when D is a square there is an extra term besides $P_{k,D}$ in the expression of the even part of the period polynomial.

Kohnen and Zagier proved in [6] that the functions $D^{k-1/2} f_{k,D}(z)$ are the D-th Fourier coefficients of the kernel function for the Shimura–Shintani correspondence.

Recently, Bringmann, Kane and Kohnen gave a new proof of the fact that $P_{k,D}(X)$ is the even part of the period polynomial of $f_{k,D}(z)$ for positive non-square discriminants D using new modular objects related to the theory of harmonic weak Maass forms [2].

Zagier proved that $P_{k,D}(X)$ belongs to W^+_{2k-2} by showing that the function $A_{k,D}(x)$ is periodic and satisfies $A_{k,D}|(1-S) = -P_{k,D}$. The same argument he used applies to $A_{k,\mathcal{A}}(x)$ if $\mathcal{A}$ is a Γ-equivalence class in $\mathcal{Q}$, so $A_{k,\mathcal{A}}|(1-S) = -P_{k,\mathcal{A}}$ and for even $k \geq 2$ the polynomial $-P_{k,\mathcal{A}}(X)$ belongs to W^+_{2k-2}. Hence $-P_{k,\mathcal{A}}$ is (modulo $X^{2k-2}-1$) the even part of a period polynomial and $A_{k,\mathcal{A}}(x)$ is the corresponding even part of the Eichler integral on $\mathbb{R}$. Then, by Corollary 4.2, $P^{\Gamma}_{k,\mathcal{A}}(x) = A_{k,\mathcal{A}}(x)$ for all $x \in \mathbb{R}$. Together with Theorem 3.4, we obtain

Corollary 4.3. *For a Γ-equivalence class $\mathcal{A}$ of $\mathcal{Q}$ and an even integer $k \geq 2$, the following identities hold:*

$$A_{k,\mathcal{A}}(x) = \sum_{Q\in\mathcal{A}^{\mathrm{Sim}}} \sum_{\gamma\in\Gamma(x)} (Q|\gamma)(x)^{k-1} = \sum_{Q\in\mathcal{A}^{\mathrm{Sim}}} \sum_{\substack{\gamma\in\Gamma(x)\\ Q(\gamma(\infty))<0\\ Q(\lfloor\gamma(x)\rfloor)>0}} (Q|\gamma)(x)^{k-1}.$$

The sum $A_{k,\mathcal{A}}(x)$ is finite if and only if $x \in \mathbb{Q}$.

Corollary 4.4. *For every positive non-square discriminant D, the functions $P^{\Gamma}_{2,D}(x)$ and $P^{\Gamma}_{4,D}(x)$ have the respective values $-5L(-1,\chi_D)$ and $L(-3,\chi_D)$, where $L(s,\chi_D)$ is the Dirichlet L-series of the character $\chi_D(n) = (D/n)$ (Kronecker symbol).*

Remark 4.5. We can also give a direct proof of the identity $P^{\Gamma}_{2,D}(x) = -5L(-1,\chi_D)$. To do this, write $P^{\Gamma}_{2,D}(x)$ as

$$P^{\Gamma}_{2,D}(x) = \sum_{i\geq 1} \sum_{\substack{(a,b,c)\in\mathbb{Z}^3\\ a>0>c\\ b^2-4ac=D}} \left(a\delta^2_{i-1} + b\delta_{i-1}\delta_i + c\delta^2_i\right).$$

If (a,b,c) appears in the sum, then $(-c,-b,-a)$ appears too. Hence

$$P^{\Gamma}_{2,D}(x) = \left(\sum_{i=1}^{\infty}\left(\delta^2_{i-1} - \delta^2_i\right)\right) \cdot \left(\sum_{\substack{a,b,c\in\mathbb{Z}\\ a>0>c\\ b^2-4ac=D}} a\right).$$

But the first sum telescopes to 1 because the δ_i decrease to 0 and $\delta_0 = 1$, and the second sum equals $-5L(-1,\chi_D)$ by results of [10,4,12].

If we sum over the quadratic functions now in a single Γ_1-equivalence class $\mathcal{B}$, we have to symmetrize with respect to an involution on the set $\mathcal{Q}_D/\Gamma_1$ to construct functions related to modular forms. For each Γ_1-equivalence class $\mathcal{B}$, and each (not necessarily even) integer $k \geq 2$, we define

$$A^*_{k,\mathcal{B}}(x) = A_{k,\mathcal{B}}(x) + (-1)^k A_{k,-\mathcal{B}}(x),$$
$$P_{k,\mathcal{B}}(X) = \sum_{Q\in\mathcal{B}^{\mathrm{Sim}}} Q(X)^{k-1} + (-1)^k \sum_{Q\in(-\mathcal{B})^{\mathrm{Sim}}} Q(X)^{k-1},$$

where $-\mathcal{B} := \{-Q \mid Q \in \mathcal{B}\}$.

Again we can use the argument in §6 of [14] to deduce that $A^*_{k,\mathcal{B}}(x)$ is periodic and satisfies $A^*_{k,\mathcal{B}}|(1-S) = -P_{k,\mathcal{B}}$ for $k \geq 2$. Thus

$$-P_{k,\mathcal{B}}|\tilde{\gamma}_{i+1} = \left(A^*_{k,\mathcal{B}} - A^*_{k,\mathcal{B}}\middle|T^{m_i}S\right)\middle|\tilde{\gamma}_{i+1} = A^*_{k,\mathcal{B}}|\tilde{\gamma}_{i+1} - A^*_{k,\mathcal{B}}|\tilde{\gamma}_i \quad (i \geq 0)$$

and so $A^*_{k,\mathcal{B}}(x) = P^{\Gamma_1}_{k,\mathcal{B}}$. Hence we obtain

Corollary 4.6. *For a Γ_1-equivalence class $\mathcal{B}$ of $\mathcal{Q}$ and an integer $k \geq 2$, the following identities hold*

$$\begin{aligned} A^*_{k,\mathcal{B}}(x) &= \sum_{\gamma\in\Gamma_1(x)} \left(\sum_{Q\in\mathcal{B}^{\mathrm{Sim}}} (Q|\gamma)(x)^{k-1} + (-1)^k \sum_{Q\in(-\mathcal{B})^{\mathrm{Sim}}} (Q|\gamma)(x)^{k-1} \right) \\ &= \sum_{\gamma\in\Gamma_1(x)} \left(\sum_{\substack{Q\in\mathcal{B}^{\mathrm{Sim}} \\ Q(\gamma(\infty))<0 \\ Q(\gamma(x))>0}} (Q|\gamma)(x)^{k-1} + (-1)^k \sum_{\substack{Q\in(-\mathcal{B})^{\mathrm{Sim}} \\ Q(\gamma(\infty))<0 \\ Q(\gamma(x))>0}} (Q|\gamma)(x)^{k-1} \right). \end{aligned}$$

5. Remarks on the quartic case

We could try to give a similar construction to the function $A_{k,D}(x)$ for sums taken over quartic polynomials. Quartic polynomials can have 0, 2 or 4 real roots; in the second case the discriminant is negative, otherwise it is positive. The analogous case to the quadratic construction is to consider polynomials with at least two real roots, because for such a polynomial Q satisfying $Q(\infty) < 0$, the set of real numbers x on which $Q(x) \geq 0$ is compact, as for the quadratic case.

The discriminant can be written in terms of the two Γ_1-invariants I and J associated to a quartic polynomial $aX^4 + bX^3 + cX^2 + dX + e$:

$$I = 12ae - 3bd + c^2,$$
$$J = 72ace + 9bcd - 27ad^2 - 27eb^2 - 2c^3,$$
$$D = \frac{1}{27}\left(4I^3 - J^2\right).$$

A naive generalization of the function $A_{k,D}(x)$ would be taking the sum over the $k-1$-st powers of quartic polynomials with integer coefficients, fixed I and J, that are negative at ∞ and positive at x. But this sum diverges. In fact by Theorems 3.1 and 3.3, for a Γ-equivalence class $\mathcal{A}$ in $\mathcal{F}_4$ and an even integer $k \geq 1$,

$$\sum_{P\in\mathcal{A}\langle x\rangle} P(x)^{k-1} = \sum_{P\in\mathcal{A}^{\mathrm{Sim}}} \sum_{\substack{\gamma\in\Gamma(x)\\ P(\lfloor\gamma(x)\rfloor)>0\\ P(\gamma(\infty))<0}} (P|\gamma)(x)^{k-1} = \sum_{P\in\mathcal{A}^{\mathrm{Red}}} \sum_{\substack{\gamma\in\Gamma(x)'\\ P(\gamma(\infty))<0}} (P|\gamma)(x)^{k-1}$$

where neither the set $\mathcal{A}^{\mathrm{Sim}}$ nor $\mathcal{A}^{\mathrm{Red}}$ is finite.

We could try to modify the naive generalization modifying the left or the right hand side of the equation above. On the right hand side, we should replace $\mathcal{A}^{\mathrm{Sim}}$ by a finite set $\mathcal{A}^{\mathrm{Fin}}$ of polynomials in $\mathcal{A}$ such that $\sum_{P\in\mathcal{A}^{\mathrm{Fin}}} P(X)^{k-1}$ is the even part of a period polynomial. If we look at the left hand side, we should add some linear inequalities for the coefficients of the polynomials $P(X)$ to make the sum converge. But any linear inequality involving other coefficients of $P(X)$ than $P(\infty)$ would probably break the invariance by T of the sum, because the only invariants by T for a quartic $P(X)$ are $P(\infty)$, I, J and $P(X)$ itself. So the new sum would not be the even part of an Eichler integral anymore.

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