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Abstract

The Riemann zeta identity at even integers of Lettington, along with his other Bernoulli and zeta relations, are generalized. Other corresponding recurrences and determinant relations are illustrated. Another consequence is the application to sums of double zeta values. A set of identities for the Ramanujan and generalized Ramanujan polynomials is presented. An alternative proof of Lettington's identity is provided, together with its generalizations to the Hurwitz and Lerch zeta functions, hence to Dirichlet L series, to Eisenstein series, and to general Mellin transforms.

The Hurwitz numbers $\tilde{H}_n$ occur in the Laurent expansion about the origin of a certain Weierstrass $\wp$ function for a square lattice, and are highly analogous to the Bernoulli numbers. An integral representation of the Laurent coefficients about the origin for general $\wp$ functions, and for these numbers in particular, is presented. As a Corollary, the asymptotic form of the Hurwitz numbers is determined. In addition, a series representation of the Hurwitz numbers is given, as well as a new recurrence. Other results concern the Matter numbers of the equianharmonic case of the $\wp$ function.

Key words and phrases

Bernoulli number, Bernoulli polynomial, Riemann zeta function, Euler number, Euler polynomial, alternating zeta function, double zeta values, Hurwitz zeta function, Lerch zeta function, polygamma function, Ramanujan polynomial, Bernoulli relations, zeta identities, Eisenstein series, recurrence, Hessenberg determinant, integral representation, Mellin transform, Hurwitz numbers, Matter numbers

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1. Introduction and statement of results

Let $\zeta(s)$ denote the Riemann zeta function and $B_n(x)$ the nth degree Bernoulli polynomial [15, 26], such that $B_n = B_n(0) = (-1)^{n-1}n\zeta(1-n)$ is the nth Bernoulli number (e.g., [8, 20]). This relation is extended, for instance when n is even, to $B_n(x) = -n\zeta(1-n,x)$ where $\zeta(s,x) = \sum_{n=0}^{\infty}(n+x)^{-s}$ (Re $s > 1$) is the Hurwitz zeta function. There is the well known relation (explicit evaluation)

$$\zeta(2m) = \frac{(-1)^{m+1}}{(2m)!}2^{2m-1}\pi^{2m}B_{2m}. \tag{1.1}$$

The Bernoulli numbers have the well known exponential generating function

$$\frac{x}{e^x-1} = \sum_{n\geq 0}\frac{B_n}{n!}x^n, \qquad |x| < 2\pi,$$

while the ordinary generating series $\beta(x) = \sum_{n\geq 0} B_n x^n = 1 - x/2 + x^2/6 - x^4/30 + x^6/42 + \ldots$ is divergent. However the continued fractions for the generating series $\sum_{n\geq 1} B_{2n}(4x)^n$, $\beta(x)$, $\sum_{n\geq 1}(2n+1)B_{2n}x^n$, and $\sum_{n\geq 1}(4^n-2^n)|B_n|x^{2n-1}/n$ are convergent [14].

In Lettington [22] was presented an identity previously known by Song [28] for the Riemann zeta function at even integers,

$$\zeta(2j) = (-1)^{j+1}\left[\frac{j\pi^{2j}}{(2j+1)!} + \sum_{k=1}^{j-1}\frac{(-1)^k\pi^{2j-2k}}{(2j-2k+1)!}\zeta(2k)\right]. \tag{1.2}$$

This is hardly an isolated identity, and we show how to systematically derive many related ones. Other very recent interest in the subject of zeta function-value recurrences is evidenced by [25], which extends work of Song [28]. The very recent paper of Merca [25] uses a standard generating function of the Bernoulli polynomials, while Song used a Fourier series approach. All such identities may be further generalized using the methods of this paper. As a quick example, an application of Theorem 9 of Section 3 of this paper generalizes Corollary 2.2 of [25] in terms of values of the Hurwitz zeta function.

As an initial illustration of our results, we have the identities given in the following.

Theorem 1. (a)

$$(3^{1-2j}-1)\zeta(2j) = \frac{(-1)^j j(2\pi)^{2j}}{(2j)!3^{2j-1}} + (-1)^{j+1}(2\pi)^{2j}\sum_{m=0}^{j}\frac{(-1)^{m+1}}{(2j-2m)!2^{2m-1}3^{2j-2m}\pi^{2m}}\zeta(2m),$$

(b)

$$(4^{1-2j}-2^{1-2j})\zeta(2j) = \frac{(-1)^j j(2\pi)^{2j}}{(2j)!4^{2j-1}} + (-1)^{j+1}(2\pi)^{2j}\sum_{m=0}^{j}\frac{(-1)^{m+1}}{(2j-2m)!2^{2m-1}4^{2j-2m}\pi^{2m}}\zeta(2m),$$

(c)

$$(6^{1-2j}-3^{1-2j}-2^{1-2j}+1)\zeta(2j) = \frac{(-1)^j j(2\pi)^{2j}}{(2j)!6^{2j-1}} + (-1)^{j+1}(2\pi)^{2j}\sum_{m=0}^{j}\frac{(-1)^{m+1}}{(2j-2m)!2^{2m-1}6^{2j-2m}\pi^{2m}}\zeta(2m),$$

(d)

$$-\frac{j}{12^{2j-1}}(1+7^{2j-1}) + \sum_{m=0}^{j}\frac{(2j)!}{(2j-2m)!}\frac{(-1)^{m+1}}{2^{2m-1}\pi^{2m}}(1+7^{2j-2m})\frac{\zeta(2m)}{12^{2j-2m}}$$
$$= \frac{(6^{1-2j}-3^{1-2j}-2^{1-2j}+1)(2j)!(-1)^{j+1}}{2^{4j-1}\pi^{2j}}\zeta(2j),$$

and (e)

$$-\frac{j}{8^{2j-1}}(1+5^{2j-1}) + \sum_{m=0}^{j}\frac{(2j)!}{(2j-2m)!}\frac{(-1)^{m+1}}{2^{2m-1}\pi^{2m}}(1+5^{2j-2m})\frac{\zeta(2m)}{8^{2j-2m}} = \frac{(4^{1-2j}-2^{1-2j})(2j)!(-1)^{j+1}}{2^{4j-1}\pi^{2j}}\zeta(2j).$$

Moreover, in a separate section we give an alternative proof of (1.2), and its extensions to $\zeta(s,a)$ and so to Dirichlet L-functions.

Now define functions

$$\phi_j(s) = \frac{\zeta(s)}{j^s}, \qquad \theta_3(s) = \left(1-\frac{1}{3^{s-1}}\right)\zeta(s),$$

$$\theta_4(s) = \left(\frac{1}{2^{s-1}}-\frac{1}{4^{s-1}}\right)\zeta(s), \quad \theta_6(s) = \left(-1+\frac{1}{2^{s-1}}+\frac{1}{3^{s-1}}-\frac{1}{6^{s-1}}\right)\zeta(s). \qquad (1.3)$$

Theorem 1 and similar results have numerous implications for these and other functions. In particular, Theorem 1(a) leads to Theorem 2(b).

Theorem 2. (a)

$$4\frac{j^{2s}}{2^{2s}}\phi_j(2s) = \frac{\pi^{2s}(2s-1)}{(2s+1)!} + \sum_{n=1}^{s-1}\frac{(-1)^{s-n}\pi^{2n}}{(2n+1)!}4\frac{j^{2(s-n)}}{2^{2(s-n)}}\phi_j(2s-2n),$$

(b)

$$\theta_3(2j) = (-1)^j \pi^{2j} \frac{2^{2j}}{3^{2j}} \frac{(1-3j)}{(2j)!} + 2\sum_{m=0}^{j-1} \frac{(-1)^{m-1}}{(2m)!} \left(\frac{2\pi}{3}\right)^{2m} \left(1 - \frac{3}{3^{2(j-m)}}\right)^{-1} \theta_3(2j-2m),$$

(c)

$$\theta_4(2j) = (-1)^j \pi^{2j} \frac{1}{2^{2j}} \frac{(1-4j)}{(2j)!} + 2\sum_{m=0}^{j-1} \frac{(-1)^{m-1}}{(2m)!} \left(\frac{\pi}{2}\right)^{2m} \left(\frac{2}{2^{2(j-m)}} - \frac{4}{4^{2(j-m)}}\right)^{-1} \theta_4(2j-2m),$$

and (d)

$$\theta_6(2j) = (-1)^j \pi^{2j} \frac{1}{3^{2j}} \frac{(1-6j)}{(2j)!} + 2\sum_{m=0}^{j-1} \frac{(-1)^{m-1}}{(2m)!} \left(\frac{\pi}{3}\right)^{2m} \left(-1 + \frac{2}{2^{2(j-m)}} + \frac{3}{3^{2(j-m)}} - \frac{6}{4^{2(j-m)}}\right)^{-1} \times \theta_6(2j-2m).$$

We next recall from [22, 23] the definition of a half-weighted minor corner layered determinant $\Psi_s(\vec{h}, \vec{H})$, using the vectors $\vec{h} = (h_1, h_2, h_3, \ldots)$ and $\vec{H} = (H_1, H_2, H_3, \ldots)$:

$$\Psi_s(\vec{h}, \vec{H}) = (-1)^s \begin{vmatrix} H_1 & 1 & 0 & 0 & \ldots & 0 \\ H_2 & h_1 & 1 & 0 & \ldots & 0 \\ H_3 & h_2 & h_1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{s-1} & h_{s-2} & h_{s-3} & h_{s-4} & \ldots & 1 \\ H_s & h_{s-1} & h_{s-2} & h_{s-3} & \ldots & h_1 \end{vmatrix}.$$

Theorem 3. (a) Define vectors $\vec{u}$ and $\vec{U}_2$ with entries $u_s = 1/(2s+1)!$ and $U_{2s} = (2s-1)/(2s+1)!$. Then

$$4\frac{j^{2s}}{2^{2s}}\phi_j(2s) = (-1)^s \pi^{2s} \Psi_s(\vec{u}, \vec{U}_2).$$

(b) Define vectors $\vec{U}_3$ and $\vec{H}_3$ with entries $U_{3s} = 1/(2s)!$ and

$$H_{3s} = -\left[\frac{1-3s}{(2s)!} + \theta_3(2s)(-1)^{s-1}\left(\frac{2\pi}{3}\right)^{-2s}\right].$$

Then

$$(-1)^s \left(\frac{2\pi}{3}\right)^{2s} \Psi_s(\vec{U}_3, \vec{H}_3) = 2\theta_3(2s)\left(1 - \frac{3}{3^{2s}}\right)^{-1}.$$

Let $\binom{t}{d_1,d_2,\ldots,d_s}$ denote the multinomial coefficient, the coefficient of the expansion of the sum of s terms to the tth power. [1] [2]

Corollary 1.

$$4\phi_j(2s) = \frac{(2\pi)^{2s}}{j^{2s}}\frac{1}{(2^{2s-1}-1)}\sum_{t=1}^{s}\sum_{d_i\geq 0}\binom{t}{d_1,d_2,\ldots,d_s}\frac{(-1)^{t+s}}{3!^{d_1}5!^{d_2}\cdots(2s+1)!^{d_s}},$$

and

$$4\theta_3(2s) = \left(1-\frac{1}{3^{2s-1}}\right)\frac{(2\pi)^{2s}}{(2^{2s-1}-1)}\sum_{t=1}^{s}\sum_{d_i\geq 0}\binom{t}{d_1,d_2,\ldots,d_s}\frac{(-1)^{t+s}}{3!^{d_1}5!^{d_2}\cdots(2s+1)!^{d_s}}.$$

Here the sums are such that $d_1+d_2+\ldots+d_s=t$ and $d_1+2d_2+\ldots+sd_s=s$.

In fact, the minor corner layered determinants of [22, 23] are very special cases of lower Hessenberg determinants. Following the proof of Theorem 3, we recall a much more general determinantal result.

Define, for integers $a\geq 2$ and $b\geq 1$, the double zeta values

$$\zeta(a,b)=\sum_{n=1}^{\infty}\sum_{m=1}^{n-1}\frac{1}{n^a m^b}.$$

From manipulating series, it follows that

$$\zeta(a,b)+\zeta(b,a)=\zeta(a)\zeta(b)-\zeta(a+b).$$

More broadly, multiple zeta values have found applications in knot theory and quantum field theory (e.g., [6]). We note that any recurrence of Riemann zeta values (or of Bernoulli numbers) of the form $\sum_j f(s,j)\zeta(2j)\zeta(2s-2j)$ for some function f then leads to a sum formula for double zeta values. For we have, for instance,

$$\sum_j f(s,j)[\zeta(2j,2s-2j)+\zeta(2s-2j,2j)] = \sum_j f(s,j)\zeta(2j)\zeta(2s-2j)-\zeta(2s)\sum_j f(s,j).$$

[1] On pp. 22 and 23 of [23], the following typographical errors occur. The upper index for the multinomial coefficient for the summations for $\eta(2s)$, $\phi(2s)$, and $\theta(2s)$ in Lemma 3.3 should be t. In the display equation for the proof of Theorem 1.3, an "=" should be inserted after x^{2s-2}.

[2] On p. 17 of [23], $(-1)^k$ should be $(-1)^{s-k}$ in the summand on the right side of the second display equation, and vice versa for the summand of the right side of the third display equation. At the bottom of p. 8, Ψ should read Ψ_n (twice).

As an illustration of how Bernoulli relations of this article carry over to double zeta series, we have the following.

Theorem 4. For integers $s > 1$,

$$\sum_{k=1}^{s-1} \frac{1}{2^{2(s-k)}} \left(1 - \frac{1}{2^{2k}}\right) [\zeta(2k, 2s-2k) + \zeta(2s-2k, 2k)] = \frac{2^{-2s-1}}{3}(4^s + 6s - 1)\zeta(2s).$$

Let $E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k}$, with $E_{2n+1} = 0$ and $E_k = 2^k E_k(1/2)$ denote the Euler polynomial. It has a well known exponential generating function

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \qquad |t| < \pi. \tag{1.4}$$

We recall a connection with the alternating zeta function

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s).$$

We have the evaluation $\eta(-j) = (-1)^j E_j(0)/2$ for $j \geq 0$. The values $E_n(0) = (-1)^n E_n(1)$ are 0 for $n \geq 2$ and $E_0(0) = 1$, while otherwise ([1], p. 805) $E_n(0) = -2(n+1)^{-1}(2^{n+1} - 1)B_{n+1}$ for $n \geq 1$. Many results analogous to Theorem 1 are possible, and the following provides a brief example.

Theorem 5. For integers $n > 0$,

$$-\frac{1}{2n}(1 - 3^{1-2n})(2^{2n} - 1)B_{2n} = \sum_{m=0}^{n-1} \binom{2n-1}{2m} \frac{E_{2m}}{(2m)!} \left(-\frac{1}{6}\right)^{2(n-m)-1}.$$

Theorem 6. For integers $j \geq 0$,

$$2\sum_{m=0}^{j} (1 - 2^{1-2m})(1 - 2^{2(m-j)+1})\zeta(2m)\zeta(2j - 2m) = -(1 - 2j)\zeta(2j).$$

Theorem 1.3 of [23] gives equivalent forms of the recurrence

$$\zeta(2s + 2) = \frac{2}{2^{2s+2} - 1} \sum_{k=0}^{s-1} (2^{2k+2} - 1)\zeta(2s - 2k)\zeta(2k + 2). \tag{1.5}$$

Hence, if we introduce functions $\tilde{\theta}_j(s) = (1 - j^{-s})\zeta(s)$, we may write

$$\frac{(2^{2s+2} - 1)}{(1 - j^{-(2s+2)})} \tilde{\theta}_j(2s + 2) = 2j^{2(s+1)} \sum_{k=0}^{s-1} \frac{(2^{2k+2} - 1)}{(j^{2(k+1)} - 1)} \phi_j(2s - 2k) \tilde{\theta}_j(2k + 2).$$

In fact, (1.5) is proved in [3] (p. 406) as Theorem 3.4. In addition, we may note that besides the well known relation (e.g., [29, 32])

$$\zeta(2n) = \frac{2}{2n+1}\sum_{k=1}^{n-1}\zeta(2k)\zeta(2n-2k), \tag{1.6}$$

Williams [32] some time ago proved

$$\sum_{k=1}^{n}\mathcal{L}(2k-1)\mathcal{L}(2n-2k+1) = \left(n-\frac{1}{2}\right)(1-2^{-2n})\zeta(2n),$$

where $\mathcal{L}$ is the function

$$\mathcal{L}(s) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^s} = 2^{-s}\Phi\left(-1, s, \frac{1}{2}\right) = 4^{-s}\left[\zeta\left(s, \frac{1}{4}\right) - \zeta\left(s, \frac{3}{4}\right)\right],$$

and Φ is the Lerch zeta function (see Theorem 10).

We point out that *each* part of Theorem 1 permits the identification of candidate pseudo-characteristic polynomials for the Riemann zeta and other related functions. For instance, if we separate the $m = 0$ and $m = j$ terms of the sum of the right side of Theorem 1(a), we obtain

$$3(3^{-2j}-1)\zeta(2j) = (-1)^j\frac{(3j-1)}{(2j)!}\left(\frac{2\pi}{3}\right)^{2j} + \sum_{m=1}^{j-1}\frac{(-1)^m 2^{2m+1}\pi^{2m}\zeta(2j-2m)}{(2m)!3^{2m}}.$$

We then put

$$p_j^{(3)}(x) = \frac{1}{3(3^{-2j}-1)}\sum_{m=1}^{j-1}\frac{(-1)^m 2^{2m+1}\pi^{2m}}{(2m)!3^{2m}}x^{2m}$$

and

$$z_j^{(3)}(x) = (-1)^j\frac{(3j-1)}{(2j)!3(3^{-2j}-1)}\left(\frac{2\pi}{3}\right)^{2j} + p_j^{(3)}(x).$$

We recognize that

$$p_j^{(3)}(x) = \frac{2}{3(3^{-2j}-1)}\left[-1+\cos\left(\frac{2\pi}{3}x\right) - \sum_{m=j}^{\infty}\frac{(-1)^m}{(2m)!}\left(\frac{2\pi}{3}x\right)^{2m}\right].$$

It then appears that inequalities of the following sort hold, where k is either $2s$ or $2s-1$.

Conjecture 1. For integers $s \geq 4$,

$$\zeta(k) - \{\zeta(k)\}^2 \leq z_s^{(3)}(\zeta(k)) \leq \zeta(k) + \{\zeta(k)\}.$$

Here $\{x\} = x - [x]$ denotes the fractional part of x.

We provide new identities for the Ramanujan $R_{2s+1}(z)$ and generalized Ramanujan $R_{2s}(z)$ polynomials, and for other functions. For this we require the following definitions. Let the numbers B_s^* for $s \geq 2$ be given by

$$B_s^* = -\frac{1}{s+1}\sum_{k=0}^{s-1}\binom{s+1}{k}2^{k-s}B_k,$$

which are then such that $B_{2s}^* = B_{2s}$ and $B_{2s-1}^* = (1-2^{-2s})2B_{2s}/s$. To these values are prepended the initial values $B_0^* = 1$ and $B_1^* = 1/4$. The Ramanujan polynomials are given by

$$R_{2s+1}(z) = \sum_{k=0}^{s+1}\frac{B_{2k}B_{2s+2-2k}}{(2k)!(2s+2-2k)!}z^{2k},$$

while the generalized Ramanujan polynomials $Q_r(z)$ [23] are given by

$$Q_r(z) = \sum_{k=0}^{[(r+1)/2]}\frac{B_{r+1-2k}^* B_{2k}^*}{(r+1-2k)!(2k)!}z^{2k}.$$

The Pochhammer symbol $(a)_j = \Gamma(a+j)/\Gamma(a)$, with Γ denoting the Gamma function, $\psi(z) = \Gamma'(z)/\Gamma(z)$ denotes the digamma function, and $H_n = \sum_{k=1}^n 1/k$ the nth harmonic number.

Theorem 7. (a) The Ramanujan polynomials satisfy the identities

$$[1-(-1)^n]R_{2s+1}^{(n)}(1) + \sum_{j=1}^{n-1}\binom{n}{j}\left[\frac{(n-1)!}{(j-1)!} - (-1)^j(2s+2)_{n-j}\right]R_{2s+1}^{(j)}(1)$$
$$= (2s+2)_n R_{2s+1}(1),$$

and (b), with $R_{2s}(z) = Q_{2s}(z)$, these generalized Ramanujan polynomials satisfy the identities

$$[1-(-1)^n]\left[R_{2s}^{(n)}(1) - \frac{1}{2^n}R_{2s}^{(n)}\left(\frac{1}{2}\right)\right]$$
$$+\sum_{j=1}^{n-1}\binom{n}{j}\left[\frac{(n-1)!}{(j-1)!} - (-1)^j(2s+2)_{n-j}\right]\left[R_{2s}^{(j)}(1) - \frac{1}{2^j}R_{2s}^{(j)}\left(\frac{1}{2}\right)\right]$$
$$= (2s+2)_n\left[R_{2s}(1) - R_{2s}\left(\frac{1}{2}\right)\right].$$

(c) Suppose the following functional equation, as appears in Grosswald's generalization [16] of Ramanujan's formula pertaining to the Riemann zeta function at odd integer argument ([17], p. 945). For analytic functions F and S,

$$F\left(-\frac{1}{z}\right) - (-1)^\delta \left(\frac{z}{i}\right)^r F(z) = S\left(\frac{z}{i}\right),$$

where z is in the upper half plane and r is real. Then

$$[(-1)^n - (-1)^\delta]F^{(n)}(i) - (-1)^\delta(-r)_n i^n F(i)$$
$$+\sum_{j=1}^{n-1}\binom{n}{j}\left[(-1)^n\frac{(n-1)!}{(j-1)!} - (-1)^\delta(-1)^{n-j}(-r)_{n-j}\right] i^{j-n}F^{(j)}(i) = \frac{1}{i^n}S^{(n)}(1).$$

(d) Let F be Zagier's function [33] (p. 164)

$$F(x) = \sum_{n=1}^\infty \frac{1}{n}[\psi(nx) - \ln(nx)] = \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right)\ln(1-e^{-xt})dt.$$

Then

$$F'(x) - \frac{1}{x^2}F\left(\frac{1}{x}\right) = -\frac{\pi^2}{6} + \frac{\pi^2}{6x^2} + \frac{\ln x}{x},$$

and for $n \geq 2$

$$[1+(-1)^n]F^{(n)}(1) + (-1)^n\sum_{j=1}^{n-1}\binom{n}{j}\frac{(n-1)!}{(j-1)!}F^{(j)}(1) = (-1)^{n-1}(n-1)!\left[\frac{\pi^2}{6}n - H_{n-1}\right].$$

The form of the expressions in (a) and (b) shows that no new information is included from the even order derivative relations, while the form of (d) shows that no new information is included for odd n. Explicitly in terms of Bernoulli summations, the $n = 1$ case of (a) is the equality of

$$(s+1)R_{2s+1}(1) = (s+1)\sum_{k=0}^{s+1}\frac{B_{2k}B_{2s+2-2k}}{(2k)!(2s+2-2k)!}$$

and

$$R'_{2s+1}(1) = \sum_{k=1}^{s+1}\frac{B_{2k}B_{2s+2-2k}}{(2k-1)!(2s+2-2k)!}.$$

The following theorem provides a vast generalization of (1.2). Not only is a much broader class of functions f applicable, but part (b) shows that arguments $f(mk)$ for integer m may be summed in place of just $f(2k)$.

Theorem 8. (General Mellin transform result.) Suppose that an analytic function f has an integral representation

$$f(s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}g(t)dt, \qquad \mathrm{Re}\ s>1,$$

for some function g. (a) Putting

$$c_j^M(b)=\frac{\sqrt{b}}{2(2j)!}\int_0^\infty g(t)[(\sqrt{b}t-1)^{2j}-(\sqrt{b}t+1)^{2j}]dt,$$

we have

$$b^j f(2j)=-c_j^M(b)-\sum_{k=1}^{j-1}\frac{b^k}{(2j-2k+1)!}f(2k).$$

Section 3 provides important cases of this result. (b) Let $m\geq 2$ be an integer. Then there are constants $c_j^{m,m-1}(b)$ such that

$$b^j f(mj)=c_j^{m,m-1}(b)-\sum_{k=1}^{j-1}\frac{b^k}{(mj-mk+1)!}f(mk).$$

In partial sum, we have generalized the zeta identity of [22, 28], and the Bernoulli and zeta relations of [23]. We have noted the relevance of a subset of these results for obtaining summation formulas for double zeta values. Additionally in this paper, beginning with Section 3, we give an alternative proof of (1.2) and several other generalizations by using integral representations. Lettington and Song's identity may then be viewed as a very special case of a recurrence among values of polygamma functions, and beyond this context, among values of Dirichlet L series, values of the Lerch zeta function, or values of lattice Dirichlet series.

2. Proof of Theorems

Theorem 1. (a) We recall the expression for Bernoulli polynomials $B_s(x)=\sum_{k=0}^s\binom{s}{k}B_{s-k}x^k$, so that

$$B_j(x)=\sum_{k=0}^{j}\binom{j}{k}B_kx^{j-k}.$$

For $n = 2j$ even we have the relation $B_n(1/3) = B_n(2/3) = (3^{1-n} - 1)B_n/2$ and obtain

$$\begin{aligned}\frac{1}{2}(3^{1-2j} - 1)B_{2j} &= \sum_{k=0}^{2j} \binom{2j}{k} B_k \frac{1}{3^{2j-k}} \\ &= \frac{B_0}{3^{2j}} + 2j\frac{B_1}{3^{2j-1}} + \sum_{k=2}^{2j} \binom{2j}{k} B_k \frac{1}{3^{2j-k}} \\ &= \frac{1}{3^{2j}} - \frac{j}{3^{2j-1}} + \sum_{m=1}^{j} \binom{2j}{2m} B_{2m} \frac{1}{3^{2(j-m)}} \\ &= -\frac{j}{3^{2j-1}} + \sum_{m=0}^{j} \binom{2j}{2m} B_{2m} \frac{1}{3^{2(j-m)}}.\end{aligned}$$

The relation (1.1) is then applied.

(b) is based upon the relation, for $n = 2j$ even, $B_n(1/4) = B_n(3/4) = (4^{1-n} - 2^{1-n})B_n/2$, so that

$$\frac{1}{2}(4^{1-2j} - 2^{1-2j})B_{2j} = -\frac{j}{4^{2j-1}} + \sum_{m=0}^{j} \binom{2j}{2m} B_{2m} \frac{1}{4^{2(j-m)}}.$$

Then relation (1.1) is again used.

(c) is based upon the relation for $n = 2j$ even

$$B_n\left(\frac{1}{6}\right) = B_n\left(\frac{5}{6}\right) = \frac{1}{2}(6^{1-n} - 3^{1-n} - 2^{1-n} + 1)B_n.$$

(d) is based upon the relation

$$B_n\left(\frac{1}{12}\right) + B_n\left(\frac{7}{12}\right) = 2^{1-n}B_n\left(\frac{1}{6}\right) = 2^{-n}(6^{1-n} - 3^{1-n} - 2^{1-n} + 1)B_n.$$

(e) is based upon the relation

$$B_n\left(\frac{1}{8}\right) + B_n\left(\frac{5}{8}\right) = 2^{1-n}B_n\left(\frac{1}{4}\right) = 2^{-n}(4^{1-n} - 2^{1-n})B_n.$$

$\square$

Remark. The relations that we have employed follow from the symmetry $B_n(1 - x) = (-1)^n B_n(x)$ and the multiplication formula $B_n(mx) = m^{n-1}\sum_{k=0}^{m-1} B_n(x+k/m)$. Hence many more identities may be developed.

Theorem 2. (a) We recall the recurrence for Bernoulli numbers

$$B_s = -\frac{1}{s+1}\sum_{k=0}^{s-1}\binom{s+1}{k}B_k.$$

Then

$$\begin{aligned}
\phi_j(2s) &= \frac{1}{j^{2s}}\zeta(2s) = \frac{(-1)^{s+1}2^{2s-1}\pi^{2s}}{j^{2s}(2s)!}B_{2s} \\
&= -\frac{1}{j^{2s}}\frac{(-1)^{s+1}2^{2s-1}\pi^{2s}}{(2s)!(2s+1)}\sum_{k=0}^{2s-1}\binom{2s+1}{k}B_k \\
&= \frac{1}{j^{2s}}\frac{(-1)^s(2\pi)^{2s}}{2(2s+1)!}\left[\binom{2s+1}{1}B_1 + \sum_{k=0}^{2s-2}\binom{2s+1}{k}B_k\right] \\
&= \frac{1}{j^{2s}}\frac{(-1)^s(2\pi)^{2s}}{2(2s+1)!}\left[(2s+1)B_1 + \sum_{n=0}^{s-1}\binom{2s+1}{2n}B_{2n}\right] \\
&= \frac{1}{j^{2s}}\frac{(-1)^s(2\pi)^{2s}}{2}\left[-\frac{1}{2(2s)!} + \sum_{n=0}^{s-1}\frac{1}{(2s-2n+1)!}\frac{B_{2n}}{(2n)!}\right].
\end{aligned}$$

The relation (1.1) is used so that

$$\begin{aligned}
\phi_j(2s) &= \frac{(-1)^{s+1}2^{2s}}{2j^{2s}}\left[\frac{\pi^{2s}}{2(2s)!} + \sum_{n=1}^{s}\frac{(-1)^{s-n}}{(2n+1)!}\frac{\pi^{2n}}{2^{2(s-n)-1}}\zeta(2s-2n)\right] \\
&= \frac{(-1)^{s+1}2^{2s}}{2j^{2s}}\left[\frac{\pi^{2s}}{2(2s)!} + \sum_{n=1}^{s}\frac{(-1)^{s-n}}{(2n+1)!}\frac{\pi^{2n}j^{2(s-n)}}{2^{2(s-n)-1}}\phi_j(2s-2n)\right] \\
&= \frac{(-1)^{s+1}2^{2s}}{2j^{2s}}\left[\frac{\pi^{2s}(2s-1)}{2(2s+1)!} + \sum_{n=1}^{s-1}\frac{(-1)^{s-n}}{(2n+1)!}\frac{\pi^{2n}j^{2(s-n)}}{2^{2(s-n)-1}}\phi_j(2s-2n)\right].
\end{aligned}$$

(b) Theorem 1(a) and relation (1.1) are first used so that

$$\begin{aligned}
-\theta_3(2j)\frac{(2j)!(-1)^{j+1}}{(2\pi)^{2j}} &= -\frac{j}{3^{2j-1}} + \frac{1}{3^{2j}} + \sum_{m=1}^{j}\binom{2j}{2m}\frac{(-1)^{m+1}(2m)!}{\pi^{2m}3^{2(j-m)}2^{2m-1}}\zeta(2m) \\
&= -\frac{j}{3^{2j-1}} + \frac{1}{3^{2j}} + \sum_{m=0}^{j-1}\binom{2j}{2m}\frac{(-1)^{j-m+1}[2(j-m)]!}{\pi^{2(j-m)}3^{2m}2^{2(j-m)-1}}\zeta(2j-2m).
\end{aligned}$$

The relation $\zeta(2s) = (1-3/3^{2s})^{-1}\theta_3(2s)$ is then employed.

(c) and (d) follow similarly from Theorem 1(b) and 1(c) and relation (1.1). □

Theorem 3. This follows from Theorem 2(a) and (b) and the result that Ψ_s satisfies the recurrence ([23], Lemma 3.1)

$$\Psi_s(\vec{h}, \vec{H}) = -H_s - \sum_{k=1}^{s-1} h_{s-k} \Psi_k(\vec{h}, \vec{H}).$$

□

Corollary 1 follows from the multinomial expression for $\zeta(2s)$ of [23], Lemma 3.3. □

A lower Hessenberg determinant has entries $a_{ij} = 0$ for $j - i > 1$,

$$A_n = \begin{vmatrix} a_{11} & a_{12} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & a_{n-1,4} & \dots & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & a_{n,4} & \dots & a_{n,n} \end{vmatrix}.$$

With $|A_0| = 1$ and $|A_1| = a_{11}$, these determinants satisfy the recurrence (e.g., [7])

$$\det(A_n) = a_{n,n}\det(A_{n-1}) + \sum_{r=1}^{n-1} \left[(-1)^{n-r} a_{n,r} \det(A_{r-1}) \prod_{j=r}^{n-1} a_{j,j+1} \right].$$

The minor corner layered determinants of [22, 23] with superdiagonal of all 1's and comprised of at most 3 vectors are obviously special cases. Any time that a recurrence can be made to take the form just above, a determinant expression may then be written.

The $s \times s$ minor corner layered determinant of [22, 23] is given by

$$\Delta_s(\vec{h}) = (-1)^s \begin{vmatrix} h_1 & 1 & 0 & 0 & \dots & 0 \\ h_2 & h_1 & 1 & 0 & \dots & 0 \\ h_3 & h_2 & h_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{s-1} & h_{s-2} & h_{s-3} & h_{s-4} & \dots & 1 \\ h_s & h_{s-1} & h_{s-2} & h_{s-3} & \dots & h_1 \end{vmatrix},$$

and we may note that it inverts symmetrically as

$$h_s(\vec{\Delta}) = (-1)^s \begin{vmatrix} \Delta_1 & 1 & 0 & 0 & \dots & 0 \\ \Delta_2 & \Delta_1 & 1 & 0 & \dots & 0 \\ \Delta_3 & \Delta_2 & \Delta_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Delta_{s-1} & \Delta_{s-2} & \Delta_{s-3} & \Delta_{s-4} & \dots & 1 \\ \Delta_s & \Delta_{s-1} & \Delta_{s-2} & \Delta_{s-3} & \dots & \Delta_1 \end{vmatrix}.$$

The recurrence for $\Delta_s(\vec{h})$,

$$\Delta_s(\vec{h}) = -\sum_{k=0}^{s-1} h_{s-k}\Delta_k(\vec{h}),$$

is of a standard convolution form. Indeed, if we let $N(t) = \sum_{j=0}^{\infty} \Delta_j t^j$, with $\Delta_0 = 1$,

$$d(t) = 1 + h_1 t + h_2 t^2 + \dots + h_n t^n \equiv \sum_{j=0}^{n} h_j t^j,$$

with $h_0 = 1$, such that $N(t)d(t) = 1$, then we have

$$N(t)d(t) = \sum_{m=0}^{\infty}\sum_{j=0}^{m} \Delta_j h_{m-j} t^m.$$

This implies the set of equations for $m \geq 0$

$$\sum_{j=0}^{m} \Delta_j h_{m-j} = \delta_{m0},$$

wherein δ_{jk} is the Kronecker delta symbol. Rewriting this equation, we obtain

$$\Delta_m = -\sum_{j=0}^{m-1} \Delta_j h_{m-j} + \delta_{m0}.$$

Thus for $m \geq 1$,

$$\Delta_m = -\sum_{j=0}^{m-1} \Delta_j h_{m-j},$$

just as in [23], and we next make the connection with the multinomial expansion.

Let $\vec{j} = (j_1, j_2, \dots, j_n)$, $|\vec{j}| = j_1 + j_2 + \dots + j_n$, and $p(\vec{j}) = j_1 + 2j_2 + \dots + nj_n$. Then, by the use of geometric series and multinomial expansion,

$$\frac{1}{d(t)} = \sum_{k=0}^{\infty} (-1)^k (h_1 t + h_2 t^2 + \dots + h_n t^n)^k$$

$$= \sum_{k=0}^{\infty} (-1)^k \sum_{|\vec{j}|=k} \binom{k}{j_1, j_2, \ldots, j_k} h_1^{j_1} h_2^{j_2} \cdots h_n^{j_n} t^{p(\vec{j})}$$

$$= \sum_{\ell=0}^{\infty} \sum_{p(\vec{j})=\ell} \binom{|\vec{j}|}{j_1, j_2, \ldots, j_k} (-1)^{|\vec{j}|} h_1^{j_1} h_2^{j_2} \cdots h_n^{j_n} t^{\ell}.$$

Additionally, we note that the recurrence for Δ_m may be written in terms of the companion matrix

$$C_d = \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots & 0 & -h_n \\ 1 & 0 & 0 & 0 & \ldots & 0 & -h_{n-1} \\ 0 & 1 & 0 & 0 & \ldots & 0 & -h_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & -h_2 \\ 0 & 0 & 0 & 0 & \ldots & 1 & -h_1 \end{bmatrix}.$$

Theorem 4. This follows by writing (1.32) of [23] in the form

$$\frac{1}{2}\left(1 - \frac{1}{2^{2s}}\right)\zeta(2s) = \sum_{k=1}^{s-1} \frac{1}{2^{2(s-k)}}\left(1 - \frac{1}{2^{2k}}\right)\zeta(2s-2k)\zeta(2k).$$

□

Theorem 5. We apply the relation

$$E_{2n-1}\left(\frac{1}{3}\right) = -E_{2n-1}\left(\frac{2}{3}\right) = -\frac{1}{2n}(1 - 3^{1-2n})(2^{2n} - 1)B_{2n},$$

so that

$$\begin{aligned} E_{2n-1}\left(\frac{1}{3}\right) &= -\frac{1}{2n}(1 - 3^{1-2n})(2^{2n} - 1)B_{2n} \\ &= \sum_{k=0}^{2n-1} \binom{2n-1}{k} \frac{E_k}{2^k} \left(-\frac{1}{6}\right)^{2n-k-1} \\ &= \sum_{m=0}^{n-1} \binom{2n-1}{2m} \frac{E_{2m}}{(2m)!} \left(-\frac{1}{6}\right)^{2(n-m)-1}. \end{aligned}$$

□

Remark. Similarly, for instance, we could apply the relation

$$E_{2n}\left(\frac{1}{6}\right) = E_{2n}\left(\frac{5}{6}\right) = \frac{(1 + 3^{-2n})}{2^{2n} + 1} E_{2n}.$$

Theorem 6. The result follows from relation (1.1) and an identity attributed to Gosper,

$$\sum_{i=0}^{n} \frac{(1-2^{1-i})(1-2^{i-n+1})}{(n-i)!i!} B_{n-i}B_i = \frac{(1-n)}{n!}B_n.$$

In order to keep this paper self contained, we provide a proof of this identity. We use the exponential generating function of Bernoulli polynomials,

$$\frac{te^{xt}}{e^t-1} = \sum_{n\geq 0} \frac{B_n(x)}{n!}t^n, \qquad |t| < 2\pi, \tag{2.1}$$

so that

$$\frac{te^{t/2}}{e^t-1} = \sum_{n\geq 0} B_n\left(\frac{1}{2}\right)\frac{t^n}{n!},$$

and recall that $B_n(1/2) = (2^{1-n}-1)B_n$. Then

$$\frac{t^2e^t}{(e^t-1)^2} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B_n\left(\frac{1}{2}\right)B_m\left(\frac{1}{2}\right)\frac{t^{n+m}}{n!m!}$$

$$= \sum_{j=0}^{\infty}\sum_{n=0}^{j} B_n\left(\frac{1}{2}\right)B_{j-n}\left(\frac{1}{2}\right)\frac{t^j}{n!(j-n)!}.$$

We also have

$$\frac{t^2e^t}{(e^t-1)^2} = \frac{2t}{e^t-1} - \frac{d}{dt}\frac{t^2}{(e^t-1)}$$

$$= 2\sum_{n=0}^{\infty} B_n\frac{t^n}{n!} - \frac{d}{dt}\sum_{n=0}^{\infty} B_n\frac{t^{n+1}}{n!}$$

$$= \sum_{n=0}^{\infty}[2B_n - (n+1)B_n]\frac{t^n}{n!},$$

and we conclude that

$$\sum_{n=0}^{j} \frac{B_n\left(\frac{1}{2}\right)B_{j-n}\left(\frac{1}{2}\right)}{n!(j-n)!} = (1-j)\frac{B_j}{j!}.$$

Gosper's identity then follows. □

Remark. Using the value $\zeta(0) = -1/2$, the $m = 0$ term of the left side of Theorem 6 may be separated and moved to the right side. The result is then seen to correspond with a case presented at the top of p. 406 of [3]. We have given a distinct method of proof.

Theorem 7. The Ramanujan and generalized Ramanujan polynomials respectively satisfy the functional equations

$$R_{2s+1}(z) = z^{2s+2} R_{2s+1}\left(\frac{1}{z}\right)$$

and

$$R_{2s}(z) - R_{2s}\left(\frac{z}{2}\right) = z^{2s+2}\left[R_{2s}\left(\frac{1}{z}\right) - R_{2s}\left(\frac{1}{2z}\right)\right].$$

We then follow the procedure of [11], noting the relation $(d/dz)^j z^r = (-1)^j (-r)_j z^{r-j}$, using the product rule for differentiation and the derivative of a composition of functions. We arrive at, for instance for part (a),

$$\frac{(-1)^n}{z^n} \sum_{j=1}^{n} \binom{n}{j} \frac{(n-1)!}{(j-1)!} \frac{1}{z^j} R_{2s+1}^{(j)}\left(\frac{1}{z}\right)$$

$$= \sum_{j=0}^{n} \binom{n}{j} (-1)^j (2s+2)_j z^{-2s-j-2} R_{2s+1}^{(n-j)}(z).$$

We then set $z = 1$, shift the summation index on the right side as $j \to n - j$, and separate the $j = 0$ and $j = n$ terms. (b) proceeds similarly.

For (c) we apply the specified functional equation, follow similar steps, and then evaluate at $z = i$.

(d) One of the functional equations satisfied by F is [33] (p. 171)

$$F(x) + F\left(\frac{1}{x}\right) = -\frac{\pi^2}{6}x - \frac{\pi^2}{6x} + \frac{1}{2}\ln^2 x + C_1,$$

where $C_1 = \pi^2/3 + 2F(1)$ is a constant involving the first Stieltjes constant γ_1 (e.g., [12]), as $F(1) = -\gamma^2/2 - \pi^2/12 - \gamma_1$, $\gamma = -\psi(1)$ being the Euler constant. We again use the derivatives of a composite function and the product rule, and then evaluate at $x = 1$. In the process the relation [3]

$$\left(\frac{d}{dx}\right)^j \ln^2 x = 2(\ln x - H_{j-1})(-1)^{j-1}\frac{(j-1)!}{x^j}$$

is employed. □

Remarks. We note that part (c) of the Theorem suffices to write the set of identities implied by (2.3) of [23], wherein [16]

$$F_s(z) = \sum_{n=1}^{\infty} \sigma_{-s}(n) e^{2\pi i n z} = \sum_{n=1}^{\infty} \frac{\sigma_s(n)}{n^s} e^{2\pi i n z} = -\zeta(s) - F_s(-z),$$

with σ the sum of divisors function.

δ takes only the values 0 and 1 in [17], but that is not necessary in part (c) of the Theorem.

We could write highly related identities at other special points, including $z = -1$ in parts (a) and (b) and $z = -i$ in part (c).

Identities other than Theorem 7(d) could be developed for Zagier's functions $P(x, y)$, $a(x)$, $A(x, s)$, and $R(x, y)$.

We may mention the work of Leopoldt [21], who generalized the Bernoulli numbers to include weightings by Dirichlet characters and proved a von Staudt-Clausen theorem.

Theorem 8. (a) Is immediate upon applying the integral representation. (b) is similar, and we note that $c_j^{m,m-1}(b)$ depends upon the integration of a certain function $t^{m-1}{}_mF_{m-1}[(-1)^m bt^m]$ for $0 \le t < \infty$:

$$c_j^{m,m-1}(b) = \frac{1}{(m-1)!}\frac{b}{[mj-(m-1)]!}$$

$$\times \int_0^\infty g(t) t^{m-1}{}_mF_{m-1}\left[\frac{m-1}{m} - j, 1-j, \frac{m+1}{m} - j, \frac{m+2}{m} - j, \ldots, \frac{2(m-1)}{m} - j;\right.$$
$$\left.\frac{m+1}{m}, \frac{m+2}{m}, \ldots, \frac{2m-1}{m}; (-1)^m bt^m\right] dt.$$

[3]We may recognize the constants $s(n, 2) = (-1)^n (n-1)! H_{n-1}$ as a special case of the Stirling numbers $s(n, k)$ of the first kind.

$\square$

Let $K_s(y)$ be the modified Bessel function of the second kind.

Corollary 2. Define the function for $b \neq 0$ and Re $y > 0$

$$c_j^M(b,y) = \frac{\sqrt{b}}{4(2j)!}\int_0^\infty \exp\left[-\frac{y}{2}\left(t+\frac{1}{t}\right)\right][(\sqrt{b}t-1)^{2j}-(\sqrt{b}t+1)^{2j}]dt.$$

Then

$$\frac{b^j}{(2j-1)!}K_{2j}(y) = -c_j^M(b,y) - \sum_{k=1}^{j-1}\frac{b^k}{(2j-2k+1)!}\frac{K_{2k}(y)}{(2k-1)!}.$$

Proof. This follows from the representation for Re $y > 0$ and $s \in \mathbb{C}$

$$\frac{1}{\Gamma(s)}K_s(y) = \frac{1}{2\Gamma(s)}\int_0^\infty \exp\left[-\frac{y}{2}\left(t+\frac{1}{t}\right)\right]t^{s-1}dt.$$

$\square$

Corollary 3. Define the function for $b \neq 0$ and Re $a > 2j$,

$$c_j^M(b,a) = \frac{\sqrt{b}}{2(2j)!}\int_0^\infty (1+t)^{-a}[(\sqrt{b}t-1)^{2j}-(\sqrt{b}t+1)^{2j}]dt.$$

Then for Re $a > 2j$,

$$b^j\Gamma(a-2j) = -\Gamma(a)c_j^M(b,a) - \sum_{k=1}^{j-1}\frac{b^k\Gamma(a-2k)}{(2j-2k+1)!}.$$

In fact, this is an identity for a ${}_3F_2(b)$ hypergeometric function, and we have

$$c_j^M(b,a) = -\frac{b}{(2j-1)!}\frac{\Gamma(a-2)}{\Gamma(a)}{}_3F_2\left(1,\frac{1}{2}-j,1-j;\frac{3-a}{2},2-\frac{a}{2};b\right).$$

Proof. We use a representation for a case of the Beta function $B(x,y)$,

$$\int_0^\infty \frac{t^{x-1}}{(1+t)^a}dt = \frac{\Gamma(x)\Gamma(a-x)}{\Gamma(a)} = B(x,a-x),$$

valid for $0 <$ Re $x < a$, such conditions resulting from maintaining convergence at the lower and upper limits of integration, respectively. We then apply Theorem 8 and multiply through by $\Gamma(a)$.

The hypergeometric form for the sum on k may be obtained by shifting the summation index $k \to k+1$, and then using the duplication formula for Pochhammer symbols, further details being omitted. □

Remarks. We may note that a similar result follows for the derivatives with respect to s of the function $K_s(y)/\Gamma(s)$.

For Re $z > 0$ and Re $x > 0$,

$$x^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty e^{-xt} t^{z-1} dt.$$

There follows the identity

$$b^j x^{-2j} = -c_j^M(b,x) - \sum_{k=1}^{j-1} \frac{b^k x^{-2k}}{(2j-2k+1)!},$$

where

$$c_j^M(b,x) = \frac{\sqrt{b}}{2(2j)!} \int_0^\infty e^{-xt} [(\sqrt{b}t - 1)^{2j} - (\sqrt{b}t + 1)^{2j}] dt.$$

Upon evaluating c_j^M using binomial expansion,

$$c_j^M(b,x) = -\frac{1}{(2j)!} \sum_{n=1}^{j} \binom{2j}{2n-1} (2n-1)! \frac{b^n}{x^{2n}},$$

this identity is explicitly verified. As such, putting $x \to 1/x$ and summing with various coefficients, this relation becomes a generating identity of, for example, families of polynomials.

3. Another proof of the identity (1.2), and other generalizations

In giving a different proof of Lettington's identity (1.2), we make use of a standard integral representation of the Riemann zeta function,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt, \quad \text{Re } s > 1, \tag{3.1}$$

where Γ denotes the Gamma function, and Hermite's formula for the Hurwitz zeta function

$$\zeta(s,a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + 2\int_0^\infty (a^2+y^2)^{-s/2} \sin[s \tan^{-1}(y/a)] \frac{dy}{e^{2\pi y} - 1},$$

holding for all complex $s \neq 1$.

Lemma 1. For integers $j \geq 0$,

$$\int_0^\infty \frac{(\pi^2+t^2)^j}{e^t-1} \sin\left(2j\tan^{-1}\left(\frac{t}{\pi}\right)\right) dt = \frac{j\pi^{2j+1}}{2j+1}.$$

Proof. The $j=0$ case is obvious, and in the following we take $j \geq 1$. With $v = 2\pi y$ and $s = -2j$ in Hermite's formula we obtain

$$\int_0^\infty \left(4a^2 + \frac{v^2}{\pi^2}\right)^j \sin\left(2j\tan^{-1}\frac{v}{2\pi a}\right)\frac{dv}{e^v-1} = \pi 4^j \left[\frac{a^{2j}}{2} - \frac{a^{2j+1}}{2j+1} - \zeta(-2j,a)\right].$$

For $a = 1/2$, $\zeta(-2j, 1/2) = (2^{-2j}-1)\zeta(-2j) = 0$ owing to the trivial zeros of the Riemann zeta function. We then find that

$$\int_0^\infty \left(1 + \frac{v^2}{\pi^2}\right)^j \sin\left(2j\tan^{-1}\frac{v}{\pi}\right)\frac{dv}{e^v-1} = \frac{\pi j}{2j+1},$$

and the Lemma follows. □[4]

Proof of (1.2). We first note the following sum, based upon manipulation of binomial expansions,

$$\sum_{k=1}^{j} \frac{(-1)^k \pi^{2j-2k} t^{2k-1}}{(2j-2k+1)!(2k-1)!} = \frac{\pi^{2j-1}}{(2j)!}\sum_{k=1}^{j}(-1)^k \binom{2j}{2k-1}\left(\frac{t}{\pi}\right)^{2k-1}$$
$$= -\frac{i[(\pi-it)^{2j} - (\pi+it)^{2j}]}{2\pi(2j)!}.$$

Also as a prelude, with $y = a\tan^{-1} x$,

$$\sin y = \frac{1}{2i}\left[\left(\frac{1+ix}{1-ix}\right)^{a/2} - \left(\frac{1+ix}{1-ix}\right)^{-a/2}\right].$$

We now use the integral representation (3.1) so that

$$\sum_{k=1}^{j-1} \frac{(-1)^k \pi^{2j-2k}}{(2j-2k+1)!}\zeta(2k) = \sum_{k=1}^{j-1} \frac{(-1)^k \pi^{2j-2k}}{(2j-2k+1)!(2k-1)!}\int_0^\infty \frac{t^{2k-1}}{e^t-1}dt$$
$$= -\int_0^\infty \left[\frac{2(-1)^j j\pi t^{2j} + t(\pi^2+t^2)^j \sin(2j\tan^{-1}(t/\pi))}{\pi t(2j)!}\right]\frac{dt}{e^t-1}.$$

[4]The reader is invited to provide an alternative proof of Lemma 1 using induction.

The first term on the right side evaluates to $(-1)^{j-1}\zeta(2j)$ according to (3.1), while the second term is evaluated by Lemma 1, and (1.2) is shown. $\square$

Remarks. Following steps similar to the proof of Lemma 1, we also find for integers $j \geq 0$

$$-\int_0^\infty \frac{(\pi^2+t^2)^{-j}}{e^t-1} \sin\left(2j\tan^{-1}\left(\frac{t}{\pi}\right)\right) dt = \pi\left[\frac{1}{2} - \frac{1}{2}\frac{1}{(1-2j)} - (1-2^{-2j})\zeta(2j)\right].$$

The method of this section may also be applied to the sums of Theorem 1.

Theorem 9. (a) For Re $a > 0$, define

$$c_j(a) = \frac{1}{\pi(2j)!}\int_0^\infty (\pi^2+t^2)^j \sin(2j\tan^{-1}(t/\pi))\frac{e^{-(a-1)t}dt}{e^t-1}.$$

Then

$$\zeta(2j,a) = (-1)^{j-1}\left[c_j(a) + \sum_{k=1}^{j-1}\frac{(-1)^k\pi^{2j-2k}}{(2j-2k+1)!}\zeta(2k,a)\right].$$

(b) Define for Re $a > 0$ and $b \neq 0$

$$c_j(a,b) = \frac{\sqrt{b}}{\pi(2j)!}\int_0^\infty (\pi^2+bt^2)^j \sin\left(2j\tan^{-1}\left(\frac{\sqrt{b}t}{\pi}\right)\right)\frac{e^{-(a-1)t}dt}{e^t-1}.$$

Then

$$b^j\zeta(2j,a) = (-1)^{j-1}\left[c_j(a,b) + \sum_{k=1}^{j-1}\frac{(-1)^k\pi^{2j-2k}b^k}{(2j-2k+1)!}\zeta(2k,a)\right].$$

Proof. We use the integral representation for Re $a > 0$

$$\zeta(s,a) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-(a-1)t}}{e^t-1}dt, \quad \text{Re } s > 1,$$

and follow steps similar to the above in reproving (1.2). $\square$

Remark. We suspect that the constants $c_j(a = 1/2)$ may be evaluated explicitly.

We now introduce Dirichlet L-functions $L(s,\chi)$ (e.g., [19], Ch. 16), that are known to be expressible as linear combinations of Hurwitz zeta functions. We let χ_k be a Dirichlet character modulo k, and have

$$L(s,\chi) = \sum_{n=1}^\infty \frac{\chi_k(n)}{n^s}, \qquad \text{Re } s > 1.$$

This equation holds for at least Re $s > 1$. If χ_k is a nonprincipal character, then convergence obtains for Re $s > 0$. The L functions are extendable to the whole complex plane, satisfy functional equations, and have integral representations.

Corollary 4. Define the constants

$$b_j^{(k)} = \frac{1}{k^s} \sum_{m=1}^{k} \chi_k(m) c_j \left(\frac{m}{k}\right).$$

Then for L functions of characters modulo k,

$$L(2j, \chi) = (-1)^{j-1} \left[b_j^{(k)} + \sum_{\ell=1}^{j-1} \frac{(-1)^k \pi^{2j-2\ell}}{(2j-2\ell+1)!} L(2\ell, \chi) \right].$$

Proof. This follows from Theorem 7(a) and the relation

$$L(s, \chi) = \frac{1}{k^s} \sum_{m=1}^{k} \chi_k(m) \zeta\left(s, \frac{m}{k}\right).$$

□

We may now emphasize the Lettington relation (1.2) as a recurrence among special values of polygamma functions $\psi^{(j)}$. [5] For we have the relation

$$\zeta(2j, a) = \frac{1}{(2j-1)!} \psi^{2j-1}(a),$$

and the example evaluations for $j > 0$

$$\psi^{(j)}(1) = (-1)^j (j-1)! \zeta(j)$$

and

$$\psi^{(j)}\left(\frac{1}{2}\right) = (-1)^j (j-1)! (2^j - 1) \zeta(j).$$

The Lerch zeta function is given by (e.g., [15] p. 1075)

$$\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}.$$

[5] We recall the definitions $\psi(z) = \Gamma'(z)/\Gamma(z)$ and $\psi^{(j)}(z) = (d^j/dz^j)\psi(z)$, and the functional equation $\psi^{(j)}(x+1) = (-1)^j j!/x^{j+1} + \psi^{(j)}(x)$.

This series converges for $a \in \mathbb{C}$ not a negative integer and all $s \in \mathbb{C}$ when $|z| < 1$, and for Re $s > 1$ when $|z| = 1$ and has the integral representation generalizing Hermite's formula

$$\Phi(z,s,a) = \frac{a^{-s}}{2} + \int_0^\infty \frac{z^t}{(t+a)^s} dt - 2\int_0^\infty (a^2+y^2)^{-s/2} \sin[y \ln z - s \tan^{-1}(y/a)] \frac{dy}{e^{2\pi y}-1},$$

for Re $a > 0$. It satisfies the functional equation

$$\Phi(z,s,a) = z^n \Phi(z,s,n+a) + \sum_{k=0}^{n-1} \frac{z^k}{(k+a)^s}.$$

Theorem 10. Define for Re $a > 0$ and $b \neq 0$

$$c_j(a,b,z) = \frac{\sqrt{b}}{\pi(2j)!} \int_0^\infty (\pi^2 + bt^2)^j \sin\left(2j \tan^{-1}\left(\frac{\sqrt{b}t}{\pi}\right)\right) \frac{e^{-(a-1)t}dt}{e^t - z}.$$

Then

$$b^j \Phi(z,2j,a) = (-1)^{j-1}\left[c_j(a,b,z) + \sum_{k=1}^{j-1} \frac{(-1)^k \pi^{2j-2k} b^k}{(2j-2k+1)!} \Phi(z,2k,a)\right].$$

Proof. We use the integral representation for Re $a > 0$

$$\Phi(z,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-(a-1)t}}{e^t - z} dt, \quad \text{Re } s > 1.$$

□

We let ${}_pF_q$ be the generalized hypergeometric function and

$$\mathrm{Li}_s(z) = \sum_{k=1}^\infty \frac{z^k}{k^s} = z\Phi(z,s,1),$$

for $s \in \mathbb{C}$ and $|z| < 1$ or Re $s > 1$ and $|z| = 1$ be the polylogarithm function.

Corollary 5. (a) For Re $a > 0$ and $b \neq 0$,

$$b^j a^{-2j} {}_{2j+1}F_{2j}(1,a,\ldots,a;a+1,\ldots,a+1;z) = (-1)^{j-1}\left[c_j(a,b,z) + \sum_{k=1}^{j-1} \frac{(-1)^k \pi^{2j-2k} b^k}{(2j-2k+1)!} a^{-2k} {}_{2k+1}F_{2k}(1,a,\ldots,a;a+1,\ldots,a+1;z)\right],$$

and (b) for $|z| \leq 1$,

$$b^j \mathrm{Li}_{2j}(z) = (-1)^{j-1} \left[z c_j(1, b, z) + \sum_{k=1}^{j-1} \frac{(-1)^k \pi^{2j-2k} b^k}{(2j-2k+1)!} \mathrm{Li}_{2k}(z) \right].$$

Proof. (a) follows from the relation for integers $k \geq 1$

$$\Phi(z, k, a) = a^{-k}{}_{k+1}F_k(1, a, \ldots, a; a+1, \ldots, a+1; z)$$

and (b) follows from $\Phi(z, 2j, 1) = \mathrm{Li}_{2j}(z)/z$. □

Other functions will have a recurrence similar to, and in several cases again generalizing, (1.2), and we conclude this section noting that this applies to Eisenstein series. Let

$$f_1(\tau, t) = \frac{\cosh^2(\tau t/2)}{1 - 2e^{-t}\cosh(\tau t) + e^{-2t}}, \tag{3.2a}$$

$$f_2(\tau, t) = \frac{\cos^2(t/2)}{1 - 2e^{i\tau t}\cos t + e^{2i\tau t}}, \tag{3.2b}$$

and

$$\tilde{E}_s = \lim_{K \to \infty} \sum_{|m|,|n| \leq K} \frac{1}{(m\mu + n\nu)^s}, \tag{3.3}$$

where $\mu, \nu \in \mathbb{C}$ and $\tau = \nu/\mu \notin \mathbb{R}$. Then there is a recurrence for values $\tilde{E}_{2j}$, owing to the integral representation ([13], Theorem 5, $\mu = 1$, $\nu = \tau$)

$$\tilde{E}_s(\tau) = \cos\left(\frac{\pi}{2}s\right) \frac{4}{\Gamma(s)} \int_0^\infty t^{s-1}[e^{-is\pi/2}e^{-t}f_1(\tau, t) + e^{i\tau t}f_2(\tau, t)]dt, \qquad \mathrm{Re}\ s > 2. \tag{3.4}$$

Here the ratio τ is in the fundamental region $-1/2 < \mathrm{Re}\ \tau \leq 1/2$, $\mathrm{Im}\ \tau > 0$, $|\tau| \geq 1$, and if $|\tau| = 1$, then $\mathrm{Re}\ \tau \geq 0$. By Corollary 6 of [13], the summatory conditionally convergent case of $s = 2$ is also given by this integral. Walker [30] found a remarkable formula for $\tilde{E}_2(\tau)$ in terms of the Dedekind $\eta(\tau) = e^{i\pi\tau/12}\prod_{n=1}^\infty(1 - e^{2\pi i\tau n})$ function, when summing over increasing disks.

Omitting further details, we arrive at:

Theorem 11. Define the function, for τ in the fundamental region in the upper half plane, and $b \neq 0$,

$$C_j^E(b,\tau) = \frac{4\sqrt{b}}{\pi(2j)!}\int_0^\infty \left[(\pi^2+bt^2)^j \sin\left(2j\tan^{-1}\left(\frac{\sqrt{b}t}{\pi}\right)\right)e^{-t}f_1(\tau,t)\right.$$
$$\left. + i(\pi^2-bt^2)^j\sin\left(2j\tan^{-1}\left(\frac{\sqrt{-b}t}{\pi}\right)\right)e^{it\tau}f_2(\tau,t)\right]dt.$$

Then for τ in the fundamental region,

$$b^j\tilde{E}_{2j}(\tau) = (-1)^{j-1}\left[C_j^E(b,\tau)+\sum_{k=1}^{j-1}\frac{(-1)^k\pi^{2j-2k}b^k}{(2j-2k+1)!}\tilde{E}_{2k}(\tau)\right].$$

Remarks. It appears that a generalized Hermite formula for $\tilde{E}_s$ should exist, and we may guess that it contains a half-line integral term something like

$$\int_0^\infty [(\pi^2+t^2)^{-s/2}\sin(s\tan^{-1}(t/\pi))e^{-t}f_1(\tau,t)+i(\pi^2-t^2)^{-s/2}\sin(s\tan^{-1}(t/\pi))e^{it\tau}f_2(\tau,t)]dt.$$

In fact, such a formula should follow from the contour integral representation of Theorem 7 of [13], and as a precursor we have the following result for the Hurwitz zeta function. Hermite-type formulas are usually found via Plana summation. However, the following shows that this is not necessary.

Theorem 12. For Re $s>1$, Re $a>0$, and $0<c<1$,

$$\zeta(s,a) = -\int_0^\infty \left[\cos\left(s\tan^{-1}\left(\frac{t}{c+a-1}\right)\right)\cos\pi c\sin\pi c\right.$$
$$\left. -\sinh\pi t\cosh\pi t\sin\left(s\tan^{-1}\left(\frac{t}{c+a-1}\right)\right)\right]$$
$$\times\left[(c+a-1)^2+t^2\right]^{s/2}\left(\cosh^2\pi t-\cos^2\pi t\right)\Big]^{-1}dt.$$

Proof. We recall the partial fractions form of the cotangent function,

$$\cot z = \sum_{n=-\infty}^{\infty}\frac{1}{z-n\pi}.$$

Then for Re $s>1$, we have

$$\zeta(s,a) = \pi\sum_{k=0}^\infty \text{Res}\,(t+a)^{-s}\cot\pi t\big|_{t=k}.$$

Closing a semicircular contour encompassing the positive integers along the real-t axis leads to

$$\zeta(s,a) = \frac{i}{2}\int_{c-i\infty}^{c+i\infty} (t+a-1)^{-s} \cot \pi t \; dt$$
$$= -\frac{1}{2}\int_{-\infty}^{\infty} (c+a-1+it)^{-s} \cot[\pi(c+it)] dt.$$

The contributions for negative and positive t are then combined, using for instance

$$\cot \pi(c-it) = \frac{\cos \pi c \cosh \pi t - i \sin \pi c \sinh \pi t}{\cosh \pi t \sin \pi c + i \cos \pi c \sinh \pi t}.$$

$\square$

We may similarly consider certain lattice Dirichlet series (Kronecker series)

$$G(s,\chi) = \sum_{\omega \in \Lambda \backslash \{0\}} \frac{\chi(\omega)}{|\omega|^{2s}}, \qquad \chi(\omega) = e^{i(m\mu\alpha + n\nu\beta)},$$

with α and β real and $\Lambda \in \mathbb{C}$ a lattice. Letting

$$|\omega_{m,n}|^2 = |m\mu + n\nu|^2 = Q(m,n),$$

there is the integral representation [13]

$$G(s,\chi) = \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1} \sum_{\omega \in \Lambda \backslash \{0\}} \chi(\omega) e^{-tQ(m,n)} dt,$$

leading to the following result, whose proof is omitted.

Theorem 13. Define for $b \neq 0$

$$C_j^G(b,\chi) = \frac{\sqrt{b}}{\pi(2j)!} \sum_{\omega \in \Lambda \backslash \{0\}} \chi(\omega) \int_0^{\infty} (\pi^2 + bt^2)^j \sin\left(2j \tan^{-1}\left(\frac{\sqrt{b}t}{\pi}\right)\right) e^{-tQ(m,n)} dt.$$

Then

$$b^j G(2j,\chi) = (-1)^{j-1}\left[C_j^G(b,\chi) + \sum_{k=1}^{j-1} \frac{(-1)^k \pi^{2j-2k} b^k}{(2j-2k+1)!} G(2k,\chi)\right].$$

Let $\sigma_k(n)$ be the sum of divisors function, the sum of the powers d^k of the positive divisors of n, i.e., $\sigma_k(n) = \sum_{d|n} d^k$. With nome $q = e^{i\pi\tau}$, there are the series representations

$$\tilde{E}_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

$$= 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \frac{n^{2k-1}q^n}{1-q^n}.$$

These series provide another means by which to establish the equivalent of Theorem 11, and of the various generalizations of Theorem 1 for $\tilde{E}_{2k}(\tau)$.

4. Integral and series representations for the Hurwitz numbers $\tilde{H}_n$

The Hurwitz numbers $\tilde{H}_n$ [9, 18] occur in the Laurent expansion of a certain Weierstrass $\wp$ function about the origin, and are highly analogous to the Bernoulli numbers. The $\wp$ function in question has periods $\tilde{\omega}$ and $\tilde{\omega}i$, where $\tilde{\omega}$ is the Beta function value

$$\tilde{\omega} = 2\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{2}B\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}\frac{\Gamma(1/4)}{\Gamma(3/4)},$$

and satisfies the nonlinear differential equations

$$\wp'(z)^2 = 4\wp(z)^3 - 4\wp(z), \qquad \wp''(z) = 6\wp(z)^2 - 2.$$

Then this $\wp$ function expands about $z = 0$ as

$$\wp(z) = \frac{1}{z^2} + \sum_{n=2}^{\infty} \frac{2^n \tilde{H}_n}{n} \frac{z^{n-2}}{(n-2)!}. \tag{4.1}$$

From the property $\wp(iz) = -\wp(z)$, and the evenness of $\wp(z)$, it follows that $\tilde{H}_n = 0$ unless n is a multiple of 4, the first few values being $\tilde{H}_4 = 1/10$, $\tilde{H}_8 = 3/10$, and $\tilde{H}_{12} = 567/130$. The analogous expansion for Bernoulli numbers is

$$\frac{1}{\sin^2 x} = \frac{1}{x^2} + \sum_{n=2}^{\infty} \frac{(-1)^{n/2-1}2^n B_n}{n} \frac{x^{n-2}}{(n-2)!}. \tag{4.2}$$

The analogy between B_{2n} and $\tilde{H}_{4n}$ is furthered by comparing the Riemann zeta function values

$$\sum_{r\in\mathbb{Z}^+} \frac{1}{r^{2n}} = (-1)^{n-1}\frac{(2\pi)^{2n}}{(2n)!}B_{2n}, \quad n \geq 1,$$

with the sum over Gaussian integers

$$\sum_{\lambda\in\mathbb{Z}+i\mathbb{Z}\setminus(0,0)} \frac{1}{\lambda^{4n}} = \frac{(2\tilde{\omega})^{4n}}{(4n)!}\tilde{H}_{4n}.$$

In addition, Hurwitz [18] proved a Clausen-von Staudt type theorem for $\tilde{H}_n$.

An integral representation for $\tilde{H}_n$ would have many applications, including the development of various recurrences, again in analogy to the previous sections of this paper. Therefore, we present the following.

Theorem 14. For $k \geq 1$ and the function f_1 defined in (3.2a) of the previous section,

$$\tilde{H}_{2k+2} = \frac{2(k+1)}{4^k \tilde{\omega}^{2k+2}} [1 + (-1)^{k+1}] \int_0^\infty e^{-t} f_1(i,t) t^{2k+1} dt.$$

Proof. We make use of Theorem 10 of [13], such that [6]

$$\wp(z,\tau) = \frac{1}{z^2} + 8 \int_0^\infty t \left[e^{-t} \sinh^2\left(\frac{zt}{2}\right) f_1(\tau,t) + e^{it\tau} \sin^2\left(\frac{zt}{2}\right) f_2(\tau,t) \right] dt,$$

with τ in the fundamental region and z in a domain containing the origin. With the Maclaurin series

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) = \sum_{k=1}^\infty (-1)^{k+1} \frac{2^{2k-1} x^{2k}}{(2k)!},$$

and

$$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1) = \sum_{k=1}^\infty \frac{2^{2k-1} x^{2k}}{(2k)!},$$

we may write

$$\wp(z) - \frac{1}{z^2} = 4 \sum_{k=1}^\infty \frac{z^{2k}}{(2k)!} \int_0^\infty [e^{-t} f_1(\tau,t) + e^{it\tau}(-1)^{k+1} f_2(\tau,t)] t^{2k+1} dt. \tag{4.3}$$

For the square lattice with $\tau = i$, $f_1(i,t) = f_2(i,t)$. We then equate like powers of z with those of the expansion (4.1) and apply the scaling $\wp(x|\mu,\nu) = \wp(x/\mu|1,\tau)/\mu^2$ [13] (p. 145). □

Corollary 6. Define the constants

$$c_{\ell,j}^{(k)} = \int_0^\infty e^{-\ell t} t^{4k-1} \cos^{\ell-(2j+1)} t \; dt.$$

[6]Note that in (33) and (34) of [13], the τ and λ arguments of the functions f_1 and f_2 need to be reversed.

Then

$$\tilde{H}_{4k} = \frac{k}{4^{2(k-1)}} \frac{1}{\tilde{\omega}^{4k}} \sum_{\ell=1}^{\infty} \sum_{j=0}^{(\ell-1)/2} 2^{\ell-(2j+1)} \binom{\ell-j-1}{j} (-1)^j (c_{\ell,j}^{(k)} + c_{\ell+1,j}^{(k)}).$$

Proof. In the integral representation for $\tilde{H}_{4n}$, we may note the simplified product

$$e^{-t} f_1(i,t) = \frac{\cos t + 1}{4(\cosh t - \cos t)}.$$

From the identity

$$\sum_{k=1}^{\infty} e^{-kt} \sin kx = \frac{\sin x}{2(\cosh t - \cos x)},$$

we have

$$\tilde{H}_{4k} = \frac{k}{4^{2(k-1)}} \frac{1}{\tilde{\omega}^{4k}} \sum_{\ell=1}^{\infty} \int_0^{\infty} \frac{(\cos t + 1)}{\sin t} e^{-\ell t} t^{4k-1} \sin \ell t \; dt.$$

We then apply the identity

$$\begin{aligned} \sin nx &= \sin x \sum_{j=0}^{(n-1)/2} (-1)^j \binom{n}{2j+1} \cos^{n-(2j+1)} x \sin^{2j} x \\ &= \sin x \sum_{j=0}^{(n-1)/2} (-1)^j 2^{n-(2j+1)} \binom{n-j-1}{j} \cos^{n-(2j+1)} x. \end{aligned}$$

□

Remarks. The form of Theorem 14 verifies that $\tilde{H}_{4j+2} = 0$ and $\tilde{H}_{4n} \neq 0$. The recurrence relation for $\tilde{H}_{4n}$ obtained from the differential equation for $\wp(z)$ for the square lattice is

$$(2n-3)(4n-1)(4n+1)\tilde{H}_{4n} = 3 \sum_{j=1}^{n-1} (4j-1)(4n-4j-1) \binom{4n}{4j} \tilde{H}_{4j} \tilde{H}_{4(n-j)}.$$

By rewriting the result of Theorem 14, we have

$$\tilde{H}_{4k} = \frac{8k}{4^{2k-1}\tilde{\omega}^{4k}} \int_0^{\infty} e^{-t} f_1(i,t) t^{4k-1} dt,$$

and this highly suggests the study of sums of the form

$$\sum_{k=1}^{n-1} \binom{4n+p}{4k} \frac{4^{2k-1}}{8k} \tilde{H}_{4k},$$

with $p = 0, 1, 2, 3$. Indeed, we obtain the following.

Theorem 15. Define the constants

$$C_n^H = \int_0^\infty \frac{1}{t} e^{-t} f_1(i,t) \left[-1 + {}_4F_3 \left(\frac{1}{4} - n, \frac{1}{2} - n, \frac{3}{4} - n, -n; \frac{1}{4}, \frac{1}{2}, \frac{3}{4}; \frac{t^4}{\tilde{\omega}^4} \right) \right] dt.$$

Then

$$\frac{4^{2n-1}}{8n} \tilde{H}_{4n} = C_n^H - \sum_{k=1}^{n-1} \binom{4n}{4k} \frac{4^{2k-1}}{8k} \tilde{H}_{4k}.$$

Proof. We apply the integral representation of Theorem 14 to obtain:,

$$\sum_{k=1}^{n-1} \binom{4n}{4k} \frac{4^{2k-1}}{8k} \tilde{H}_{4k} = \sum_{k=1}^{n-1} \binom{4n}{4k} \frac{1}{\tilde{\omega}^{4k}} \int_0^\infty e^{-t} f_1(i,t) t^{4k-1} dt,$$

with

$$\binom{4n}{4k} = \frac{(-4n)_{4k}}{(4k)!}.$$

In order to reach hypergeometric form, we then apply the quadriplication formulas for the Pochhammer symbol and Gamma function, so that

$$(-4n)_{4k} = 4^{4k} (-n)_k \left(-n + \frac{1}{4} \right)_k \left(-n + \frac{1}{2} \right)_k \left(-n + \frac{3}{4} \right)_k,$$

and

$$\begin{aligned} (4k)! = \Gamma(4k+1) &= (2\pi)^{-3/2} 4^{4k+1/2} \Gamma\left(k + \frac{1}{4}\right) \Gamma\left(k + \frac{1}{2}\right) \Gamma\left(k + \frac{3}{4}\right) k! \\ &= 4^{4k} \left(\frac{1}{4}\right)_k \left(\frac{1}{2}\right)_k \left(\frac{3}{4}\right)_k k!. \end{aligned}$$

□

Another very useful application of an integral representation can be in developing asymptotic relations. Indeed, we have the following. This result and an asymptotic formula given later (Theorem 19) for the Matter numbers make use of the Γ function integral, for Re $\mu > 0$, Re $\beta > |\text{Im } \delta|$, ([15], p. 490)

$$\int_0^\infty x^{\mu-1} e^{-\beta x} \cos \delta x \, dx = (\delta^2 + \beta^2)^{-\mu/2} \cos \left(\mu \tan^{-1} \frac{\delta}{\beta} \right). \tag{4.4}$$

Corollary 7. As $n \to \infty$,

$$\tilde{H}_{4n} \sim 4\frac{(4n)!}{(2\tilde{\omega})^{4n}},$$

with the refinement

$$\tilde{H}_{4n} \sim 4\frac{(4n)!}{2^{6n}\tilde{\omega}^{4n}}[(-1)^k + 2^{2n}].$$

Proof. We have

$$\begin{aligned}
\tilde{H}_{4k} &\sim \frac{8k}{4^{2k-1}\tilde{\omega}^{4k}} \int_0^\infty e^{-t}\cos^2(t/2)t^{4k-1}dt \\
&= \frac{4k}{4^{2k-1}\tilde{\omega}^{4k}} \int_0^\infty e^{-t}(\cos t + 1)t^{4k-1}dt \\
&= \frac{8k}{4^{2k-1}\tilde{\omega}^{4k}}\frac{1}{4^{2k+1}}[(1-i)^{4k} + (1+i)^{4k} + 2^{4k+1}]\Gamma(4k) \\
&= \frac{(4k)!}{4^{2k-1}2^{2k}\tilde{\omega}^{4k}}[(-1)^k + 2^{2k}].
\end{aligned}$$

□

The following collects various integral representations for lattice sums $S_r(\Lambda) = \sum_{\omega\in\Lambda\setminus(0,0)} 1/\omega^r$ and the g_2 and g_3 invariants of the Weierstrass $\wp$ function.

Corollary 8. Let the functions f_1 and f_2 be defined as in the previous section. Then

(a)

$$S_{2k}(\Lambda) = \frac{4}{(2k+1)!}\int_0^\infty [e^{-t}f_1(\tau,t) + (-1)^{k+1}e^{i\tau t}f_2(\tau,t)]t^{2k+1}dt,$$

(b)

$$g_2 = \frac{40}{\mu^4}\int_0^\infty [e^{-t}f_1(\tau,t) + e^{i\tau t}f_2(\tau,t)]t^3dt,$$

and (c)

$$g_3 = \frac{14}{3\mu^6}\int_0^\infty [e^{-t}f_1(\tau,t) - e^{i\tau t}f_2(\tau,t)]t^5dt.$$

The discriminant of the polynomial $4x^3 - g_2x - g_3$ is given by $\Delta(\Lambda) = g_2(\Lambda)^3 - 27g_3(\Lambda)^2$ and the j-invariant of the lattice Λ is $j(\Lambda) = 1728g_2(\Lambda)^3/\Delta(\Lambda)$.

Proof. (a) follows from the work in proving Theorem 14. We have $S_4 = g_2/60$ and $S_6 = g_3/140$ (e.g., [2], p. 166), so that $\wp(z,\Lambda) - 1/z^2 = g_2 z^2/20 + g_3 z^4/28 + O(z^6)$. Then for (b),

$$\left(\wp(z,\Lambda) - \frac{1}{z^2}\right)''_{z=0} = \frac{g_2}{10},$$

while for (c),

$$\left(\wp(z,\Lambda) - \frac{1}{z^2}\right)^{(iv)}_{z=0} = \frac{6g_3}{7}.$$

□

Remarks. For the square lattice with $\tau = i$ and $\mu = \tilde{\omega}$ (and the scaling of [13]) we obtain as expected $g_2 = 4$ and $g_3 = 0$.

We also obtain integral representations for the coefficients of the Laurent expansion about $z = 0$ for the Weierstrass zeta function ζ_w from Corollary 8 since $\zeta_w(z,\tau) = -\int^z \wp(s,\tau)ds = 1/z - S_4 z^3 - S_6 z^5 + O(z^7)$.

Our results are also relevant to series expansions of the solutions of the first Painlevé equation $\frac{d^2u}{dz^2} = 6u^2 + z$. This is because Boutroux [5] showed that, for large $|z|$, with the scaled variables $U = z^{-1/2}u$ and $Z = \frac{4}{5}z^{5/4}$, the solution of this equation behaves asymptotically like the Weierstrass function, $U \sim \wp$, which satisfies the second order differential equation $\wp'' = 6\wp^2 - g_2/2$.

The following two Theorems provide sets of identities satisfied by the Hurwitz numbers.

Theorem 16. Let the Laurent expansion of the Weierstrass $\wp$ function be written as

$$\wp(z) = \frac{1}{z^2} + \sum_{k=2}^{\infty} d_k z^{2k-2}.$$

Then (a) for generic $\wp$ functions, i.e., such functions with arbitrary lattices,

$$2(k-1)kd_k + \frac{(4-4^k)}{2k-1}d_k + \sum_{j=2}^{k-2}(j-1)\frac{(4^{k-j}-4)}{[2(k-j)-1]}d_j d_{k-j} = 0.$$

(b) For the square lattice of periods $\tilde{\omega}$ and $\tilde{\omega}i$, we identify

$$d_k = \frac{4^k}{2k}\frac{\tilde{H}_{2k}}{(2k-2)!},$$

in which case

$$\frac{(k-1)}{(2k-2)!}\tilde{H}_{2k} + \frac{(4-4^k)}{(2k)!}\tilde{H}_{2k} + \frac{1}{4}\sum_{j=2}^{k-2}(j-1)\frac{(4^{k-j}-4)}{[2(k-j)-1]!}\frac{\tilde{H}_{2j}\tilde{H}_{2(k-j)}}{j(k-j)(2j-2)!} = 0.$$

Proof. Introduce the Weierstrass ζ-function via $\zeta'(z) = -\wp(z)$. This function has a simple pole at the origin, is analytic in a neighborhood of that point, and $\zeta(z) - z^{-1}$ vanishes at $z = 0$. Accordingly, it has the Laurent expansion

$$\zeta(z) = \frac{1}{z} - \sum_{k=2}^{\infty}\frac{d_k}{2k-1}z^{2k-1}.$$

Now it holds that

$$\wp'(z)\zeta(2z) = 2\wp'(z)\zeta(z) + \frac{1}{2}\wp''(z).$$

We use

$$\wp'(z) = -\frac{2}{z^3} + \sum_{k=2}^{\infty}(2k-2)d_k z^{2k-3},$$

$$\wp''(z) = \frac{6}{z^4} + \sum_{k=2}^{\infty}(2k-2)(2k-3)d_k z^{2k-4},$$

and manipulation of the product of two infinite series. As it must be, the $O(1/z^4)$ polar terms cancel. Then the coefficients of z^{2k-4} on both sides are equated, giving the result. $\square$

Theorem 17. Let

$$c_j \equiv \sum_{n=4}^{j-3}\frac{2^j}{n(j-n)}\frac{\tilde{H}_n\tilde{H}_{j-n}}{(n-3)!(j-n-3)!}.$$

Then for $\ell \geq 4$,

$$-9\frac{2^\ell}{\ell}\frac{\tilde{H}_\ell}{(\ell-3)!} + \frac{9}{4}c_\ell + 4\times 2^\ell(2^{\ell-2}+2)\frac{\tilde{H}_\ell}{\ell(\ell-2)!} - 4\sum_{n=4}^{\ell-3}(2\times 2^\ell + 2^{n-2})\frac{\tilde{H}_{\ell-n}\tilde{H}_n}{(\ell-n)(\ell-n-3)!n(n-2)!}$$

$$+\sum_{j=8}^{\ell-2} \frac{\tilde{H}_{\ell-j}c_j}{(\ell-j)(\ell-j-2)!}2^{\ell-j}(2+2^{\ell-j-2})$$

$$=3\frac{2^\ell}{\ell}\frac{\tilde{H}_\ell}{(\ell-4)!}+\frac{1}{4}\sum_{n=4}^{\ell-4}\frac{2^\ell}{n(\ell-n)}\frac{\tilde{H}_n\tilde{H}_{\ell-n}}{(n-4)!(\ell-n-4)!}.$$

The initial identities are given by:

$$-30\tilde{H}_4^2+\tilde{H}_8=0, \qquad 6\tilde{H}_4^3-\frac{9}{5}\tilde{H}_4\tilde{H}_8+\frac{52}{4725}\tilde{H}_{12}=0,$$

and

$$68\tilde{H}_4^2\tilde{H}_8-\frac{16}{5}\tilde{H}_8^2-\frac{1408}{945}\tilde{H}_4\tilde{H}_{12}+\frac{901}{315315}\tilde{H}_{16}=0.$$

The initial values of the c_j's are: $c_8=4/25$, $c_{12}=8/125$, and $c_{16}=236/24375$.

Proof. We write the duplication formula of the $\wp$ function in the form

$$\wp(2z)[\wp'(z)]^2=\frac{1}{4}[\wp''(z)]^2-2[\wp'(z)]^2\wp(z),$$

wherein polar terms of $O(1/z^8)$ and $O(1/z^4)$ cancel. Then comparing coefficients of the terms $O(1)$, $O(z^4)$, and $O(z^8)$ gives the initial relations. For the general result, we give various intermediate expressions based upon the use of (4.1):

$$(\wp'(z))^2=\frac{4}{z^6}-\frac{4}{z^3}\sum_{n=3}^{\infty}\frac{2^n}{n}\frac{\tilde{H}_n}{(n-3)!}z^{n-3}+\sum_{j=6}^{\infty}c_jz^{j-6},$$

$$[\wp''(z)]^2=\frac{36}{z^8}+\frac{12}{z^4}\sum_{n=4}^{\infty}\frac{2^n}{n}\frac{\tilde{H}_n}{(n-4)!}z^{n-4}+\sum_{j=8}^{\infty}\sum_{n=4}^{j-4}\frac{2^j}{n(j-n)}\frac{\tilde{H}_n\tilde{H}_{j-n}}{(n-4)!(j-n-4)!}z^{j-8},$$

with resulting expressions for $\wp(kz)[\wp'(z)]^2$, $k=1,2$ being omitted. Again the polar terms in the duplication formula cancel. We compare the coefficients of $z^{\ell-8}$ on both sides of this formula, and then combine like terms. $\square$

Remarks. The denominators of the Hurwitz numbers form a divisibility sequence, so it could be posed whether the c_j's or a closely related set of numbers also has interesting arithmetic properties. In fact, it does appear that the denominators of $\{c_j\}$ form a divisibility sequence, and that other properties may hold as well.

There are several other relations between the Weierstrass ζ, σ, and $\wp$ functions that could be used to develop identities amongst the coefficients d_k and so among the Hurwitz numbers. Here $\sigma'(z)/\sigma(z) = \zeta(z)$ is an entire function which vanishes at the origin. Some of these relations, for instance, relate functions at argument $3z$ to values at argument z. However, this is not further pursued in this article.

The elliptic function analogs of the trigonometric functions $\sin z$, $\cot z$, and $\csc^2 z$ are $\sigma(z)$, $\zeta(z)$, and $\wp(z)$, respectively. For example, $\cot z = (d/dz)\sin z$ and $-(d/dz)\cot z = \csc^2 z$.

Theorem 18. (Series representation of the Hurwitz numbers). For $m \geq 1$,

$$\tilde{H}_{4m} = \left(\frac{\pi}{\tilde{\omega}}\right)^{4m}\left[-B_{4m} + 8m\sum_{n=1}^{\infty}\frac{n^{4m-1}}{e^{2\pi n}-1}\right].$$

Proof. The $\wp$ function has the Fourier expansion [4]

$$\wp(z,\tau) = -2\left(\frac{1}{6} + \sum_{n=1}^{\infty}\frac{1}{\sin^2(n\pi\tau)}\right) + \frac{\pi^2}{\sin^2\pi z} - 8\pi^2\sum_{n=1}^{\infty}\frac{n\cos(2\pi nz)}{e^{-2\pi i\tau n}-1}.$$

This expression is expanded in powers of z with $\tau = i$, appropriately scaling in terms of $\tilde{\omega}$, and the use of (4.2) and the following.

Lemma 2.

$$\sum_{n=1}^{\infty}\frac{1}{\sin^2(n\pi i)} = \frac{1}{\pi^2}\sum_{n=1}^{\infty}\Gamma^2(in)\Gamma^2(1-in) = -\frac{1}{6} + \frac{1}{2\pi}.$$

Proof of Lemma 2. There is the expansion

$$\csc^2\pi z = \frac{1}{\sin^2\pi z} = \frac{1}{\pi^2 z^2} + \frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{z^2+k^2}{(z^2-k^2)^2}.$$

Then

$$\sum_{n=1}^{\infty}\frac{1}{\sin^2(n\pi i)} = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(k^2-n^2)}{(k^2+n^2)^2}$$

$$= -\frac{1}{6} + \frac{2}{\pi^2}\sum_{k=1}^{\infty}\left[-\frac{1}{2k^2} + \frac{\pi^2}{2}\operatorname{csch}^2(\pi k)\right].$$

□

There follows

$$\sum_{m=1}^{\infty} \frac{2^{4m}\tilde{H}_{4m}}{4m(4m-2)!} z^{4m-2} = \frac{1}{\tilde{\omega}^2}\left[-\pi + 2B_2\pi^2 - \sum_{m=1}^{\infty} \frac{2^{4m}B_{4m}\pi^{4m}}{4m(4m-2)!}\frac{z^{4m-2}}{\tilde{\omega}^{4m-2}}\right.$$

$$\left. -8\pi^2 \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!}\frac{(2\pi)^{2j}}{\tilde{\omega}^{2j}} \sum_{n=1}^{\infty} \frac{n^{2j+1}}{(e^{2\pi n}-1)} z^{2j}\right].$$

Equating the coefficients of z^{4m-2} gives the Theorem. □

5. The equianharmonic case for $\wp$

We have covered the lemniscatic case of the $\wp$ function, for which the lattice is a certain square. We are concerned in this section with the equianharmonic case, for which the lattice is composed of equilateral triangles. [7] So we now consider a $\wp$ function for which $g_2 = 0$, $g_3 = 1$, with half periods

$$\omega_1 = \int_{4^{1/3}}^{\infty} \frac{dx}{\sqrt{4x^3-1}} = \frac{\Gamma^3(1/3)}{4\pi},$$

and $\omega_2 = e^{i\pi/3}\omega_1$. Then $\tau = \omega_2/\omega_1 = e^{i\pi/3}$ is a 6th root of unity.

We write the expansion

$$\wp(z;0,1) = \frac{1}{z^2} + \sum_{n=1}^{\infty} \frac{(6n-1)}{(2\omega_1)^{6n}} S_{6n}(e^{i\pi/3}) z^{6n-2}, \tag{5.1}$$

such that $S_{2n}(e^{i\pi/3}) = 0$ unless $n \equiv 0 \bmod 3$. For this case, early investigation of the coefficients was performed by Matter [10, 24, 27].

Theorem 19. Let the functions f_1 and f_2 be as given in section 3. Then (a)

$$S_{6n}(e^{i\pi/3}) = \frac{4}{(6n-1)!}\int_0^{\infty} [e^{-t} f_1(e^{i\pi/3}, t) + e^{it\exp(i\pi/3)}(-1)^n f_2(e^{i\pi/3}, t)] t^{6n-1} dt,$$

and (b)

$$S_{6n}(e^{i\pi/3}) \sim 6 + 2\left(-\frac{1}{27}\right)^n.$$

Proof. Part (a) follows by using the intermediate result (4.3) in the proof of Theorem (14), the expansion (5.1), and then equating the coefficients of z^{6n-2}.

[7] Further background on these two cases may be found in section 7.5 of [31].

The asymptotic form of (b) is given by

$$S_{6n}(e^{i\pi/3}) \sim \frac{4}{(6n-1)!}\int_0^\infty [e^{-t}\cosh^2(e^{i\pi t/3}/2) + (-1)^n e^{it\exp(i\pi/3)}\cos^2(t/2)]t^{6n-1}dt.$$

The integral may be evaluated in terms of $\Gamma(6n) = (6n-1)!$ by using (4.4). Then after a number of steps of simplification, we obtain the stated form. □

Remark. A systematic development of the denominators of the functions f_1 and f_2 in terms of decreasing exponential functions may be performed by using the generating function of the Chebyshev polynomials of the second kind $U_n(x)$, with $U_0(x) = 1$, $U_1(x) = 2x$, and $U_2(x) = 4x^2 - 1$. We have, for $t > 0$,

$$\frac{1}{1 - 2e^{-t}\cosh(\tau t) + e^{-2t}} = \sum_{k=0}^\infty U_k(\cosh \tau t)e^{-kt}.$$

The following is an analog of Theorem 15 for Matter numbers.

Theorem 20. Let

$${}_6F_{11}^{\pm} \equiv {}_6F_{11}\left(1-n, \frac{7}{6}-n, \frac{4}{3}-n, \frac{3}{2}-n, \frac{5}{3}-n, \frac{11}{6}-n; \frac{7}{6}, \frac{7}{6}, \frac{4}{3}, \frac{4}{3}, \frac{3}{2}, \frac{3}{2}, \frac{5}{3}, \frac{5}{3}, \frac{11}{6}, \frac{11}{6}, 2; \pm\frac{t^6}{6^6}\right),$$

and define the constants

$$C_n^S = \frac{4}{5!}\binom{6n}{6}\int_0^\infty t^5\left[e^{-t}f_1(e^{i\pi/3}, t)_6F_{11}^+ - e^{it\exp(i\pi/3)}f_2(e^{i\pi/3}, t)_6F_{11}^-\right]dt.$$

Then

$$S_{6n}(e^{i\pi/3}) = C_n^S - \sum_{k=1}^{n-1}\binom{6n}{6k}S_{6k}(e^{i\pi/3}).$$

Proof. We apply the integral representation of part (a) of the previous Theorem.

□

6. Discussion: other Bernoulli relations

Proposition 1. For integers $k \geq 0$,

$$B_{2k+2} = \frac{(2k+2)(2k+1)}{4(2^{2k+2}-1)}\int_0^1 E_{2k}(x)dx.$$

Proof. We integrate the generating function (1.4), so that

$$\frac{z}{2}\tanh\left(\frac{z}{2}\right) = \sum_{n=0}^{\infty}\frac{z^n}{n!}\int_0^1 E_n(x)dx.$$

We then compare with the generating function

$$\tanh x = \sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)}{(2k)!}B_{2k}x^{2k-1}, \qquad |x| < \frac{\pi}{2},$$

and the result follows. □

Similarly, if we integrate the generating function (2.1) for Bernoulli polynomials, we obtain the well known relations $\int_0^1 B_0(x)dx = 1$ and for $n > 0$, $\int_0^1 B_n(x)dx = 0$.

Lehmer obtained the recurrences [8, 20]

$$\sum_{k=0}^{n}\binom{6n+3}{6k}B_{6k} = 2n+1$$

and

$$\sum_{k=0}^{n}\binom{6n+5}{6k+2}B_{6k+2} = \frac{1}{3}(6n+5).$$

In terms of the Riemann zeta function at even integers congruent to 0 and 2 modulo 6, we thus have

$$\sum_{k=0}^{n}\frac{(6n+3)!}{(6n-6k+3)!}\frac{(-1)^{k+1}}{2^{6k-1}\pi^{6k}}\zeta(6k) = 2n+1$$

and

$$\sum_{k=0}^{n}\frac{(6n+5)!}{(6n-6k+3)!}\frac{(-1)^{k+1}}{2^{6k+1}\pi^{6k+2}}\zeta(6k+2) = \frac{1}{3}(6n+5).$$

These expressions then provide a starting point for generalization according to the previous sections of this paper. Furthermore, we have determined the following recurrences, which may then be expressed in terms of sums of values $\zeta(6k+q)$, being special cases of $\zeta(6k+q,a)$, $\Phi(z,6k+q,a)$, and of other functions.

Proposition 2.

$$\sum_{k=0}^{n}\binom{6n+7}{6k+4}B_{6k+4} = -\left(n+\frac{7}{6}\right),$$

$$\sum_{k=0}^{n}\binom{6n+9}{6k+6}B_{6k+6}=2(n+1),$$

$$\sum_{k=0}^{n}\binom{6n+11}{6k+8}B_{6k+8}=-\frac{1}{2}(6n+11)(n+1),$$

$$\sum_{k=0}^{n}\binom{6n+13}{6k+10}B_{6k+10}=\frac{1}{60}(6n+13)(18n^2+81n+100),$$

$$\sum_{k=0}^{n}\binom{6n+15}{6k+12}B_{6k+12}=-\frac{1}{420}(648n^4+6156n^3+22266n^2+36765n+24185)(n+2)(n+1),$$

$$\sum_{k=0}^{n}\binom{6n+17}{6k+14}B_{6k+14}=\frac{1}{8400}(6n+17)(n+2)(n+1)$$
$$\times(1944n^5+23652n^4+116046n^3+288423n^2+366675n+196000),$$

and

$$\sum_{k=0}^{n}\binom{6n+19}{6k+16}B_{6k+16}=-\frac{1}{55440}(6n+19)(n+2)(n+1)$$
$$\times(11664n^7+209952n^6+1623240n^5+6998400n^4+182132n^3+28919198n^2+25568993n+10026324).$$

These successive sums may be obtained from the initial ones by shifting the summation index k and upper limit n.

Owing to the particular binomial coefficient in the summand in these relations, they are closely connected with the properties of various ${}_6F_5$ hypergeometric functions when integral representations are applied. Similarly, other ${}_6F_5$ functions appear if we use a summation representation such as

$$\frac{B_{2k}}{2k}=2\sum_{m=1}^{\infty}\frac{m^{2k-1}}{e^{2\pi m}-1}-\frac{2\pi}{k}[1+(-1)^k]\sum_{m=1}^{\infty}\frac{m^{2k}e^{2\pi m}}{(e^{2\pi m}-1)^2}.$$

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