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Euler's factorial series at algebraic integer points

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ABSTRACT

We study a linear form in the values of Euler's series $F(t) = \sum_{n=0}^{\infty} n!t^n$ at algebraic integer points $\alpha_j \in \mathbb{Z}_{\mathbb{K}}$, $j = 1, \dots, m$, belonging to a number field \mathbb{K} . In the two main results it is shown that there exists a non-Archimedean valuation $v|p$ of the field \mathbb{K} such that the linear form $\Lambda_v = \lambda_0 + \lambda_1 F_v(\alpha_1) + \dots + \lambda_m F_v(\alpha_m)$, $\lambda_i \in \mathbb{Z}_{\mathbb{K}}$, does not vanish. The second result contains a lower bound for the v -adic absolute value of Λ_v , and the first one is also extended to the case of primes in residue classes. On the way to the main results, we present explicit Padé approximations to the generalised factorial series $\sum_{n=0}^{\infty} (\prod_{k=0}^{n-1} P(k)) t^n$, where $P(x)$ is a polynomial of degree one.

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1. Introduction

Leonhard Euler laid the foundations for the study of diverging series in the eighteenth century. It is because of him that we are still today interested in the series

$$F(t) := {}_2F_0(1, 1 \mid t) = \sum_{n=0}^{\infty} n!t^n. \quad (1)$$

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(The substitution $t = -1$ yields Euler's object of interest, the series he called *divergent series par excellence*. On how he managed to associate the value $0.5963\dots$ to it, see [14].) In the v -adic metric (where v extends p for some prime p) of a number field \mathbb{K} , the series $F(t)$ converges to a point in the v -adic closure \mathbb{K}_v when $t \in \mathbb{K}$ is such that $|t|_v < p^{\frac{1}{p-1}}$. Thus we write $\sum_{n=0}^{\infty} n!t^n =: F_v(t)$ when treating the series as a function in the v -adic domain \mathbb{K} .

Euler's series (1) is a member of the class of F -series (series of the form $\sum_{n=0}^{\infty} a_n n!z^n$, with certain conditions on the coefficients a_n) introduced by V. G. Chirskiĭ in [3,4]. In those papers Chirskiĭ answered the problem of the existence of global relations¹ between members of the class of F -functions. As he points out in [5], the results can be refined in terms of estimating the prime p for which there exists a valuation $v|p$ breaking the global relation. These estimates were made entirely effective by Bertrand, Chirskiĭ, and Yebbou in [2]. In [2, Theorem 1.1] they describe an infinite collection of intervals each containing a prime number p such that for some valuation $v|p$ it holds

$$h_1 f_1(\xi) + \dots + h_m f_m(\xi) \neq 0, \quad (2)$$

where $h_i \in \mathbb{Z}_{\mathbb{K}}$ and $f_1(t) \equiv 1, f_2(t), \dots, f_m(t)$ are F -series that are linearly independent over $\mathbb{K}(z)$ and constitute a solution to a differential system D , and $\xi \in \mathbb{K} \setminus \{0\}$ is an ordinary point of the system D . What is more, the non-vanishing in (2) is replaced by a lower bound for the expression $|h_1 f_1(\xi) + \dots + h_m f_m(\xi)|_v$.

In their recent paper [11], T. Matala-aho and W. Zudilin studied the irrationality of the p -adic value of Euler's series, $F_p(\xi)$, at a point $\xi \in \mathbb{Z} \setminus \{0\}$ (i.e., global relations of the numbers 1 and $F_p(\xi)$). In Theorem 3.1 of this paper, we generalise their idea to a non-trivial linear form

$$\Lambda_v := \lambda_0 + \lambda_1 F_v(\alpha_1) + \dots + \lambda_m F_v(\alpha_m), \quad \lambda_i \in \mathbb{Z}_{\mathbb{K}},$$

in the values of Euler's series at m given pairwise distinct algebraic integer points $\alpha_1, \dots, \alpha_m \in \mathbb{Z}_{\mathbb{K}} \setminus \{0\}$. Theorem 3.1 states that in any collection V of non-Archimedean valuations of \mathbb{K} satisfying a certain condition, there exists a valuation $v' \in V$ such that $\Lambda_{v'} \neq 0$. The result can also be extended to the case of primes in arithmetic progressions, generalising the recent result of Ernvall-Hytönen et al. [8]. This is done in Theorem 10.2 in the last section.

In the second main result, Theorem 3.4, we characterise an interval $I(m, H)$ (where H is an upper bound for the height of the coefficients λ_i) from which one can find a prime p such that there exists a valuation $v'|p$ for which

$$\|\Lambda_{v'}\|_{v'} > H^{-(m+1) - 114m^2 \cdot \frac{\log \log \log H}{\log \log H}}.$$

¹ Let $P \in \mathbb{K}[x_1, \dots, x_m]$ be a polynomial of m variables and suppose $F_1(t), \dots, F_m(t) \in \mathbb{K}[[t]]$ are power series. Take a $\xi \in \mathbb{K}$. A relation $P(F_1(\xi), \dots, F_m(\xi)) = 0$ is called *global* if it holds in all the fields \mathbb{K}_v where all the series $F_1(\xi), \dots, F_m(\xi)$ converge.

Our method is grounded on explicit Padé approximations, whereas Bertrand, Chirskii, and Yebbou [2] rely on Siegel's lemma. In addition, the functional dependence on H in the error term $114m^2 \cdot \frac{\log \log \log H}{\log \log H}$ of our lower bound is improved compared to Bertrand et al. [2]: theirs is of the form $\frac{c(m)}{\sqrt{\log \log H}}$, where $c(m)$ is a constant depending on m . See Remark 8.5 for further discussion.

The proofs of both our main results are based on Padé approximations which are used to construct small approximation forms for the values $F_v(\alpha_j)$, $j = 1, \dots, m$. Therefore, before moving to the proofs of the theorems, we shall present explicit Padé approximations (with the orders of the remainders as free parameters) to the generalised factorial series

$$G(t) = \sum_{n=0}^{\infty} [P]_n t^n, \quad (3)$$

where $P(x)$ is a polynomial of degree one and $[P]_n := \prod_{k=0}^{n-1} P(k)$ (see Theorem 4.2).

A brief outline of the proofs is presented right after the formulation of the main results in Section 3. Section 9 contains some examples of the use of the main results. We shall study the sum

$$\sum_{n=0}^{\infty} n! f_n,$$

where $(f_n)_{n=0}^{\infty}$ is the sequence of Fibonacci numbers, and show that for any rational number $\frac{a}{b} \in \mathbb{Q}$ there exists a valuation of the field $\mathbb{Q}(\sqrt{5})$ such that

$$\sum_{n=0}^{\infty} n! f_n \neq \frac{a}{b}.$$

2. Preliminaries: number fields and valuations

Let $\mathbb{K} = \mathbb{Q}(\gamma)$ be an algebraic number field of degree κ , and let $\mathbb{Z}_{\mathbb{K}}$ be its ring of integers (the algebraic integers contained in \mathbb{K}). All the absolute values of \mathbb{K} are extensions of the absolute values of \mathbb{Q} . When $p \in \mathbb{P} \cup \{\infty\}$, where \mathbb{P} is the set of prime numbers, there are as many distinct extensions of $|\cdot|_p$ to \mathbb{K} as there are irreducible factors of the minimal polynomial of γ in $\mathbb{Q}_p[x]$ (see [1, Chapter V]). Here \mathbb{Q}_p denotes the completion of \mathbb{Q} with respect to the metric $|\cdot|_p$, so that $\mathbb{Q}_{\infty} = \mathbb{R}$. When $p \in \mathbb{P}$, we have $|p|_p = \frac{1}{p}$.

If $|\cdot|_v$ extends the standard p -adic metric $|\cdot|_p$ to \mathbb{K} , it is customary to write $v|_p$, and similarly, when extending the Archimedean absolute value $|\cdot| = |\cdot|_{\infty}$, we write $v|_{\infty}$. The collection of non-Archimedean valuations of \mathbb{K} is denoted by V_0 , and the collection of Archimedean valuations of \mathbb{K} by V_{∞} .

The completion of \mathbb{K} with respect to the metric $|\cdot|_v$ is denoted by \mathbb{K}_v . We also denote $\kappa_v = [\mathbb{K}_v : \mathbb{Q}_p]$ (local degree), so that $\sum_v \kappa_v = \kappa = [\mathbb{K} : \mathbb{Q}]$.

2.1. Normalisation and product formula

It is convenient to use the normalisation

$$\|\cdot\|_v = |\cdot|_v^{\frac{\kappa_v}{\kappa}}.$$

Since $\frac{\kappa_v}{\kappa} \leq 1$, the triangle inequality is valid also for the normalised Archimedean absolute value.

The following product formula holds for any $x \in \mathbb{K} \setminus \{0\}$:

$$\prod_v \|x\|_v = 1, \quad (4)$$

where the product is taken over all normalised, pairwise non-equivalent valuations of \mathbb{K} . Note that if $x \in \mathbb{Q}$, then

$$\prod_{v|p} \|x\|_v = |x|_p \quad (5)$$

for any $p \in \mathbb{P} \cup \{\infty\}$.

For more details on valuations, the reader should consult Bachman [1], for example.

3. Results

Let \mathbb{K} be any number field and let $F_v(t)$ denote the value of Euler's series (1) at a point t in the v -adic domain \mathbb{K} . Let $m \in \mathbb{Z}_{\geq 1}$ and choose m pairwise distinct, non-zero algebraic integers $\alpha_1, \dots, \alpha_m \in \mathbb{Z}_{\mathbb{K}} \setminus \{0\}$. Denote $\bar{\alpha} = (\alpha_1, \dots, \alpha_m)^T$. We define

$$c_1 = c_1(\bar{\alpha}) = \prod_{v \in V_{\infty}} \left(\left(\max_{1 \leq j \leq m} \{1, \|\alpha_j\|_v\} \right)^m \prod_{i=1}^m \left(\|\alpha_i\|_v + \max_{1 \leq j \leq m} \{1, \|\alpha_j\|_v\} \right) \right)$$

and

$$c_2 = c_2(\bar{\alpha}, V) = c_1 \prod_{v \in V} \max_{1 \leq j \leq m} \{\|\alpha_j\|_v\}$$

for any $V \subseteq V_0$

Theorem 3.1. *Let $\lambda_0, \lambda_1, \dots, \lambda_m \in \mathbb{Z}_{\mathbb{K}}$ be such that $\lambda_j \neq 0$ for at least one j . Suppose $V \subseteq V_0$ is a collection of non-Archimedean valuations of \mathbb{K} such that*

$$\limsup_{l \rightarrow \infty} c_2^l(m! + m)^{\kappa}(m! + m)! \prod_{v \in V} \|(m!)!!\|_v = 0. \quad (6)$$

Then there exists a valuation $v' \in V$ for which

$$\lambda_0 + \lambda_1 F_{v'}(\alpha_1) + \dots + \lambda_m F_{v'}(\alpha_m) \neq 0.$$

Remark 3.2. Let us show that any collection $V \subseteq V_0$ whose complement in V_0 is finite satisfies condition (6). Choose $v_1, \dots, v_k \in V_0$ and let $V = V_0 \setminus \{v_1, \dots, v_k\}$. Suppose in addition that $v_i | p_i$ for some $p_i \in \mathbb{P}$, $i = 1, \dots, k$. Then, by recalling that

$$|n!|_p \geq p^{-\frac{n}{p-1}}, \quad (7)$$

and using the product formula (4), we get

$$\begin{aligned} & c_2^l (ml + m)^\kappa (ml + m)! \prod_{v \in V} \|(ml)!l!\|_v \\ &= \frac{c_2^l (ml + m)^\kappa (ml + m)!}{\left(\prod_{i=1}^k \|(ml)!l!\|_{v_i} \right) \prod_{v \in V_\infty} \|(ml)!l!\|_v} \\ &= \frac{c_2^l (ml + m)^\kappa (ml + m)!}{\left(\prod_{i=1}^k |(ml)!l!|_{p_i}^{\frac{\kappa v_i}{p_i}} \right) (ml)!l!} \\ &\leq \frac{c_2^l \left(\prod_{i=1}^k p_i^{\frac{\kappa v_i}{p_i} \cdot \frac{ml+l}{p_i-1}} \right) (ml + m)^\kappa (ml + m)!}{(ml)!l!} \\ &= \frac{\left(c_2 \prod_{i=1}^k p_i^{\frac{\kappa v_i}{p_i} \cdot \frac{m+1}{p_i-1}} \right)^l (ml + m)^\kappa (ml + 1) \cdots (ml + m)}{l!} \\ &\xrightarrow{l \rightarrow \infty} 0. \end{aligned}$$

Remark 3.3. From the previous remark it follows that there are infinitely many valuations $v \in V_0$ such that $\Lambda_v \neq 0$.

Theorem 3.4. Let $\log H \geq se^s$ with $s = \max \{e^\kappa + 1, c_1 + 1, (m + 3)^2 + 1\}$, $\kappa = [\mathbb{K} : \mathbb{Q}]$. Suppose that $\lambda_0, \lambda_1, \dots, \lambda_m \in \mathbb{Z}_{\mathbb{K}}$ are such that at least one of them is non-zero and

$$\prod_{v \in V_\infty} \max_{0 \leq i \leq m} \{|\lambda_i|_v\} \leq H.$$

Then there exists a prime

$$p \in \left] \log \left(\frac{\log H}{\log \log H} \right), \frac{17m \log H}{\log \log H} \right[$$

and a valuation $v' | p$ for which

$$\|\lambda_0 + \lambda_1 F_{v'}(\alpha_1) + \dots + \lambda_m F_{v'}(\alpha_m)\|_{v'} > H^{-(m+1)-114m^2 \cdot \frac{\log \log \log H}{\log \log H}}. \quad (8)$$

Remark 3.5. Letting H have values in a very rapidly increasing sequence, something like $H_{i+1} = e^{e^{H_i}}$, the intervals $I(m, H_i)$ will be distinct.

3.1. Outline of the proofs

The idea behind the following proofs is to use Padé approximations to construct small linear forms

$$s_{l,\mu,j} = b_{l,\mu,0}F_v(\alpha_j) - b_{l,\mu,j}, \quad b_{l,\mu,0}, b_{l,\mu,j} \in \mathbb{Z}_{\mathbb{K}}, \quad j = 1, \dots, m,$$

in the numbers $F(\alpha_j)$. (Here $l \in \mathbb{Z}_{\geq 1}$ and $\mu \in \{0, 1, \dots, m\}$ are auxiliary parameters.) With these equations the linear form

$$\Lambda_v = \lambda_0 + \lambda_1 F_v(\alpha_1) + \dots + \lambda_m F_v(\alpha_m)$$

under study can be written as

$$b_{l,\mu,0}\Lambda_v = W + \lambda_1 s_{l,\mu,1} + \dots + \lambda_m s_{l,\mu,m}, \quad (9)$$

where $W = W(l, \mu) = \sum_{i=0}^m \lambda_i b_{l,\mu,i}$ is an integer element in \mathbb{K} . In case it is non-zero, the product formula implies

$$1 = \prod_v \|W\|_v. \quad (10)$$

In the proof of Theorem 3.1, we shall assume that $\Lambda_v = 0$ for all $v \in V$ (whence equation (9) gives W another representation as a linear combination of $s_{l,\mu,i}$), and then aim at a contradiction by estimating the product $\prod_v \|W\|_v$ from above. For this we need estimates for the Padé coefficients $b_{l,\mu,i}$, $s_{l,\mu,i}$, expressed in terms of the auxiliary parameter l . These are very roughly

$$\begin{aligned} \|b_{l,\mu,i}\|_v &\approx \|(ml)!\|_v, \quad i = 0, 1, \dots, m, \quad v|\infty, \\ \|s_{l,\mu,j}\|_v &\approx \|(ml)!!\|_v, \quad j = 1, \dots, m, \quad v \in V_0. \end{aligned}$$

The contradiction with (10) is reached via the condition (6) when l is taken to infinity.

When the target is a precise lower bound for $\|\Lambda_v\|_v$, the use of the parameter l also becomes more subtle: We define the number ℓ so that it is the largest l for which the expression

$$N(l) \sim \log H + ml \log \log l - l \log l$$

is still positive. Then we make the assumption that

$$\|b_{\ell+1,\mu,0}\Lambda_v\|_v < \|\lambda_1 s_{\ell+1,\mu,1} + \dots + \lambda_m s_{\ell+1,\mu,m}\|_v$$

for all $v|p$, $p \in [\log(\ell + 1), m(\ell + 2)] \cap \mathbb{P}$. This leads to the estimate

$$0 \leq \log \left(\prod_v \|W(\ell + 1, \mu)\|_v \right) \approx \log H + m(\ell + 1) \log \log(\ell + 1) - (\ell + 1) \log(\ell + 1) < 0,$$

giving the desired contradiction. It follows that there exists a prime

$$p \in [\log(\ell + 1), m(\ell + 2)] \quad (11)$$

and a valuation $v'|p$ such that

$$\|W(\ell + 1, \mu)\|_{v'} \leq \|\Lambda_{v'}\|_{v'},$$

leading to

$$1 \leq \left(\prod_{v \in V_\infty} \|W(\ell + 1, \mu)\|_v \right) \|\Lambda_{v'}\|_{v'}.$$

This is the key to the lower bound for $\|\Lambda_{v'}\|_{v'}$, and the final step is to give an estimate for the product $\prod_{v \in V_\infty} \|W(\ell + 1, \mu)\|_v$. Approximately it is

$$\log \left(\prod_{v \in V_\infty} \|W(\ell + 1, \mu)\|_v \right) \approx (m + 1) \log H + m^2 \ell \log \log \ell. \quad (12)$$

The definition of ℓ gives a connection between ℓ and H , enabling us to write the bound (12) and the interval (11) solely in terms of H :

$$\ell \log \log \ell \approx \frac{\log \log \log H}{\log \log H} \cdot \log H.$$

As the attentive reader may have noted, one crucial point in the proofs is the non-vanishing of the quantity $W(l, \mu)$. This is the part where the auxiliary parameter μ is needed. A non-vanishing determinant of the Padé polynomials will ensure that for each $l \in \mathbb{Z}_{\geq 1}$, there exists a $\mu \in \{0, 1, \dots, m\}$ such that $W(l, \mu) \neq 0$.

4. Padé approximations

Let $m \in \mathbb{Z}_{\geq 1}$, $\bar{l} = (l_1, \dots, l_m)^T \in \mathbb{Z}_{\geq 1}^m$, and $L := \sum_{j=1}^m l_j$. For a given vector $\bar{\beta} = (\beta_1, \dots, \beta_m)^T$, define the numbers $\sigma_i = \sigma_i(\bar{l}, \bar{\beta})$ by the equation

$$\prod_{j=1}^m (\beta_j - w)^{l_j} = \sum_{i=0}^L \sigma_i w^i. \quad (13)$$

Then, by the binomial theorem,

$$\sigma_i(\bar{l}, \bar{\beta}) = (-1)^i \sum_{i_1 + \dots + i_m = i} \binom{l_1}{i_1} \dots \binom{l_m}{i_m} \cdot \beta_1^{l_1 - i_1} \dots \beta_m^{l_m - i_m}.$$

Lemma 4.1. *We have*

$$\sum_{i=0}^L \sigma_i i^k \beta_j^i = 0 \quad (14)$$

for all $j \in \{1, \dots, m\}$ and $k \in \{0, 1, \dots, l_j - 1\}$. Moreover, when $\bar{\beta} = (\beta_1, \dots, \beta_m)^T \in \mathbb{K}^m$ and $\|\cdot\|_v$ is any Archimedean absolute value of the field \mathbb{K} , it holds

$$\sum_{i=0}^L \|\sigma_i\|_v t^i \leq \prod_{j=1}^m (\|\beta_j\|_v + t)^{l_j} \quad (15)$$

for all $t \geq 0$.

Proof. It is not too hard to deduce that

$$\left(x \frac{d}{dx}\right)^n f(x) = \sum_{i=1}^n a_{n,i} x^i \left(\frac{d}{dx}\right)^i f(x), \quad (16)$$

where the coefficients $a_{n,i}$ satisfy the recursions

$$\begin{cases} a_{n,1} = 1; \\ a_{n,i} = a_{n-1,i-1} + i a_{n-1,i}, & i = 2, \dots, n-1; \\ a_{n,n} = 1 \end{cases}$$

for all $n \in \mathbb{Z}_{\geq 1}$. (Actually, these are the Stirling numbers of the second kind.) Let now $j \in \{1, \dots, m\}$. For $k = 0$, the claim (14) follows directly from the definition (13). For $k \in \{1, \dots, l_j - 1\}$, we use (13) and (16):

$$\sum_{i=0}^L \sigma_i i^k \beta_j^i = \left(w \frac{d}{dw}\right)^k \prod_{i=1}^m (\beta_i - w)^{l_i} \Big|_{w=\beta_j} = 0$$

because $\left(\frac{d}{dw}\right)^h \prod_{i=1}^m (\beta_i - w)^{l_i} \Big|_{w=\beta_j} = 0$ for all $h \in \{0, 1, \dots, k\}$.

Property (15) follows simply from the expansion of σ_i and the triangle inequality:

$$\sum_{i=0}^L \|\sigma_i\|_v t^i \leq \sum_{i=0}^L \left(\sum_{i_1 + \dots + i_m = i} \left\| \binom{l_1}{i_1} \dots \binom{l_m}{i_m} \right\|_v \cdot \|\beta_1\|_v^{l_1 - i_1} \dots \|\beta_m\|_v^{l_m - i_m} \right) t^i$$

$$\begin{aligned} &\leq \sum_{i=0}^L \left(\sum_{i_1+\dots+i_m=i} \binom{l_1}{i_1} \cdots \binom{l_m}{i_m} \cdot \|\beta_1\|_v^{l_1-i_1} \cdots \|\beta_m\|_v^{l_m-i_m} \right) t^i \\ &= \prod_{j=1}^m (\|\beta_j\|_v + t)^{l_j} \end{aligned}$$

when $t \geq 0$. \square

4.1. Generalised factorial series

When $l_1 = l_2 = \dots = l_m$, the following theorem is a particular case of Theorem 2.2 in [10]. Due to the special nature of the function (3), however, we don't need to restrict the parameters l_j .

Theorem 4.2. Let $\mu \in \mathbb{Z}_{\geq 0}$. Let $G(t) = \sum_{n=0}^{\infty} [P]_n t^n$, where $P(x)$ is a polynomial of degree one and $P(k) \neq 0$ for $k = 0, 1, \dots, L + \mu - 1$. Define $[P]_n := \prod_{k=0}^{n-1} P(k)$, $n \in \mathbb{Z}_{\geq 1}$, and $[P]_0 := 1$. Set

$$A_{\bar{l}, \mu, 0}(t) = \sum_{i=0}^L \frac{\sigma_i(\bar{l}, \bar{\beta})}{[P]_{i+\mu}} t^{L-i}.$$

Then there exist polynomials $A_{\bar{l}, \mu, j}(t)$ and remainders $R_{\bar{l}, \mu, j}(t)$, $j = 1, \dots, m$, such that

$$A_{\bar{l}, \mu, 0}(t)G(\beta_j t) - A_{\bar{l}, \mu, j}(t) = R_{\bar{l}, \mu, j}(t), \quad (17)$$

where

$$\begin{cases} \deg A_{\bar{l}, \mu, 0}(t) = L, \\ \deg A_{\bar{l}, \mu, j}(t) \leq L + \mu - 1, \\ \text{ord}_{t=0} R_{\bar{l}, \mu, j}(t) \geq L + \mu + l_j. \end{cases} \quad (18)$$

Proof. Writing

$$A_{\bar{l}, \mu, 0}(t) = \sum_{h=0}^L \frac{\sigma_{L-h}(\bar{l}, \bar{\beta})}{[P]_{L-h+\mu}} t^h,$$

we have

$$A_{\bar{l}, \mu, 0}(t)G(\beta_j t) = \sum_{N=0}^{\infty} r_{N,j} t^N,$$

where

$$r_{N,j} = \sum_{n+h=N} \sigma_{L-h}(\bar{l}, \bar{\beta}) \cdot \frac{[P]_n}{[P]_{L-h+\mu}} \cdot \beta_j^n = \sum_{h=0}^{\min\{L,N\}} \sigma_{L-h}(\bar{l}, \bar{\beta}) \cdot \frac{[P]_{N-h}}{[P]_{L-h+\mu}} \cdot \beta_j^{N-h}. \quad (19)$$

When $N = L + \mu + a$, $0 \leq a \leq l_j - 1$, then

$$\begin{aligned} r_{N,j} &= \beta_j^{\mu+a} \sum_{h=0}^L \sigma_{L-h}(\bar{l}, \bar{\beta}) \left(\prod_{k=1}^a P(L + \mu - h - 1 + k) \right) \beta_j^{L-h} \\ &= \beta_j^{\mu+a} \sum_{i=0}^L \sigma_i(\bar{l}, \bar{\beta}) \left(\prod_{k=1}^a P(i + \mu - 1 + k) \right) \beta_j^i. \end{aligned}$$

(Note that the product above equals 1 when $a = 0$.) Since $\deg P(x) = 1$, we may write

$$\prod_{k=1}^a P(i + \mu - 1 + k) = \sum_{k=0}^a p_k i^k,$$

where the coefficients p_k do not depend on i . Hence

$$r_{N,j} = \beta_j^{\mu+a} \sum_{i=0}^L \sigma_i(\bar{l}, \bar{\beta}) \left(\sum_{k=0}^a p_k i^k \right) \beta_j^i = \beta_j^{\mu+a} \sum_{k=0}^a p_k \sum_{i=0}^L \sigma_i(\bar{l}, \bar{\beta}) i^k \beta_j^i = 0$$

due to (14). Thus we can choose

$$A_{\bar{l}, \mu, j}(t) = \sum_{N=0}^{L+\mu-1} r_{N,j} t^N \quad (20)$$

and

$$R_{\bar{l}, \mu, j}(t) = \sum_{N=L+\mu+l_j}^{\infty} r_{N,j} t^N. \quad \square \quad (21)$$

4.2. Euler's series

To prove Theorem 3.1, we need approximations to the series $F(\alpha_j t)$. Hence we choose $P(x) = 1 + x$ and $\bar{\beta} = \bar{\alpha} = (\alpha_1, \dots, \alpha_m)^T$, and set $l_j = l \in \mathbb{Z}_{\geq 1}$ for all $j \in \{1, \dots, m\}$. Theorem 4.2 gives

$$A_{\bar{l}, \mu, 0}(t) = \sum_{i=0}^{ml} \frac{\sigma_i}{(i + \mu)!} t^{ml-i}, \quad \sigma_i = \sigma_i(\bar{l}, \bar{\alpha}),$$

and, directly by (20) and (19),

$$A_{\bar{l},\mu,j}(t) = \sum_{N=0}^{ml+\mu-1} t^N \sum_{h=0}^{\min\{ml,N\}} \sigma_{ml-h} \cdot \frac{(N-h)!}{(ml-h+\mu)!} \cdot \alpha_j^{N-h}$$

when $j = 1, \dots, m$. Similarly by (21) and (19), for $N = (m+1)l + \mu + k$, $k \in \mathbb{Z}_{\geq 0}$, we have

$$\begin{aligned} r_{N,j} &= \sum_{h=0}^{ml} \sigma_{ml-h} \cdot \frac{((m+1)l + \mu + k - h)!}{(ml - h + \mu)!} \cdot \alpha_j^{(m+1)l + \mu + k - h} \\ &= \alpha_j^{l+\mu+k} \sum_{i=0}^{ml} \sigma_i \cdot \frac{(i + \mu + l + k)!}{(i + \mu)!} \cdot \alpha_j^i \\ &= l!k! \binom{l+k}{k} \alpha_j^{l+\mu+k} \sum_{i=0}^{ml} \sigma_i \binom{i + \mu + l + k}{i + \mu} \alpha_j^i, \end{aligned}$$

so that

$$R_{\bar{l},\mu,j}(t) = l!t^{(m+1)l+\mu} \sum_{k=0}^{\infty} t^k k! \binom{l+k}{k} \alpha_j^{l+\mu+k} \sum_{i=0}^{ml} \sigma_i \binom{i + \mu + l + k}{i + \mu} \alpha_j^i$$

when $j = 1, \dots, m$.

To make the polynomials belong to $\mathbb{Z}_{\mathbb{K}}[t]$, we multiply everything by $(ml + \mu)!$ and denote

$$B_{l,\mu,0}(t) := (ml + \mu)! A_{\bar{l},\mu,0}(t) = \sum_{i=0}^{ml} \sigma_i \cdot \frac{(ml + \mu)!}{(i + \mu)!} \cdot t^{ml-i},$$

$$\begin{aligned} B_{l,\mu,j}(t) &:= (ml + \mu)! A_{\bar{l},\mu,j}(t) \\ &= (ml + \mu)! \sum_{N=0}^{ml+\mu-1} t^N \sum_{h=0}^{\min\{ml,N\}} \sigma_{ml-h} \cdot \frac{(N-h)!}{(ml-h+\mu)!} \cdot \alpha_j^{N-h}, \end{aligned}$$

$$\begin{aligned} S_{l,\mu,j}(t) &:= (ml + \mu)! R_{\bar{l},\mu,j}(t) \\ &= (ml + \mu)! l! t^{(m+1)l+\mu} \sum_{k=0}^{\infty} k! \binom{l+k}{k} \alpha_j^{l+\mu+k} t^k \sum_{i=0}^{ml} \sigma_i \binom{i + \mu + l + k}{i + \mu} \alpha_j^i. \end{aligned} \quad (22)$$

In this notation, the Padé approximation formula in (17) may be rewritten as

$$B_{l,\mu,0}(t)F(\alpha_j t) - B_{l,\mu,j}(t) = S_{l,\mu,j}(t), \quad j = 1, \dots, m. \quad (23)$$

5. Linear form and product formula

Let $\lambda_0, \lambda_1, \dots, \lambda_m \in \mathbb{Z}_{\mathbb{K}}$ be such that at least one of them is non-zero, and denote

$$\Lambda_v := \lambda_0 + \lambda_1 F_v(\alpha_1) + \dots + \lambda_m F_v(\alpha_m)$$

when $v \in V_0$. Equation (23) gives

$$s_{l,\mu,i} = b_{l,\mu,0} F_v(\alpha_i) - b_{l,\mu,i},$$

where

$$b_{l,\mu,i} = B_{l,\mu,i}(1), \quad i = 0, 1, \dots, m; \quad s_{l,\mu,i} = S_{l,\mu,i}(1), \quad i = 1, \dots, m.$$

Assume that $\Lambda_v = 0$ for all $v \in V$, where the collection V satisfies condition (6). Then also

$$0 = b_{l,\mu,0} \Lambda_v = W + \lambda_1 s_{l,\mu,1} + \dots + \lambda_m s_{l,\mu,m},$$

where

$$W = W(l, \mu) := \lambda_0 b_{l,\mu,0} + \lambda_1 b_{l,\mu,1} + \dots + \lambda_m b_{l,\mu,m} \in \mathbb{Z}_{\mathbb{K}}.$$

If $W \neq 0$, then

$$\begin{aligned} 1 &= \prod_v \|W\|_v \\ &\leq \left(\prod_{v \in V_\infty} \|W\|_v \right) \prod_{v \in V} \|W\|_v \\ &\leq \left(\prod_{v \in V_\infty} \|\lambda_0 b_{l,\mu,0} + \lambda_1 b_{l,\mu,1} + \dots + \lambda_m b_{l,\mu,m}\|_v \right) \\ &\quad \cdot \prod_{v \in V} \|\lambda_1 s_{l,\mu,1} + \dots + \lambda_m s_{l,\mu,m}\|_v \\ &\leq \left(\prod_{v \in V_\infty} \left(\sum_{i=0}^m \|\lambda_i\|_v \right) \max_{0 \leq i \leq m} \{\|b_{l,\mu,i}\|_v\} \right) \prod_{v \in V} \max_{1 \leq i \leq m} \{\|s_{l,\mu,i}\|_v\}. \end{aligned} \tag{24}$$

Next we shall see that such a non-zero $W(l, \mu)$ actually exists.

6. Determinant

Lemma 6.1. *When the numbers α_j , $j = 1, \dots, m$, are pairwise different and non-zero, we have*

$$\Delta(t) := \begin{vmatrix} B_{l,0,0}(t) & B_{l,0,1}(t) & \cdots & B_{l,0,m}(t) \\ B_{l,1,0}(t) & B_{l,1,1}(t) & \cdots & B_{l,1,m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ B_{l,m,0}(t) & B_{l,m,1}(t) & \cdots & B_{l,m,m}(t) \end{vmatrix} \neq 0.$$

Proof. By (18), the degrees of the entries are at most

$$\begin{pmatrix} ml & ml-1 & \cdots & ml-1 \\ ml & ml & \cdots & ml \\ \vdots & \vdots & \ddots & \vdots \\ ml & ml+m-1 & \cdots & ml+m-1 \end{pmatrix}.$$

Hence

$$\deg \Delta(t) \leq (m+1)ml + \frac{(m-1)m}{2}.$$

Column operations together with (23) yield the representation

$$\Delta(t) = \begin{vmatrix} B_{l,0,0}(t) & -S_{l,0,1}(t) & \cdots & -S_{l,0,m}(t) \\ B_{l,1,0}(t) & -S_{l,1,1}(t) & \cdots & -S_{l,1,m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ B_{l,m,0}(t) & -S_{l,m,1}(t) & \cdots & -S_{l,m,m}(t) \end{vmatrix}. \quad (25)$$

According to (18), the orders of the entries in (25) are at least

$$\begin{pmatrix} 0 & (m+1)l & \cdots & (m+1)l \\ 0 & (m+1)l+1 & \cdots & (m+1)l+1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & (m+1)l+m & \cdots & (m+1)l+m \end{pmatrix}.$$

By expanding (25) by the first column we see that

$$\text{ord } \Delta(t) \geq \sum_{i=0}^{m-1} ((m+1)l + i) = m(m+1)l + \frac{(m-1)m}{2}.$$

Thus

$$\Delta(t) = bt^{m(m+1)l + \frac{(m-1)m}{2}},$$

where the coefficient b is an $m \times m$ determinant formed from the lowest term coefficients of the remainders $-S_{l,\mu,j}$, $\mu = 0, 1, \dots, m-1$, $j = 1, \dots, m$ (corresponding to $k = 0$ in (22)), multiplied by the lowest term coefficient of the polynomial $B_{l,m,0}(t)$ which is $\sigma_{ml} = (-1)^{ml}$:

$$b = (-1)^{ml} \cdot (-1)^m (l!)^m \left(\prod_{\mu=0}^{m-1} (ml + \mu)! \right) \left(\prod_{j=1}^m \alpha_j^l \right) \cdot \begin{vmatrix} \sum_{i=0}^{ml} \sigma_i \binom{i+l}{i} \alpha_1^i & \sum_{i=0}^{ml} \sigma_i \binom{i+l}{i} \alpha_2^i & \cdots & \sum_{i=0}^{ml} \sigma_i \binom{i+l}{i} \alpha_m^i \\ \alpha_1 \sum_{i=0}^{ml} \sigma_i \binom{i+1+l}{i+1} \alpha_1^i & \alpha_2 \sum_{i=0}^{ml} \sigma_i \binom{i+1+l}{i+1} \alpha_2^i & \cdots & \alpha_m \sum_{i=0}^{ml} \sigma_i \binom{i+1+l}{i+1} \alpha_m^i \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{m-1} \sum_{i=0}^{ml} \sigma_i \binom{i+m-1+l}{i+m-1} \alpha_1^i & \alpha_2^{m-1} \sum_{i=0}^{ml} \sigma_i \binom{i+m-1+l}{i+m-1} \alpha_2^i & \cdots & \alpha_m^{m-1} \sum_{i=0}^{ml} \sigma_i \binom{i+m-1+l}{i+m-1} \alpha_m^i \end{vmatrix}.$$

It remains to show that $b \neq 0$.

Since

$$\binom{i+\mu+l}{i+\mu} = \frac{(i+\mu+l)!}{(i+\mu)!l!} = \frac{1}{l!} (i+\mu+1) \cdots (i+\mu+l) = \frac{1}{l!} \left(i^l + \sum_{k=0}^{l-1} p_k i^k \right)$$

for any $\mu \in \{0, 1, \dots, m-1\}$, where the coefficients p_k do not depend on i , we get

$$\sum_{i=0}^{ml} \sigma_i \binom{i+\mu+l}{i+\mu} \alpha_j^i = \frac{1}{l!} \left(\sum_{i=0}^{ml} \sigma_i i^l \alpha_j^i + \sum_{k=0}^{l-1} p_k \sum_{i=0}^{ml} \sigma_i i^k \alpha_j^i \right) = \frac{1}{l!} \sum_{i=0}^{ml} \sigma_i i^l \alpha_j^i$$

for all $j = 1, \dots, m$, $\mu = 0, 1, \dots, m-1$ by the property (14). Hence

$$b = (-1)^{m(l+1)} \left(\prod_{\mu=0}^{m-1} (ml + \mu)! \right) \left(\prod_{j=1}^m \alpha_j^l \right) \cdot \begin{vmatrix} \sum_{i=0}^{ml} \sigma_i i^l \alpha_1^i & \sum_{i=0}^{ml} \sigma_i i^l \alpha_2^i & \cdots & \sum_{i=0}^{ml} \sigma_i i^l \alpha_m^i \\ \alpha_1 \sum_{i=0}^{ml} \sigma_i i^l \alpha_1^i & \alpha_2 \sum_{i=0}^{ml} \sigma_i i^l \alpha_2^i & \cdots & \alpha_m \sum_{i=0}^{ml} \sigma_i i^l \alpha_m^i \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{m-1} \sum_{i=0}^{ml} \sigma_i i^l \alpha_1^i & \alpha_2^{m-1} \sum_{i=0}^{ml} \sigma_i i^l \alpha_2^i & \cdots & \alpha_m^{m-1} \sum_{i=0}^{ml} \sigma_i i^l \alpha_m^i \end{vmatrix} \\ = (-1)^{m(l+1)} \left(\prod_{\mu=0}^{m-1} (ml + \mu)! \right) \left(\prod_{j=1}^m \alpha_j^l \right) \left(\prod_{j=1}^m \left(\sum_{i=0}^{ml} \sigma_i i^l \alpha_j^i \right) \right) \cdot \prod_{1 \leq i < j \leq m} (\alpha_j - \alpha_i)$$

by the Vandermonde determinant formula. Here, using (13) and (16),

$$\sum_{i=0}^{ml} \sigma_i i^l \alpha_j^i = \left(w \frac{d}{dw} \right)^l \prod_{i=1}^m (\alpha_i - w)^l \Big|_{w=\alpha_j} = (-1)^l l! \alpha_j^l \prod_{\substack{i=1 \\ i \neq j}}^m (\alpha_i - \alpha_j)^l \neq 0$$

for all $j = 1, \dots, m$. \square

Lemma 6.2. *For any given $l \in \mathbb{Z}_{\geq 1}$, there exists a $\mu \in \{0, 1, \dots, m\}$ such that $W(l, \mu) \neq 0$.*

Proof. From Lemma 6.1 it follows in particular that

$$\begin{vmatrix} b_{l,0,0} & b_{l,0,1} & \cdots & b_{l,0,m} \\ b_{l,1,0} & b_{l,1,1} & \cdots & b_{l,1,m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{l,m,0} & b_{l,m,1} & \cdots & b_{l,m,m} \end{vmatrix} = \Delta(1) \neq 0.$$

We assumed that $(\lambda_0, \lambda_1, \dots, \lambda_m)^T \neq \bar{0}$, so by linear algebra it follows that the quantity $W(l, \mu) = \lambda_0 b_{l,\mu,0} + \lambda_1 b_{l,\mu,1} + \dots + \lambda_m b_{l,\mu,m}$ must be non-zero for some $\mu \in \{0, 1, \dots, m\}$. \square

7. Estimates for the polynomials and remainders and proof of Theorem 3.1

As the last step in proving Theorem 3.1, we give upper bounds for the Padé polynomials and remainders. Now, using the triangle inequality and property (15) with $v|\infty$,

$$\begin{aligned} \|b_{l,\mu,0}\|_v &= \|B_{l,\mu,0}(1)\|_v \\ &= \left\| \sum_{i=0}^{ml} \sigma_i \frac{(ml+\mu)!}{(i+\mu)!} \right\|_v \\ &\leq \left\| (ml)! \binom{ml+\mu}{\mu} \right\|_v \sum_{i=0}^{ml} \|\sigma_i\|_v \\ &\leq \left\| (ml)! \binom{ml+\mu}{\mu} \right\|_v \prod_{j=1}^m (\|\alpha_j\|_v + 1)^t \end{aligned}$$

and

$$\begin{aligned} \|b_{l,\mu,j}\|_v &= \|B_{l,\mu,j}(1)\|_v \\ &= \left\| (ml+\mu)! \sum_{N=0}^{ml+\mu-1} \sum_{h=0}^{\min\{ml,N\}} \frac{(N-h)!}{(ml-h+\mu)!} \sigma_{ml-h} \alpha_j^{N-h} \right\|_v \\ &\leq \|(ml+\mu)!\|_v \sum_{N=0}^{ml+\mu-1} \sum_{h=0}^{\min\{ml,N\}} \left\| \frac{(N-h)!}{(ml-h+\mu)!} \right\|_v \|\sigma_{ml-h}\|_v \|\alpha_j\|_v^{N-h} \\ &\leq \|(ml+\mu)!\|_v \sum_{N=0}^{ml+\mu-1} \sum_{h=0}^{\min\{ml,N\}} \|\sigma_{ml-h}\|_v (\max\{1, \|\alpha_j\|_v\})^{N-h} \end{aligned}$$

$$\begin{aligned}
&\leq \|(ml + \mu)!\|_v \sum_{N=0}^{ml+\mu-1} \sum_{h=0}^{\min\{ml, N\}} \|\sigma_{ml-h}\|_v (\max\{1, \|\alpha_j\|_v\})^{ml+m-1-h} \\
&\leq \|(ml + \mu)!\|_v (\max\{1, \|\alpha_j\|_v\})^{m-1} (ml + m) \\
&\quad \cdot \sum_{h=0}^{\min\{ml, N\}} \|\sigma_{ml-h}\|_v (\max\{1, \|\alpha_j\|_v\})^{ml-h} \\
&\leq \|(ml + \mu)!\|_v (\max\{1, \|\alpha_j\|_v\})^{ml} (ml + m) \cdot \prod_{i=1}^m (\|\alpha_i\|_v + \max\{1, \|\alpha_j\|_v\})^l
\end{aligned}$$

for all $j = 1, \dots, m$, $\mu = 0, 1, \dots, m$.

We still need non-Archimedean estimates for the remainders, so let now $v \in V_0$. Then

$$\begin{aligned}
\|s_{l,\mu,j}\|_v &= \|S_{l,\mu,j}(1)\|_v \\
&= \left\| (ml + \mu)! \sum_{k=0}^{\infty} k! \binom{l+k}{k} \alpha_j^{l+k+\mu} \sum_{i=0}^m \sigma_i \binom{i+\mu+l+k}{i+\mu} \alpha_j^i \right\|_v \\
&\leq \|(ml + \mu)!\|_v \|\alpha_j\|_v^l,
\end{aligned}$$

for all $j = 1, \dots, m$, $\mu = 0, 1, \dots, m$.

So, recalling property (5) of our normalised valuations, the expression in (24) becomes

$$\begin{aligned}
&\left(\prod_{v \in V_\infty} \left(\sum_{i=0}^m \|\lambda_i\|_v \right) \max_{0 \leq i \leq m} \{\|b_{l,\mu,i}\|_v\} \right) \prod_{v \in V} \max_{1 \leq i \leq m} \{\|s_{l,\mu,i}\|_v\} \\
&\leq \left(\prod_{v \in V_\infty} \left(\sum_{i=0}^m \|\lambda_i\|_v \right) (ml + m) \|(ml + m)!\|_v \cdot \left(\max_{1 \leq j \leq m} \{1, \|\alpha_j\|_v\} \right)^{ml} \right. \\
&\quad \cdot \left. \prod_{i=1}^m \left(\|\alpha_i\|_v + \max_{1 \leq j \leq m} \{1, \|\alpha_j\|_v\} \right)^l \right) \cdot \prod_{v \in V} \|(ml)!\|_v \left(\max_{1 \leq j \leq m} \{\|\alpha_j\|_v\} \right)^l \quad (26) \\
&\leq \left(\prod_{v \in V_\infty} \left(\sum_{i=0}^m \|\lambda_i\|_v \right) \right) c_2^l (ml + m)^\kappa (ml + m)! \cdot \prod_{v \in V} \|(ml)!\|_v,
\end{aligned}$$

where

$$\begin{aligned}
c_2 &= \left(\prod_{v \in V_\infty} \left(\left(\max_{1 \leq j \leq m} \{1, \|\alpha_j\|_v\} \right)^m \prod_{i=1}^m \left(\|\alpha_i\|_v + \max_{1 \leq j \leq m} \{1, \|\alpha_j\|_v\} \right) \right) \right) \\
&\quad \cdot \prod_{v \in V} \max_{1 \leq j \leq m} \{\|\alpha_j\|_v\}.
\end{aligned}$$

Proof of Theorem 3.1. In Section 6 we saw that for every $l \in \mathbb{Z}_{\geq 1}$, there exists a $\mu \in \{0, 1, \dots, m\}$ such that $W = W(l, \mu) \neq 0$. Hence the estimate in (24) holds for infinitely

many $W(l, \mu)$, so that our assumption $\Lambda_v = 0$ for all $v \in V$ and estimates (24) and (26) lead to

$$1 \leq \left(\prod_{v \in V_\infty} \left(\sum_{i=0}^m \|\lambda_i\|_v \right) \right) c_2^l (ml + m)^\kappa (ml + m)! \prod_{v \in V} \|(ml)!!\|_v$$

which holds for infinitely many l . This is a contradiction with condition (6), and thus there must exist a valuation $v' \in V$ such that $\Lambda_{v'} \neq 0$. \square

Remark 7.1. For a fixed linear form $\lambda_0 + \lambda_1 F_v(\alpha_1) + \dots + \lambda_m F_v(\alpha_m)$, the existence of one integer l such that

$$\left(\prod_{v \in V_\infty} \left(\sum_{i=0}^m \|\lambda_i\|_v \right) \right) c_2^l (ml + m)^\kappa (ml + m)! \prod_{v \in V} \|(ml)!!\|_v < 1$$

would be enough. However, we don't want condition (6) to be dependent on the linear form.

8. Lower bound: proof of Theorem 3.4

8.1. Product formula again

The fundamental product formula (4) is the starting point for the proof of our second theorem as well. We repeat Section 5 with a slightly more refined assumption. First we need some notation though.

Let $m \in \mathbb{Z}_{\geq 1}$ and $\log H \geq se^s$, where

$$s = \max \{ e^\kappa + 1, c_1 + 1, (m + 3)^2 + 1 \}, \quad (27)$$

$$\kappa = [\mathbb{K} : \mathbb{Q}],$$

$$c_1 = \prod_{v \in V_\infty} \left(\left(\max_{1 \leq j \leq m} \{1, \|\alpha_j\|_v\} \right)^m \prod_{i=1}^m \left(\|\alpha_i\|_v + \max_{1 \leq j \leq m} \{1, \|\alpha_j\|_v\} \right) \right).$$

Suppose that $\lambda_0, \lambda_1, \dots, \lambda_m \in \mathbb{Z}_{\mathbb{K}}$ are such that at least one of them is non-zero and

$$\prod_{v \in V_\infty} \max_{0 \leq i \leq m} \{\|\lambda_i\|_v\} \leq H.$$

Define

$$N(l) := \log H + \left(2(m + 1) + \frac{2m}{l} + \frac{\log c_1}{\log \log l} + \frac{1}{\log \log l} + \frac{(\kappa - \frac{1}{2}) \log l}{l \log \log l} \right. \\ \left. + \frac{\kappa \log m}{l \log \log l} + \frac{\kappa \log(m + 1)}{l \log \log l} + \frac{\kappa}{l^2 \log \log l} \right) l \log \log l - l \log l \quad (28)$$

and let

$$\ell := \max \{l \in \mathbb{Z}_{\geq 2} \mid N(l) \geq 0\}. \quad (29)$$

Denote, as before,

$$\Lambda_v = \lambda_0 + \lambda_1 F_v(\alpha_1) + \dots + \lambda_m F_v(\alpha_m).$$

We saw in Section 5 that

$$b_{l,\mu,0}\Lambda_v = W + \lambda_1 s_{l,\mu,1} + \dots + \lambda_m s_{l,\mu,m},$$

where

$$W = W(l, \mu) = \lambda_0 b_{l,\mu,0} + \lambda_1 b_{l,\mu,1} + \dots + \lambda_m b_{l,\mu,m} \in \mathbb{Z}_{\mathbb{K}}. \quad (30)$$

By Lemma 6.2 we know that $W(\ell+1, \mu) \neq 0$ for some $\mu \in \{0, 1, \dots, m\}$. Assume that

$$\|b_{\ell+1,\mu,0}\Lambda_v\|_v < \|\lambda_1 s_{\ell+1,\mu,1} + \dots + \lambda_m s_{\ell+1,\mu,m}\|_v$$

for all $v|p$, $p \in [\log(\ell+1), m(\ell+2)] \cap \mathbb{P}$. (The intersection certainly is non-empty due to Bertrand's postulate. As for the choice of this interval, see Remark 8.5.) Then

$$\begin{aligned} \|W(\ell+1, \mu)\|_v &= \|b_{\ell+1,\mu,0}\Lambda_v - (\lambda_1 s_{\ell+1,\mu,1} + \dots + \lambda_m s_{\ell+1,\mu,m})\|_v \\ &= \|\lambda_1 s_{\ell+1,\mu,1} + \dots + \lambda_m s_{\ell+1,\mu,m}\|_v \end{aligned} \quad (31)$$

for all $v|p$, $p \in [\log(\ell+1), m(\ell+2)] \cap \mathbb{P}$.

Now we use the product formula and plug in the two representations (30) and (31):

$$\begin{aligned} 1 &= \prod_v \|W(\ell+1, \mu)\|_v \\ &\leq \left(\prod_{v \in V_\infty} \|W\|_v \right) \prod_{p \in [\log(\ell+1), m(\ell+2)]} \prod_{v|p} \|W\|_v \\ &= \left(\prod_{v \in V_\infty} \left\| \sum_{i=0}^m \lambda_i b_{\ell+1,\mu,i} \right\|_v \right) \prod_{p \in [\log(\ell+1), m(\ell+2)]} \prod_{v|p} \left\| \sum_{i=1}^m \lambda_i s_{\ell+1,\mu,i} \right\|_v. \end{aligned} \quad (32)$$

On the Archimedean side we have

$$\left\| \sum_{i=0}^m \lambda_i b_{\ell+1,\mu,i} \right\|_v \leq (m+1) \max_{0 \leq i \leq m} \{\|\lambda_i\|_v\} \max_{0 \leq i \leq m} \{\|b_{\ell+1,\mu,i}\|_v\}, \quad v \in V_\infty,$$

so, using the estimates made in Section 7 together with property (5),

$$\begin{aligned}
\prod_{v \in V_\infty} \left\| \sum_{i=0}^m \lambda_i b_{\ell+1, \mu, i} \right\|_v &\leq (m+1)^\kappa H \left(\prod_{v \in V_\infty} \left((m(\ell+1) + m) \|(m(\ell+1) + \mu)\|_v \right. \right. \\
&\quad \cdot \left. \left(\max_{1 \leq j \leq m} \{1, \|\alpha_j\|_v\} \right)^{m(\ell+1)} \right. \\
&\quad \cdot \left. \left. \prod_{i=1}^m \left(\|\alpha_i\|_v + \max_{1 \leq j \leq m} \{1, \|\alpha_j\|_v\} \right)^{\ell+1} \right) \right) \\
&\leq (m+1)^\kappa (m(\ell+1) + m)^\kappa H c_1^{\ell+1} (m(\ell+1) + \mu)!.
\end{aligned} \tag{33}$$

(Recall that $\#V_\infty \leq \kappa = [\mathbb{K} : \mathbb{Q}]$.)

As for the case $v|p$, we get

$$\left\| \sum_{i=1}^m \lambda_i s_{\ell+1, \mu, i} \right\|_v \leq \max_{1 \leq i \leq m} \{\|s_{\ell+1, \mu, i}\|_v\},$$

so, by the estimates of Section 7 and property (5),

$$\begin{aligned}
\prod_{p \in [\log(\ell+1), m(\ell+2)]} \prod_{v|p} \left\| \sum_{i=1}^m \lambda_i s_{\ell+1, \mu, i} \right\|_v &\leq \prod_{p \in [\log(\ell+1), m(\ell+2)]} \prod_{v|p} \|(m(\ell+1) + \mu)!(\ell+1)\|_v \\
&= \prod_{p \in [\log(\ell+1), m(\ell+2)]} |(m(\ell+1) + \mu)!(\ell+1)!|_p.
\end{aligned} \tag{34}$$

Let us now combine estimates (32), (33), and (34):

$$\begin{aligned}
1 &\leq \left(\prod_{v \in V_\infty} \left\| \sum_{i=0}^m \lambda_i b_{\ell+1, \mu, i} \right\|_v \right) \prod_{p \in [\log(\ell+1), m(\ell+2)]} \prod_{v|p} \left\| \sum_{i=1}^m \lambda_i s_{\ell+1, \mu, i} \right\|_v \\
&\leq (m+1)^\kappa (m(\ell+1) + m)^\kappa H c_1^{\ell+1} (m(\ell+1) + \mu)! \\
&\quad \cdot \prod_{p \in [\log(\ell+1), m(\ell+2)]} |(m(\ell+1) + \mu)!(\ell+1)!|_p \\
&= \frac{(m+1)^\kappa (m(\ell+2))^\kappa H c_1^{\ell+1} \prod_{p \in [\log(\ell+1), m(\ell+2)]} |(\ell+1)!|_p}{\prod_{p < \log(\ell+1)} |(m(\ell+1) + \mu)!|_p} =: \Omega.
\end{aligned} \tag{35}$$

Note that $m(\ell+1) + \mu \leq m(\ell+2)$, whence the equality on the last row.

8.2. Deriving contradiction

We are working to establish a contradiction with (35), so let us study the expression $\log \Omega$ more closely. First of all, we have

$$\begin{aligned}
\Omega &= \frac{(m+1)^\kappa (m(\ell+2))^\kappa H c_1^{\ell+1} \prod_{p \in [\log(\ell+1), m(\ell+2)]} |(\ell+1)!|_p}{\prod_{p < \log(\ell+1)} |(m(\ell+1) + \mu)!|_p} \\
&= \frac{(m+1)^\kappa (m(\ell+2))^\kappa H c_1^{\ell+1}}{(\ell+1)! \prod_{p < \log(\ell+1)} |(m(\ell+1) + \mu)!|_p} \\
&\leq \frac{(m+1)^\kappa (m(\ell+2))^\kappa H c_1^{\ell+1} \prod_{p < \log(\ell+1)} p^{\frac{m(\ell+1) + \mu + (\ell+1)}{p-1}}}{(\ell+1)!}
\end{aligned}$$

because of the product formula and property (7). Recall also the Stirling formula (see [12])

$$\log n! = \left(n + \frac{1}{2}\right) \log n - n + \log \sqrt{2\pi} + \frac{\theta(n)}{12}, \quad 0 < \theta(n) < 1.$$

With these equations and estimate $\mu \leq m$ we get

$$\begin{aligned}
\log \Omega &\leq \log \left(\frac{(m+1)^\kappa (m(\ell+2))^\kappa H c_1^{\ell+1} \prod_{p < \log(\ell+1)} p^{\frac{m(\ell+1) + \mu + (\ell+1)}{p-1}}}{(\ell+1)!} \right) \\
&\leq \kappa \log(m+1) + \kappa \log(m(\ell+2)) + \log H + (\ell+1) \log c_1 + \\
&\quad \sum_{p < \log(\ell+1)} \log p^{\frac{m(\ell+2) + (\ell+1)}{p-1}} - \left((\ell+1) + \frac{1}{2} \right) \log(\ell+1) + (\ell+1) \\
&\leq \kappa \log(m+1) + \kappa \log m + \kappa \log(\ell+2) + \log H + (\ell+1) \log c_1 \\
&\quad + (m(\ell+2) + (\ell+1)) \sum_{p < \log(\ell+1)} \frac{\log p}{p-1} - (\ell+1) \log(\ell+1) \\
&\quad - \frac{1}{2} \log(\ell+1) + (\ell+1) \\
&\leq \log H + \kappa \log m + \kappa \log(m+1) + \frac{\kappa}{\ell+1} + \left(\kappa - \frac{1}{2} \right) \log(\ell+1) \\
&\quad + \left(\log c_1 + \left(m+1 + \frac{m}{\ell+1} \right) \sum_{p < \log(\ell+1)} \frac{\log p}{p-1} + 1 \right) (\ell+1) \\
&\quad - (\ell+1) \log(\ell+1),
\end{aligned} \tag{36}$$

where

$$\log(\ell+2) < \log(\ell+1) + \frac{1}{\ell+1}$$

by the mean value theorem.

To be able to continue, we need to know how the sum

$$\sum_{p < x} \frac{\log p}{p-1}$$

behaves. Help is found from [13] (see the corollary of Theorem 6):

Lemma 8.1. [13]

$$\sum_{p \leq x} \frac{\log p}{p} < \log x, \quad x > 1.$$

Since $p-1 \geq \frac{p}{2}$ for all primes p , it follows that

$$\sum_{p < x} \frac{\log p}{p-1} \leq 2 \sum_{p < x} \frac{\log p}{p} < 2 \log x. \quad (37)$$

Combining estimates (35), (36), and (37), we have

$$0 \leq \log \Omega$$

$$\begin{aligned} &\leq \log H + \kappa \log m + \kappa \log(m+1) + \frac{\kappa}{\ell+1} + \left(\kappa - \frac{1}{2}\right) \log(\ell+1) \\ &\quad + \left(\log c_1 + \left(m+1 + \frac{m}{\ell+1}\right) \sum_{p < \log(\ell+1)} \frac{\log p}{p-1} + 1 \right) (\ell+1) - (\ell+1) \log(\ell+1) \\ &< \log H + \kappa \log m + \kappa \log(m+1) + \frac{\kappa}{\ell+1} + \left(\kappa - \frac{1}{2}\right) \log(\ell+1) \\ &\quad + \left(\log c_1 + 2 \left(m+1 + \frac{m}{\ell+1}\right) \log \log(\ell+1) + 1 \right) (\ell+1) - (\ell+1) \log(\ell+1) \\ &< \log H + \left(2(m+1) + \frac{2m}{\ell+1} + \frac{\log c_1}{\log \log(\ell+1)} + \frac{1}{\log \log(\ell+1)} \right. \\ &\quad + \frac{(\kappa - \frac{1}{2}) \log(\ell+1)}{(\ell+1) \log \log(\ell+1)} + \frac{\kappa \log m}{(\ell+1) \log \log(\ell+1)} + \frac{\kappa \log(m+1)}{(\ell+1) \log \log(\ell+1)} \\ &\quad \left. + \frac{\kappa}{(\ell+1)^2 \log \log(\ell+1)} \right) (\ell+1) \log \log(\ell+1) - (\ell+1) \log(\ell+1) \\ &= N(\ell+1) < 0, \end{aligned}$$

a contradiction with (29). Thus there must exist a prime

$$p \in [\log(\ell+1), m(\ell+2)] \quad (38)$$

and a valuation $v'|p$ such that

$$\|b_{\ell+1,\mu,0}\Lambda_{v'}\|_{v'} \geq \|\lambda_1 s_{\ell+1,\mu,1} + \dots + \lambda_m s_{\ell+1,\mu,m}\|_{v'}.$$

Then, for this valuation v' ,

$$\|W\|_{v'} = \|b_{\ell+1,\mu,0}\Lambda_{v'} - (\lambda_1 s_{\ell+1,\mu,1} + \dots + \lambda_m s_{\ell+1,\mu,m})\|_{v'} \leq \|b_{\ell+1,\mu,0}\Lambda_{v'}\|_{v'} \leq \|\Lambda_{v'}\|_{v'},$$

and

$$1 = \prod_v \|W\|_v \leq \left(\prod_{v \in V_\infty} \|W\|_v \right) \|W\|_{v'} \leq \left(\prod_{v \in V_\infty} \|W\|_v \right) \|\Lambda_{v'}\|_{v'}. \quad (39)$$

8.3. Bounds for ℓ

For the final stages of the proof, we need to express the number ℓ in terms of the height H . In order to do this, we introduce the inverse function of the function $y(z) = z \log z$, $z \geq 1/e$, considered in [9].

Lemma 8.2. [9] *The inverse function $z(y)$ of the function $y(z) = z \log z$, $z \geq \frac{1}{e}$, is strictly increasing. When $y > e$, the inverse function may be given by the infinite nested logarithm fraction*

$$z(y) = \lim_{n \rightarrow \infty} z_n(y) = \frac{y}{\log \frac{y}{\log \frac{y}{\log \dots}}}.$$

Let $z_0(y) = y$ and $z_n(y) = \frac{y}{\log z_{n-1}(y)}$ for all $n \in \mathbb{Z}_{\geq 1}$. Then we also have

$$z_1 < z_3 < \dots < z < \dots < z_2 < z_0.$$

Another little lemma from [7] gives a useful upper estimate:

Lemma 8.3. [7] *If $y \geq re^r$, where $r \geq e$, then*

$$z(y) \leq \left(1 + \frac{\log r}{r}\right) \frac{y}{\log y}.$$

Proof. Denote $z := z(y)$ with $y \geq re^r$. Then

$$z = \frac{y}{\log z} = \frac{y}{\log y} \frac{\log y}{\log z} = \frac{y}{\log y} \left(1 + \frac{\log \log z}{\log z}\right) \leq \frac{y}{\log y} \left(1 + \frac{\log r}{r}\right),$$

because $\log z \geq r \geq e$. \square

Now, $N(\ell + 1) < 0$ implies

$$(\ell + 1) \log(\ell + 1) > \log H \geq se^s, \quad (40)$$

so that, by applying the z -function, $\ell + 1 > e^s$. According to (27), we have

$$\ell > e^s - 1 \geq \max \left\{ e^{e^\kappa}, e^{c_1}, e^{(m+3)^2} \right\}. \quad (41)$$

Hence, using the lower bound (41) and the fact that $m \geq 1$, we may estimate from the definition of $N(\ell)$ in (28):

$$\begin{aligned} 0 &\leq N(\ell) \\ &< \log H + \left(2(m+1) + \frac{2m}{e^{(m+3)^2}} + 1 + \frac{1}{2 \log(m+3)} + \frac{(\kappa - \frac{1}{2})(m+3)^2}{e^{(m+3)^2} \cdot \kappa} \right. \\ &\quad \left. + \frac{\kappa \log m}{e^{(m+3)^2} \cdot \kappa} + \frac{\kappa \log(m+1)}{e^{(m+3)^2} \cdot \kappa} + \frac{\kappa}{e^{2(m+3)^2} \cdot \kappa} \right) \ell \log \log \ell - \ell \log \ell \\ &\leq \log H + (2(m+1) + 1 + 0.360674 + 3 \cdot 10^{-6}) \ell \log \log \ell - \ell \log \ell \\ &< \log H + (2m + 3.361) \ell \log \log \ell - \ell \log \ell \end{aligned} \quad (42)$$

since $\frac{1}{2 \log(m+3)} \leq 0.360674$ when $m \geq 1$. Thus

$$\ell \log \ell \left(1 - \frac{(2m + 3.361) \log \log \ell}{\log \ell} \right) \leq \log H, \quad (43)$$

where

$$\frac{\log \log \ell}{\log \ell} < \frac{2 \log(m+3)}{(m+3)^2}$$

by (41), and so

$$1 - \frac{(2m + 3.361) \log \log \ell}{\log \ell} > 1 - \frac{(2m + 3.361) \cdot 2 \log(m+3)}{(m+3)^2} > 0 \quad (44)$$

for all $m \geq 1$.

By inequalities (43) and (44) and the lower bound in (41), we have

$$\frac{(m+3)^2}{(m+3)^2 - (2m + 3.361) \cdot 2 \log(m+3)} \cdot \log H > \ell \log \ell > (m+3)^2 e^{(m+3)^2},$$

so we may apply Lemma 8.3 with $r = (m+3)^2$:

$$\begin{aligned} \ell &< z \left(\frac{(m+3)^2}{(m+3)^2 - (2m+3.361) \cdot 2 \log(m+3)} \cdot \log H \right) \\ &\leq \left(1 + \frac{2 \log(m+3)}{(m+3)^2} \right) \frac{\frac{(m+3)^2}{(m+3)^2 - (2m+3.361) \cdot 2 \log(m+3)} \cdot \log H}{\log \log H}. \end{aligned} \quad (45)$$

8.4. Measure

To get the measure from (39), we need an upper estimate for the product $\prod_{v \in V_\infty} \|W\|_v$. Back in (33) we estimated that

$$\prod_{v \in V_\infty} \|W\|_v \leq (m+1)^\kappa (m(\ell+2))^\kappa H c_1^{\ell+1} (m(\ell+2))!$$

(taking into account that $\mu \leq m$). From estimate (42) it follows that

$$\ell \log \ell < (2m + 3.361) \ell \log \log \ell + \log H$$

and by the mean value theorem we have

$$\log(\ell+2) < \frac{2}{\ell} + \log \ell.$$

With these estimates we get

$$\begin{aligned} \log \left(\prod_{v \in V_\infty} \|W\|_v \right) &\leq \log ((m+1)^\kappa (m(\ell+2))^\kappa H c_1^{\ell+1} (m(\ell+2))!) \\ &\leq \kappa \log(m+1) + \kappa \log m + \kappa \log(\ell+2) + \log H \\ &\quad + (\ell+1) \log c_1 + (m(\ell+2)) \log(m(\ell+2)) \\ &= \kappa \log(m+1) + \kappa \log m + \kappa \log(\ell+2) + \log H + \ell \log c_1 + \log c_1 \\ &\quad + (m \log m) \ell + m \ell \log(\ell+2) + 2m \log m + 2m \log(\ell+2) \\ &\leq \kappa \log(m+1) + \kappa \log m + \frac{2\kappa}{\ell} + \kappa \log \ell + \log H + \ell \log c_1 + \log c_1 \\ &\quad + (m \log m) \ell + m \ell \log \ell + 2m + 2m \log m + 2m \log \ell + \frac{4m}{\ell} \\ &< \kappa \log(m+1) + \kappa \log m + \frac{2\kappa}{\ell} + \kappa \log \ell + \log H + \ell \log c_1 + \log c_1 \\ &\quad + (m \log m) \ell + m ((2m + 3.361) \ell \log \log \ell + \log H) + 2m \\ &\quad + 2m \log m + 2m \log \ell + \frac{4m}{\ell} \\ &= (m+1) \log H + \left(\frac{\kappa \log(m+1)}{\ell \log \log \ell} + \frac{\kappa \log m}{\ell \log \log \ell} + \frac{2\kappa}{\ell^2 \log \log \ell} \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{\kappa \log \ell}{\ell \log \log \ell} + \frac{\log c_1}{\log \log \ell} + \frac{\log c_1}{\ell \log \log \ell} + \frac{m \log m}{\log \log \ell} + 2m^2 \\
& + 3.361m + \frac{2m}{\ell \log \log \ell} + \frac{2m \log m}{\ell \log \log \ell} + \frac{2m \log \ell}{\ell \log \log \ell} \\
& + \frac{4m}{\ell^2 \log \log \ell} \Big) \ell \log \log \ell.
\end{aligned}$$

The coefficient of $\ell \log \log \ell$ can be estimated using the bound (41) and the fact that $m \geq 1$: we have

$$\frac{m \log m}{\log \log \ell} < \frac{m \log m}{2 \log(m+3)} < \frac{m}{2}, \quad \frac{\log c_1}{\log \log \ell} < 1,$$

and the rest of the fractions together are less than 0.0000034. Hence

$$\log \left(\prod_{v \in V_\infty} \|W\|_v \right) \leq (m+1) \log H + (2m^2 + 3.861m + 1.0000034) \ell \log \log \ell. \quad (46)$$

By (45) and the assumption $\log H \geq se^s > (m+3)^2 e^{(m+3)^2}$, we have

$$\begin{aligned}
\ell & < \frac{\left(1 + \frac{2 \log(m+3)}{(m+3)^2}\right) (m+3)^2}{(m+3)^2 - (2m + 3.361) \cdot 2 \log(m+3)} \cdot \frac{\log H}{2 \log(m+3) + (m+3)^2} \\
& = \frac{1}{(m+3)^2 - (2m + 3.361) \cdot 2 \log(m+3)} \cdot \log H \\
& < \log H.
\end{aligned}$$

Thus

$$\log \log \ell < \log \log \log H. \quad (47)$$

Let us next estimate $(2m^2 + 3.861m + 1.0000034) \ell$, again using (45):

$$\begin{aligned}
& (2m^2 + 3.861m + 1.0000034) \ell \\
& \leq \frac{(2m^2 + 3.861m + 1.0000034) \left(1 + \frac{2 \log(m+3)}{(m+3)^2}\right) (m+3)^2}{(m+3)^2 - (2m + 3.361) \cdot 2 \log(m+3)} \cdot \frac{\log H}{\log \log H} \\
& = \frac{m^2 \left(2 + \frac{3.861}{m} + \frac{1.0000034}{m^2}\right) \left(1 + \frac{2 \log(m+3)}{(m+3)^2}\right)}{1 - \frac{2 \cdot 2m \log(m+3)}{(m+3)^2} - \frac{2 \cdot 3.361 \log(m+3)}{(m+3)^2}} \cdot \frac{\log H}{\log \log H} \\
& < 114m^2 \cdot \frac{\log H}{\log \log H}
\end{aligned} \quad (48)$$

since $m \geq 1$.

Combining estimates (46), (47), and (48), yields

$$\log \left(\prod_{v \in V_\infty} \|W\|_v \right) < \left((m+1) + 114m^2 \cdot \frac{\log \log \log H}{\log \log H} \right) \log H,$$

so that inequality (39) implies

$$\|\Lambda_{v'}\|_{v'} \geq \frac{1}{\prod_{v \in V_\infty} \|W\|_v} > H^{-(m+1) - 114m^2 \cdot \frac{\log \log \log H}{\log \log H}}.$$

8.5. Infinitely many intervals

We still need an upper estimate for $m(\ell+1)$ in terms of the height H in order to write the interval (38) with respect to H . Once more we use (45) and the assumption $\log H > (m+3)^2 e^{(m+3)^2}$:

$$\begin{aligned} m(\ell+2) &\leq m \cdot \frac{\left(1 + \frac{2 \log(m+3)}{(m+3)^2}\right) (m+3)^2}{(m+3)^2 - (2m + 3.361) \cdot 2 \log(m+3)} \cdot \frac{\log H}{\log \log H} + 2m \\ &= m \left(\frac{1 + \frac{2 \log(m+3)}{(m+3)^2}}{1 - \frac{2 \cdot 2m \log(m+3)}{(m+3)^2} - \frac{2 \cdot 3.361 \log(m+3)}{(m+3)^2}} + \frac{2 \log \log H}{\log H} \right) \cdot \frac{\log H}{\log \log H} \\ &\leq m \left(\frac{1 + \frac{2 \log(m+3)}{(m+3)^2}}{1 - \frac{2 \cdot 2m \log(m+3)}{(m+3)^2} - \frac{2 \cdot 3.361 \log(m+3)}{(m+3)^2}} + \frac{4 \log(m+3) + 2(m+3)^2}{(m+3)^2 e^{(m+3)^2}} \right) \quad (49) \\ &\quad \cdot \frac{\log H}{\log \log H} \\ &< 17m \cdot \frac{\log H}{\log \log H} \end{aligned}$$

since $m \geq 1$.

By (40) and Lemma 8.3 we have

$$\log(\ell+1) > \log(z(\log H)) > \log(z_1(\log H)) = \log \left(\frac{\log H}{\log \log H} \right).$$

Combining this with (49) above leads to

$$[\log(\ell+1), m(\ell+2)] \subseteq \left[\log \left(\frac{\log H}{\log \log H} \right), \frac{17m \log H}{\log \log H} \right] =: I(m, H).$$

This ends the proof of Theorem 3.4. \square

Remark 8.4. The constants 114 and 17 can be improved by adjusting the lower bound of $\log H$, i.e., the choice of s in (27). For instance, taking $(m+3)^3$ instead of $(m+3)^2$ will reduce them considerably.

Remark 8.5. There is a connection between the width of the interval $I(m, H)$ and the error term in the lower bound (8). Our choice of $\log(\ell+1)$ in the interval (38) results in the term $\log \log \log H$ in (8) (see (47)), improving the corresponding lower bound of Bertrand et al. in [2] for this function. This is done at a cost, though, since our interval $I(m, H)$ is wider than theirs. Had we chosen $e^{\sqrt{\log(\ell+1)}}$ instead of $\log(\ell+1)$, we would have ended up with $\sqrt{\log \log H}$ instead of $\log \log \log H$. Then the dependence on H in the error term of (8) would have been $\frac{1}{\sqrt{\log \log H}}$, just as it is in [2], and the interval $I(m, H)$ would have had $\exp\left(\sqrt{\log\left(\frac{\log H}{\log \log H}\right)}\right)$ as its lower bound, very much like in [2] and [15].

The best lower bound (in terms of H) would have been achieved by considering an interval of the form $[2, ml]$ with no dependence on l in the lower bound, because the empty sum $\sum_{p < 2} \frac{\log p}{p-1}$ would not then cause an extra term in our estimates. The disappearing of $\log \log l$ from the estimates would mean that we would have $\frac{1}{\log \log H}$ instead of $\frac{\log \log \log H}{\log \log H}$ in the error term. This is in line with the exponential function (see [7]). However, this result won't give us infinitely many distinct primes when H grows, like Theorem 3.4 does.

9. Corollaries and examples

9.1. The field of rationals

When $\mathbb{K} = \mathbb{Q}$, Theorem 3.1 reduces to:

Corollary 9.1. Let $m \in \mathbb{Z}_{\geq 1}$ and $\lambda_0, \lambda_1, \dots, \lambda_m \in \mathbb{Z}$, where $\lambda_j \neq 0$ for at least one j . Choose m pairwise distinct, non-zero integers $\alpha_j \in \mathbb{Z} \setminus \{0\}$, $j = 1, \dots, m$. Suppose P is a subset of the prime numbers such that

$$\limsup_{l \rightarrow \infty} c_2^l(ml+m)(ml+m)! \prod_{p \in P} |(ml)!!!|_p = 0,$$

where

$$c_2 = \left(\max_{1 \leq j \leq m} \{1, |\alpha_j|\} \right)^m \left(\prod_{i=1}^m \left(|\alpha_i| + \max_{1 \leq j \leq m} \{1, |\alpha_j|\} \right) \right) \prod_{p \in P} \max_{1 \leq j \leq m} \{|\alpha_j|_p\}.$$

Then there exists a prime $p' \in P$ for which

$$\lambda_0 + \lambda_1 F_{p'}(\alpha_1) + \dots + \lambda_m F_{p'}(\alpha_m) \neq 0.$$

Example 9.2. For instance, take $\alpha_1 = 1$ and $\alpha_2 = -1$. Then, if $P \subseteq \mathbb{P}$ is such that

$$\limsup_{l \rightarrow \infty} 4^l (2l+2)(2l+2)! \prod_{p \in P} |(2l)!|!|_p = 0,$$

there exists a prime $p' \in P$ for which

$$\lambda_0 + \lambda_1 F_{p'}(1) + \lambda_2 F_{p'}(-1) \neq 0.$$

In particular, taking $\lambda_0 = 2a \in \mathbb{Z}$ and $\lambda_1 = \lambda_2 = -b \in \mathbb{Z}$, it follows that there exists a prime $p' \in P$ such that

$$a - b \sum_{n=0}^{\infty} (2n)! \neq 0,$$

i.e., $\sum_{n=0}^{\infty} (2n)! \neq \frac{a}{b}$ for some $p' \in P$.

9.2. Linear recurrences

A sequence $(x_n)_{n=0}^{\infty}$ satisfies a *k*th order homogeneous linear recurrence with constant coefficients, if, for all $n \in \mathbb{Z}_{\geq k}$,

$$x_n = c_1 x_{n-1} + c_2 x_{n-2} + \dots + c_k x_{n-k}$$

for some $c_1, \dots, c_k \in \mathbb{C}$ with $c_k \neq 0$. If the characteristic polynomial $x^k - c_1 x^{k-1} - \dots - c_k \in \mathbb{C}[x]$ of this recurrence has k distinct zeros $\alpha_1, \dots, \alpha_k \in \mathbb{C}$, then the solution $(x_n)_{n=0}^{\infty}$ is given by the linear combination

$$x_n = a_1 \alpha_1^n + \dots + a_k \alpha_k^n, \quad n \in \mathbb{Z}_{\geq 0},$$

where the coefficients $a_1, \dots, a_k \in \mathbb{C}$ are determined by given initial conditions. (More about recurrences in [6].)

Suppose now that $c_1, \dots, c_k \in \mathbb{Z}$. Then the roots $\alpha_1, \dots, \alpha_k$ lie in a number field \mathbb{K} of degree at most $k!$, and so do the coefficients a_1, \dots, a_k . Furthermore, if $\alpha_1, \dots, \alpha_k \in \mathbb{Z}_{\mathbb{K}}$, then $F(\alpha_i)$, $i = 1, \dots, k$, converges for any non-Archimedean valuation v of \mathbb{K} , and we have

$$\sum_{i=1}^k a_i F_v(\alpha_i) = \sum_{i=1}^k a_i \sum_{n=0}^{\infty} n! \alpha_i^n = \sum_{n=0}^{\infty} n! \sum_{i=1}^k a_i \alpha_i^n = \sum_{n=0}^{\infty} n! x_n.$$

Multiplying both sides by $d := \text{lcm}_{1 \leq i \leq k} \{\text{den } \alpha_i\}$ ² we get a linear form with coefficients $b_i := da_i \in \mathbb{Z}_{\mathbb{K}}$:

² The denominator $\text{den } \alpha$ of an algebraic number $\alpha \in \mathbb{K}$ is the smallest positive rational integer n such that $n\alpha$ is an algebraic integer.

$$\sum_{i=1}^k b_i F_v(\alpha_i) = d \sum_{n=0}^{\infty} n! x_n.$$

If at least one of the coefficients a_i is non-zero, it follows from Theorem 3.1 that for any $a, b \in \mathbb{Z}_{\mathbb{K}}$ there exists a non-Archimedean valuation v' of \mathbb{K} such that

$$\sum_{n=0}^{\infty} n! x_n \neq \frac{a}{b}.$$

Example 9.3 (*The Fibonacci numbers*). The Fibonacci numbers are given by the sequence

$$f_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n), \quad \alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}, \quad n \in \mathbb{Z}_{\geq 0}.$$

Let us work in $\mathbb{Q}(\sqrt{5})$ and study the series $\sum_{n=0}^{\infty} n! f_n$. The minimal polynomial of α and β is $x^2 - x - 1$, so α and β are algebraic integers and thus $\|\alpha\|_v, \|\beta\|_v \leq 1$ for any non-Archimedean valuation v of the field $\mathbb{Q}(\sqrt{5})$. Actually, as $\alpha\beta = -1$, we get $\|\alpha\|_v = \|\beta\|_v = 1$ for all $v \in V_0$. Hence both series $\frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} n! \alpha^n$ and $-\frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} n! \beta^n$ converge v -adically and their sum is

$$\begin{aligned} \frac{1}{\sqrt{5}} (F_v(\alpha) - F_v(\beta)) &= \frac{1}{\sqrt{5}} \left(\sum_{n=0}^{\infty} n! \alpha^n - \sum_{n=0}^{\infty} n! \beta^n \right) \\ &= \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} n! (\alpha^n - \beta^n) = \sum_{n=0}^{\infty} n! f_n. \end{aligned} \tag{50}$$

Because $x^2 - 5 = (x - \sqrt{5})(x + \sqrt{5})$ in $\mathbb{R}[x]$, the Archimedean absolute value of \mathbb{Q} has two extensions to $\mathbb{Q}(\sqrt{5})$. These are given by

$$\left| a + b\sqrt{5} \right|_1 = \left| a + b\sqrt{5} \right|, \quad \left| a + b\sqrt{5} \right|_2 = \left| a - b\sqrt{5} \right|,$$

where now $|\cdot|$ is the unique Archimedean extension of the Archimedean absolute value of \mathbb{Q} to \mathbb{C} , the algebraic closure of the Archimedean completion of \mathbb{Q} . Further,

$$\begin{aligned} \left\| a + b\sqrt{5} \right\|_1 &= \left| a + b\sqrt{5} \right|_1^{\frac{1}{2}} = \sqrt{\left| a + b\sqrt{5} \right|}, \\ \left\| a + b\sqrt{5} \right\|_2 &= \left| a + b\sqrt{5} \right|_2^{\frac{1}{2}} = \sqrt{\left| a - b\sqrt{5} \right|}. \end{aligned}$$

Let $a, b \in \mathbb{Z}$, $b \neq 0$ and choose $\alpha_1 = \alpha$, $\alpha_2 = \beta$. Then

$$\begin{aligned} c_2((\alpha, \beta), V) &= (\max\{1, \|\alpha\|_1, \|\beta\|_1\})^2 (\max\{1, \|\alpha\|_2, \|\beta\|_2\})^2 \\ &\quad \cdot (\|\alpha\|_1 + \max\{1, \|\alpha\|_1, \|\beta\|_1\}) (\|\beta\|_1 + \max\{1, \|\alpha\|_1, \|\beta\|_1\}) \end{aligned}$$

$$\begin{aligned}
& \cdot (\|\alpha\|_2 + \max\{1, \|\alpha\|_2, \|\beta\|_2\}) (\|\beta\|_2 + \max\{1, \|\alpha\|_2, \|\beta\|_2\}) \\
& \cdot \prod_{v \in V} \max\{\|\alpha\|_v, \|\beta\|_v\} \\
& = 4 \left(\frac{1 + \sqrt{5}}{2} \right)^3 \left(\sqrt{\frac{-1 + \sqrt{5}}{2}} + \sqrt{\frac{1 + \sqrt{5}}{2}} \right)^2 \approx 72.
\end{aligned}$$

By taking $\lambda_0 = 5a \in \mathbb{Z}$, $\lambda_1 = -b\sqrt{5} \in \mathbb{Z}_{\mathbb{K}} \setminus \{0\}$, $\lambda_2 = b\sqrt{5} \in \mathbb{Z}_{\mathbb{K}} \setminus \{0\}$, and considering the linear form

$$\lambda_0 + \lambda_1 F_v(\alpha) + \lambda_2 F_v(\beta), \quad v \in V_0,$$

equation (50) and Theorem 3.1 give:

Corollary 9.4. *If V is any collection of non-Archimedean valuations of $\mathbb{Q}(\sqrt{5})$ such that*

$$\limsup_{l \rightarrow \infty} c_2^l (2l+2)(2l+2)! \prod_{v \in V} \|(2l)!!\|_v = 0,$$

then there exists a valuation $v' \in V$ for which

$$a - b \sum_{n=0}^{\infty} n! f_n \neq 0.$$

10. Arithmetic progressions

Ernvall-Hytönen et al. [8] have proved

Proposition 10.1. [8, Theorem 5] *Let $a \in \mathbb{Z}$, $b, \xi \in \mathbb{Z} \setminus \{0\}$, and $n \in \mathbb{Z}_{\geq 3}$ be given. Assume that R is any union of the primes in r residue classes in the reduced residue system modulo n , where $r > \frac{\varphi(n)}{2}$. Then there are infinitely many primes $p \in R$ such that $a - bF_p(\xi) \neq 0$.*

Because each non-Archimedean valuation of the number field \mathbb{K} is attached to the prime it extends, the division of primes into $\varphi(n)$ residue classes induces a division of the non-Archimedean valuations into $\varphi(n)$ classes. How many of these classes are needed to fulfil condition (6)?

Theorem 10.2. *Let $m \in \mathbb{Z}_{\geq 1}$ and $\lambda_0, \lambda_1, \dots, \lambda_m \in \mathbb{Z}_{\mathbb{K}}$ where $\lambda_j \neq 0$ for at least one j . Choose m pairwise distinct, non-zero algebraic integers $\alpha_1, \dots, \alpha_m \in \mathbb{Z}_{\mathbb{K}}$. Let $n \in \mathbb{Z}_{\geq 3}$ be given. Assume that R is a union of the primes in r residue classes in the reduced residue system modulo n , where $r > \frac{m\varphi(n)}{m+1}$, and let $V = \{v \in V_0 \mid v|p \text{ for some } p \in R\}$. Then there exists a valuation $v' \in V$ such that*

$$\lambda_0 + \lambda_1 F_{v'}(\alpha_1) + \dots + \lambda_m F_{v'}(\alpha_m) \neq 0.$$

Proof. Let us show that the collection V satisfies condition (6). We shall follow the method of Ernvall-Hytönen et al. [8]. By [8, Lemma 9] we have

$$\log \left(\prod_{p \equiv a \pmod{n}} |l|_p \right) = -\frac{l \log l}{\varphi(n)} + O(l \log \log l)$$

when $n \in \mathbb{Z}_{\geq 3}$ and $\gcd(a, n) = 1$. Using this, the fact

$$\log((ml + m)!) = ml \log l + O(l),$$

and property (5), we get

$$\begin{aligned} & \log \left(c_2^l (ml + m)^\kappa (ml + m)! \prod_{v \in V} \|(ml)!!\|_v \right) \\ &= l \log c_2 + \kappa \log(m(l + 1)) + \log((ml + m)!) + \sum_{v \in V} \log \|(ml)!!\|_v \\ &= ml \log l + O(l) + \sum_{p \in R} \sum_{v|p} \log \|(ml)!!\|_v \\ &= ml \log l + O(l) + \sum_{p \in R} \log |(ml)!!|_p \\ &= ml \log l + O(l) - \frac{rml \log l}{\varphi(n)} - \frac{rl \log l}{\varphi(n)} + O(l \log \log l) \\ &= \left(m - \frac{r(m+1)}{\varphi(n)} \right) l \log l + O(l \log \log l) \xrightarrow{l \rightarrow \infty} -\infty, \end{aligned}$$

because the coefficient $\left(m - \frac{r(m+1)}{\varphi(n)} \right)$ is negative. The result follows from Theorem 3.1. \square

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