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# Classification of some Global Integrals related to groups of type $A_n$



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## ABSTRACT

In this paper we start a classification of certain global integrals. First, we use the language of unipotent orbits to write down a family of global integrals. We then classify all those integrals which satisfy the dimension equation we set. After doing so, we check which of these integrals are global unipotent integrals. We do all this for groups of type  $A_n$ , and using all this we derive a certain interesting conjecture about the length of these integrals.

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## 1. Introduction

In this paper we begin a classification of what we refer to as global unipotent integrals. Constructing global integrals is one of the ways in which one can study the Langlands conjectures related to  $L$  functions. In this method, one constructs a global integral which depends on a complex variable  $s$ , and the goal is to determine if this integral is Eulerian. To do so one carries out the process of unfolding, which consists mainly of a series of Fourier expansions. At the end we obtain an integral which involves an integration

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over a quotient of the type  $N(\mathbf{A}) \backslash G(\mathbf{A})$  where  $N$  and  $G$  are two groups. The integral involves a set of functionals or bi-linear forms etc. Then the task is to determine if this integral is factorizable. This is the case if, for example, the functional or the bi-linear forms etc. which appear in the integral satisfies some uniqueness properties. Maybe the most difficult part in this program is to determine a way of how to actually construct the initial global integral. In [G1] we describe a way using the language of unipotent orbits to construct such global integrals. One of the most general constructions is given by integral (2) which appears in the next section. However, as mentioned at the end of that section, this is not the most general construction. Nevertheless, the vast majority of global Eulerian integrals which appear in the literature are of the type of integral (2).

Thus, one of the main problems is to determine all global integrals (2) which, for  $\text{Re}(s)$  large, after the unfolding process, unfold to the global integral (4). We refer to such integrals as global unipotent integrals. As mentioned above, the process of unfolding involves mainly certain Fourier expansions. Therefore, a good knowledge of the Fourier coefficients of the representations in question is crucial. In our context, for a representation  $\pi$ , this is captured by the set of unipotent orbits  $\mathcal{O}(\pi)$ . This notion is defined in [G2], Definition 2.1. This, in turn, leads to the definition of the dimension of the representation  $\pi$ , and to the definition of what we refer to as the dimension equation of a global integral. See equation (5). Roughly speaking, this equation states that the sum of the dimensions of the representations involved in the integral is equal to the sum of the dimensions of the groups which are involved in the integration. There are several reasons which motivate to set up this dimension equation. Maybe the main reason is simply because all known integrals which unfold to integrals involving the Whittaker coefficient of at least some of the representations do satisfy this equation. See [G1] and [G3].

To summarize, a main aspect of this theory is to classify all nonzero global unipotent integrals which are given by integral (2), and which satisfies the dimension equation (5).

In this paper we consider this classification for the group  $G = GL_m$ . We do not do it for the most general case, but rather restrict things to the case where the embedding of the group  $G$  inside the groups  $G_j$ , see section 2, is such that the center of  $G$  coincides with the center of  $G_j$ . In section 3 we list all possible Fourier coefficients, defined on an arbitrary reductive group  $H$ , such that the stabilizer of this coefficient inside some Levi part of a certain parabolic subgroup is the group  $G$  embedded in  $H$  as mentioned above. Once we classify all such Fourier coefficients, we obtain a list of global integrals defined by integral (2). The first step is to write down the dimension equation for these integrals, and to study it. This is done in section 4. There are two main consequences which arise from the study of this equation. The first one is a list of global integrals which, in the cases of  $m = 2, 3$ , is a complete list of all possible integrals. For  $m = 2$  this list is given in Tables 1 and 2, and for  $m = 3$  it is given in Tables 3–7. When  $m \geq 4$  we only get a partial list, given in Table 8. We emphasize that these tables list global integrals of the type of integral (2) which satisfies the dimension equation (5). It does not guarantee that these integrals are nonzero, and if nonzero that they are a global *unipotent* integral.

An interesting result regarding what we define as the length of the integral follows from these tables. Given a global integral of the type of integral (2), we define its length to be the number of representations involved in the integral. Conjecture 1 states that for all  $m \geq 2$ , if a global integral is a nonzero global unipotent integral with  $G = GL_m$ , then its length is at most three. In section 4 we prove this conjecture for  $m = 2, 3$  and preliminary computations indicate that this conjecture is true for all  $m$ . We hope to study this problem in the near future. We should mention that while this conjecture is obvious for  $m = 2$ , for  $m = 3$  it requires a nontrivial result about cuspidal representations of the exceptional group  $E_6$ . This result is stated in Lemma 1 and proved in the last section of the paper.

Sections 5, 6 and 7 consist of unfolding the global integrals for the case when  $m = 2$ . As mentioned above, the lists we obtain in section 4 do not guarantee that the global integral in question is nonzero, or if it is a global unipotent integral. For that we need the process of the unfolding. After some preparations which are done in sections 5 and 6, in section 7 we prove two theorems, summarized in Theorem 1, which state in which cases the global integral is actually a nonzero global unipotent integral. For that we define the notion of an odd Eisenstein series, see Definition 2, and prove in Theorems 2 and 3 that a global integral which appears in Tables 1 or 2 is a nonzero global unipotent integral if and only if at least one of the representations involved in the integral is an odd Eisenstein series.

## 2. The basic setup

Let  $\mathbf{A}$  be the ring of adeles of a global field  $F$ . Let  $\psi$  denote a nontrivial additive character of  $F \backslash \mathbf{A}$ . For basic facts and notation about unipotent orbits we refer to [C] and [C-M].

We first recall some basic facts about unipotent orbits. As explained in [G2], section 2, given a reductive group  $H$ , and a unipotent orbit  $\mathcal{O}$  of  $H$ , one can associate with this orbit a set of Fourier coefficients. Thus, to  $\mathcal{O}$ , we can associate a certain unipotent subgroup  $U(\mathcal{O})$  of  $H$ , and a character  $\psi_{U(\mathcal{O})}$  of  $U(\mathcal{O})(F) \backslash U(\mathcal{O})(\mathbf{A})$ . It is possible that to a given unipotent orbit there will correspond infinite number of Fourier coefficients. Hence the choice of the character  $\psi_{U(\mathcal{O})}$  is not always unique.

Given an irreducible automorphic representation  $\pi$  of  $H(\mathbf{A})$ , one can associate with it a set of unipotent orbits of  $H$ , which we denote by  $\mathcal{O}_H(\pi)$ . This set is defined in [G2] Definition 2.1. As mentioned in that reference, it is conjectured that this set consists of a unique unipotent orbit. Henceforth, we shall assume that this is the case. Another notion we need is the notion of the dimension of  $\pi$ . We define  $\dim \pi = \frac{1}{2} \dim \mathcal{O}_H(\pi)$ . Notice that this notion is well defined only if we assume that  $\mathcal{O}_H(\pi)$  consists of one unipotent orbit.

For  $1 \leq i \leq l$ , let  $G_i$  denote  $l$  reductive groups. For  $1 \leq j \leq l$ , let  $\pi_j$  denote an automorphic representation defined on  $G_j(\mathbf{A})$ . Let  $\mathcal{O}_{G_i}$  denote a unipotent orbit of the group  $G_i$ . As explained above, we let  $U(\mathcal{O}_{G_i})$  denote the corresponding unipotent group,

and we choose a corresponding character  $\psi_{U(\mathcal{O}_{G_i})}$  of  $U(\mathcal{O}_{G_i})(F) \backslash U(\mathcal{O}_{G_i})(\mathbf{A})$ . Assume, that the stabilizer of  $\psi_{U(\mathcal{O}_{G_i})}$  inside a suitable Levi subgroup of  $G_i$  contains the *same* reductive group  $G$ . Then we can form the global integral

$$\int_{Z(\mathbf{A})G(F) \backslash G(\mathbf{A})} \varphi_{\pi_1}^{U_1, \psi_{U_1}}(g) \varphi_{\pi_2}^{U_2, \psi_{U_2}}(g) \cdots \varphi_{\pi_{l-1}}^{U_{l-1}, \psi_{U_{l-1}}}(g) \varphi_{\pi_l}^{U_l, \psi_{U_l}}(g) dg \quad (1)$$

Here  $Z$  is the center of  $G$ , and we assume that the embedding of  $G$  is such that we can divide by  $Z$ . For each  $i$  we have

$$\varphi_{\pi_i}^{U_i, \psi_{U_i}}(g) = \int_{U(\mathcal{O}_{G_i})(F) \backslash U(\mathcal{O}_{G_i})(\mathbf{A})} \varphi_{\pi_i}(ug) \psi_{U(\mathcal{O}_{G_i})}(u) du$$

Allowing the vectors  $\varphi_{\pi_i}$  to vary in the space of the representation  $\pi_i$ , the above set of Fourier coefficients defines an automorphic representation  $\sigma_i$  of  $G(\mathbf{A})$ . Henceforth, we will assume that for all  $i$ , the representation  $\sigma_i$  is not a one dimensional representation.

We assume that the representations  $\pi_j$  are such that the integral (1) converges. This will be the case if we assume that one of the representations  $\pi_i$  is an irreducible cuspidal representation. After reordering, we may assume that  $\pi_1$  is cuspidal. We also want the above integral to depend on a complex variable  $s$ . To do so, we assume that  $\pi_l = E_\tau$  is an Eisenstein series defined on the group  $G_l(\mathbf{A})$ . Thus, the global integral we study is given by

$$\int_{Z(\mathbf{A})G(F) \backslash G(\mathbf{A})} \varphi_{\pi_1}^{U_1, \psi_{U_1}}(g) \varphi_{\pi_2}^{U_2, \psi_{U_2}}(g) \cdots \varphi_{\pi_{l-1}}^{U_{l-1}, \psi_{U_{l-1}}}(g) E_\tau^{U_l, \psi_{U_l}}(g, s) dg \quad (2)$$

As mentioned above, and as defined in [G2], to each representation  $\pi_j$ , one can associate a unipotent orbit  $\mathcal{O}_{G_j}(\pi_j)$ . Similarly, we define the set  $\mathcal{O}_{G_l}(E_\tau(\cdot, s))$ . See [G2] section 5. Thus, for  $1 \leq j \leq l-1$ , we may associate with each representation  $\pi_j$ , a unipotent subgroup of  $G_j$ , which we shall denote by  $V_j(\pi_j)$ , and a character  $\psi_{V_j(\pi_j)}$  of  $V_j(\pi_j)(F) \backslash V_j(\pi_j)(\mathbf{A})$  such that the Fourier coefficient

$$L_{\pi_i}(g_i) = \int_{V_i(\pi_i)(F) \backslash V_i(\pi_i)(\mathbf{A})} \varphi_{\pi_i}(v_i g_i) \psi_{V_i(\pi_i)}(v_i) dv_i \quad (3)$$

is not zero for some choice of data. Similarly, for the Eisenstein series  $E_\tau(\cdot, s)$  we can associate an integral  $L_\tau$  which is defined in a similar way as above.

Suppose that for  $\text{Re}(s)$  large, after unfolding integral (2), we obtain the integral

$$\int_{Z(\mathbf{A})M(\mathbf{A}) \backslash G(\mathbf{A})} L_{\pi_1}^{R_1}(g) L_{\pi_2}^{R_2}(g) \cdots L_{\pi_{l-1}}^{R_{l-1}}(g) f_{L_\tau^{R_l}}(w_0 g) dg \quad (4)$$

Here,  $M$  is a certain subgroup of  $G$ , and

$$L_{\pi_i}^{R_i}(g) = \int_{R_i(\mathbf{A})} L_{\pi_i}(rg) \psi_{R_i}(r) dr$$

where  $R_i$  is a certain unipotent subgroup of  $G_i$ , and  $\psi_{R_i}$  is an additive character defined on this group.

To make things clear we give an example. Consider the integral which represents the exterior square  $L$  function defined on a cuspidal representation  $\pi_1$  of  $GL_4(\mathbf{A})$ . This integral was introduced in [J-S]. Here  $G = GL_2$ , and integral (2) is given by

$$\int_{Z(\mathbf{A})GL_2(F) \backslash GL_2(\mathbf{A})} \varphi_{\pi_1}^{U_1, \psi_{U_1}}(g) E^{U_2, \psi_{U_2}}(g, s) dg$$

Here  $U_1$  is the subgroup of  $GL_4$  consisting of all unipotent matrices of the form  $\begin{pmatrix} I_2 & X \\ & I_2 \end{pmatrix}$ , and  $\psi_{U_1}$  is the character given by  $\psi(\text{tr} X)$ . Also,  $E(g, s)$  is the standard Eisenstein series defined on the group  $GL_2(\mathbf{A})$ , and  $U_2$  is the trivial group. Unfolding this integral one obtains

$$\int_{Z(\mathbf{A})N(\mathbf{A}) \backslash GL_2(\mathbf{A})} L_{\pi_1}^{R_1}(g) f(g, s) dg$$

where

$$L_{\pi_1}^{R_1}(g_1) = \int_{\mathbf{A}} W_{\pi_1}(k(r_1)g_1) dr_1$$

Here  $W_{\pi_1}$  is the Whittaker coefficient of  $\pi_1$ , and  $k(r_1) = I_4 + r_1 e_{3,2}$  where  $e_{3,2}$  is the matrix of size four which has one at the  $(3, 2)$  entry and zero elsewhere.

**Definition 1.** In the above notations, if, for  $\text{Re}(s)$  large, after an unfolding process, integral (2) is equal to integral (4), then we refer to integral (2) as to a unipotent global integral. The number  $l$  is the length of the integral, and if all the functionals  $L_{\pi_i}^{R_i}$  and  $L_{\tau_i}^{R_i}$  are factorizable, then we say that integral (2) is an Eulerian unipotent integral.

The motivation to referring to such an integral as an unipotent global integral follows from the fact that each functional  $L_{\pi_i}(g_i)$ , defined in (3), is defined by an integration over a unipotent group only. There are many cases when a given global integral unfolds to functionals which involves integration over reductive groups. In this paper we will not consider these cases.

In [G1] we give a general overview about the motivation for the above construction and definition. We also motivate some of the discussions below. As explained in [G1] and also in [G3], we are mainly interested in Eulerian unipotent integrals which satisfy the dimension equation

$$\sum_{i=1}^l (\dim \pi_i - \dim U(\mathcal{O}_{G_i})) = \dim G - \dim Z \quad (5)$$

where we write  $\pi_l$  for the Eisenstein series  $E_\tau$ . Our goal is to study the following

**Problem 1.** Classify all Eulerian unipotent integrals which satisfy the dimension equation (5).

In this paper we study the above problem for the group  $G = GL_m$ . Moreover, we consider those cases where the center  $Z$  of  $GL_m$  coincides with the center of each of the groups  $G_i$ . This restricts the choice of the groups  $G_i$ , and also the relevant unipotent orbits  $\mathcal{O}_{G_i}$ . We classify these cases in the next section.

Thus, for these cases we study the following

**Conjecture 1.** Assume that  $G = GL_m$ . Suppose that the integral (2) is a nonzero Eulerian unipotent integral which satisfies the dimension equation (5). Assume that the representations  $\sigma_i$  generated by the Fourier coefficient  $\varphi_{\pi_i}^{U_i, \psi_{U_i}}(g)$  are not a one dimensional representation of  $G(\mathbf{A})$ . Then the length  $l$  of the integral (2) is less than or equal to three.

The motivation for this Conjecture is the fact that in the literature there is not a single example of a nonzero Eulerian unipotent integral which satisfies the dimension equation (5), and such that  $l > 3$ .

It is easy to construct examples with  $l = 3$ . The Rankin product integral given by

$$\int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} \varphi_{\pi_1}(g)\varphi_{\pi_2}(g)E(g,s)dg$$

is an Eulerian unipotent integral. Here  $\pi_i$  are irreducible cuspidal representations of  $G(\mathbf{A})$ , and  $E(g,s)$  is a suitable Eisenstein series defined on this group. We also mention that in [G3] there is a classification of all integrals as above, when now  $\pi_2$  is an arbitrary automorphic representation, and  $E(g,s)$  is an arbitrary degenerate Eisenstein series.

Notice that since the group  $G = GL_m$  contains a nontrivial unipotent subgroup, then  $\dim \pi_i > \dim U(\mathcal{O}_{G_i})$ . This follows from our assumption about the groups  $\sigma_i$ . Indeed, since we assume that  $\pi_i$  has a nonzero Fourier coefficient corresponding to the unipotent orbit  $\mathcal{O}_{G_i}$ , then clearly  $\dim \pi_i \geq \dim U(\mathcal{O}_{G_i})$ . If there is an equality, then we have  $\mathcal{O}_{G_i}(\pi_i) = \mathcal{O}_{G_i}$ . This means that the representation  $\sigma_i$  is a one dimensional representation of the group  $G(\mathbf{A})$ . To see this, let  $x_\alpha(r)$  denote the one dimensional unipotent subgroup of  $G = GL_m$  which corresponds to some simple root of this group. Then we consider the integral

$$\int_{F\backslash\mathbf{A}} \varphi_{\pi_i}^{U_i, \psi_{U_i}}(x_\alpha(r)g)\psi(r)dr$$

It is not hard to show that this Fourier coefficient corresponds to a unipotent orbit which is strictly greater than  $\mathcal{O}_{G_i}$ . In section 7 we will prove this when  $m = 2$ . The general case is similar. So, if we assume that  $\mathcal{O}_{G_i}(\pi_i) = \mathcal{O}_{G_i}$ , then we conclude that the above integral must be zero for all choice of data. But then, if we consider the group  $SL_2$  which is generated by  $x_{\pm\alpha}$ , we deduce that  $\varphi_{\pi_i}^{U_i, \psi_{U_i}}(mg) = \varphi_{\pi_i}^{U_i, \psi_{U_i}}(g)$  for all  $m \in SL_2(\mathbf{A})$ . This implies that as a representation of  $GL_m(\mathbf{A})$ , the representation  $\sigma_i$  is a one dimensional representation. Since, see Conjecture 1, we assume that this is not the case, we deduce that  $\dim \pi_i > \dim U(\mathcal{O}_{G_i})$ .

Notice that this proves that Conjecture 1 holds when  $G$  is a group of type  $A_1$ . Indeed, in this case  $\dim G - \dim Z = 3$  and hence  $l \leq 3$ .

A few general remarks. First, consider the case when  $l = 1$ . Since we want the global integral to depend on a complex variable then, excluding the simple cases of Hecke type integrals, we assume that the only representation appearing in integral (2) is an Eisenstein series. Since we require that the integral converges, then we must take  $G$  to be trivial. Hence integral (2) reduces to the Fourier coefficient  $E_{\tau}^{U_l, \psi_{U_l}}(g_l, s)$ . These types of integrals were studied in various references, see [S], and are known as the Langlands–Shahidi type integrals. Henceforth, we consider integrals as integral (2) such that  $l \geq 2$ .

As a second remark, we mention that the integrals of the form (2) are not the only known Eulerian unipotent integrals. An extension of these integrals is when the unipotent groups are defined by a diagonal embedding. For example it is not hard to show that for a suitable Eisenstein series on  $GL_4(\mathbf{A})$ , the integral

$$\int \varphi_{\sigma}(h) E_{\tau_1} \left( \begin{pmatrix} I & X \\ & I \end{pmatrix} \begin{pmatrix} h & \\ & h \end{pmatrix}, s_1 \right) E_{\tau_2} \left( \begin{pmatrix} I & X \\ & I \end{pmatrix} \begin{pmatrix} h & \\ & h \end{pmatrix}, s_2 \right) \psi(\mathrm{tr} X) dX dh$$

is an Eulerian unipotent integral. Here  $\sigma$  is an irreducible cuspidal representation of  $GL_2(\mathbf{A})$ . This issue is discussed in [G1].

### 3. The relevant unipotent orbits

In this section we classify the relevant unipotent orbits whose stabilizer is the group  $GL_m$ . We only consider those unipotent orbits such that the center  $Z$  of  $GL_m$  coincides with the center of  $G_i$ . We describe the orbits, the corresponding unipotent groups, and their characters. We also collect some information needed later.

1) Consider the group  $GL_{km}$  with  $k, m \geq 2$ , and denote  $\mathcal{O} = (k^m)$ . The corresponding unipotent groups are

$$U_{k,m}(\mathcal{O}) = \left\{ \begin{pmatrix} I & X_1 & * & * & * \\ & I & X_2 & * & * \\ & & I & * & * \\ & & & \ddots & X_{k-1} \\ & & & & I \end{pmatrix}, X_j \in Mat_{m \times m} \right\}$$

The character is given by

$$\psi_{U_{k,m}(\mathcal{O})}(u) = \psi(\operatorname{tr}(X_1 + X_2 + \cdots + X_{k-1}))$$

The stabilizer of  $\psi_{U_{k,m}(\mathcal{O})}$  inside the group  $GL_m \times \cdots \times GL_m$ , counted  $k$  times, is the diagonal embedding of  $GL_m$  which we denote by  $GL_m^\Delta$ . Notice that the center of the stabilizer coincides with the center of  $GL_{km}$ .

For the classical groups, it follows from [C-M] page 88, and for the exceptional groups it follows from [C] page 400, that if  $m \geq 4$ , then the above cases are the only cases where the stabilizer is  $GL_m$  whose center  $Z$  is the center of the group  $G_i$ .

**2)** In case when  $m = 3$ , beside the cases discussed in **1)** there is another case in the exceptional groups. Let  $GE_6$  denote the similitude group of the exceptional group  $E_6$ . We can realize this group as a subgroup of  $E_8$ . We will use the notations from [G4]. It follows from [C] page 402 that for the unipotent orbit whose label is  $D_4$ , its stabilizer inside a suitable Levi subgroup is the group of type  $A_2$ . In [G4] page 106, one defines the unipotent group  $U$  with character  $\psi_U$  whose stabilizer is the group  $GL_3$ . As explained in that reference the embedding of  $GL_3$  is such that its center  $Z$  is the center of  $GE_6$ .

**3)** Finally we consider the case of  $m = 2$ , that is the group  $GL_2$ . Beside the cases in **1)** we have other cases in other groups. Let  $GSp_{2(2n+1)}$  denote the similitude group of the symplectic group  $Sp_{2(2n+1)}$ . In matrices we take the symplectic form  $\begin{pmatrix} & J_{2n+1} \\ -J_{2n+1} & \end{pmatrix}$  where  $J_{2n+1}$  is the  $2n+1$  matrix with ones on the other diagonal and zeros elsewhere. Let  $\mathcal{O} = ((2n+1)^2)$ , and let  $U_n(\mathcal{O})$  denote the standard unipotent radical subgroup of the parabolic subgroup of  $GSp_{2(2n+1)}$ , whose Levi part is  $GL_2 \times \cdots \times GL_2$  counted  $n$  times. We take  $U_n(\mathcal{O})$  to consist of upper unipotent matrices. Identify the quotient  $U_n(\mathcal{O})/[U_n(\mathcal{O}), U_n(\mathcal{O})]$  with  $L = Mat_2 \oplus \cdots \oplus Mat_2$ , counted  $n$  times. Then the group  $GL_2 \times \cdots \times GL_2$  acts on  $U_n(\mathcal{O})/[U_n(\mathcal{O}), U_n(\mathcal{O})]$  by conjugation, and there is an open orbit for this action. For  $(X_1, X_2, \dots, X_n) \in L$  define the character  $\psi(\operatorname{tr}(X_1 + X_2 + \cdots + X_n))$ , and extend it trivially to a character  $\psi_{U_n(\mathcal{O})}$  of  $U_n(\mathcal{O})(F) \backslash U_n(\mathcal{O})(\mathbf{A})$ . The stabilizer of  $\psi_{U_n(\mathcal{O})}$  inside  $GL_2 \times \cdots \times GL_2$  is  $GL_2$  embedded diagonally. Its not hard to check that the center of  $GL_2$  coincides with the center of  $GSp_{2(2n+1)}$ .

A similar situation occurs in the similitude group  $GSO_{4n}$ . Let  $\mathcal{O} = ((2n)^2)$ . Let  $U_n(\mathcal{O})$  denote the standard unipotent radical of the parabolic subgroup whose Levi part is  $GL_2 \times \cdots \times GL_2$  counted  $n$  times. Identify  $U_n(\mathcal{O})/[U_n(\mathcal{O}), U_n(\mathcal{O})]$  with  $L = Mat_2 \oplus \cdots \oplus Mat_2 \oplus Mat_2^0$ , where  $Mat_2$  appears  $n$  times and  $Mat_2^0 = \left\{ \begin{pmatrix} y & \\ & -y \end{pmatrix} : y \in \mathbf{A} \right\}$ . Then the group  $GL_2 \times \cdots \times GL_2$  acts on  $U_n(\mathcal{O})/[U_n(\mathcal{O}), U_n(\mathcal{O})]$  by conjugation, and there is an open orbit for this action. For  $(X_1, X_2, \dots, X_{n-1}, y) \in L$  define the character  $\psi(\operatorname{tr}(X_1 + X_2 + \cdots + X_{n-1}) + y)$ , and extend it trivially to a character  $\psi_{U_n(\mathcal{O})}$  of  $U_n(\mathcal{O})(F) \backslash U_n(\mathcal{O})(\mathbf{A})$ . The stabilizer of  $\psi_{U_n(\mathcal{O})}$  inside  $GL_2 \times \cdots \times GL_2$  is  $GL_2$  embedded diagonally.

Finally, consider the similitude group  $GE_7$ . We use the notations as in [G4]. In the notations of [C], let  $\mathcal{O} = E_6$ . Let  $U(\mathcal{O})$  denote the unipotent radical subgroup of the



parabolic subgroup whose Levi part contains the group  $GL_2 \times GL_2 \times GL_2$  embedded in  $GE_7$  as follows. In terms of the roots of  $E_7$ , the group  $GL_2 \times GL_2 \times GL_2$  contains the group  $SL_2 \times SL_2 \times SL_2$  generated by  $x_{\pm 0100000}$ ;  $x_{\pm 0000100}$ ;  $x_{\pm 0000001}$ . Thus  $\dim U(\mathcal{O}) = 60$ . Define a character  $\psi_{U(\mathcal{O})}$  of the group  $U(\mathcal{O})(F) \backslash U(\mathcal{O})(\mathbf{A})$  as follows. For  $u \in U(\mathcal{O})$  write

$$u = x_{1000000}(r_1)x_{0010000}(r_2)x_{0101000}(r_3)x_{0001100}(r_4)x_{0000110}(r_5)x_{0000011}(r_6)u'$$

Here  $u'$  is an element of  $U(\mathcal{O})$  which is a product of one parameter unipotent subgroups none of which are among the above six roots. Define  $\psi_{U(\mathcal{O})}(u) = \psi(r_1 + r_2 + \cdots + r_6)$ . Then the stabilizer of  $\psi_{U(\mathcal{O})}$  inside  $GL_2 \times GL_2 \times GL_2$  is the diagonal group  $GL_2$ . It follows from [G4] that the center of this copy of  $GL_2$  is the center of  $GE_7$ .

#### 4. On some global integrals

In this section we study global integrals of the type (2) which we assume to satisfy the dimension equation (5). The result we obtain in this section is a set of integrals which satisfies the dimension equation. We emphasize that the global integrals we obtain are not necessarily nonzero or that they are an Eulerian unipotent integrals. We consider these issues for some cases, in the next sections. We can summarize the content of this section in the following

**Proposition 1.** *Consider an integral of type (2) such that  $G = GL_m$ , and assume that it satisfies the dimension equation (5). If  $m = 2, 3$ , then Conjecture 1 holds. In this case the unipotent orbits attached to the various representations involved in the integral are given in Tables 1–7. When  $m \geq 4$  then  $k = 1, 2$ . See Subsection 4.3 for details.*

We recall from section 2 that we assume that the representation  $\pi_1$  is an irreducible cuspidal representation of the group  $G_1(\mathbf{A})$ , and that the representation  $\pi_l$  is an Eisenstein series. From section 3 we deduce that the group  $G_1$  is one of the following groups. First,  $G_1 = GL_{k_1 m}$  with  $k_1 \geq 1$  and  $m \geq 2$ . In addition, if  $m = 3$ , then we can also have  $G_1 = GE_6$ , and if  $m = 2$ , then we also have that  $G_1$  is one of the groups  $GSp_{2(n+1)}$ ,  $GSO_{4n}$  or  $GE_7$ . In the first case we will write  $k$  for  $k_1$ . Also, we denote  $G = GL_m$ , and by  $Z$  the center of  $G$ . In the first subsection we deal with the group  $G = GL_2$ , then with  $G = GL_3$  and in the last subsection we consider the group  $G = GL_m$  for  $m \geq 4$ .

##### 4.1. The case when $G = GL_2$

Assume that  $G = GL_2$ . The dimension equation is then

$$\sum_{i=1}^l (\dim \pi_i - \dim U(\mathcal{O}_{G_i})) = 3$$

**Table 1**  
The  $l = 2$  case.

$\mathcal{O}(\pi_1)$	$(4)_{GL}$ $((2n+4)(2n-2))_{GSp}$ $((2n+3)(2n-3))_{GSO}$ $E_7(a_1)$	$(2)_{GL}$ $((2n+2)(2n))_{GSp}$ $((2n+1)(2n-1))_{GSO}$ $E_7(a_2)$
$\mathcal{O}(E_\tau)$	$((p+1)(p-1))_{GL}$ $((2p+2)(2p))_{GSp}$ $((2p+1)(2p-1))_{GSO}$ $E_7(a_2)$	$((p+2)(p-2))_{GL}$ $((2p+4)(2p-2))_{GSp}$ $((2p+3)(2p-3))_{GSO}$ $E_7(a_1)$

It follows from the dimension equation that we have  $l = 2, 3$ . We will use the notation  $\lambda_H$  for the partition  $\lambda$  of the classical group  $H$ .

In Table 1 we consider all integrals (2) with  $l = 2$ . There are two cases to consider. In the first case we have  $\dim \pi_1 - \dim U(\mathcal{O}_{G_1}) = 2$  and  $\dim E_\tau - \dim U(\mathcal{O}_{G_2}) = 1$ , and in the second case we have  $\dim \pi_1 - \dim U(\mathcal{O}_{G_1}) = 1$  and  $\dim E_\tau - \dim U(\mathcal{O}_{G_2}) = 2$ . The first case is listed in the second column of Table 1, and the second case in the third column of Table 1.

Consider the first case. If  $\pi_1$  is defined on  $GL_{2k}(\mathbf{A})$ , then we have

$$2 = \dim \pi_1 - \dim U(\mathcal{O}_{G_1}) = \frac{1}{2}(\dim \mathcal{O}_{GL}(\pi_1) - \dim(k^2)_{GL})$$

The only partition that satisfies this equation is  $\mathcal{O}_{GL}(\pi_1) = ((k+2)(k-2))_{GL}$ . But  $\pi_1$  is cuspidal, and hence generic. The only cuspidal representation  $\pi_1$  such that  $\mathcal{O}_{GL}(\pi_1) = ((k+2)(k-2))_{GL}$  is when  $k = 2$ , and we obtain  $\mathcal{O}_{GL}(\pi_1) = (4)_{GL}$ . As another example from this case, assume that  $\pi_1$  is defined on  $GSp_{2(2n+1)}(\mathbf{A})$ . In this case the only partition which satisfies

$$2 = \dim \pi_1 - \dim U(\mathcal{O}_{G_1}) = \frac{1}{2}(\dim \mathcal{O}_{GSp}(\pi_1) - \dim((2n+1)^2)_{GSp})$$

is the partition  $\mathcal{O}_{GSp}(\pi_1) = ((2n+4)(2n-2))_{GSp}$ . In contrast to the case when  $\pi_1$  was defined on  $GL_{2k}$ , in this case cuspidal representations of  $GSp_{2(2n+1)}(\mathbf{A})$  which satisfy  $\mathcal{O}(\pi_1) = ((2n+4)(2n-2))_{GSp}$  do exist. These are some CAP representations, which can be constructed, for example, using the method described in [G5]. The other two cases, which appear in the second column of Table 1 are constructed in a similar way.

As for the Eisenstein series, the situation is similar. In this case we have  $\dim E_\tau - \dim U(\mathcal{O}_{G_2}) = 1$ . If, for example,  $E_\tau$  is defined on the group  $GSO_{4p}(\mathbf{A})$ , then the only partition that satisfies this last equation is  $((2p+1)(2p-1))$ . This explains the last entry in the second column of Table 1.

When  $l = 3$  we have

$$\dim \pi_1 - \dim U(\mathcal{O}_{G_1}) = \dim \pi_2 - \dim U(\mathcal{O}_{G_2}) = \dim E_\tau - \dim U(\mathcal{O}_{G_3}) = 1$$

In Table 2 we list all relevant possible cases.

**Table 2**  
The  $l = 3$  case.

$\mathcal{O}(\pi_1)$	$(2)_{GL}$ $((2n+2)(2n))_{GSp}$ $((2n+1)(2n-1))_{GSO}$ $E_7(a_2)$
$\mathcal{O}(\pi_2)$	$((q+1)(q-1))_{GL}$ $((2q+2)(2q))_{GSp}$ $((2q+1)(2q-1))_{GSO}$ $E_7(a_2)$
$\mathcal{O}(E_\tau)$	$((p+1)(p-1))_{GL}$ $((2p+2)(2p))_{GSp}$ $((2p+1)(2p-1))_{GSO}$ $E_7(a_2)$

As an example, consider the case when all three representations are defined on the exceptional group  $E_7(\mathbf{A})$ . Since, see [C-M],  $E_7(a_2)$  is the only unipotent orbit which satisfies

$$1 = \dim \pi_1 - \dim U(\mathcal{O}_{G_1}) = \frac{1}{2}(\dim \mathcal{O}_{GE_7}(\pi_1) - \dim E_6)$$

this explains the relevant entry in the second table. The entries for  $\pi_2$  and  $E_\tau$  are the same. Thus, for a corresponding integral to exist, we need to prove that there is a cuspidal representation  $\pi_1$ , and an Eisenstein series  $E_\tau$ , both defined on  $GE_7(\mathbf{A})$  such that  $\mathcal{O}(\pi_1) = \mathcal{O}(E_\tau) = E_7(a_2)$ .

We emphasize that the entries in each table are independent. For example, in Table 2 there are  $4^3 = 64$  cases to consider. This means that using the data from Table 2, we can construct 64 types of integrals, as defined by integral (2). Clearly, we still need to check which of these integral is well defined, and which is a nonzero Eulerian unipotent integral.

#### 4.2. The case when $G = GL_3$

Assume that  $G = GL_3$ . In this section all partitions are partitions of the group  $GL$ . It follows from section 3 that  $\pi_1$  is defined on  $GL_{3k}$  for some  $k \geq 1$ , or defined on the group  $GE_6(\mathbf{A})$ .

In this case, the dimension equation is given by

$$\sum_{i=1}^l (\dim \pi_i - \dim U(\mathcal{O}_{G_i})) = 8$$

First, we claim that for any irreducible representation  $\pi$  of the group  $H = GL_{3k}(\mathbf{A})$  such that  $\mathcal{O}(\pi) > (k^3)$ , or for the group  $H = GE_6(\mathbf{A})$  such that  $\mathcal{O}(\pi) > E_6$ , we have  $\dim \pi - \dim U(\mathcal{O}_H) \geq 2$ . Indeed, if  $H = GL_{3k}$ , then  $\dim \pi - \dim U(\mathcal{O}_H) = \frac{1}{2}(\mathcal{O}(\pi) - \dim(k^3))$ . The first partition which is strictly greater than  $(k^3)$  is  $((k+1)k(k-1))$ . Hence  $\dim \pi - \dim U(\mathcal{O}_H) \geq \frac{1}{2}(\dim((k+1)k(k-1)) - \dim(k^3)) = 2$ , and the claim

**Table 3**  
The  $l = 2$  case.

$\mathcal{O}(\pi_1)$	(6)
$\mathcal{O}(E_\tau)$	$((p+1)p(p-1))$ $D_5(a_1)$

**Table 4**  
The  $l = 2$  case.

$\mathcal{O}(\pi_1)$	(3)
$\mathcal{O}(E_\tau)$	$((p+2)(p+1)(p-3))$ $((p+3)(p-1)(p-2))$ $E_6(a_1)$

follows. In the case of  $H = GE_6$ , we note that the first partition which is strictly greater than  $D_4$  is  $D_5(a_1)$ , and it satisfies  $\frac{1}{2}(\dim D_5(a_1) - \dim D_4) = 2$ . See [C-M].

We need the following result which we shall prove in the last section.

**Lemma 1.** *There are no irreducible nonzero cuspidal representations  $\pi$  defined on  $GE_6(\mathbf{A})$ , such that  $\mathcal{O}_{GE_6}(\pi)$  is equal to  $D_5$ , or to  $D_5(a_1)$ .*

Next we claim that if  $\pi$  is an irreducible cuspidal representation of  $H = GL_{3k}(\mathbf{A})$  or of the group  $H = GE_6(\mathbf{A})$  such that  $\mathcal{O}(\pi) > E_6$ , then we have  $\dim \pi - \dim U(\mathcal{O}_H) \geq 3$ . For the exceptional group this claim follows from Lemma 1. For the group  $GL_{3k}$ , it follows from the fact that every cuspidal representation is generic.

Returning to the global integral (2) which satisfies the dimension equation, we assume first that  $\pi_1$  is a cuspidal representation of the group  $GL_{3k}(\mathbf{A})$ . It follows from Proposition 3, which we state and prove in the next subsection, that  $k = 1, 2$ . Assume first that  $k = 2$ . Then  $\pi_1$  is defined on  $GL_6(\mathbf{A})$ , and hence  $\mathcal{O}(\pi_1) = (6)$ . Hence  $\dim \pi_1 - \dim U(\mathcal{O}_{G_1}) = 6$ . Thus, we deduce that  $l = 2$ , and  $\dim E_\tau - \dim U(\mathcal{O}_{G_2}) = 2$ . From this we obtain Table 3.

Next consider the case when  $k = 1$ . This means that  $\pi_1$  is a cuspidal representation of  $GL_3(\mathbf{A})$ . Hence its dimension is 3, and we obtain

$$\sum_{i=2}^l (\dim \pi_i - \dim U(\mathcal{O}_{G_i})) = 5$$

Since we proved that for any representation  $\pi$  as above  $\dim \pi - \dim U(\mathcal{O}_H) \geq 2$  then  $l = 2, 3$ . If  $l = 2$ , then  $\dim E_\tau - \dim U(\mathcal{O}_{G_2}) = 5$ . We then obtain Table 4.

Assume that  $l = 3$ . Thus  $\dim E_\tau - \dim U(\mathcal{O}_{G_3}) = 2, 3$ . Assume first that it is equal to three. Then  $E_\tau$  is defined either on  $GL_{3p}(\mathbf{A})$  for some  $p \geq 1$ , and satisfies  $\mathcal{O}(E_\tau) = ((p+1)^2(p-2))$ ,  $((p+2)(p-1)^2)$ , or  $E_\tau$  is defined on  $GE_6(\mathbf{A})$  and satisfies  $\mathcal{O}(E_\tau) = E_6(a_3)$ . Since  $l = 3$ , then we have a third representation, denoted by  $\pi_2$

**Table 5**  
The  $l = 3$  case.

$\mathcal{O}(\pi_1)$	(3)	(3)
$\mathcal{O}(\pi_2)$	$((q+1)q(q-1))$ $D_5(a_1)$	$((q+1)^2(q-2))$ $((q+2)(q-1)^2)$ $E_6(a_3)$
$\mathcal{O}(E_\tau)$	$((p+1)^2(p-2))$ $((p+2)(p-1)^2)$ $E_6(a_3)$	$((p+1)p(p-1))$ $D_5(a_1)$

**Table 6**  
The  $l = 2$  case.

$\mathcal{O}(\pi_1)$	$E_6$	$E_6(a_1)$	$E_6(a_3)$
$\mathcal{O}(E_\tau)$	$((p+1)p(p-1))$ $D_5(a_1)$	$((p+1)^2(p-2))$ $((p+2)(p-1)^2)$ $E_6(a_3)$	$((p+2)(p+1)(p-3))$ $((p+3)(p-1)(p-2))$ $E_6(a_1)$

which satisfies  $\dim \pi_2 - \dim U(\mathcal{O}_{G_2}) = 2$ . Hence, the representation is defined either on  $GL_{3q}(\mathbf{A})$  for some  $q \geq 1$ , and satisfies  $\mathcal{O}(\pi_2) = ((q+1)q(q-1))$  or defined on  $GE_6(\mathbf{A})$  and satisfies  $\mathcal{O}(\pi_2) = D_5(a_1)$ . From this we obtain the second column of Table 5.

The last column in Table 5 is obtained by interchanging the roles of the representations  $\pi_2$  and  $E_\tau$ . Thus, the third column corresponds to the case when  $\dim E_\tau - \dim U(\mathcal{O}_{G_3}) = 2$ .

The final case to consider in this subsection is when  $\pi_1$  defines an irreducible cuspidal representation of  $GE_6(\mathbf{A})$ . Since  $\varphi_{\pi_1}^{U_1, \psi_{U_1}}(g)$  is not zero, it follows from section 3 that  $\mathcal{O}_{GE_6}(\pi_1) > D_4$ . Hence, the possibilities for  $\mathcal{O}_{GE_6}(\pi_1)$  are  $E_6$ ,  $E_6(a_1)$ ,  $D_5$ ,  $E_6(a_3)$  or  $D_5(a_1)$ . However, the representation  $\pi_1$  is cuspidal. Hence, from Lemma 1 it follows that  $\mathcal{O}_{GE_6}(\pi_1) = E_6$ ,  $E_6(a_1)$  or  $E_6(a_3)$ .

In all cases we have that  $\dim U(\mathcal{O}_{G_1}) = 30$ . Hence, it follows from [C-M] page 129 that  $\dim \pi_1 - \dim U(\mathcal{O}_{G_1}) = 6, 5, 3$ . Thus, we have the corresponding three possibilities

$$\sum_{i=2}^l (\dim \pi_i - \dim U(\mathcal{O}_{G_i})) = 2, 3, 5$$

From the fact that  $\dim \pi_i - \dim U(\mathcal{O}_{G_i}) \geq 2$  we deduce that when  $\mathcal{O}_{GE_6}(\pi_1) = E_6$  then we have  $l = 2$  and in the other two cases we have  $l = 2, 3$ . We summarize all possible cases in Tables 6 and 7.

For example, the last column in Table 6 corresponds to the case when  $l = 2$  and  $\mathcal{O}(\pi_1) = E_6(a_3)$ . In this case we have  $\dim \pi_1 - \dim U(\mathcal{O}_{G_1}) = 3$ , and since  $l = 2$ , we deduce that  $\dim E_\tau - \dim U(\mathcal{O}_{G_2}) = 5$ . If  $E_\tau$  is defined on  $GL_{3p}(\mathbf{A})$  for some  $p \geq 1$ , then the only options are  $\mathcal{O}(E_\tau) = ((p+2)(p+1)(p-3))$  or  $\mathcal{O}(E_\tau) = ((p+3)(p-1)(p-2))$ . If  $E_\tau$  is defined on  $GE_6(\mathbf{A})$ , then the only possibility is  $\mathcal{O}(E_\tau) = E_6(a_1)$ .

**Table 7**The  $l = 3$  case.

$\mathcal{O}(\pi_1)$	$E_6(a_3)$	$E_6(a_3)$
$\mathcal{O}(\pi_2)$	$((q+1)q(q-1))$ $D_5(a_1)$	$((q+1)^2(q-2))$ $((q+2)(q-1)^2)$ $E_6(a_3)$
$\mathcal{O}(E_\tau)$	$((p+1)^2(p-2))$ $((p+2)(p-1)^2)$ $E_6(a_3)$	$((p+1)p(p-1))$ $D_5(a_1)$

This completes the case when  $G = GL_3$ . Notice that we proved

**Proposition 2.** *Given a global integral of the form (2), where  $G = GL_3$ , which satisfies the dimension equation (5), then  $l \leq 3$ .*

#### 4.3. The case when $G = GL_m$ with $m \geq 4$

Let  $G = GL_m$  with  $m \geq 4$ . It follows from section 3 that we may assume that every automorphic representation  $\pi_i$  which appears in integral (2) is defined on  $GL_{k_i m}(\mathbf{A})$  for some  $k_i \geq 1$ . As before, we assume that  $\pi_1$  is an irreducible cuspidal representation of  $GL_{km}$  where we write  $k$  for  $k_1$ .

The following proposition is valid for all  $m \geq 2$ .

**Proposition 3.** *Suppose that  $m \geq 2$ . Then  $k = 1, 2$ .*

**Proof.** From the dimension equation (5) we obtain

$$\sum_{i=2}^l (\dim \pi_i - \dim U(\mathcal{O}_{G_i})) + \dim \pi_1 - \dim U(\mathcal{O}_{G_1}) = \dim G - \dim Z$$

Since  $\pi_1$  is cuspidal, then it is generic and  $\dim \pi_1 = \frac{1}{2} \dim(km) = \frac{1}{2} km(km-1)$ . We also have  $\dim U_{k,m}(\mathcal{O}) = \frac{1}{2} k(k-1)m^2$ . Hence the dimension equation is

$$\sum_{i=2}^l (\dim \pi_i - \dim U(\mathcal{O}_{G_i})) + \frac{1}{2} km(km-1) - \frac{1}{2} k(k-1)m^2 = m^2 - 1$$

This is the same as

$$\sum_{i=2}^l (\dim \pi_i - \dim U(\mathcal{O}_{G_i})) + (\frac{1}{2} km - m - 1)(m-1) = 0$$

If  $k \geq 3$  then the left hand side is a positive number. Hence, we must have  $k = 1, 2$ .  $\square$

**Table 8**  
The  $l = 2$  case.

$\mathcal{O}(\pi_1)$	$(2m)$
$\mathcal{O}(E_\tau)$	$((p+1)p^{m-2}(p-1))$

Assume first that  $k = 2$ . Then the above equation becomes

$$\sum_{i=2}^l (\dim \pi_i - \dim U(\mathcal{O}_{G_i})) = m - 1 \quad (6)$$

Next consider the Fourier coefficient  $E_\tau^{U_l, \psi_{U_l}}(g, s)$  which appears in integral (2). This Eisenstein series is defined on the group  $G_l(\mathbf{A})$ . Assume that  $G_l = GL_{pm}$  for some  $p \geq 1$ . The unipotent orbit attached to the Fourier coefficient of the Eisenstein series is  $(p^m)$ , hence  $\mathcal{O}_{GL_{pm}}(E_\tau) > (p^m)$ , or  $\mathcal{O}_{GL_{pm}}(E_\tau) \geq ((p+1)p^{m-2}(p-1))$ . Thus, from the formula for the dimension of a partition, see [C-M], we obtain

$$\begin{aligned} \dim E_\tau - \dim U_{p,m}(\mathcal{O}) &= \frac{1}{2}(\dim \mathcal{O}_{GL_{pm}}(E_\tau) - \dim(p^m)) \geq \\ &\geq \frac{1}{2}(\dim((p+1)p^{m-2}(p-1)) - \dim(p^m)) = m - 1 \end{aligned}$$

Combining this with equation (6), we deduce that when  $k = 2$ , we also have  $l = 2$ .

Thus, the case when  $\pi_1$  is an irreducible cuspidal representation of  $GL_{2m}(\mathbf{A})$  produces Table 8.

Assume that  $k = 1$ . In this case  $\pi_1$  is a cuspidal representation of  $GL_m(\mathbf{A})$ . Assuming that integral (2) satisfies the dimension equation, does not by itself limit the possibilities as in the previous cases. In some more details, it follows from the proof of Proposition 3 that the dimension equation is

$$\sum_{i=2}^l (\dim \pi_i - \dim U(\mathcal{O}_{G_i})) = \left(\frac{1}{2}m + 1\right)(m - 1)$$

It is not hard to produce examples of representations which satisfy this equation. For example, when  $m = 4$ , the right hand side of the above equation is equal to 9. It is not hard to construct an Eisenstein series  $E_\tau$  on  $GL_4(\mathbf{A})$  such that  $\dim E_\tau = 3$ . Indeed, let  $\tau$  be the trivial representation and assume that  $E_\tau$  is the Eisenstein series associated with the induced representation  $\text{Ind}_{P(\mathbf{A})}^{GL_4(\mathbf{A})} \delta_P^s$ . Here  $P$  is the maximal parabolic subgroup of  $GL_4$  whose Levi part is  $GL_3 \times GL_1$ . Hence, the integral

$$\int_{Z(\mathbf{A})GL_4(F) \backslash GL_4(\mathbf{A})} \varphi_{\pi_1}(g) E_\tau(g, s_1) E_\tau(g, s_2) E_\tau(g, s_3) dg$$

satisfies the dimension equation. Notice that this does not necessarily mean that Conjecture 1 is not true for  $m \geq 4$ , since in that conjecture we assume that the global integral

is nonzero. And indeed this is what happens in the above integral. A simple unfolding implies that it is identically zero.

## 5. On some Eisenstein series

In this section we study some Eisenstein series needed to construct integrals of the type (2). More precisely, the tables produced in the previous section assume the existence of certain Eisenstein series with certain Fourier coefficients. In this section we indicate how to construct such Eisenstein series.

Given a reductive classical group  $H$ , it follows from [C-M] that unipotent orbits of  $H$  are parameterized by certain partitions. Given such a partition  $\lambda$ , we emphasize the dependence on  $H$  by writing  $\lambda_H$  instead of  $\lambda$ . Given two partitions  $\mathbf{b}_1 = (k_1 k_2 \dots k_p)$  and  $\mathbf{b}_2 = (m_1 m_2 \dots m_q)$  of the numbers  $n$  and  $r$ , we set  $\mathbf{b}_1 + \mathbf{b}_2 = ((k_1 + m_1)(k_2 + m_2) \dots)$ . We also write  $2\mathbf{b} = \mathbf{b} + \mathbf{b}$ .

Let  $H$  denote a reductive group, and let  $P = MU$  denote a maximal parabolic subgroup of  $H$ . Let  $\tau$  denote an automorphic representation of  $M(\mathbf{A})$ , and denote by  $E_\tau(\cdot, s)$  the Eisenstein series associated with the induced representation  $\text{Ind}_{P(\mathbf{A})}^{H(\mathbf{A})} \tau \delta_P^s$ . Notice that by induction of stages, this covers all possible Eisenstein series. We are interested in the set  $\mathcal{O}_H(E_\tau(\cdot, s))$  for  $\text{Re}(s)$  large. By that we mean in the domain where the Eisenstein series is defined by a convergent series. For the classical groups we have the following

**Proposition 4.** *With the above notations, for  $\text{Re}(s)$  large, we have*

- 1) *For  $H = GL_n$ , assume that  $M = GL_a \times GL_{n-a}$ , and  $\tau = \tau_1 \otimes \tau_2$ . Then we have*

$$\mathcal{O}_{GL_n}(E_\tau(\cdot, s)) = \mathcal{O}_{GL_a}(\tau_1) + \mathcal{O}_{GL_{n-a}}(\tau_2)$$

- 2) *For  $H = GSp_{2n}$ , assume that  $M = GL_a \times GSp_{2(n-a)}$ , and  $\tau = \tau_1 \otimes \tau_2$ . Then we have*

$$\mathcal{O}_{GSp_{2n}}(E_\tau(\cdot, s)) = 2\mathcal{O}_{GL_a}(\tau_1) + \mathcal{O}_{GSp_{2(n-a)}}(\tau_2)$$

- 3) *For  $H = GSO_{2n}$ , assume that  $M = GL_a \times GSO_{2(n-a)}$ , and  $\tau = \tau_1 \otimes \tau_2$ . Then we have*

$$\mathcal{O}_{GSO_{2n}}(E_\tau(\cdot, s)) = 2\mathcal{O}_{GL_a}(\tau_1) + \mathcal{O}_{GSO_{2(n-a)}}(\tau_2)$$

*In particular, if  $H$  is one of the above classical groups, then  $\dim \mathcal{O}_H(E_\tau(\cdot, s)) = \dim \tau + \dim U$ .*

**Proof.** The proof of this proposition is a straightforward computation of the relevant unipotent orbit, which is done by unfolding the Eisenstein series. The computations in general are very similar to the computations done in [G3] Proposition 1. We omit the details.

The last equation in the statement of the proposition follows from the fact that from the above parts 1)–3), we deduce that  $\mathcal{O}_H(E_\tau(\cdot, s))$  is the induced orbit as de-



defined in [C-M] section 7. Then the statement about the dimension follows from [C-M] lemma 7.2.5.  $\square$

Since we are mainly interested in the case when  $m = 2$ , we work out the relevant Eisenstein series in this case only. In other words, the Eisenstein series which appear in Tables 1 and 2. From the above proposition we deduce:

**Lemma 2.** *Let  $E_\tau(\cdot, s)$  denote the Eisenstein series which was defined right before Proposition 4. Assume this Eisenstein series appears in Table 1 or 2. Then the unipotent orbits of the representations  $\tau_i$  appearing in the induction data, see also Proposition 4, are described as follows.*

A) For the group  $GL_{2p}$  we have the following cases:

- 1) Suppose that  $\mathcal{O}(E_\tau(\cdot, s)) = ((p+1)(p-1))$ . Then, there is  $0 \leq i \leq 2$  such that  $M = GL_{2(a-1)+i} \times GL_{2(p-a+1)-i}$ ;  $\mathcal{O}(\tau_1) = (a(a-2+i))$  and  $\mathcal{O}(\tau_2) = ((p-a+1)(p-a+1-i))$ .
- 2) Suppose that  $\mathcal{O}(E_\tau(\cdot, s)) = ((p+2)(p-2))$ . Then, there is  $0 \leq i \leq 4$  such that  $M = GL_{2(a-2)+i} \times GL_{2(p-a+2)-i}$ ;  $\mathcal{O}(\tau_1) = (a(a-4+i))$  and  $\mathcal{O}(\tau_2) = ((p-a+2)(p-a+2-i))$ .

B) For the group  $GSp_{4p+2}$  we have the following cases:

- 3) Suppose that  $\mathcal{O}(E_\tau(\cdot, s)) = ((2p+2)(2p))$ . Then, there is  $0 \leq i \leq 1$  such that  $M = GL_{2a-i} \times GSp_{2(2p-2a+i+1)}$ ;  $\mathcal{O}(\tau_1) = (a(a-i))$  and  $\mathcal{O}(\tau_2) = ((2p-2a+2)(2p-2a+2i))$ .
- 4) Suppose that  $\mathcal{O}(E_\tau(\cdot, s)) = ((2p+4)(2p-2))$ . Then, there is  $0 \leq i \leq 3$  such that  $M = GL_{2a-i} \times GSp_{2(2p-2a+i+1)}$ ;  $\mathcal{O}(\tau_1) = (a(a-i))$  and  $\mathcal{O}(\tau_2) = ((2p-2a+4)(2p-2a+2i-2))$ .

C) For the group  $GSO_{4p}$  we have the following cases:

- 5) Suppose that  $\mathcal{O}(E_\tau(\cdot, s)) = ((2p+1)(2p-1))$ . Then, there is  $0 \leq i \leq 1$  such that  $M = GL_{2a-i} \times GSO_{2(2p-2a+i)}$ ;  $\mathcal{O}(\tau_1) = (a(a-i))$  and  $\mathcal{O}(\tau_2) = ((2p-2a+1)(2p-2a+2i-1))$ .
- 6) Suppose that  $\mathcal{O}(E_\tau(\cdot, s)) = ((2p+3)(2p-3))$ . Then, there is  $0 \leq i \leq 3$  such that  $M = GL_{2a-i} \times GSO_{2(2p-2a+i)}$ ;  $\mathcal{O}(\tau_1) = (a(a-i))$  and  $\mathcal{O}(\tau_2) = ((2p-2a+3)(2p-2a+2i-3))$ .

**Proof.** The proof follows immediately from Proposition 4. We give some details about the first case.

Assume that  $\mathcal{O}(E_\tau(\cdot, s)) = ((p+1)(p-1))$ . Assume that  $\mathcal{O}(\tau_1) = (\alpha_1\beta_1)$  and that  $\mathcal{O}(\tau_2) = (\alpha_1\beta_2)$  then it follows from Proposition 4 that  $\alpha_1 + \alpha_2 = p+1$  and  $\beta_1 + \beta_2 = p-1$ . Assume that  $\tau_1$  is an automorphic representation of  $GL_{2(a-1)+i}(\mathbf{A})$  for some  $a$  and  $i = 1, 2$ . Then  $\tau_2$  is an automorphic representation of  $GL_{2(p-a+1)-i}(\mathbf{A})$ . This means that  $\alpha_1 + \beta_1 = 2a+i-2$  and that  $\alpha_2 + \beta_2 = 2p-2a-i+2$ . Also, we have  $\alpha_1 \geq \beta_1$  and  $\alpha_2 \geq \beta_2$ . From these six relations the claim follows. Indeed, we obtain the relations  $\alpha_1 = p+1-\alpha_2$ ,

then  $\beta_1 = 2a + i - p + \alpha_2 - 3$  and  $\beta_2 = 2p - 2a - i - \alpha_2 + 2$ . The relation  $\alpha_1 \geq \beta_1$  implies  $2p + 4 \geq 2a + i + 2\alpha_2$ , and the second inequality implies  $2a + i + 2\alpha_2 \geq 2p + 2$ . Hence,  $2a + i + 2\alpha_2 = 2p + 2, 2p + 3, 2p + 4$ . If  $i = 1$  then we must have  $2a + i + 2\alpha_2 = 2p + 3$  from which it follows that  $\alpha_1 = a$ . From this the claim follows. Similar result happens when  $i = 2$ . We omit the details.  $\square$

For the exceptional groups we proceed as follows. We use the following lemma which is a version of proposition 5.16 in [G2].

**Lemma 3.** *Let  $H$  denote an exceptional group, and let  $E_\tau(\cdot, s)$  denote an Eisenstein series attached to  $\text{Ind}_{P(\mathbf{A})}^{H(\mathbf{A})} \tau \delta_P^s$ . Here  $P$  is a maximal parabolic subgroup of  $H$  with Levi decomposition  $P = MU$ . Let  $\tau$  denote an automorphic representation of  $M(\mathbf{A})$ . Then, for  $\text{Re}(s)$  large, we have  $\dim \mathcal{O}_H(E_\tau(\cdot, s)) = \dim \tau + \dim U$ .*

## 6. On some Fourier expansions

Let  $\pi$  denote an automorphic representation of the group  $H(\mathbf{A})$ , where  $H$  is one of the groups  $GL_{2k}$ ,  $GSp_{2(2k+1)}$ ,  $GSO_{4k}$ , or  $GE_7$ . Let  $V$  be any one of the unipotent subgroups defined in section 3 part 1) with  $m = 2$ , or part 3). Let  $\psi_V$  denote the character of  $V(F) \backslash V(\mathbf{A})$  defined in that section in each case. Then the stabilizer of  $\psi_V$  contains the group  $GL_2$ .

For  $g \in GL_2$ , define

$$f(g) = \int_{V(F) \backslash V(\mathbf{A})} \varphi_\pi(vg) \psi_V(v) dv$$

In this section we compute the following integral

$$\int_{F \backslash \mathbf{A}} f \left( \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} g \right) \psi(ay) dy \quad (7)$$

where  $a = 0, 1$ .

**Lemma 4. a)** *When  $a = 1$ , integral (7) corresponds to the Fourier coefficient of  $\pi$  associated with the unipotent orbit*

- 1)  $((k+1)(k-1))$  if  $H = GL_{2k}$ .
- 2)  $((2k+2)(2k))$  if  $H = GSp_{2(2k+1)}$ .
- 3)  $((2k+1)(2k-1))$  if  $H = GSO_{4k}$ .
- 4)  $E_7(a_2)$  if  $H = GE_7$ .

**b)** *When  $a = 0$ , the constant term of  $f(g)$  corresponds to a sum of Fourier coefficients associated with every unipotent orbit of  $H$  which is strictly greater than the unipotent orbit written in part a), and a certain Fourier coefficient which contains the constant term specified below, as an inner integration.*

**Proof.** We work out the details for the case where  $H = GL_{2k}$ . The other cases are similar. We compute the integral

$$\int_{F \backslash \mathbf{A}} \int_{V(F) \backslash V(\mathbf{A})} \varphi_{\pi}(v\mu(y)) \psi_V(v) \psi(ay) dy dv$$

where

$$\mu(y) = I_{2k} + ye_{1,2} + ye_{3,4} + \cdots + ye_{2k-1,2k}$$

Here, we denote by  $e_{i,j}$  the matrix of order  $2k$  with one at the  $(i, j)$  entry and zero elsewhere. Expand the above integral along the subgroup

$$V_1 = \{v_1(r_2, r_3, \dots, r_k) = I_{2n} + r_2 e_{3,4} + r_3 e_{5,6} + \cdots + r_k e_{2k-1,2k} \quad : \quad r_i \in \mathbf{A}\}$$

Then the above integral is equal to

$$\int \sum_{\xi_i \in F} \int \varphi_{\pi}(v_1(r_2, r_3, \dots, r_k) v\mu(y)) \psi_V(v) \psi(ay + \sum_{i=2}^k \xi_i r_i) dy dv dr_i$$

Let

$$L_1 = \{l_1(z_2, z_3, \dots, z_k) = I_{2k} + z_2 e_{2,3} + z_3 e_{4,5} + \cdots + z_k e_{2k-2,2k-1}\}$$

Then  $L_1$  is a subgroup of  $V$ . Since  $\varphi_{\pi}$  is an automorphic function, then we have  $\varphi_{\pi}(m) = \varphi_{\pi}(l_1(\xi_2, \xi_3, \dots, \xi_k)m)$ . Conjugating this discrete matrix from left to right, collapsing summation with integration, the last integral is equal to

$$\int_{L_1(\mathbf{A})} \int_{V_2(F) \backslash V_2(\mathbf{A})} \varphi_{\pi}(vl_1) \psi_{a,V_2}(v) dv dl_1$$

Here

$$V_2 = \{u \in U \quad : \quad u_{i,i+1} = 0; \quad i = 2, 4, \dots, 2k-2\}$$

where  $U$  is the maximal standard unipotent subgroup of  $GL_{2k}$ . Also, for  $v \in V_2$  we have

$$\psi_{a,V_2}(v) = \psi(av_{1,2} + v_{1,3} + v_{2,4} + v_{3,5} + \cdots + v_{2k-2,2k})$$

Assume that  $a = 1$ . Then, using the correspondence between unipotent orbits and Fourier coefficients as described in [G2] section 2, we deduce that the integration over  $V_2$  in the above integral, is a Fourier coefficient associated with the unipotent orbit  $((k+1)(k-1))$ .

Next, assume  $a = 0$ . For this section let  $U$  denote the standard maximal unipotent subgroup of  $GL_{2k}$ . Let  $w$  denote the Weyl element of  $GL_{2k}$  defined as follows. For all  $1 \leq i \leq k$  set  $w_{i,2i-1} = w_{k+i,2i} = 1$ , and all other entries of  $w$  are zeros. Since  $w$  is a discrete element, then  $\varphi_{\pi}$  is left invariant by it. Conjugating  $w$  from left to right, the

above integral is equal to

$$\int_{L_1(\mathbf{A})} \int_{L_2(F) \backslash L_2(\mathbf{A})} \int_{V_3(F) \backslash V_3(\mathbf{A})} \varphi_\pi(vl_2wl_1)\psi_{V_3}(v)dvdl_2dl_1$$

Here  $V_3$  is the subgroup of  $U$  defined by

$$V_3 = \{v \in U : u_{i,j} = 0; 2 \leq i \leq k \text{ and } k \leq j \leq 2k-1\}$$

and  $L_2$  is the group of all lower unipotent matrices defined by

$$L_2 = \{l_2 = \begin{pmatrix} I_k & \\ X & I_k \end{pmatrix} : X_{i,j} = 0; j \leq i+1\}$$

Also, the character  $\psi_{V_3}$  is defined as follows

$$\psi_{V_3}(v) = \psi(v_{1,2} + v_{2,3} + \cdots + v_{k-1,k} + v_{k+1,k+2} + v_{k+2,k+3} + \cdots + v_{2k-1,2k}) \quad (8)$$

Let  $L_3$  denote the subgroup of  $U$  defined by

$$L_3 = \{l_3 = \begin{pmatrix} I_k & Y \\ & I_k \end{pmatrix} : Y_{i,j} = 0; i \leq j \text{ and } i = k\}$$

Next we expand the above integral along the group  $L_3(F) \backslash L_3(\mathbf{A})$ . Carrying out a similar process as in the previous expansion in this section, and doing it one variable at the time, we obtain that the above integral is equal to

$$\int_{L_1(\mathbf{A})} \int_{L_2(\mathbf{A})} \int_{V_4(F) \backslash V_4(\mathbf{A})} \varphi_\pi(vl_2wl_1)\psi_{V_4}(v)dvdl_2dl_1 \quad (9)$$

Here  $V_4$  is the subgroup of  $U$  defined by

$$V_4 = \{u \in U : u_{k,j} = 0; k \leq j \leq 2k-1\}$$

The character  $\psi_{V_4}$  is the trivial extension of  $\psi_{V_3}$  to  $\psi_{V_4}$ .

Next, consider the expansion of the above integral along the one parameter unipotent subgroup of  $U$  consisting of all matrices of the form  $x_\alpha(r) = I_{2k} + re_{k,2k-1}$ . First consider the contribution from the nontrivial orbit. It is a sum over  $\xi \in F^*$  of the Fourier coefficients

$$\int_{L_1(\mathbf{A})} \int_{L_2(\mathbf{A})} \int_{F \backslash \mathbf{A}} \int_{V_4(F) \backslash V_4(\mathbf{A})} \varphi_\pi(x_\alpha(r)vl_2wl_1)\psi_{V_4}(v)\psi(\xi r)drdvdl_2dl_1$$

Using the corresponding between unipotent orbits and Fourier coefficients, as described in [G2] section 2, we deduce that the above Fourier coefficient corresponds to the unipotent

orbit  $((k+2)(k-2))$ . The second contribution to integral (9) from the above expansion is from the constant term, and it is equal to

$$\int_{L_1(\mathbf{A})} \int_{L_2(\mathbf{A})} \int_{V_5(F) \backslash V_5(\mathbf{A})} \varphi_\pi(vl_2wl_1)\psi_{V_5}(v)dvdl_2dl_1$$

where

$$V_5 = \{u \in U : u_{k,j} = 0; k \leq j \leq 2k-2\}$$

We further expand this integral along  $x_\alpha(r) = I_{2k} + re_{k,2k-2}$ . Again we get two contributions. The first, from the nontrivial orbit, contributes a sum of Fourier coefficients, each corresponds to the unipotent orbit  $((k+3)(k-3))$ . The second is the constant term. Arguing by induction we eventually expand along the group  $x_\alpha(r) = I_{2k} + re_{k,k+1}$ . The contribution from the nontrivial orbit will produce a sum of Fourier coefficients which corresponds to the unipotent orbit  $(2k)$ , and the trivial orbit will produce the integral

$$\int_{L_1(\mathbf{A})} \int_{L_2(\mathbf{A})} \int_{U(F) \backslash U(\mathbf{A})} \varphi_\pi(ul_2wl_1)\psi_U(v)dudl_2dl_1$$

where  $\psi_U$  is the character defined by (8) extended trivially to  $U$ . Notice that this last integral contains the constant term of  $\pi$  along the unipotent radical of the maximal parabolic group whose Levi part is  $GL_k \times GL_k$ . To summarize, we expressed integral (7) as a sum of Fourier coefficients which corresponds to unipotent orbits which are greater than  $(k+1)(k-1)$ , and an integral containing a constant term as an inner integration. This is the statement in part **b)** of the lemma.

We finish the proof of the lemma with the description of the constant term which is obtained in the other cases. First, in the classical groups. In the case when  $H = GSp_{2(2k+1)}$  we obtain the constant term along the unipotent radical of the maximal parabolic subgroup whose Levi part is  $GL_{2k+1}$ . When  $H = GSO_{4k}$  we get the unipotent radical of the maximal parabolic subgroup whose Levi subgroup is  $GL_{2k}$ . Finally, when  $H = GE_7$  we obtain the unipotent radical of the maximal parabolic subgroup whose Levi part is  $E_6$ .  $\square$

## 7. Unfolding global integrals with $G = GL_2$

It follows from Tables 1 and 2 that there are two cases to consider for the group  $G = GL_2$ . First, if  $l = 2$  the global integral (2) is

$$\int_{Z(\mathbf{A})G(F) \backslash G(\mathbf{A})} \varphi_{\pi_1}^{U_1, \psi_{U_1}}(g) E_\tau^{U_2, \psi_{U_2}}(g, s) dg \quad (10)$$

and second, if  $l = 3$  we have,

$$\int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} \varphi_{\pi_1}^{U_1, \psi_{U_1}}(g) \varphi_{\pi_2}^{U_2, \psi_{U_2}}(g) E_{\tau}^{U_3, \psi_{U_3}}(g, s) dg \quad (11)$$

For  $1 \leq j \leq 3$ , let  $G_j$  denote one of the groups  $GL_{2p}$ ,  $GSp_{2(2p+1)}$ ,  $GSO_{4p}$  or  $GE_7$ . In integrals (10) and (11), we assume that  $\pi_1$  is a cuspidal representation. For  $j = 1, 2$ , the sets  $\mathcal{O}_{G_j}(\pi_j)$  are listed in Tables 1 and 2, and similarly for  $j = 2, 3$  for  $\mathcal{O}_{G_j}(E_{\tau})$ . As in Section 5, the Eisenstein series  $E_{\tau}(\cdot, s)$  is associated with the induced representation  $Ind_{P(\mathbf{A})}^{H(\mathbf{A})} \tau \delta_P^s$ . Here  $H = G_2$  when we consider integral (10), and  $H = G_3$  when we consider integral (11). We also assume that  $P$  is a maximal parabolic subgroup of  $H$ . The representation  $\tau$  is an automorphic representation of  $M(\mathbf{A})$  where  $M$  is the Levi part of  $P$ .

In this section we determine which of the above integrals, assuming that the representations involved in it satisfy the requirements of Tables 1 and 2, is a nonzero global unipotent integral. To do that we carry out the unfolding process. As it turns out not all the integrals we consider are unipotent global integrals. To describe which of the above integrals are unipotent global integrals we start with

**Definition 2.** We say that  $E_{\tau}(\cdot, s)$ , defined on  $GL_{2p}$ , is an odd Eisenstein series, if, using induction by stages, there is a maximal parabolic subgroup such that  $E_{\tau}(\cdot, s)$  is induced from that parabolic subgroup, and the value of  $i$  as appears in Lemma 2 is an odd number.

Similarly, in the other classical groups  $H$ , we define  $E_{\tau}(\cdot, s)$  to be odd if we can induce from a maximal parabolic subgroup whose Levi part is  $GL_{\alpha} \times L$  where  $\alpha$  is an odd number. Here  $L$  is a classical group of the same type of  $H$ . In the case of the exceptional groups, we define the notion of an odd Eisenstein series right before Theorem 3.

We can now state the main result of this paper,

**Theorem 1.** Assume that  $G = GL_2$ . Then the global integrals (10) and (11) are nonzero global unipotent integrals if and only if at least one of the Eisenstein series appearing in Tables 1 or 2 is an odd Eisenstein series.

We start with the unfolding of the Eisenstein series. Let  $V = U_2$  or  $V = U_3$  be one of the unipotent groups introduced in section 3 with  $m = 2$ . Thus, for the group  $H = GL_{2p}$ , then  $V = U_{p,2}(\mathcal{O})$ , and for the other classical groups we let  $V = U_n(\mathcal{O})$ . For  $H = GE_7$  this group was denoted in section 3 by  $U(\mathcal{O})$ . By  $\psi_V$  we denote the corresponding character which was defined in section 3.

Denote by  $U(P)$  the unipotent radical of  $P$ . We denote by  $U$  the unipotent maximal subgroup of  $GL_{2p}$  consisting of upper triangular matrices. We have

$$\begin{aligned}
E_{\tau}^{V, \psi_V}(h, s) &= \int_{V(F) \backslash V(\mathbf{A})} \sum_{\gamma \in P(F) \backslash H(F)} f_{\tau}(\gamma v h, s) \psi_V(v) dv = \\
&= \sum_{\gamma \in P(F) \backslash H(F) / V(F)} \int_{V^{\gamma}(F) \backslash V(\mathbf{A})} f_{\tau}(\gamma v h, s) \psi_V(v) dv \quad (12)
\end{aligned}$$

where  $V^{\gamma} = V \cap \gamma^{-1} P \gamma$ .

We call an element  $\gamma \in P(F) \backslash H(F) / V(F)$  not admissible, if there exists an element  $v \in V$  such that  $\gamma v \gamma^{-1} \in U(P)$ , and such that  $\psi_V(v) \neq 1$ . Otherwise we say that  $\gamma$  is admissible. From the definition it follows that a non-admissible element contributes zero to the above summation. Our goal is to characterize all the admissible elements. We shall write all details for the group  $H = GL_{2p}$ . For the other classical groups and for  $GE_7$  the process is similar.

It follows from the Bruhat decomposition that every element in  $P(F) \backslash H(F) / V(F)$  can be written as  $\gamma = w v_w$ . Here  $w$  is a Weyl element of  $GL_{2p}$ , and  $v_w = I_{2p} + z_1 e_{1,2} + z_2 e_{3,4} + \dots + z_p e_{2p-1,2p}$  where  $z_i \in F$  and  $e_{i,j}$  is the matrix of size  $2p$  with one at the  $(i, j)$  entry, and zero elsewhere. We claim that if  $w$  is not admissible, then  $w v_w$  is also not admissible. This follows from the action by conjugation of  $v_w$  on the group  $V$ . Indeed, if there is a  $v \in V$  such that  $w v w^{-1} \in U(P)$ , and  $\psi_V(v) \neq 1$ , then we can find an element  $v' \in V$  such that  $\psi_V(v') \neq 1$ , and that  $v_w v' v_w^{-1} = v$ . From this the claim follows.

Assume that the Levi part of  $P$  is  $GL_r \times GL_{2p-r}$  with  $p \leq r$ . Then  $2p-r \leq r$ . Assume that  $w$  is admissible. We shall write  $w[i, j]$  for its  $(i, j)$  entry. Thus  $w[i, j] = 0, 1$ . By a suitable multiplication from the left by an element of  $GL_r \times GL_{2p-r}$ , we may assume that there is a maximal number  $q'$  such that  $0 \leq q' \leq 2p-r$  and such that  $w[r+i, i] = 1$  for all  $1 \leq i \leq q'$ . From the maximality of  $q'$  we obtain that  $w[j, q'+1] = 0$  for all  $r+1 \leq j \leq 2p$ . Hence, adjusting by an element of  $GL_r \times GL_{2p-r}$ , we may assume that  $w[1, q'+1] = 1$ . It is convenient to consider the cases  $q'$  even or odd separately. Assume that  $q' = 2q$ . Then  $w[1, 2q+1] = 1$ . Consider  $v = I_{2p} + z e_{2q+1, 2q+3}$ . Then  $\psi_V(v) \neq 1$ . Consider the matrix  $w v w^{-1}$ . A simple matrix multiplication implies that if  $w[j, 2q+3] = 1$  for some  $r+1 \leq j \leq 2p$ , then  $w v w^{-1} \in U(P)$  and hence  $w$  is not admissible. Thus, we have  $w[j', 2q+3] = 1$  for some  $2 \leq j' \leq r$ . By a suitable multiplication from the left by elements in  $GL_r \times GL_{2p-r}$ , we may assume that  $w[2, 2q+3] = 1$ . The process is inductive, namely using the same argument we deduce that  $w[3, 2q+5] = 1$ , and so on until  $w[p-q, 2p-1] = 1$ . Thus we have determined the first  $p-q$  rows of  $w$ .

Consider the next  $r-p+q$  rows. Multiplication from the left by elements in  $GL_r \times GL_{2p-r}$ , we let  $q_0$  be the smallest positive number such that  $w[p-q+1, 2(q+q_0)] = 1$ . Then arguing as above we deduce that  $w$  is admissible if and only if, after a suitable multiplication by  $GL_r \times GL_{2p-r}$ , we have  $w[p-q+2, 2(q+q_0+1)] = 1$ , and so on. Let  $q_1$  be such that  $w[r, 2(q+q_1)] = 1$ . Using the same argument,  $w$  is admissible if and only if  $2(q+q_1) = 2p$ . In other words,  $w$  is admissible if and only if

$$w[r, 2p] = w[r-1, 2(p-1)] = \dots = w[p-q+1, 2(2p-r-q+1)] = 1$$

Thus, so far we determined the first  $r + 2q$  rows of  $w$ . But, up to multiplication by  $GL_r \times GL_{2p-r}$ , this determines also the last  $2p - 2q - r$  rows. In other words we have  $w[r + 2q + 1, 2q + 2] = w[r + 2q + 2, 2q + 4] = \dots = w[2p, 2(2p - q - r)] = 1$ .

A similar construction holds when  $q' = 2q + 1$ . Writing  $q$  for  $q'$ , we can parameterize the admissible Weyl elements by the elements  $w_q$  with  $0 \leq q \leq 2p - r$ , where

$$w_q = \begin{pmatrix} L'_q & & \\ & I_{2(r-p)+q} & \\ I_q & & L''_q \end{pmatrix}$$

Here  $I_k$  is the identity matrix of size  $k$ , and

$$\begin{aligned} L'_q &= \{x \in \text{Mat}_{(2p-r-q) \times 2(2p-r-q)} : x_{i,2i-1} = 1 \text{ for } 1 \leq i \leq 2p - r - q \\ &\quad \text{and } x_{i,j} = 0 \text{ elsewhere}\} \\ L''_q &= \{y \in \text{Mat}_{(2p-r-q) \times 2(2p-r-q)} : y_{i,2i} = 1 \text{ for } 1 \leq i \leq 2p - r - q \\ &\quad \text{and } y_{i,j} = 0 \text{ elsewhere}\} \end{aligned}$$

Notice that

$$w_{2p-r} = \begin{pmatrix} & I_r \\ I_{2p-r} & \end{pmatrix}$$

From all this we conclude that we may consider those double cosets whose representatives are of the form  $w_q z(r_1, r_2, \dots, r_p)$  where  $r_i \in F$  and

$$z(r_1, r_2, \dots, r_p) = I_{2p} + r_1 e_{1,2} + r_2 e_{3,4} + \dots + r_p e_{2p-1,2p}$$

To eliminate more double cosets, we specify the Eisenstein series as in [Lemma 2](#) part **A**. Thus we assume that  $\mathcal{O}(E_\tau(\cdot, s))$  is equal to  $((p+1)(p-1))$  or  $((p+2)(p-2))$ . Also, we use our assumption that the Eisenstein series is odd. See [Definition 2](#). In other words, we assume that, using induction by stages, there is a number  $a$  and an odd number  $i$  such that  $E_\tau(\cdot, s)$  is induced from that parabolic subgroup. Thus, we have three cases. First, if  $\mathcal{O}(E_\tau(\cdot, s)) = ((p+1)(p-1))$  then we assume that  $M = GL_{2a-1} \times GL_{2(p-a)+1}$  and  $\mathcal{O}(\tau_1) = (a(a-1))$  and  $\mathcal{O}(\tau_2) = ((p-a+1)(p-a))$ . In the second case  $\mathcal{O}(E_\tau(\cdot, s)) = ((p+2)(p-2))$  and there are two possible induction data. The first possibility is  $M = GL_{2a-3} \times GL_{2(p-a)+3}$  and  $\mathcal{O}(\tau_1) = (a(a-3))$  and  $\mathcal{O}(\tau_2) = ((p-a+2)(p-a+1))$ . The second possibility is  $M = GL_{2a-1} \times GL_{2(p-a)+1}$  and  $\mathcal{O}(\tau_1) = (a(a-1))$  and  $\mathcal{O}(\tau_2) = ((p-a+2)(p-a-1))$ . However, changing  $a$  in the first possibility with  $p-a+2$  gives us the second possibility.

To summarize, if  $\mathcal{O}(E_\tau(\cdot, s)) = ((p+1)(p-1))$  then the induction data is  $\mathcal{O}(\tau_1) = (a(a-1))$  and  $\mathcal{O}(\tau_2) = ((p-a+1)(p-a))$ . If  $\mathcal{O}(E_\tau(\cdot, s)) = ((p+2)(p-2))$ , then the induction data is  $\mathcal{O}(\tau_1) = (a(a-1))$  and  $\mathcal{O}(\tau_2) = ((p-a+2)(p-a-1))$ .

We return to the computation of  $E_\tau^{V, \psi_V}(h, s)$ . Suppose that  $V$  contains a subgroup  $V_1$  such that  $\gamma^{-1}V_1\gamma \subset M$ . Then the right most integral in identity [\(12\)](#) contains the



integral

$$\int_{V_1(F) \backslash V_1(\mathbf{A})} f_\tau(\gamma^{-1}v_1\gamma h') \psi_V(v_1) dv_1$$

as inner integration. This integral defines certain Fourier coefficients of the automorphic functions  $\varphi_{\tau_1}$  and  $\varphi_{\tau_2}$ . If, for some  $\gamma$  the unipotent orbit corresponding to one of these Fourier coefficients is strictly greater than  $\mathcal{O}(\tau_1)$  or  $\mathcal{O}(\tau_2)$ , then the above integral is zero, and hence the contribution to (12) from this representative is zero.

Let  $\gamma = w_q z(r_1, r_2, \dots, r_p)$ . To handle the elements  $z(r_1, r_2, \dots, r_p)$ , we consider the subgroup  $V'$  of  $V$  defined by  $V' = \{v \in V : v_{2j, 2j-1} = 0; 1 \leq j \leq p-1\}$ . Notice that  $z(r_1, r_2, \dots, r_p)$  normalizes  $V'$ , and if by restriction, we consider the character  $\psi_V$  as a character of  $V'$ , then  $\psi_V(z(r_1, r_2, \dots, r_p)^{-1}v'z(r_1, r_2, \dots, r_p)) = \psi_V(v')$ . Therefore, if we replace  $V$  by  $V'$  and take  $V_1$  to be a subgroup of  $V'$ , then we may ignore the unipotent part of  $\gamma$ . Recall that  $i$ , as defined in Lemma 2 is odd. This means that both numbers  $r$  and  $2p-r$  are odd. We recall that  $GL_r \times GL_{2p-r}$  is the Levi part of  $P$ , the parabolic subgroup we used to construct the Eisenstein series. The Weyl elements which we still need to consider are given by  $w_q$  where we take  $q = 2t, 2t+1$  with  $0 \leq t \leq \frac{2p-r-1}{2}$ . It follows from matrix multiplication that after conjugating by  $w_{2t}$  and by  $w_{2t+1}$  we obtain on  $\varphi_{\tau_1}$  the Fourier coefficient corresponding to the unipotent orbit  $((p-t)(r-p+t))$ , and on  $\varphi_{\tau_2}$  the Fourier coefficient corresponding to the unipotent orbit  $((2p-r-t)t)$ .

Because of the induction data of the Eisenstein series, as given in Lemma 2, these unipotent orbits must satisfy  $((2p-r-t)t) \leq (a(a-1))$  or  $((p-t)(r-p+t)) \leq (a(a-1))$ . For otherwise, the above integral will be zero. But  $(a(a-1))$  is the smallest unipotent orbit of  $GL_{2p}$  of the form  $(n_1 n_2)$ , and hence either  $((2p-r-t)t) = (a(a-1))$  or  $((p-t)(r-p+t)) = (a(a-1))$ . In both cases we obtain  $t = \frac{2p-r-1}{2}$ . Thus, in the factorization of (12) we are left with two possible nonzero contributions corresponding to the Weyl elements  $w_{2p-r-1}$  and  $w_{2p-r}$ .

To continue we now unfold the global integrals (10) and (11). First, unfolding the Eisenstein series, we notice that we need to consider representatives of the space of double cosets  $P \backslash H/V \cdot G$ . Using the above discussion, we only need to consider two types of representatives. They are  $w_{2p-r-1}z(r_1, r_2, \dots, r_p)$  and  $w_{2p-r}z(r_1, r_2, \dots, r_p)$ . However, it is not hard to check that all of these representatives, which were distinct when we considered  $P \backslash H/V$ , are now reduced to one element, which we choose to be  $w_{2p-r-1}$ . We denote this element by  $w_0$ .

Thus, for  $\text{Re}(s)$  large, integral (10) is equal to

$$\int_{Z(\mathbf{A})B(F) \backslash G(\mathbf{A})} \varphi_{\pi_1}^{U_1, \psi_{U_1}}(g) \int_{V^{w_0}(\mathbf{A}) \backslash V(\mathbf{A})} f_\tau^{V^{w_0}, \psi_{w_0}}(w_0 v g, s) \psi_V(v) dv dg \quad (13)$$

and integral (11) is equal to

$$\int_{Z(\mathbf{A})B(F)\backslash G(\mathbf{A})} \varphi_{\pi_1}^{U_1, \psi_{U_1}}(g) \varphi_{\pi_2}^{U_2, \psi_{U_2}}(g) \int_{V^{w_0}(\mathbf{A})\backslash V(\mathbf{A})} f_{\tau}^{V^{w_0}, \psi_{w_0}}(w_0 v g, s) \psi_V(v) dv dg \quad (14)$$

Here  $B$  is the Borel subgroup of  $G = GL_2$  which consists of all upper unipotent matrices, and

$$f_{\tau}^{V^{w_0}, \psi_{w_0}}(h, s) = \int_{V^{w_0}(F)\backslash V^{w_0}(\mathbf{A})} f_{\tau}(v_0 h, s) \psi_{w_0}(v_0) dv_0$$

If the Levi part of  $P$  is  $GL_{2a-1} \times GL_{2(p-a)+1}$ , then this Fourier coefficient corresponds to the unipotent orbit  $(a(a-1))$  of  $GL_{2a-1}$ , and corresponds to the unipotent orbit  $((p-a+1)(p-a))$  of  $GL_{2(p-a)+1}$ .

Let  $U(B)$  denote the unipotent radical of the Borel group  $B$ . Consider first the case when  $\mathcal{O}(E_{\tau}(\cdot, s)) = ((p+1)(p-1))$ . Then the induction data is  $\mathcal{O}(\tau_1) = (a(a-1))$  and  $\mathcal{O}(\tau_2) = ((p-a+1)(p-a))$ . Arguing in a similar way as in the proof of the first part of Lemma 4, we deduce that for all  $u \in U(B)(\mathbf{A})$  we have  $f_{\tau}^{V^{w_0}, \psi_{w_0}}(w_0^{-1} u w_0 h, s) = f_{\tau}^{V^{w_0}, \psi_{w_0}}(h, s)$ . Thus, integrals (13) and (14) are equal to

$$\int_{Z(\mathbf{A})T(F)U(B)(\mathbf{A})\backslash G(\mathbf{A})} \int_{U(B)(F)\backslash U(B)(\mathbf{A})} \varphi_{\pi_1}^{U_1, \psi_{U_1}}(ug) R_{\tau}(g) du dg \quad (15)$$

and

$$\int_{Z(\mathbf{A})T(F)U(B)(\mathbf{A})\backslash G(\mathbf{A})} \int_{U(B)(F)\backslash U(B)(\mathbf{A})} \varphi_{\pi_1}^{U_1, \psi_{U_1}}(ug) \varphi_{\pi_2}^{U_2, \psi_{U_2}}(ug) R_{\tau}(g) du dg \quad (16)$$

Here

$$R_{\tau}(g) = \int_{V^{w_0}(\mathbf{A})\backslash V(\mathbf{A})} f_{\tau}^{V^{w_0}, \psi_{w_0}}(w_0 v g, s) \psi_V(v) dv$$

and  $T$  is defined as all matrices of the form  $T = \{\text{diag}(c, 1) : c \in F^*\}$ . Consider first integral (15). We apply Lemma 4 part **b)** to obtain that the integral

$$\int_{U(B)(F)\backslash U(B)(\mathbf{A})} \varphi_{\pi_1}^{U_1, \psi_{U_1}}(ug) du$$

is a sum of terms which are related to all unipotent orbits which are strictly greater than the ones listed in that lemma part **a)**, and to a certain constant term. By cuspidality of  $\pi_1$  we may ignore the summand with the constant term. Also, the case we consider now corresponds to the first column in Table 1. Thus  $\mathcal{O}_{G_1}(\pi_1)$  consists of the unipotent orbit

specified in the first row of that table. From this, and from the computations done in the proof of Lemma 4, we obtain

$$\int_{U(B)(F) \backslash U(B)(\mathbf{A})} \varphi_{\pi_1}^{U_1, \psi_{U_1}}(ug) du = \sum_{t \in T(F)} L_{\pi_1}^{R_1}(tg)$$

Here  $L_{\pi_1}$  is defined in the beginning of section 2 and is given by

$$L_{\pi_i}(g_i) = \int_{V_i(\pi_i)(F) \backslash V_i(\pi_i)(\mathbf{A})} \varphi_{\pi_i}(v_i g_i) \psi_{V_i(\pi_i)}(v_i) dv_i$$

where this Fourier coefficient corresponds to the unipotent orbit as specified in the first row in the second column of Table 1. Plugging the above identity into integral (15), collapsing summation with integration, we deduce that in this case, integral (2) is a global unipotent integral.

Next consider integral (16). Expand the function  $\varphi_{\pi_1}^{U_1, \psi_{U_1}}(ug)$  along the unipotent group  $U(B)$ . By Lemma 4 part b), and by the cuspidality of  $\pi_1$ , we may ignore the contribution from the constant term. From part a) of that lemma we obtain

$$\varphi_{\pi_1}^{U_1, \psi_{U_1}}(ug) = \sum_{t \in T(F)} \int_{U(B)(F) \backslash U(B)(\mathbf{A})} \varphi_{\pi_1}^{U_1, \psi_{U_1}}(u_1 t u g) \psi_{U(B)}(u_1) du_1 du \quad (17)$$

Plug this into integral (16). Use the fact that  $\pi_2$  and  $f_\tau$  are left invariant by  $T(F)$  to collapse summation and integration. Thus we obtain

$$\int_{Z(\mathbf{A})U(B)(\mathbf{A}) \backslash G(\mathbf{A})} \int_{U(B)(F) \backslash U(B)(\mathbf{A})} L_{\pi_1}^{R_1}(ug) \varphi_{\pi_2}^{U_2, \psi_{U_2}}(ug) R_\tau(g) dudg \quad (18)$$

From the above expansion we deduce that  $L_{\pi_1}^{R_1}(ug) = \psi_{U(B)}(u_1) L_{\pi_1}^{R_1}(g)$ . Thus, using Lemma 4 part a), integral (18) is equal to

$$\int_{Z(\mathbf{A})U(B)(\mathbf{A}) \backslash G(\mathbf{A})} L_{\pi_1}^{R_1}(g) L_{\pi_2}^{R_2}(g) R_\tau(g) dudg \quad (19)$$

where  $L_{\pi_j}^{R_j}$  is defined as above and corresponds to the unipotent orbits appearing in the first and second row of Table 2. Thus, we deduce that also in this case integral (2) is a global unipotent integral.

Finally, we consider integral (10) when  $\mathcal{O}(E_\tau(\cdot, s)) = ((p+2)(p-2))$ . Then the induction data is  $\mathcal{O}(\tau_1) = (a(a-1))$  and  $\mathcal{O}(\tau_2) = ((p-a+2)(p-a-1))$ . Starting with integral (13), we obtain

$$\int_{Z(\mathbf{A})T(F)U(B)(\mathbf{A}) \backslash G(\mathbf{A})} \int_{U(B)(F) \backslash U(B)(\mathbf{A})} \varphi_{\pi_1}^{U_1, \psi_{U_1}}(ug) R_\tau(ug) dudg$$

Notice that in this case the function  $R_\tau(g)$  is not left invariant under  $U(B)(\mathbf{A})$ . Now we consider the expansion given by identity (17). Using cuspidality of  $\pi_1$  we may ignore the contribution from the constant term, and collapsing summation with integration, we obtain

$$\int_{Z(\mathbf{A})U(B)(\mathbf{A}) \backslash G(\mathbf{A})} \int_{U(B)(F) \backslash U(B)(\mathbf{A})} L_{\pi_1}^{R_1}(ug) R_\tau(ug) du dg$$

As before, we have  $L_{\pi_1}^{R_1}(ug) = \psi_{U(B)}(u_1) L_{\pi_1}^{R_1}(g)$ , and hence we obtain

$$\int_{Z(\mathbf{A})U(B)(\mathbf{A}) \backslash G(\mathbf{A})} \int_{U(B)(F) \backslash U(B)(\mathbf{A})} L_{\pi_1}^{R_1}(g) R_\tau^{U(B), \psi_{U(B)}}(g) dg$$

Using a variation of Lemma 4 part a), we obtain that  $R_\tau^{U(B), \psi_{U(B)}}(g)$  is a Fourier coefficient of the representation  $\tau_1$  which corresponds to the unipotent orbit  $(a(a-1))$  of  $GL_{2a-1}$ , and a Fourier coefficient of  $\tau_2$  which corresponds to the unipotent orbit  $((p-a+2)(p-a-1))$  of  $GL_{2(p-a)+1}$ . Hence, in this case, integral (2) is also a global unipotent integral.

To complete the proof we will consider the Classical groups case first,

**Theorem 2.** Assume that  $G = GL_2$ , and that all the groups  $G_j$  are classical groups. Then the global integral (2) is a nonzero global unipotent integral if and only if one of the representations appearing in Table 1 or 2 is an odd Eisenstein series.

**Proof.** The case when one of the Eisenstein series is odd was considered above. Hence, we may assume that none of the Eisenstein series appearing in integral (2) is odd. As before we treat the case where the Eisenstein series is defined on  $H = GL_{2p}$ . Assume that  $E_\tau(\cdot, s)$  is associated with the induced representation  $Ind_{P(\mathbf{A})}^{H(\mathbf{A})} \tau \delta_P^s$ , where  $P$  is the maximal parabolic subgroup whose Levi part is  $GL_{2r} \times GL_{2(p-r)}$ . Consider the Weyl element

$$w_0 = \begin{pmatrix} & I_{2(p-r)} \\ I_{2r} & \end{pmatrix}$$

Unfolding the Eisenstein series, we consider the contribution to integral (2) from the double coset representative  $w_0$ . A simple matrix conjugation implies that, as an inner integration, we obtain an integral which involves the period integral

$$\int_{Z(\mathbf{A})G(F) \backslash G(\mathbf{A})} \varphi_{\pi_1}^{U_1, \psi_{U_1}}(g) \varphi_{\pi_2}^{U_2, \psi_{U_2}}(g) \cdots \varphi_{\pi_{l-1}}^{U_{l-1}, \psi_{U_{l-1}}}(g) \varphi_{\tau_1}^{U_r, \psi_{U_r}}(g) \varphi_{\tau_2}^{U_{p-r}, \psi_{U_{p-r}}}(g) dg \quad (20)$$

Here,  $U_r = U_{2r,2}$ , the unipotent subgroup which was defined in section 3, and  $\psi_{U_r}$  is the character of this group as defined in that section. Similarly for  $U_{p-r}$ . This process is

inductive. Namely, if any other representation is an Eisenstein series, then by assumption it is not an odd Eisenstein series, and we can further unfold it. Taking the right double coset representative, similar to  $w_0$ , we obtain as inner integration which is similar to the one given by integral (20). From this we conclude that eventually we will obtain a global integral which involves a period integral of the type given by integral (20) where none of the representations is an Eisenstein series. This integral, if not zero, involves also integration over a reductive group. Therefore, in this case, integral (2) is not a global *unipotent* integral.  $\square$

To complete the classification for the group  $G = GL_2$ , we need to consider integrals (10) and (11) where the Eisenstein series  $E_\tau(\cdot, s)$  is defined on the exceptional group  $GE_7(\mathbf{A})$ . We say that  $E_\tau(\cdot, s)$  is an odd Eisenstein series if the Levi part  $M$  of the parabolic subgroup from which we form the Eisenstein series does not contain all three roots  $\alpha_2$ ,  $\alpha_5$  and  $\alpha_7$ . In other words,  $E_\tau(\cdot, s)$  is odd if  $M$  contains a subgroup of the type  $E_6$  or  $A_6$  or  $A_4 \times A_2$ . Here we label the roots of  $GE_7$  as in [G4]. Thus,  $E_\tau(\cdot, s)$  is odd if  $M$  does not contain the diagonal copy of  $GL_2$  which stabilizes the character  $\psi_{U(\mathcal{O})}$  as defined in section 3. Indeed, from the definition of this character, it follows that this copy of  $GL_2$ , contains the group  $SL_2$  generated by  $x_{\alpha_2}(r)x_{\alpha_5}(-r)x_{\alpha_7}(r)$  and  $x_{-\alpha_2}(r)x_{-\alpha_5}(-r)x_{-\alpha_7}(r)$ . We prove a similar result to Theorem 2. We have

**Theorem 3.** *Assume that  $G = GL_2$ , and that the Eisenstein series  $E_\tau(\cdot, s)$  is defined on the exceptional group  $GE_7(\mathbf{A})$ . Then the global integral (2) is a nonzero global unipotent integral if and only if the Eisenstein series appearing in Table 1 or 2 is an odd Eisenstein series.*

**Proof.** The idea is the same as in the classical groups. First, if the Eisenstein series is not odd, then a similar argument as in the classical groups proves that the integral (2) is not a global unipotent integral. More precisely, assume that none of the Eisenstein series appearing in integral (2) is odd, either on the classical groups or on  $GE_7$ . Then, it is not hard to produce a Weyl element, which can be taken as a representative of the double cosets  $P \backslash H/V \cdot GL_2$ , such that we obtain an integral of the type of integral (20), as inner integration. Thus we conclude that integral (2) is not a nonzero global unipotent integral.

Next we consider the case where the Eisenstein series  $E_\tau(\cdot, s)$  is an odd Eisenstein series defined on  $GE_7(\mathbf{A})$ . Thus there are three cases to consider. In each of them, we first write a certain Weyl element  $w_0$ , which will be the only double coset representative of  $P \backslash H/V \cdot GL_2$  which will contribute a nonzero term in the unfolding process. For that element we also write down the group  $w_0^{-1}(V \cdot GL_2)w_0 \cap M$ . To obtain  $w_0$ , we first write down  $w_1$  which is the shortest Weyl element in  $M \backslash GE_7$ . Then we consider the Weyl element  $w_1 w[257]$  where  $w[257]$  is the unique reflection in the group  $GL_2$  as embedded above in  $GE_7$ . Let  $w_0$  denote the shortest Weyl element which is in the same coset

$M \backslash GE_7$  as  $w_1 w [257]$ . Thus, ignoring the contributions to integral (2) from the other terms, we then consider the integral

$$\int_{Z(\mathbf{A})B(F) \backslash G(\mathbf{A})} \varphi_{\pi_1}^{U_1, \psi_{U_1}}(g) \varphi_{\pi_2}^{U_2, \psi_{U_2}}(g) \int_{V^{w_0}(\mathbf{A}) \backslash V(\mathbf{A})} f_{\tau}^{V^{w_0}, \psi_{w_0}}(w_0 v g, s) \psi_V(v) dv dg \quad (21)$$

Here  $B$  is the Borel subgroup of  $G = GL_2$  which consists of all upper unipotent matrices, and

$$f_{\tau}^{V^{w_0}, \psi_{w_0}}(h, s) = \int_{V^{w_0}(F) \backslash V^{w_0}(\mathbf{A})} f_{\tau}(v_0 h, s) \psi_{w_0}(v_0) dv_0$$

Integral (21) corresponds to the case of integral (11). However, if we assume that  $\varphi_{\pi_2}^{U_2, \psi_{U_2}}(g)$  is one for all  $g$ , then it covers also the case of integral (10). Notice that this is exactly as in integral (14) where we considered the classical groups. To complete the study of these cases, we need to determine the groups  $V^{w_0} \cap M$  and the sets  $\mathcal{O}_M(\tau)$ . Then, using Lemma 4, we argue in a similar way as in the case of the classical groups. To determine the unipotent orbit  $\mathcal{O}_M(\tau)$  we use the dimension identity  $\dim E_{\tau} = \dim \tau + \dim U(P)$  established in Lemma 3. Here  $U(P)$  is the unipotent radical of  $P$ . It follows from Tables 1 and 2 that  $\mathcal{O}_{GE_7}(E_{\tau})$  is  $E_7(a_2)$  or  $E_7(a_1)$ . Hence,  $\dim E_{\tau} = 61, 62$ . From this it is easy to determine  $\dim \tau$ , and hence to determine  $\mathcal{O}_M(\tau)$ . There are three types of odd Eisenstein series, and we consider each one of them.

1) Suppose that  $M$  is of type  $A_6$ . Then  $\dim U(P) = 42$ , and hence  $\dim \tau = 19$  if  $\mathcal{O}_{GE_7}(E_{\tau}) = E_7(a_2)$  and  $\dim \tau = 20$  if  $\mathcal{O}_{GE_7}(E_{\tau}) = E_7(a_1)$ . Hence  $\mathcal{O}_M(\tau) = (52)$  in the first case and  $\mathcal{O}_M(\tau) = (61)$  in the second case. As for the group  $V^{w_0} \cap M$  in this case, it is defined as follows. Let  $U$  denote the maximal unipotent subgroup of  $GL_7$ . Then  $V^{w_0} \cap M = \{u \in U : u_{1,2} = u_{3,4} = 0\}$ . The character  $\psi_{w_0}$  is defined as

$$\psi_{w_0}(v) = \psi(v_{1,3} + v_{2,4} + v_{4,5} + v_{5,6} + v_{6,7})$$

The corresponding Fourier coefficient is associated with the unipotent orbit (52) of  $GL_7$ .

2) Suppose that  $M$  is of type  $E_6$ . In this case we have  $\dim \tau = 34, 35$ , and hence  $\mathcal{O}_M(\tau) = D_5, E_6(a_1)$ . The group  $V^{w_0} \cap M$  in this case is defined as follows. Let  $Q$  denote the parabolic subgroup of  $GE_7$  whose Levi part is  $T(GE_7) \cdot (SL_2 \times SL_2)$  where the group  $SL_2 \times SL_2$  is generated by  $x_{\pm\alpha_2}$  and  $x_{\pm\alpha_3}$ . Also,  $T(GE_7)$  is the maximal torus of  $GE_7$ . Let  $U(Q)$  denote the unipotent radical of  $Q$ . Then  $V^{w_0} \cap M = U(Q)$ . The character  $\psi_{w_0}$  in the case is defined as follows. For  $u \in U(Q)$ , write

$$u = x_{\alpha_1}(r_1) x_{\alpha_3+\alpha_4}(r_2) x_{\alpha_2+\alpha_4}(r_3) x_{\alpha_5}(r_4) x_{\alpha_6}(r_5) u'$$

Here  $u'$  is an element in  $U(Q)$  which is a product of one dimensional unipotent subgroups of  $U(Q)$  corresponding to positive roots of  $E_7$  and does not include any one of the above five roots. Then, we define  $\psi_{w_0}(u) = \psi(r_1 + r_2 + \cdots + r_5)$ . It is not hard to check that the corresponding Fourier coefficient  $f_{\tau}^{V^{w_0}, \psi_{w_0}}$  is associated to the unipotent orbit  $D_5$  of  $E_7$ .

3) Suppose that  $M$  is of type  $A_4 \times A_2$ . In this case  $\dim \tau = 11, 12$ . The unipotent group  $V^{w_0} \cap M$ , viewed as a subgroup of  $GL_5 \times GL_3$  is defined as follows. Let  $U_5$  denote the standard maximal unipotent subgroup of  $GL_5$ , and similarly define  $U_3$ . Define  $V_5 = \{v \in U_5 : u_{1,2} = 0\}$  and  $V_3 = \{v \in U_3 : u_{1,2} = 0\}$ . Then we have  $V^{w_0} \cap M = V_5 \times V_3$ . The character  $\psi_{w_0}$  is then a product of  $\psi_{w_0,5}$  and  $\psi_{w_0,3}$  defined on the groups  $V_5$  and  $V_3$ . Here,  $\psi_{w_0,5}(v) = \psi(v_{2,3} + v_{3,4} + v_{4,5})$  and  $\psi_{w_0,3}(v) = \psi(v_{2,3})$ . Thus, on  $GL_5$  this Fourier coefficient corresponds to the unipotent orbit (41), and on  $GL_3$  it corresponds to the orbit (21). From this we can determine the sets  $\mathcal{O}(\tau_i)$ . If  $\dim \tau = 11$ , then the only option is  $\mathcal{O}(\tau_1) = (41)$  and  $\mathcal{O}(\tau_2) = (21)$ . There are other cases with  $\dim \tau = 11$  which we ignore since at least one of the sets  $\mathcal{O}(\tau_i)$  does not support  $f_\tau^{V^{w_0}, \psi_{w_0}}$ . When  $\dim \tau = 12$  there are two options. The first option is  $\mathcal{O}(\tau_1) = (5)$  and  $\mathcal{O}(\tau_2) = (21)$ , and the second option is  $\mathcal{O}(\tau_1) = (41)$  and  $\mathcal{O}(\tau_2) = (3)$ .

Next we proceed as in the case of the classical groups. Assume first that  $\mathcal{O}_{GE_7}(E_\tau) = E_7(a_2)$ . Then we obtain that  $f_\tau^{V^{w_0}, \psi_{w_0}}(uh, s) = f_\tau^{V^{w_0}, \psi_{w_0}}(h, s)$  for all  $u \in U(B)(\mathbf{A})$ , and now we proceed exactly as in integrals (15) and (16). In the second case, when  $\mathcal{O}_{GE_7}(E_\tau) = E_7(a_1)$  we proceed exactly as with the case of  $\mathcal{O}(E_\tau(\cdot, s)) = ((p+2)(p-2))$  in the classical groups. See right after integral (19).

Finally, we need to analyze the contribution to the unfolding process from other double cosets representatives of  $P \backslash H/V \cdot GL_2$ . We need to show that all of them contribute zero to the global integral. The process of doing it is similar to the one carried out in details for  $H = GL_{2p}$ . We omit the details of this computation.  $\square$

## 8. Proof of Lemma 1

In this section we prove Lemma 1. We will consider the case of  $D_5$  in details. The case of  $D_5(a_1)$  is similar. Let  $\pi$  denote an irreducible cuspidal representation of  $GE_6(\mathbf{A})$ . We assume that  $\mathcal{O}(\pi) = D_5$  and derive a contradiction. We describe the Fourier coefficient associated with this unipotent orbit. Let  $P = MU$  denote the parabolic subgroup of  $GE_6$  whose Levi part is  $M = T \cdot (SL_2 \times SL_2)$ . Here  $T$  is the maximal torus of  $GE_6$  and the two copies of  $SL_2$  are generated by  $x_{\pm 001000}$ ;  $x_{\pm 000010}$ . Consider the group  $U/[U, U]$ . As coset representatives we may choose the one parameter subgroups  $x_\alpha$  where  $\alpha$  is one of the following nine roots

$$(100000); (101000); (000001); (000011); (000100); (001100); (000110); \\ (001110); (010000)$$

The group  $M$  acts on these representatives as follows. On the first two it acts, up to a power of the determinant, as the standard representation of  $GL_2$  which contains the  $SL_2$  generated by  $x_{\pm 001000}$ . On the next two representatives it acts similarly, but this time the  $GL_2$  contains the group generated by  $x_{\pm 000010}$ . On the next four  $M$  acts as the tensor product of  $GL_2 \times GL_2$ , and on the last representatives, it acts as a one dimensional

representation. From this we can define the corresponding Fourier coefficient. Given  $u \in U$  write

$$u = x_{100000}(r_1)x_{001100}(r_2)x_{000110}(r_3)x_{000011}(r_4)x_{010000}(r_5)u'$$

Here  $u' \in U$  is any product of one parameter subgroups associated with positive roots of  $E_6$  which do not include the above five roots. Denote  $\psi_U(u) = \psi(r_1 + r_2 + \cdots + r_5)$  and define the Fourier coefficient associated with the unipotent orbit  $D_5$  by

$$\int_{U(F) \backslash U(\mathbf{A})} \varphi_\pi(u) \psi_U(u) du \quad (22)$$

The assumption that  $\mathcal{O}(\pi) = D_5$  asserts that this Fourier coefficient is not zero for some choice of data, but any Fourier coefficient of  $\pi$  associated with the unipotent orbits  $E_6$  or  $E_6(a_1)$  is zero for all choice of data.

For  $1 \leq i \leq 6$ , let  $w_i$  denote the simple reflection associated with the root  $\alpha_i$ . Let  $w_0 = w_6 w_5 w_4 w_3 w_2 w_4 w_5 w_1 w_3$ . We have

$$w_0 \alpha_1 = \alpha_2; w_0(001100) = \alpha_4; w_0(000110) = \alpha_1; w_0(000011) = \alpha_5; w_0 \alpha_2 = \alpha_3$$

Conjugating by  $w_0$ , the above Fourier coefficient is equal to

$$\int_{V^-(F) \backslash V^-(\mathbf{A})} \int_{V^+(F) \backslash V^+(\mathbf{A})} \int_{U(D_5)(F) \backslash U(D_5)(\mathbf{A})} \varphi_\pi(uv^+v^-w_0) \psi_{U(D_5)}(u) du dv^+ dv^-$$

where the notations are as follows. First, the group  $U(D_5)$  is the maximal unipotent subgroup of type  $D_5$  generated by the simple roots  $\alpha_i$  for  $1 \leq i \leq 5$ . The character  $\psi_{U(D_5)}$  is the Whittaker coefficient defined on  $U(D_5)$ . The group  $V^+$  consists of all unipotent elements  $x_\alpha$  where  $\alpha$  is one of the roots

$$(111211); (011221); (112211); (111221); (112221); (112321); (122321)$$

Similarly, the group  $V^-$  is defined by all  $x_{-\alpha}$  where  $\alpha$  is one of the roots

$$(101111); (011111); (001111); (010111); (000111); (000011); (000001)$$

Thus, by definition, the above integral is not zero for some choice of data.

We expand the above integral along the one parameter subgroup  $x_{111111}(r)$ . Thus, the above integral is equal to

$$\int \sum_{\xi \in F \backslash \mathbf{A}} \int \int \varphi_\pi(ux_{111111}(r)v^+v^-w_0) \psi_{U(D_5)}(u) \psi(\xi r) du dr dv^+ dv^-$$



Conjugate from left to right by the element  $x_{-(101111)}(-\xi)$ . Changing variables, first in  $U(D_5)$  and then in  $V^+$  we obtain that the integral

$$\int_{V_1^-(F) \backslash V_1^-(\mathbf{A})} \int_{V_1^+(F) \backslash V_1^+(\mathbf{A})} \int_{U(D_5)(F) \backslash U(D_5)(\mathbf{A})} \varphi_\pi(uv^+v^-) \psi_{U(D_5)}(u) dudv^+ dv^-$$

is not zero for some choice of data. Here  $V_1^+$  consists of all elements  $x_\alpha$  in  $V^+$  including  $x_{111111}$ . The group  $V_1^-$  consists of all  $x_\alpha$  in  $V^-$  without the root  $-(101111)$ . Thus  $\dim V_1^+ = \dim V^+ + 1$  and  $\dim V_1^- = \dim V^- - 1$ .

Proceed with this expansion four more times. First expand along  $x_{011211}$  and use the element  $x_{-(011111)}$ . Then expand along  $x_{101111}$  and use  $x_{-(001111)}$ , then expand along  $x_{011111}$  and use  $x_{-(010111)}$ , and finally expand along  $x_{001111}$  and use  $x_{-(000111)}$ . We deduce that the integral

$$\int_{V_2^-(F) \backslash V_2^-(\mathbf{A})} \int_{V_2^+(F) \backslash V_2^+(\mathbf{A})} \int_{U(D_5)(F) \backslash U(D_5)(\mathbf{A})} \varphi_\pi(uv^+v^-) \psi_{U(D_5)}(u) dudv^+ dv^-$$

is not zero for some choice of data. To describe the notations in the above integral, let  $R$  denote the unipotent radical of the maximal parabolic subgroup of  $E_6$  whose Levi part contains  $Spin_{10}$  which contains the group  $U(D_5)$ . Thus  $R$  is the abelian group generated by all  $x_\alpha$  such that  $\alpha = \sum_{i=1}^5 n_i \alpha_i + \alpha_6$ . Then  $V_2^+$  consists of all  $x_\alpha \in R$  not including the roots  $(000001)$ ;  $(000011)$ ;  $(000111)$ ;  $(010111)$ . Thus  $\dim V_2^+ = \dim R - 4 = 12$ . The group  $V_2^-$  consists of all  $x_{-\alpha}$  such that  $\alpha$  is one of the two roots  $(000011)$ ;  $(000001)$ .

Next we expand the above integral along the unipotent subgroup  $x_{010111}(r)$ . Consider first the contribution from the nontrivial character. We claim that it contributes zero to the expansion. Indeed, in this case after a conjugation by the Weyl element  $w_5 w_6$  it is not hard to check that we obtain the Fourier coefficient of  $\pi$  which is associated with the unipotent orbit  $E_6(a_1)$ . By the assumption that  $\mathcal{O}(\pi) = D_5$ , we deduce that this Fourier coefficient is zero. Thus, we are left with the contribution from the trivial character, and we obtain that the integral

$$\int_{V_2^-(F) \backslash V_2^-(\mathbf{A})} \int_{V_3^+(F) \backslash V_3^+(\mathbf{A})} \int_{U(D_5)(F) \backslash U(D_5)(\mathbf{A})} \varphi_\pi(uv^+v^-) \psi_{U(D_5)}(u) dudv^+ dv^-$$

is not zero for some choice of data. Here  $V_3^+$  is the group generated by  $V_2^+$  and  $x_{010111}$ . Now we expand along  $x_{000111}$  and as before we use the element  $x_{-(000011)}$  and then expand along  $x_{000011}$  and use  $x_{-(000001)}$ . Thus we obtain that the integral

$$\int_{V_4^+(F) \backslash V_4^+(\mathbf{A})} \int_{U(D_5)(F) \backslash U(D_5)(\mathbf{A})} \varphi_\pi(uv^+) \psi_{U(D_5)}(u) dudv^+$$

is not zero for some choice of data. Here  $V_4^+$  is the group generated by  $V_3^+$  and  $x_{000111}$  and  $x_{000011}$ . Finally, we expand along  $x_{000001}$ . The contribution from the nontrivial orbit

gives zero. Indeed, in this case we obtain the Fourier coefficient of  $\pi$  associated with the unipotent orbit  $E_6$ . As argued above, it is zero. The contribution from the trivial orbit also contributes zero. Indeed, in this case we obtain as an inner integration, the constant term along the unipotent radical  $R$ . By cuspidality of  $\pi$  it is zero. Thus the above integral is zero and we derived a contradiction.

The case when  $\mathcal{O}(\pi) = D_5(a_1)$  is similar. We give some details. Let  $U'$  denote the unipotent radical of the parabolic subgroup of  $E_6$  whose Levi part is  $T \cdot SL_2$  where the  $SL_2$  is generated by  $x_{\pm\alpha_4}$ . Thus  $\dim U' = 35$ . Let  $U$  be the subgroup of  $U'$  where we omit the 3 unipotent elements  $x_{001100}$ ;  $x_{000010}$ ;  $x_{000110}$ . Thus  $\dim U = 32$ . Given  $u \in U$  write

$$u = x_{010000}(r_1)x_{101100}(r_2)x_{000011}(r_3)x_{000111}(r_4)x_{001110}(r_5)u'$$

where  $u' \in U$  is an element generated by all  $x_\alpha$  such that  $\alpha$  is not one of the above five roots. Define  $\psi_U(u) = \psi(r_1 + r_2 + \cdots + r_5)$ . Then we can form the corresponding Fourier coefficient given by the integral (22). Let  $w_0 = w_6w_5w_4w_3w_2w_4w_5w_1$ . We have  $w_0(010000) = \alpha_3$ ;  $w_0(010100) = (001100)$ ;  $w_0(101100) = (010100)$ ;  $w_0(000011) = \alpha_5$ ;  $w_0(001110) = \alpha_1$ . The next step is to expand the integral, and use the fact  $\mathcal{O}(\pi) = D_5(a_1)$ . Eventually, we obtain as inner integration, a constant term along a certain unipotent radical, which is zero by cuspidality. We omit the details.

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