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Remarks on Kato's Euler systems for elliptic curves with additive reduction

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ABSTRACT

Extending the former work for the good reduction case, we provide a numerical criterion to verify a large portion of the “Iwasawa main conjecture without p -adic L -functions” for elliptic curves with *additive* reduction at an odd prime p over the cyclotomic \mathbb{Z}_p -extension. We also deduce the corresponding p -part of the Birch and Swinnerton-Dyer formula for elliptic curves of rank zero from the same numerical criterion. We give some explicit examples at the end and specify our choice of Kato's Euler system in the appendix.

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1. Introduction

1.1. Overview

This article is a generalization of the numerical criterion for the verification of the Iwasawa main conjecture for modular forms at a good prime [KKS] to the additive reduction case. For the multiplicative reduction case, the main conjecture follows from the good ordinary case and the use of Hida theory ([Ski16]). This criterion also has an application to the p -part of the Birch and Swinnerton-Dyer (BSD) formula for elliptic curves of rank zero (cf. [JSW17, Theorem 7.2.1]). Since there are only finitely many bad reduction primes for elliptic curves, the criterion can be practically used to check the *full* Birch and Swinnerton-Dyer formula for an elliptic curve of rank zero and not necessarily square-free conductor. All the known non-CM examples have square-free conductors ([Wan18, Appendix]). We put some numerical examples of elliptic curves with additive reduction at the end.

Let p be an odd prime and E be an elliptic curve of conductor N over \mathbb{Q} . Throughout this article, we assume that E has additive reduction at p . In other words, p^2 divides N .

The construction of Kato's Euler systems [Kat04] and the formulation of the Iwasawa main conjecture without p -adic L -functions [Kat04, Conjecture 12.10] are insensitive to the reduction type of elliptic curves. See [Jac18] for the \mathbf{D}_{dR} -valued p -adic L -function of elliptic curves with additive reduction and also [Del98] and [Del02] for different attempts to understand the additive reduction case.

We expect that the reader has some familiarity with [Rub00] and [MR04].

1.2. Strategy of proof

We briefly explain the synopsis of the proof. One can see the detail in [KKS].

Let \mathbf{c} be the Kato's Euler system for E following the convention described in Appendix A, κ the Kolyvagin system associated to \mathbf{c} , and κ^∞ the Λ -adic Kolyvagin system associated to \mathbf{c} where Λ is the Iwasawa algebra. We say that

- κ is **primitive** if κ does not vanish modulo p ;
- κ^∞ is **Λ -primitive** if κ^∞ does not vanish modulo any height one prime of Λ .

In the theory of Kolyvagin systems, the concept of primitivity and Λ -primitivity play the central role to establish the equality of the p -part of the BSD formula for the rank zero case and the Iwasawa main conjecture, respectively. Roughly, we expect the following statements in general.

- (1) If κ is primitive and E has rank zero, then the p -part of the Birch and Swinnerton-Dyer formula for E holds.
- (2) If κ^∞ is Λ -primitive, then the "Iwasawa main conjecture for E " holds.

It is also known that the primitivity of κ implies the Λ -adic primitivity of κ^∞ . In the original argument [MR04, Theorems 6.2.4 and 6.2.7], the good ordinary case is only treated. We extend both statements to the additive reduction case. See §3 for the main conjecture and §4 for the p -part of the BSD formula.

Now the problem reduces to how to check κ is primitive. In order to check this, we compute the mod p reduction of the image of Kolyvagin derivatives of Kato’s Euler system under the dual exponential map, which are called Kurihara numbers. In order to consider the mod p reduction in the image of the dual exponential map, we need to compute the integral image of the dual exponential map. This computation is done in §2.

In the rest of this section, we describe the working assumptions and Kurihara numbers, review Kato’s main conjecture, and then state the main theorems.

1.3. Working assumptions and Kurihara numbers

Let

- $\text{Tam}(E)$ be the Tamagawa number of E ,
- N_{st} be the product of split multiplicative reduction primes of E , and
- N_{ns} be the product of non-split multiplicative reduction primes of E .

We assume the following conditions throughout this article.

Assumption 1.1 (*Working assumptions*).

- (1) p does not divide $\text{Tam}(E) \cdot \prod_{\ell|N_{\text{st}}} (\ell - 1) \cdot \prod_{\ell|N_{\text{ns}}} (\ell + 1)$ (cf. Remark 1.8.(4)).
- (2) The mod p Galois representation $\bar{\rho} : G_{\mathbb{Q}} \rightarrow \text{Aut}_{\mathbb{F}_p}(E[p])$ is surjective. (Thus, E is non-CM.)
- (3) The Manin constant is prime to p . (It is expected to be true always.)

Let $f \in S_2(\Gamma_0(N))$ be the newform attached to E by [BCDT01, Theorem A]. For $\frac{a}{b} \in \mathbb{Q}$, we define $\left[\frac{a}{b}\right]^+$ by

$$2\pi \int_0^\infty f\left(\frac{a}{b} + iy\right) dy = \left[\frac{a}{b}\right]^+ \cdot \Omega_E^+ + \left[\frac{a}{b}\right]^- \cdot \Omega_E^-$$

where Ω_E^\pm are the Néron periods of E . Then it is well-known that $\left[\frac{a}{b}\right]^+ \in \mathbb{Q}$. The following theorem due to G. Stevens yields the p -integrality of the value.

Theorem 1.2 ([Ste89, §3]). *Under (2) and (3) of Assumption 1.1, we have $\left[\frac{a}{b}\right]^+ \in \mathbb{Z}_{(p)}$ for $\frac{a}{b} \in \mathbb{Q}$.*

A prime ℓ is a **Kolyvagin prime for** (E, p) if $(\ell, Np) = 1$, $\ell \equiv 1 \pmod{p}$, and $a_\ell(E) \equiv \ell + 1 \pmod{p}$. We define the **Kurihara number for** (E, p) at n by

$$\tilde{\delta}_n := \sum_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} \left(\left[\frac{a}{n} \right]^+ \cdot \prod_{\ell|n} \overline{\log_{\mathbb{F}_\ell}(a)} \right) \in \mathbb{F}_p$$

where n is the square-free product of Kolyvagin primes, $\left[\frac{a}{n} \right]^+$ is the mod p reduction of $\left[\frac{a}{n} \right]^+$, and $\overline{\log_{\mathbb{F}_\ell}(a)}$ is the mod p reduction of the discrete logarithm of a modulo ℓ with a fixed primitive root modulo ℓ . The number $\tilde{\delta}_n$ itself is not well-defined, but its non-vanishing question is well-defined.

1.4. The Iwasawa main conjecture à la Kato

Let \mathbb{Q}_∞ be the cyclotomic \mathbb{Z}_p -extension of \mathbb{Q} and \mathbb{Q}_n be the subextension of \mathbb{Q} in \mathbb{Q}_∞ of degree p^n . Let $\Lambda := \mathbb{Z}_p[[\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]]$ be the Iwasawa algebra. Let $T = \text{Ta}_p E$ be the p -adic Tate module of E and $j : \text{Spec}(\mathbb{Q}_n) \rightarrow \text{Spec}(\mathcal{O}_{\mathbb{Q}_n}[1/p])$ be the natural map. Then we define the i -th **Iwasawa cohomology of** E by

$$\mathbb{H}^i(T) := \varprojlim_n \mathbb{H}_{\text{ét}}^i(\text{Spec}(\mathcal{O}_{\mathbb{Q}_n}[1/p]), j_* T)$$

where $\mathbb{H}_{\text{ét}}^i(\text{Spec}(\mathcal{O}_{\mathbb{Q}_n}[1/p]), j_* T)$ is the étale cohomology group. See Appendix A for the full cyclotomic extension.

Theorem 1.3 ([Kat04, Theorem 12.4.(1) and (3)]). *The following statements hold.*

- (1) $\mathbb{H}^2(T)$ is a finitely generated torsion module over Λ .
- (2) $\mathbb{H}^1(T)$ is free of rank one over Λ under Assumption 1.1.(2).

We recall the Iwasawa main conjecture without p -adic zeta functions à la Kato.

Conjecture 1.4 (Kato’s IMC, [Kat04, Conjecture 12.10], [Kur02, Conjecture 6.1]).

$$\text{char}_\Lambda \left(\frac{\mathbb{H}^1(T)}{\Lambda \mathbf{z}_{\text{Kato}}} \right) = \text{char}_\Lambda (\mathbb{H}^2(T))$$

where \mathbf{z}_{Kato} is Kato’s zeta element (Definition A.13).

Remark 1.5.

- (1) Following the argument of [Kur02, §6], [Kob03, Theorem 7.1.ii], if $E(\mathbb{Q}_{\infty, p})[p^\infty]$ is finite, then $\text{Sel}_{\text{str}}(\mathbb{Q}_\infty, E[p^\infty])^\vee$ and $\mathbb{H}^2(T)$ are pseudo-isomorphic as Λ -modules where $\text{Sel}_{\text{str}}(\mathbb{Q}_\infty, E[p^\infty])$ is the p -strict (“fine”) Selmer group of E over \mathbb{Q}_∞ .

- (2) Indeed, due to the argument of [Kat04, §13.13], if E has potentially good reduction at p , then $E(\mathbb{Q}_{\infty,p})[p^\infty]$ is finite.
- (3) The following statements are equivalent:
 - (a) E has potentially good reduction at p .
 - (b) The corresponding local automorphic representation at p is not (a twist of) Steinberg. (See [Roh94, §15].)
 - (c) The j -invariant of E is p -integral. (See [Liu06, Proposition 10.2.33].)

1.5. Main Theorems

Theorem 1.6 (The main conjecture). *Let E be an elliptic curve with additive reduction at $p > 7$ satisfying Assumption 1.1. If*

$$\tilde{\delta}_n \neq 0 \in \mathbb{F}_p$$

for some square-free product of Kolyvagin primes n , then we have

$$\text{char}_\Lambda \left(\frac{\mathbb{H}^1(T)}{\Lambda_{\mathbf{Z}_{\text{Kato}}}} \right) = \text{char}_\Lambda (\text{Sel}_{\text{str}}(\mathbb{Q}_\infty, E[p^\infty])^\vee).$$

If we further assume that E has potentially good reduction at p , then Kato’s IMC (Conjecture 1.4) holds, i.e.

$$\text{char}_\Lambda \left(\frac{\mathbb{H}^1(T)}{\Lambda_{\mathbf{Z}_{\text{Kato}}}} \right) = \text{char}_\Lambda (\mathbb{H}^2(T)).$$

Theorem 1.7 (The p -part of BSD formula). *Let E be an elliptic curve with additive reduction at $p > 7$ satisfying Assumption 1.1. Suppose that $L(E, 1) \neq 0$. If*

$$\tilde{\delta}_n \neq 0 \in \mathbb{F}_p$$

for some square-free product of Kolyvagin primes n , then the p -part of Birch and Swinnerton-Dyer formula for E holds, i.e.

$$\text{ord}_p (\#\text{III}(E/\mathbb{Q})[p^\infty]) = \text{ord}_p \left(\frac{L(E, 1)}{\Omega_E^+} \right).$$

Remark 1.8.

- (1) Even in the $p \leq 7$ case, Theorem 1.6 and Theorem 1.7 hold if (E, p) does **not** satisfy Assumption 2.5 below.
- (2) It is expected that there always exists a square-free product of Kolyvagin primes n such that $\tilde{\delta}_n \neq 0 \in \mathbb{F}_p$. Practically, it is easy to find such n ’s (cf. [Kur14]).

- (3) We do not know whether Theorem 1.6 directly implies Theorem 1.7 or not since there is neither a Mazur–Greenberg style main conjecture nor a control theorem for the additive reduction case.
- (4) If we replace $L(E, 1)$ by the N -imprimitive L -value $L^{(N)}(E, 1)$ in Theorem 1.7, then we can weaken Assumption 1.1.(1) by $p \nmid \text{Tam}(E)$ (cf. [MR04, Theorem 6.2.4]).

2. Computing the integral lattice

The goal of this section is to extend [Rub00, Proposition 3.5.1] to the additive reduction case over unramified extensions of \mathbb{Q}_p . More precisely, we compute the image of the logarithm map of the unramified local points of an elliptic curve with additive reduction.

In §2.1, we compute the integral image of the dual exponential map involving the Néron differential (Corollary 2.4) by computing the image of the logarithm map of the unramified local points of an elliptic curve with additive reduction (Theorem 2.1) via the Tate local duality. Then we review the formal groups of elliptic curves in §2.2, discuss the exceptional case (Assumption 2.5) when $p \leq 7$ in §2.3, and give a proof of Theorem 2.1 when $p > 7$ in §2.4.

The main idea is to replace a given elliptic curve with additive reduction by a different elliptic curve with good reduction over a ramified extension of degree 6 such that their generic fibers are isomorphic over the ramified extension following [Pan] and [KP]. Then we apply the theory of formal groups to the elliptic curve with good reduction. The restriction $p > 7$ appears here. In the $p \leq 7$ case, there are some exceptional cases as described in §2.3. We refer to [Pan] and [KP] for details. See also [SD13, Lemma 1].

2.1. The integral lattice

Let K be a finite unramified extension of \mathbb{Q}_p and k be the residue field. Since K/\mathbb{Q}_p is unramified, the base change to K does not change the reduction type of E ([Sil09, Proposition VII.5.4.(a)]).

For a given elliptic curve E/\mathbb{Q} with additive reduction, we consider an elliptic curve E/\mathbb{Q}_p by taking the base change. Let \mathcal{E} be a minimal Weierstrass model of E over \mathbb{Z}_p given by a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \quad (2.1)$$

where $a_i \in p\mathbb{Z}_p$ for each i . By the coordinate change, it is not difficult to see that such a minimal Weierstrass model always exists. See [Pan, Lemma 9] and [KP, Lemma 9], for example. Note that $\mathcal{E}(\mathcal{O}_K) = \mathcal{E}(K) = E(K)$ since \mathcal{E} is projective. Let \tilde{E}/\mathbb{F}_p be the reduction of \mathcal{E} modulo p and $\tilde{E}_{\text{ns}}/\mathbb{F}_p$ be the nonsingular locus of \tilde{E}/\mathbb{F}_p . Let $E_0(K) \subset E(K)$ be the inverse image of $\tilde{E}_{\text{ns}}(k)$ and $E_1(K) \subset E(K)$ be the inverse image of the identity of $\tilde{E}_{\text{ns}}(k)$, i.e. the kernel of the reduction map. They can also be explicitly written as

$$\begin{aligned}
 E_0(K) &= \{(x, y) \in E(K) : v_K(x) \leq 0, v_K(y) \leq 0\} \cup \{\infty\}, \\
 E_1(K) &= \{(x, y) \in E(K) : v_K(x) \leq -2, v_K(y) \leq -3\} \cup \{\infty\}
 \end{aligned}
 \tag{2.2}$$

where v_K is the normalized valuation of K .

Theorem 2.1 ([Pan, Theorem 1], [KP, Theorem 1, Corollary 2]). *Let E/\mathbb{Q}_p be an elliptic curve with additive reduction and K/\mathbb{Q}_p be a finite unramified extension. Then the extension of the formal logarithm map on E induces an isomorphism*

$$\log_E : E_0(K) \otimes \mathbb{Z}_p \simeq \mathcal{O}_K,
 \tag{2.3}$$

if (E, p) does not satisfy Assumption 2.5 below.

Proof of Theorem 2.1 when $p > 7$ is given in §2.4. We do not give a proof when $p \leq 7$ and (E, p) does not satisfy Assumption 2.5, but see §2.3. Indeed, we reproduce the proof in [Pan] and [KP] to make the isomorphism (2.3) explicit.

Proposition 2.2. *Let E/\mathbb{Q}_p be an elliptic curve with additive reduction at an odd prime p and K/\mathbb{Q}_p be a finite unramified extension. Then p does not divide $[E(K) : E_0(K)]$.*

Proof. It is well-known that $[E(K) : E_0(K)] \leq 4$ ([Sil09, Theorem VII.6.1]). Thus, the statement is automatic if $p \geq 5$. For $p = 3$, [Sil99, Table 4.1, Page 365] shows that

$$p \nmid [E(\mathbb{Q}_p^{\text{ur}}) : E_0(\mathbb{Q}_p^{\text{ur}})].$$

Since the formation of Néron models commutes with unramified (étale) base change ([BLR90, Proposition 2.(c), §1.2, Chapter 1]), we have

$$p \nmid [E(K) : E_0(K)]$$

following the argument in [BS12, Proof of Proposition 14]. \square

Corollary 2.3. *The images of $E(K) \otimes \mathbb{Z}_p$ and $E_0(K) \otimes \mathbb{Z}_p$ under the logarithm map coincide as \mathcal{O}_K .*

Due to [Sil99, Corollary IV.9.1], the identity component of the Néron model of E over \mathbb{Z}_p is isomorphic to \mathcal{E}_0 over \mathbb{Z}_p . Thus, we can choose a Néron differential ω_E as a basis of the cotangent space of \mathcal{E} over \mathcal{O}_K (up to a p -adic unit). Also, ω_E corresponds to $f(z)dz$ up to a p -adic unit via the modular parametrization under Assumption 1.1.(3).

Let $V = T \otimes \mathbb{Q}_p$ and $\mathbf{D}_{\text{dR}}(V)/\mathbf{D}_{\text{dR}}^0(V)$ be the tangent space of E over K and $\mathbf{D}_{\text{dR}}^0(V)$ be the cotangent space of E over K . Then we have a perfect pairing

$$\text{Tr} \circ \langle -, - \rangle_{\text{dR}} : \mathbf{D}_{\text{dR}}^0(V) \times \mathbf{D}_{\text{dR}}(V)/\mathbf{D}_{\text{dR}}^0(V) \rightarrow K \rightarrow \mathbb{Q}_p$$

where the first map is induced from the local Tate duality and the second map is the trace map. Let $\omega_E^* \in \mathbf{D}_{\text{dR}}(V)/\mathbf{D}_{\text{dR}}^0(V)$ be the dual basis to ω_E such that $\langle \omega_E^*, \omega_E \rangle_{\text{dR}} = 1$. The map

$$\widehat{E}(\mathcal{O}_K) \xrightarrow[\simeq]{\log_E} \mathcal{O}_K \xrightarrow[\simeq]{\cdot\omega_E^*} \mathcal{O}_K\omega_E^* \subseteq \mathbf{D}_{\text{dR}}(V)/\mathbf{D}_{\text{dR}}^0(V)$$

yields a natural integral structure on $\mathbf{D}_{\text{dR}}^0(V)$. By the local Tate duality, we have the following statement.

Corollary 2.4. *The image of $H^1(K, T)/H_f^1(K, T)$ under \exp^* is $\mathcal{O}_K\omega_E$ where $H_f^1(K, T)$ is the image of the Kummer map of $E(K) \otimes \mathbb{Z}_p$.*

2.2. Formal groups of elliptic curves

Let $\widehat{\mathcal{E}}$ be the formal group of elliptic curve \mathcal{E} over \mathbb{Z}_p and $\mathcal{F}_{\mathcal{E}}(X, Y)$ be the corresponding formal group law (as a two variable power series over \mathbb{Z}_p). Considering a_i 's in Equation (2.1) as variables, we can regard

$$\mathcal{F}_{\mathcal{E}}(X, Y) \in \mathbb{Z}_p[a_1, a_2, a_3, a_4, a_6][[X, Y]].$$

We also regard a $\mathbb{Z}_p[a_1, a_2, a_3, a_4, a_6]$ as a weighted ring with weight function $\text{wt}(a_i) = i$ for each i . Then the coefficients of $\mathcal{F}_{\mathcal{E}}(X, Y)$ in degree n are homogeneous of weight $n - 1$ by [Sil09, Proposition IV.1.1.(c)]. We also have an explicit isomorphism of topological groups

$$\psi_{K'} : E_1(K') = \mathcal{E}_1(K') \simeq \widehat{\mathcal{E}}(\mathfrak{m}_{K'})$$

defined by $(x, y) \mapsto -\frac{x}{y}$ for any finite extension K' of \mathbb{Q}_p ([Sil09, Proposition VII.2.2]).

2.3. Exceptional cases

Assumption 2.5 (*Exceptional cases*). Following the notation of Equation (2.1), if one of the following conditions

- (1) $p = 2$ and $a_1 + a_3 \equiv 2 \pmod{4}$,
- (2) $p = 3$ and $a_2 \equiv 6 \pmod{9}$,
- (3) $p = 5$ and $a_4 \equiv 10 \pmod{25}$, or
- (4) $p = 7$ and $a_6 \equiv 14 \pmod{49}$,

holds, then we call it an **exceptional case**.

In the exceptional cases, $E_0(K) = \mathcal{E}_0(K)$ admits non-trivial p -torsions and we only have

$$\log_E : \mathcal{E}_0(K) \otimes \mathbb{Z}_p \rightarrow \mathfrak{m}_K$$

as described in [Pan, Theorem 1] and [KP, Theorem 1, Corollary 2]. We do not consider the exceptional cases in this article.

2.4. General case: $p > 7$

From now on, we assume $p > 7$.

Definition 2.6 (Extended convergence of formal groups). Let $\mathcal{F}_\mathcal{E}(X, Y)$ be the formal group law for \mathcal{E} ([Sil09, Chapter IV]). Then $\mathcal{F}_\mathcal{E}(X, Y)$ converges to an element of \mathcal{O}_K for all $X, Y \in \mathcal{O}_K$ since all $a_i \in p\mathbb{Z}_p$. Thus, we can extend the formal group structure on $\widehat{\mathcal{E}}(\mathfrak{m}_K)$ to $\widehat{\mathcal{E}}(\mathcal{O}_K)$.

Note that the weighted ring structure of the formal group law in §2.2 is used in the definition.

Proof of Theorem 2.1. Let $L := K(\sqrt[p]{p})$ and $\varpi \in L$ be an element of L satisfying $\varpi^6 = p$, and \mathcal{C} be a smooth projective curve over \mathcal{O}_L embedded in $\mathbb{P}_{\mathcal{O}_L}^2$ defined by the following Weierstrass equation

$$y^2 + \frac{a_1}{\varpi^1}xy + \frac{a_3}{\varpi^3}y = x^3 + \frac{a_2}{\varpi^2}x^2 + \frac{a_4}{\varpi^4}x^4 + \frac{a_6}{\varpi^6}.$$

We also define \mathcal{C}_0 and \mathcal{C}_1 in the same manner. Note that all the coefficients lie in \mathcal{O}_L due to $\varpi^6 = p$. Then there exists a birational map ϕ over \mathcal{O}_L

$$\mathcal{E} \times_{\text{Spec}(\mathcal{O}_K)} \text{Spec}(\mathcal{O}_L) \xrightarrow{\phi} \mathcal{C}$$

given by $\phi(x, y) = (\frac{x}{\varpi^2}, \frac{y}{\varpi^3})$ pointwisely. Note that ϕ would not be defined at the singular point in the special fiber. However, ϕ becomes an isomorphism on the generic fibers over L and sends the point at infinity of \mathcal{E} to that of \mathcal{C} by definition. Thus, $\mathcal{E}(L)$ and $\mathcal{C}(L)$ are isomorphic as topological groups. Furthermore, ϕ induces a set-theoretic bijection between $\mathcal{E}_0(L)$ and $\mathcal{C}_1(L)$ due to Equation (2.2), so it is an isomorphism of topological groups. We have the following commutative diagram (*a priori* of sets)

$$\begin{CD} \mathcal{E}_0(K) @<<< \mathcal{E}_0(L) @>>> \mathcal{C}_1(L) \\ @VV \Psi_K V @VV \Psi_L V @VV \psi_L V \\ \widehat{\mathcal{E}}(\mathcal{O}_K) @<<< \widehat{\mathcal{E}}(\mathcal{O}_L) @>>> \widehat{\mathcal{C}}(\mathfrak{m}_L) \end{CD}$$

where Ψ_K and Ψ_L are defined by the assignment $(x, y) \mapsto -\frac{x}{y}$.

Let \mathcal{F}_C be the formal group law of C over \mathcal{O}_L . By looking at the Weierstrass equations and the corresponding formal group laws, it is easy to see that

$$\varpi \cdot \mathcal{F}_{\mathcal{E}, \mathcal{O}_L}(X, Y) = \mathcal{F}_C(\varpi \cdot X, \varpi \cdot Y),$$

where $\mathcal{F}_{\mathcal{E}, \mathcal{O}_L} = \mathcal{F}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathcal{O}_L$.

Thus, ϕ , $\varpi \times$, and ψ_L in the diagram are all isomorphisms of topological groups, so Ψ_L is also an isomorphism of topological groups. Thus, $\mathcal{E}_0(K) \simeq \text{Im}(\Psi_K) \subseteq \widehat{\mathcal{E}}(\mathcal{O}_L)$. Also, we can easily see $\widehat{\mathcal{E}}(\mathcal{O}_K) = \text{Im}(\Psi_K)$ by considering $\Psi_K : (x, y) \mapsto -\frac{x}{y}$. Thus, Ψ_K is also an isomorphism of topological groups.

Since we assume $p > 7$, we have $v_L(p) = 6v_K(p) = 6e(K/\mathbb{Q}_p) < p - 1$. Thus, the formal logarithm on C induces an isomorphism

$$\log_{\widehat{C}} : \widehat{C}(\mathfrak{m}_L) \simeq \mathfrak{m}_L.$$

Since $\mathcal{E}_0(K)$ embeds into $\widehat{C}(\mathfrak{m}_L)$, $\mathcal{E}_0(K)$ is also torsion-free. Since the formal logarithm map extends to $\log \otimes \mathbb{Q}_p : \mathcal{E}(K) \otimes \mathbb{Q}_p \simeq K$, we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{E}_1(K) & \longrightarrow & \mathcal{E}_0(K) & \longrightarrow & k \longrightarrow 0 \\ & & \simeq \downarrow \psi_K & & \simeq \downarrow \Psi_K & & \parallel \\ 0 & \longrightarrow & \widehat{\mathcal{E}}(\mathfrak{m}_K) & \longrightarrow & \widehat{\mathcal{E}}(\mathcal{O}_K) & \longrightarrow & k \longrightarrow 0 \\ & & \simeq \downarrow \log_{\widehat{\mathcal{E}}} & & \simeq \downarrow \log_E & & \parallel \\ 0 & \longrightarrow & \mathfrak{m}_K & \longrightarrow & \mathcal{O}_K & \longrightarrow & k \longrightarrow 0 \end{array}$$

and it proves Theorem 2.1. \square

3. The main conjecture: review of [KKS]

In this section, we present a slightly different computation from that of [KKS] due to a different normalization of Kato’s Euler system. Let E be an elliptic curve over \mathbb{Q} with additive reduction at p ; especially, the Euler factor at p of L -function $L(E, s)$ is 1.

3.1. Kato’s Euler systems and modular symbols

Let $c_{\mathbb{Q}(\mu_n)}(1) \in H^1(\mathbb{Q}(\mu_n), T)$ be Kato’s Euler system for T at $\mathbb{Q}(\mu_n)$ such that

$$\exp^*(c_{\mathbb{Q}}(1)) = \frac{L^{(Np)}(E, 1)}{\Omega_E^+}$$

where $L^{(Np)}(E, \chi, 1)$ is the Np -imprimitive L -value of E at $s = 1$ twisted by χ . See Definition A.20 for the precise choice of $c_{\mathbb{Q}(n)}(1)$.

For $a \in (\mathbb{Z}/n\mathbb{Z})^\times$, write $\sigma_{a^{-1}} \in \text{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})$ as the image of a under the global Artin map, which behaves like the inverse of the p -adic cyclotomic character.

Following [Rub00, Theorem 3.5.1] and [Kat04, Theorem 6.6 and Theorem 9.7], we have the following zeta value formula.

Theorem 3.1. *Let E be an elliptic curve over \mathbb{Q} with additive reduction at p . Let χ be a Dirichlet character of p -power exponent and conductor n with $(n, Np) = 1$. Then we have*

$$\begin{aligned} & \sum_{c \in (\mathbb{Z}/n\mathbb{Z})^\times} \chi(c) \cdot \left\langle \omega_E^*, \exp^* \left(\left(c_{\mathbb{Q}(\mu_n)}^{\chi(-1)}(1) \right)^{\sigma_c} \right) \right\rangle_{\text{dR}} \\ &= \left(\prod_{q|N_{\text{sp}}} (1 - q^{-1}\chi(q)) \right) \cdot \left(\prod_{q|N_{\text{ns}}} (1 + q^{-1}\chi(q)) \right) \cdot \frac{L^{(p)}(E, \chi, 1)}{\Omega_E^{\chi(-1)}} \end{aligned} \tag{3.1}$$

where $c_{\mathbb{Q}(\mu_n)}^{\chi(-1)}(1)$ lies in the eigenspace with respect to the complex conjugation with eigenvalue $\chi(-1)$ and $L^{(p)}(E, \chi, 1)$ is the p -imprimitive L -value of E at $s = 1$ twisted by χ .

Remark 3.2. See §A.3 for the interpolation formula for such Dirichlet characters.

We rewrite the last term in Equation (3.1) in terms of modular symbols. Following [Pol14, (2.2)], we have

$$L(E, \chi, 1) = \frac{\tau(\chi)}{n} \cdot \sum_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} \bar{\chi}(a) \cdot 2\pi i \cdot \int_{i\infty}^{-a/n} f(z) dz \tag{3.2}$$

where $\tau(\chi)$ is the Gauss sum of χ . Let $G_n = (\mathbb{Z}/n\mathbb{Z})^\times$, $G_{n,p}$ be the p -part of G_n , and G_n^p be the prime-to- p part of G_n . Expanding the Gauss sum in Equation (3.2), we have

$$\begin{aligned} \chi(-1) \cdot \frac{L(E, \chi, 1)}{\Omega_E^{\chi(-1)}} &= \frac{1}{n} \cdot \sum_{c \in (\mathbb{Z}/n\mathbb{Z})^\times} \chi(c) \cdot \sigma_c \cdot \left(\sum_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} \zeta_n^a \cdot \left[\frac{-a}{n} \right]^{\chi(-1)} \right) \\ &= \frac{1}{n} \cdot \sum_{c_1 \in G_{n,p}} \sum_{c_2 \in G_n^p} \chi(c_1 c_2) \cdot \sigma_{c_1} \cdot \sigma_{c_2} \cdot \left(\sum_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} \zeta_n^a \cdot \left[\frac{-a}{n} \right]^{\chi(-1)} \right) \\ &= \frac{1}{n} \cdot \sum_{c_1 \in G_{n,p}} \sum_{c_2 \in G_n^p} \chi(c_1) \cdot \sigma_{c_1} \cdot \sigma_{c_2} \cdot \left(\sum_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} \zeta_n^a \cdot \left[\frac{-a}{n} \right]^{\chi(-1)} \right). \end{aligned}$$

Let $\mathbb{Q}(n)$ be the maximal p -subextension of \mathbb{Q} in $\mathbb{Q}(\mu_n)$ and $\mathcal{O}_{\mathbb{Q}(n)}$ the ring of integers of $\mathbb{Q}(n)$. We define the values

$$c_{\mathbb{Q}(n)}^{\text{an},\pm} := \frac{\pm 1}{n} \cdot \left(\prod_{q|N_{\text{sp}}} (1 - q^{-1}\sigma_q^{-1}) \right) \cdot \left(\prod_{q|N_{\text{ns}}} (1 + q^{-1}\sigma_q^{-1}) \right) \cdot \sum_{c_2 \in G_n^p} \sigma_{c_2} \cdot \left(\sum_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} \zeta_n^a \cdot \left[\frac{-a}{n} \right]_f^\pm \right) \in \mathbb{Z}_p \otimes \mathcal{O}_{\mathbb{Q}(n)}$$

in order to have

$$\sum_{c_1 \in G_{n,p}} \left(\sigma_{c_1} \left(c_{\mathbb{Q}(n)}^{\text{an},\chi^{(-1)}} \right) \right) \cdot \chi(c_1) = \frac{L^{(Np)}(E, \chi, 1)}{\Omega_E^{\chi^{(-1)}}}. \tag{3.3}$$

Indeed, $c_{\mathbb{Q}(n)}^{\text{an},-} = 0$ since $\mathbb{Q}(n)$ is totally real.

Comparing the coefficients of (3.1) and (3.3), we can easily observe the following statement.

Proposition 3.3. *For any $c_1 \in G_{n,p}$, we have*

$$\left\langle \omega_E^*, \exp^* \left(\left(c_{\mathbb{Q}(n)}^{\chi^{(-1)}}(1) \right)^{\sigma_{c_1}} \right) \right\rangle_{\text{dR}} = \sigma_{c_1} \left(c_{\mathbb{Q}(n)}^{\text{an},\chi^{(-1)}} \right)$$

where $c_{\mathbb{Q}(n)}^{\chi^{(-1)}}(1)$ is Kato’s Euler system at $\mathbb{Q}(n)$.

3.2. Kolyvagin derivatives on modular symbols and Kurihara numbers

Let ℓ be a Kolyvagin prime and η_ℓ be a primitive root modulo ℓ . Let

$$D_\ell := \sum_{i=0}^{\ell-2} i \sigma_{\eta_\ell}^i \in \mathbb{Z}[\text{Gal}(\mathbb{Q}(\mu_\ell)/\mathbb{Q})]$$

be the Kolyvagin derivative operator for $\text{Gal}(\mathbb{Q}(\mu_\ell)/\mathbb{Q})$ with respect to η_ℓ and $D_n = \prod_{\ell|n} D_\ell$. Let $p^k \parallel (\ell - 1)$. Then we let

$$\bar{D}_\ell := \sum_{i=0}^{p^k-1} i \sigma_{\eta_\ell}^{\frac{\ell-1}{p^k}i} \in \mathbb{Z}[\text{Gal}(\mathbb{Q}(\ell)/\mathbb{Q})]$$

also be the Kolyvagin derivative operator for $\text{Gal}(\mathbb{Q}(\ell)/\mathbb{Q})$ with respect to $\eta_\ell^{\frac{\ell-1}{p^k}}$ and $\bar{D}_n = \prod_{\ell|n} \bar{D}_\ell$. Let $a, b \in \mathbb{Z}$ such that

$$\frac{\ell - 1}{p^k} \cdot a + p^k \cdot b = 1.$$

By rewriting

$$D_\ell = \sum_{i=0}^{\ell-2} \left(\frac{\ell-1}{p^k} a + p^k b \right) \cdot i \cdot \sigma_{\eta_\ell}^{\left(\frac{\ell-1}{p^k} a + p^k b\right) i},$$

the natural quotient map

$$\mathbb{Z}[\text{Gal}(\mathbb{Q}(\mu_\ell)/\mathbb{Q})] \rightarrow \mathbb{Z}[\text{Gal}(\mathbb{Q}(\ell)/\mathbb{Q})]$$

yields

$$D_\ell \mapsto \left(\frac{\ell-1}{p^k} \right) \cdot \left(\sum_{i=0}^{\ell-2} a \cdot i \cdot \sigma_{\eta_\ell}^{\frac{\ell-1}{p^k} ai} \right) + p^k \cdot \left(\sum_{i=0}^{\ell-2} b \cdot i \cdot \sigma_{\eta_\ell}^{\frac{\ell-1}{p^k} ai} \right)$$

since $\sigma_{\eta_\ell}^{p^k}$ maps to 1. We also have

$$\sum_{i=0}^{\ell-2} a \cdot i \cdot \sigma_{\eta_\ell}^{\frac{\ell-1}{p^k} ai} \equiv \left(\frac{\ell-1}{p^k} \right) \cdot \sum_{i=0}^{p^k-1} a \cdot i \cdot \sigma_{\eta_\ell}^{\frac{\ell-1}{p^k} ai} \pmod{p} \tag{3.4}$$

in $\mathbb{F}_p[\text{Gal}(\mathbb{Q}(\ell)/\mathbb{Q})]$. Since a is non-zero mod p , we can replace $a \cdot i$ by i in the RHS of (3.4).

Repeating this argument for each Kolyvagin prime dividing n , we have the following lemma, which compares Kolyvagin derivatives for $\text{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})$ and $\text{Gal}(\mathbb{Q}(n)/\mathbb{Q})$.

Lemma 3.4.

$$D_n \equiv u \cdot \overline{D}_n$$

for some $u \in \mathbb{F}_p^\times$ in $\mathbb{F}_p[\text{Gal}(\mathbb{Q}(n)/\mathbb{Q})]$.

Considering the mod p Taylor expansion of Mazur-Tate elements at $\prod_{i=1}^s (\sigma_{\eta_{\ell_i}} - 1)$ where $n = \prod_{i=1}^s \ell_i$, we observe the following statement.

Proposition 3.5 ([KKS, Theorem 7.5]). *Let n be a square-free product of Kolyvagin primes and D_n be “the” Kolyvagin derivative operator in $\mathbb{Z}_p[\text{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})]$. We have the following equalities in \mathbb{F}_p*

$$D_n \left(\sum_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} \zeta_n^{a'} \left[\frac{a}{n} \right]_f^\pm \right) \equiv \sum_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} \left(\prod_{\ell|n} \log_{\mathbb{F}_\ell}(a) \right) \cdot \left[\frac{a}{n} \right]_f^\pm \pmod{p}$$

where $a' = \pm a$.

Proof. Suppose that $\kappa_n \pmod{p} = 0$ in $H^1_{\mathcal{F}(n)}(\mathbb{Q}, T/pT) \otimes G_n$. Then, following [MR04, Appendix A], a straightforward computation yields

$$d_n^+ = 0 \in H^1(\mathbb{Q}(n), T)/pH^1(\mathbb{Q}(n), T).$$

Thus, $\overline{D}_n c_{\mathbb{Q}(n)}^+(1) \in pH^1(\mathbb{Q}(n), T)$. By Proposition 3.3, if $\overline{D}_n \langle \omega_E^*, \exp^* (\text{loc}_p c_{\mathbb{Q}(n)}^+(1)) \rangle_{\text{dR}} \in p\mathbb{Z}_p \otimes \mathcal{O}_{\mathbb{Q}(n)}$, then we have

$$\left(\overline{D}_n c_{\mathbb{Q}(n)}^{\text{an},+} \right) \in p\mathbb{Z}_p \otimes \mathcal{O}_{\mathbb{Q}(n)}. \tag{3.6}$$

Due to Lemma 3.4 and Proposition 3.5, (3.6) is equivalent to

$$\left(\prod_{q|N_{\text{sp}}} (1 - q^{-1}\sigma_q^{-1}) \right) \cdot \left(\prod_{q|N_{\text{ns}}} (1 + q^{-1}\sigma_q^{-1}) \right) \cdot \tilde{\delta}_n = 0$$

in $\mathbb{F}_p \otimes \mathcal{O}_{\mathbb{Q}(n)}$ and indeed, $\tilde{\delta}_n \in \mathbb{F}_p$; thus, it is equivalent to

$$\left(\prod_{q|N_{\text{sp}}} (1 - q^{-1}) \right) \cdot \left(\prod_{q|N_{\text{ns}}} (1 + q^{-1}) \right) \cdot \tilde{\delta}_n = 0 \in \mathbb{F}_p.$$

Due to Assumption 1.1.(1), it is equivalent to

$$\tilde{\delta}_n = 0 \in \mathbb{F}_p. \quad \square$$

As a result of Theorem 3.7, we obtain an analogue of Conjecture 1.4 with $\mathbf{z}_{\text{Kato}}^{(N)}$ (defined in (A.3)). By Lemma A.21, Conjecture 1.4 follows.

4. The p -part of the Birch and Swinnerton-Dyer formula for the rank zero case

The goal of this section is to prove Theorem 1.7, which generalizes [MR04, Theorem 6.2.4] to the additive reduction case when the corresponding Kolyvagin system is primitive.

Assumption 4.1 ($\text{Hyp}(\mathbb{Q}, T)$, [Rub00, §2.1]).

- (1) There exists an element $\tau \in G_{\mathbb{Q}}$ such that
 - τ acts trivially on μ_{p^∞} , and
 - $T/(\tau - 1)T$ is free of rank one over \mathbb{Z}_p .
- (2) $\overline{\rho}$ is irreducible.

We recall the “error term” in the Euler system argument. Let $W = E[p^\infty]$ and $\mathbb{Q}(W)$ be the smallest extension of \mathbb{Q} where $G_{\mathbb{Q}(W)}$ acts trivially on $E[p^\infty]$. Let

$$\begin{aligned} \mathfrak{n}_{E[p^\infty]} &:= \text{length}_{\mathbb{Z}_p} \left(H^1(\mathbb{Q}(W)/\mathbb{Q}, E[p^\infty]) \cap \text{Sel}_{\text{rel}}(\mathbb{Q}, E[p^\infty]) \right), \\ \mathfrak{n}_{E[p^\infty]}^* &:= \text{length}_{\mathbb{Z}_p} \left(H^1(\mathbb{Q}(W)/\mathbb{Q}, E[p^\infty]) \cap \text{Sel}_{\text{str}}(\mathbb{Q}, E[p^\infty]) \right). \end{aligned}$$

By [Rub00, Proposition 3.5.8.(ii)], if the p -adic representation $\rho : G_{\mathbb{Q}} \rightarrow \text{Aut}_{\mathbb{Z}_p}(T)$ is surjective, then

$$H^1(\mathbb{Q}(W)/\mathbb{Q}, E[p^\infty]) \simeq H^1(\text{GL}_2(\mathbb{Z}_p), (\mathbb{Q}_p/\mathbb{Z}_p)^2) = 0,$$

so $\mathfrak{n}_{E[p^\infty]} = \mathfrak{n}_{E[p^\infty]}^* = 0$. Furthermore, if $\bar{\rho}$ is surjective with $p > 3$, then ρ is surjective since the determinant of ρ is the cyclotomic character by [SD86, Lemma 1, §1].

The machinery of Euler and Kolyvagin systems yields the following theorem.

Theorem 4.2. *Suppose that $p > 2$ and T satisfies $\text{Hyp}(\mathbb{Q}, T)$. Let $\mathfrak{c}(1)$ be an Euler system for T and κ be the corresponding Kolyvagin system with $c_{\mathbb{Q}}(1) = \kappa_1$. Assume that $\text{loc}_p^s(c_{\mathbb{Q}}(1)) = \text{loc}_p^s(\kappa_1) \neq 0$.*

- (1) $\text{length}_{\mathbb{Z}_p}(\text{Sel}_{\text{str}}(\mathbb{Q}, E[p^\infty]))$ is finite if and only if $\kappa_1 \neq 0$.
- (2) $\text{length}_{\mathbb{Z}_p}(\text{Sel}_{\text{str}}(\mathbb{Q}, E[p^\infty])) \leq \partial^{(0)}(\kappa) := \max\{j : \kappa_1 \in p^j \text{Sel}_{\text{rel}}(\mathbb{Q}, T)\}$.
- (3) If κ is primitive, then

$$\text{length}_{\mathbb{Z}_p}(\text{Sel}_{\text{str}}(\mathbb{Q}, E[p^\infty])) = \partial^{(0)}(\kappa).$$

Proof. See [Rub00, Theorem 2.2.2] and [MR04, Corollary 5.2.13] for detail. \square

Lemma 4.3 ([CS10, 1.9. Proposition]). *If $\text{Sel}(\mathbb{Q}, E[p^\infty])$ is finite, then we have the following statements:*

- (1) $H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, E[p^\infty]) = 0$;
- (2) if we further assume $E(\mathbb{Q})[p]$ is trivial, then the global-to-local map defining Selmer groups is surjective.

Indeed, the vanishing of $H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, E[p^\infty])$ is equivalent to the finiteness of the p -strict Selmer group. See [Hac10, Lemma 3.2].

Using (the Pontryagin dual of) the second sequence of [PR00, A.3.2] with the p -strict local condition for B and the p -relaxed local condition for A , we have the following exact sequence (see also [Kob03, (7.18)])

$$H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, E[p^\infty])^\vee \supseteq \text{Sel}_{\text{rel}}(\mathbb{Q}, T) \supseteq \frac{H^1(\mathbb{Q}_p, T)}{E(\mathbb{Q}_p) \otimes \mathbb{Z}_p} \supseteq \text{Sel}(\mathbb{Q}, E[p^\infty])^\vee \supseteq \text{Sel}_{\text{str}}(\mathbb{Q}, E[p^\infty])^\vee \supseteq 0. \tag{4.1}$$

Note that here we use Lemma 4.3.(2) to have the third and the fourth terms.

Furthermore, under the finiteness of the Selmer group (due to Lemma 4.3.(1)), (4.1) becomes

$$0 \rightarrow \text{Sel}_{\text{rel}}(\mathbb{Q}, T) \rightarrow \frac{H^1(\mathbb{Q}_p, T)}{E(\mathbb{Q}_p) \otimes \mathbb{Z}_p} \rightarrow \text{Sel}(\mathbb{Q}, E[p^\infty])^\vee \rightarrow \text{Sel}_{\text{str}}(\mathbb{Q}, E[p^\infty])^\vee \rightarrow 0.$$

Taking the quotient by the \mathbb{Z}_p -module generated by $c_{\mathbb{Q}}(1)$, Kato’s Euler system at \mathbb{Q} , we have

$$0 \rightarrow \frac{\text{Sel}_{\text{rel}}(\mathbb{Q}, T)}{\mathbb{Z}_p c_{\mathbb{Q}}(1)} \rightarrow \frac{H_s^1(\mathbb{Q}_p, T)}{\mathbb{Z}_p \text{loc}_p^s c_{\mathbb{Q}}(1)} \rightarrow \text{Sel}(\mathbb{Q}, E[p^\infty])^\vee \rightarrow \text{Sel}_{\text{str}}(\mathbb{Q}, E[p^\infty])^\vee \rightarrow 0 \tag{4.2}$$

where $H_s^1(\mathbb{Q}_p, T) = \frac{H^1(\mathbb{Q}_p, T)}{E(\mathbb{Q}_p) \otimes \mathbb{Z}_p}$.

The following theorem is the generalized and sharpened version of [Rub00, Theorem 2.2.10.(ii)].

Theorem 4.4. *Suppose that $p > 2$ and T satisfies $\text{Hyp}(\mathbb{Q}, T)$. Let $\mathbf{c}(1)$ be an Euler system for T and κ be the corresponding Kolyagin system with $c_{\mathbb{Q}}(1) = \kappa_1$. Assume that $\text{loc}_p^s(\kappa_1) \neq 0$ and κ is primitive. Then $\text{Sel}(\mathbb{Q}, E[p^\infty])$ is finite and*

$$\text{length}_{\mathbb{Z}_p}(\text{Sel}(\mathbb{Q}, E[p^\infty])) = \text{length}_{\mathbb{Z}_p} H_s^1(\mathbb{Q}_p, T) / \text{loc}_p^s(\kappa_1).$$

Proof. The finiteness of Selmer groups is already proved in [Rub00, Theorem 2.2.10.(ii)]. We have

$$\begin{aligned} \partial^{(0)}(\kappa) &= \text{length}_{\mathbb{Z}_p}(\text{loc}_p^s(\text{Sel}_{\text{rel}}(\mathbb{Q}, T)) / \mathbb{Z}_p \text{loc}_p^s(\kappa_1)) && (4.2) \\ &< \infty. && (\text{loc}_p^s(\kappa_1) \neq 0) \end{aligned}$$

By Theorem 4.2.(3), we have

$$\text{length}_{\mathbb{Z}_p}(\text{Sel}_{\text{str}}(\mathbb{Q}, E[p^\infty])) = \text{length}_{\mathbb{Z}_p}(\text{loc}_p^s(\text{Sel}_{\text{rel}}(\mathbb{Q}, T)) / \mathbb{Z}_p \text{loc}_p^s(\kappa_1)) < \infty.$$

Then, by Sequence (4.2) again,

$$\text{length}_{\mathbb{Z}_p}(\text{Sel}(\mathbb{Q}, E[p^\infty])) = \text{length}_{\mathbb{Z}_p}(H_s^1(\mathbb{Q}_p, T) / \mathbb{Z}_p \text{loc}_p^s(\kappa_1)) < \infty. \quad \square$$

Applying Theorem 3.1, we extend [Rub00, Theorem 3.5.11] to the additive reduction case.

Theorem 4.5. *Let E be a non-CM elliptic curve over \mathbb{Q} with additive reduction at an odd prime $p > 7$. Assume that $L(E, 1) \neq 0$, $\bar{\rho}$ is surjective, $p \nmid \text{Tam}(E)$ and κ is primitive. Then*

$$\text{ord}_p(\#\text{III}(E/\mathbb{Q})[p^\infty]) = \text{ord}_p\left(\frac{L(E, 1)}{\Omega_E^+}\right).$$

Proof. By Corollary 2.4, we have

$$\langle \omega_E^*, \exp^*(H_s^1(\mathbb{Q}_p, T)) \rangle = \mathbb{Z}_p \subseteq \mathbb{Q}_p, \quad \langle \omega_E^*, \exp^*(\mathbb{Z}_p \text{loc}_p^s \kappa_1) \rangle = \frac{L(E, 1)}{\Omega_E^+} \mathbb{Z}_p.$$

Since $L(E, 1) \neq 0$, we have $\text{loc}_p^s(\kappa_1) \neq 0$ via Theorem 3.1. Also, the finiteness of $\text{Sel}(\mathbb{Q}, E[p^\infty])$ and the surjectivity of $\bar{\rho}$ show that

$$\#\text{Sel}(\mathbb{Q}, E[p^\infty]) = \#\text{III}(E/\mathbb{Q})[p^\infty].$$

By Theorem 4.4, we obtain

$$\text{length}_{\mathbb{Z}_p}(\text{Sel}(\mathbb{Q}, E[p^\infty])) = \text{length}_{\mathbb{Z}_p} H_s^1(\mathbb{Q}_p, T) / \text{loc}_p^s(\kappa_1).$$

Thus, the conclusion follows. \square

5. Examples

In this section, we provide three new examples of the main conjecture and the p -part of the BSD formula. The Sage code for an effective computation of Kurihara numbers due to Alexandru Ghitza is available at

https://github.com/aghitza/kurihara_numbers.

Although the original code is for good reduction, a slight modification allows us to compute the additive reduction case.

5.1. Elliptic curve of conductor 56144

Let $p = 11$ and E be the elliptic curve over \mathbb{Q} defined by

$$y^2 = x^3 - 584551x - 172021102$$

with conductor $56144 = 2^4 \cdot 11^2 \cdot 29$ as in [LMF17, Elliptic Curve 56144.w1]. The SAGE computation yields the following facts:

- The mod 11 representation is surjective;
- The Tamagawa factor of E is prime to 11;

- $a_{29}(E) = 1$, so 11 does not divide $28 = 29 - 1$.
- $\tilde{\delta}_{397,859} \neq 0$ where 397 and 859 are (the smallest) Kolyvagin primes for (E, p) .

Thus, Theorem 1.6 and Theorem 1.7 for (E, p) hold. Note that these theorems do not directly follow from the one divisibility since $\#\text{III}(E/\mathbb{Q})[11^\infty]$ is 121.

5.2. Elliptic curve of conductor 84700

Let $p = 11$ and E be the elliptic curve over \mathbb{Q} defined by

$$y^2 = x^3 - 235390375x - 1409480751250$$

with conductor $84700 = 2^2 \cdot 5^2 \cdot 7 \cdot 11^2$ as in [LMF17, Elliptic Curve 84700.bi1]. The SAGE computation yields the following facts:

- The mod 11 representation is surjective;
- The Tamagawa factor is prime to 11;
- $a_7(E) = -1$, so 11 does not divide $8 = 7 + 1$;
- $\tilde{\delta}_{23,2113} \neq 0$ where 23 and 2113 are (the smallest) Kolyvagin primes for (E, p) .

Thus, Theorem 1.6 and Theorem 1.7 for (E, p) hold. Note that these results do not directly follow from the one divisibility since $\#\text{III}(E/\mathbb{Q})[11^\infty]$ is 121.

5.3. Elliptic curve of conductor 84100

Let $p = 5$ and E be the elliptic curve over \mathbb{Q} defined by

$$y^2 = x^3 + x^2 - 28033x + 1232688$$

with conductor $84100 = 2^2 \cdot 5^2 \cdot 29^2$ as in [LMF17, Elliptic Curve 84100.b3]. The SAGE computation yields the following facts:

- The mod 5 representation is surjective;
- $a_4 = 28033 \not\equiv 10 \pmod{25}$, so it is not an exceptional case (Assumption 2.5);
- The Tamagawa factor is prime to 5;
- $\tilde{\delta}_{191,331} \neq 0$ where 191 and 331 are (the smallest) Kolyvagin primes for (E, p) .

Considering Remark 1.8.(1), Theorem 1.6 and Theorem 1.7 hold for (E, p) . Note that these results do not directly follow from the one divisibility since the rank of E is two.

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Appendix A. The choice of Kato's Euler system

The goal of this appendix is to give an *explicit* characterization of the “optimal” Kato's Euler system

$$c_F \in H^1(F, T(-1))$$

where F runs over abelian extensions of \mathbb{Q} such that

$$\exp^*(c_{\mathbb{Q}}(1)) = \frac{L^{(Np)}(E, 1)}{\Omega_E^+}$$

where $c_{\mathbb{Q}}(1)$ is the image of $c_{\mathbb{Q}}$ in $H^1(\mathbb{Q}, T)$ under the map induced from the Tate twist as described in [Kat04, Theorem 12.5]. This choice of Kato's Euler system is used in many literatures; however, it seems a bit difficult to find the explicit choice from the original construction of Kato in the literatures. Note that the Euler system in [Kat04] is constructed for $T(-1)$ due to the cohomological convention of the Galois representation ([Kat04, §14.10]).

Following [Rub98, Corollary 7.2] and [Sch98], there exists an Euler system $\{c'_F(1)\}_F$ for T such that

$$\exp^*(c'_{\mathbb{Q}}(1)) = r_E \cdot \frac{L^{(Np)}(E, 1)}{\Omega_E^+}$$

for some positive integer r_E . The goal of this appendix is to show that we can take $r_E = 1$ following [Kat04, Theorem 6.6] and [Del08, Remark in Appendix A (Page 254)] choosing the “right” Kato's Euler system.

We recall the convention for Kato's Euler systems. We assume some familiarity with [Kat04] here. We sometimes may use the notation therein from without any caution.

A.1. The cohomology classes

Let $f \in S_k(\Gamma_1(N), \psi)$ be a newform with Hecke field F over \mathbb{Q} and $S(f)$ be the rank one Hecke module generated by f over F as in [Kat04, §6.3]. Let λ be the place of F compatible with a choice of embedding $\iota_p : \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_p$. Let $m \geq 1$ be an integer.

Definition A.1. Following [Kat04, §5.1], we define ξ and S as follows:

- (1) ξ is a symbol $a(A)$ where $a, A \in \mathbb{Z}$, $A \geq 1$ and S is a non-empty finite set of primes containing $\text{prime}(mA)$, or
- (2) ξ is an element of $\text{SL}_2(\mathbb{Z})$ and S is a non-empty finite set of primes containing $\text{prime}(mN)$.

Definition A.2. Following [Kat04, (5.2.1) and (5.2.2)], we define integers r, r' and positive integers c, d as follows:

- (1) $1 \leq r \leq k - 1$, $1 \leq r' \leq k - 1$, and at least one of r, r' is $k - 1$.
- (2) c and d are positive integers with $\text{prime}(cd) \cap S = \emptyset$, and $(d, N) = 1$.

Definition A.3. Let

$$c, d z_m(f, r, r', \xi, S) \in S(f) \otimes \mathbb{Q}(\mu_m)$$

be the zeta modular form in [Kat04, §6.3] where c, d, r, r', ξ , and S satisfy Definition A.1 and Definition A.2.

For any $x \in V_F(f)$, $x^\pm := \frac{1}{2}(1 \pm \iota)x$ where ι is induced from the complex conjugation. Let $V_F(f)^\pm := V_F(f)^{\iota=\pm 1}$ and then $\dim_F V_F(f)^\pm = 1$, respectively. Let $\text{per}_f : S(f) \rightarrow V_F(f) \otimes_F \mathbb{C}$ be the period map in [Kat04, §6.3]. Let f^* be the complex conjugation of f (“the dual modular form”).

Theorem A.4 ([Kat04, Theorem 6.6]). Let χ be a character on $(\mathbb{Z}/m\mathbb{Z})^\times$. Let $\pm = (-1)^{k-r-1} \cdot \chi(-1)$ and

$$(u, v) := \begin{cases} (r + 2 - k, r) & \text{if } r' = k - 1 \\ (k - r', r') & \text{if } r = k - 1 \end{cases}$$

following [Kat04, (4.2.4)]. We add the following condition.

(*) If $\xi \in \text{SL}_2(\mathbb{Z})$, then assume $c \equiv d \equiv 1 \pmod{N}$.

Then we have

$$\sum_{b \in (\mathbb{Z}/m\mathbb{Z})^\times} \chi(b) \cdot \text{per}_f(\sigma_b(c, d z_m(f, r, r', \xi, S)))^\pm = L^{(S)}(f^*, \chi, r) \cdot (2\pi i)^{k-r-1} \cdot \gamma^\pm$$

where

$$\begin{aligned} \gamma &= c^2 d^2 \delta(f, r', a(A)) - c^u d^2 \bar{\chi}(c) \delta(f, r', ac(A)) \\ &\quad - c^2 d^v \psi(d) \delta(f, r', "a/d"(A)) + c^u d^v \bar{\chi}(cd) \psi(d) \delta(f, r', "ac/d"(A)) \end{aligned}$$

if $\xi = a(A)$, and

$$\gamma = (c^2 - c^u \bar{\chi}(c))(d^2 - d^v \bar{\chi}(d)) \delta(f, r', \xi)$$

if $\xi \in \text{SL}_2(\mathbb{Z})$.

Definition A.5. Let

$${}_{c,d}z_m^{(p)}(f, r, r', \xi, S) \in H^1(\mathbb{Z}[1/p, \zeta_m], V_{\mathcal{O}_\lambda}(f)(k-r))$$

be the zeta element in [Kat04, (8.1.3)] where $p \in S$ as well as all the “variables” satisfy Definition A.1 and Definition A.2.

By [Kat04, Proposition 8.12], the cohomology classes

$${}_{c,d}z_m^{(p)}(f, r, r', \xi, S) \in H^1(\mathbb{Z}[1/p, \zeta_m], V_{\mathcal{O}_\lambda}(f)(k-r))$$

satisfy the Euler system relation except at primes dividing cdN . See also [Kat04, §13.3]. By [Kat04, Theorem 9.7], under Definition A.2.(1), we have the following assignment

$${}_{c,d}z_m^{(p)}(f, r, r', \xi, S) \mapsto {}_{c,d}z_m(f, r, r', \xi, S)$$

under the dual exponential map. In other words, zeta modular forms (Definition A.3) and zeta elements (Definition A.5) match in the *critical* range, i.e. L -values at $s = 1, \dots, k-1$.

A.2. Kato’s modification in the p -power direction

Following [Kat04, Theorem 12.5 and §13.9], we recall Kato’s modification of zeta elements (in the p -power direction) in order to specify \mathbf{z}_{Kato} in Conjecture 1.4.

A.2.1. The elements

Let $G_\infty := \text{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$ and $j_{p^n} : \text{Spec}(\mathbb{Q}(\zeta_{p^n})) \rightarrow \text{Spec}(\mathbb{Z}[\zeta_{p^n}, 1/p])$ be the natural map. Then we define a $\mathbb{Z}_p[[G_\infty]]$ -module

$$\mathbf{H}^i(V_{\mathcal{O}_\lambda}(f)) := \varprojlim_n \mathbf{H}_{\text{ét}}^i(\text{Spec}(\mathbb{Z}[\zeta_{p^n}, 1/p], j_{p^n,*}V_{\mathcal{O}_\lambda}(f)))$$

where $\mathbf{H}_{\text{ét}}^i(\text{Spec}(\mathbb{Z}[\zeta_{p^n}, 1/p], j_{p^n,*}V_{\mathcal{O}_\lambda}(f)))$ is the étale cohomology group.

Choice A.6. Fix elements $\alpha_1, \alpha_2 \in \text{SL}_2(\mathbb{Z})$ and integers j_1, j_2 such that

- $1 \leq j_i \leq k - 1$ ($i = 1, 2$), and
- $\delta(f, j_1, \alpha_1)^+ \neq 0$ and $\delta(f, j_2, \alpha_2)^- \neq 0$ ([Kat04, (13.6)]).

Let $\gamma \in V_{F_\lambda}(f)$. Then we have

$$\gamma = b_1 \cdot \delta(f, j_1, \alpha_1)^+ + b_2 \cdot \delta(f, j_2, \alpha_2)^- \neq 0 \tag{A.1}$$

for some $b_1, b_2 \in \mathbb{Q}_{f,\lambda}$.

Choice A.7. Fix $c, d \in \mathbb{Z}$ such that

- (1) $(cd, 6p) = 1$,
- (2) $c \equiv d \equiv 1 \pmod{N}$ (needed for [Kat04, Theorem 6.6]), and
- (3) $c^2 \neq 1$ and $d^2 \neq 1$.

For a commutative ring R , let $Q(R)$ be the total quotient ring of R .

Definition A.8. We define

$$\begin{aligned} \mathbf{z}_\gamma^{(p)} &:= \left\{ \mu(c, d, j_1)^{-1} \cdot b_1 \cdot \left(c, d \mathbf{z}_{p^n}^{(p)}(f, k, j_1, \alpha_1, \text{prime}(pN)) \right)_{n \geq 1} \right\}^- \\ &\quad + \left\{ \mu(c, d, j_2)^{-1} \cdot b_2 \cdot \left(c, d \mathbf{z}_{p^n}^{(p)}(f, k, j_2, \alpha_2, \text{prime}(pN)) \right)_{n \geq 1} \right\}^+ \\ &\in \mathbf{H}^1(V_{\mathcal{O}_\lambda}(f)) \otimes Q(\mathbb{Z}_p[[G_\infty]]) \end{aligned}$$

where

$$\mu(c, d, j) := (c^2 - c^{k+1-j} \cdot \sigma_c) \cdot (d^2 - d^{j+1} \cdot \sigma_d) \cdot \prod_{\ell|N} (1 - \overline{a_\ell(f)} \ell^{-k} \sigma_\ell^{-1}) \in \mathbb{Z}_p[[G_\infty]]$$

for $j \in \mathbb{Z}$ and $\mu(c, d, j)$ is not a zero divisor in $\mathbb{Z}_p[[G_\infty]]$ for all j .

By [Kat04, §13.12], indeed, it is known that $\mathbf{z}_\gamma^{(p)} \in \mathbf{H}^1(V_{\mathcal{O}_\lambda}(f)) \otimes \mathbb{Q}_p$.

A.2.2. The submodule inverting p

Let $Z(f)$ be the $\mathcal{O}_\lambda[[G_\infty]] \otimes \mathbb{Q}_p$ -submodule of $\mathbf{H}^1(V_{\mathcal{O}_\lambda}(f)) \otimes \mathbb{Q}_p$ generated by $\mathbf{z}_\gamma^{(p)}$ for all $\gamma \in V_{F_\lambda}(f)$. Since $\mathbf{H}^1(V_{\mathcal{O}_\lambda}(f)) \otimes \mathbb{Q}_p$ is free of rank one over $\mathcal{O}_\lambda[[G_\infty]] \otimes \mathbb{Q}_p$ (by [Kat04, Theorem 12.4.(2)]), we easily observe the following independence result considering the evaluation at all the finite order characters on G_∞ .

Proposition A.9. *The cohomology class $\mathbf{z}_\gamma^{(p)}$ is independent of the choices of $\alpha_1, \alpha_2, j_1, j_2, c, d$ as above, i.e. Choice A.6 and Choice A.7.*

Furthermore, since $\mathcal{O}_\lambda[[G_\infty]] \otimes \mathbb{Q}_p$ is a product of PIDs, there exists an element $\gamma_0 \in V_f$ such that $\mathbf{z}_{\gamma_0}^{(p)}$ generates $Z(f)$ over $\mathcal{O}_\lambda[[G_\infty]] \otimes \mathbb{Q}_p$. Thus, any non-zero $\mathbf{z}_\gamma^{(p)}$ generates $Z(f)$ over $\mathcal{O}_\lambda[[G_\infty]] \otimes \mathbb{Q}_p$.

A.2.3. The submodule without inverting p

We assume the following condition and it is satisfied in most cases as described in [Kat04, Remark 12.8].

Assumption A.10. Suppose that there exists a \mathcal{O}_λ -lattice $T_f \subseteq V_{F_\lambda}(f)$ such that the image of $\rho_{T_f} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_{p^\infty})) \rightarrow \text{GL}_{\mathcal{O}_\lambda}(T_f)$ contains $\text{SL}_2(\mathbb{Z}_p)$.

Let $Z(f, T_f)$ be the $\mathbb{Z}[[G_\infty]]$ -submodule of $\mathbf{H}^1(T_f) \otimes \mathbb{Q}_p = \mathbf{H}^1(V_{\mathcal{O}_\lambda}(f)) \otimes \mathbb{Q}_p$ generated by $\mathbf{z}_\gamma^{(p)}$ for all $\gamma \in T_f \subseteq V_{F_\lambda}(f)$. Then by [Kat04, Theorem 12.5.(4)], it is known that $Z(f, T_f) \subseteq \mathbf{H}^1(T_f)$ under Assumption 1.1.(2).

Suppose that $T_f = V_{\mathcal{O}_\lambda}(f)$. Let Z be the $\mathbb{Z}_p[[G_\infty]]$ -submodule of $Z(f, V_{\mathcal{O}_\lambda}(f))$ generated by

$${}_{c,d}\mathbf{z}_{p^n}^{(p)}(f, k, j, a(A), \text{prime}(pA))_{n \geq 1} \in \mathbf{H}^1(V_{\mathcal{O}_\lambda}(f))$$

where $1 \leq j \leq k-1$, $a, A \in \mathbb{Z}$ with $A \geq 1$, $c, d \in \mathbb{Z}$ such that $(c, 6pA) = (d, 6pN) = 1$, and

$${}_{c,d}\mathbf{z}_{p^n}^{(p)}(f, k, j, \alpha, \text{prime}(pN))_{n \geq 1} \in \mathbf{H}^1(V_{\mathcal{O}_\lambda}(f))$$

where $1 \leq j \leq k-1$, $\alpha \in \text{SL}_2(\mathbb{Z})$, $c, d \in \mathbb{Z}$ such that $(c, 6pA) = (d, 6pN) = 1$. Then $Z \subseteq Z(f, V_{\mathcal{O}_\lambda}(f))$ of finite index due to [Kat04, Theorem 12.6].

Remark A.11. Indeed, Assumption A.10 is satisfied for all lattices if it is satisfied for one lattice for non-CM elliptic curves. See [Kat04, Remark 12.8].

Proposition A.12. *Under Assumption A.10, the submodule $Z(f, T_f) \subseteq \mathbf{H}^1(T_f)$ is generated by one element over $\mathcal{O}_\lambda[[G_\infty]]$.*

Proof. By Remark A.11, we may assume $V_{\mathcal{O}_\lambda}(f) = T_f$. Let $\alpha : V_{\mathcal{O}_\lambda}(f) \rightarrow \mathbf{H}^1(V_{\mathcal{O}_\lambda}(f))$ be the map defined by $\gamma \mapsto \mathbf{z}_\gamma^{(p)}$ as in [Kat04, Theorem 12.5.(1)]. Consider the $\mathcal{O}_\lambda[[G_\infty]]$ -linear map

$$\beta : V_{\mathcal{O}_\lambda}(f)^+ \otimes_{\mathcal{O}_\lambda} \mathcal{O}_\lambda[[G_\infty]]^- \oplus V_{\mathcal{O}_\lambda}(f)^- \otimes_{\mathcal{O}_\lambda} \mathcal{O}_\lambda[[G_\infty]]^+ \rightarrow \mathbf{H}^1(V_{\mathcal{O}_\lambda}(f))$$

defined by $\gamma^+ \otimes \lambda_1 + \gamma^- \otimes \lambda_2 \mapsto \lambda_1 \cdot \mathbf{z}_{\gamma^+}^{(p)} + \lambda_2 \cdot \mathbf{z}_{\gamma^-}^{(p)}$ and then it satisfies $\mathcal{O}_\lambda[[G_\infty]] \cdot \text{Im}(\alpha) = \text{Im}(\beta)$. The map β is well-defined due to the equation

$$\mathbf{z}_{i(\gamma)}^{(p)} = -\sigma_{-1} \left(\mathbf{z}_\gamma^{(p)} \right).$$

Since $V_{\mathcal{O}_\lambda}(f)^+ \otimes_{\mathcal{O}_\lambda} \mathcal{O}_\lambda[[G_\infty]]^-$ and $V_{\mathcal{O}_\lambda}(f)^- \otimes_{\mathcal{O}_\lambda} \mathcal{O}_\lambda[[G_\infty]]^+$ are free of rank one over $\mathcal{O}_\lambda[[G_\infty]]^-$ and $\mathcal{O}_\lambda[[G_\infty]]^+$, respectively, $\text{Im}(\beta)$ is free or rank one over $\mathcal{O}_\lambda[[G_\infty]]$. Thus, $\mathcal{O}_\lambda[[G_\infty]] \cdot \text{Im}(\alpha)$ is also free or rank one over $\mathcal{O}_\lambda[[G_\infty]]$; hence, the conclusion holds. \square

Definition A.13 (*Kato’s zeta element for elliptic curves*). Assume that f corresponds to an elliptic curve over \mathbb{Q} and T be its p -adic Tate module. Let $\mathbf{z}_{\gamma_0}^{(p)}$ be a generator of $Z(f, V_{\mathcal{O}_\lambda}(f)) \subseteq \mathbf{H}^1(V_{\mathcal{O}_\lambda}(f))$ over $\mathcal{O}_\lambda[[G_\infty]]$. Then we define $\mathbf{z}_{\text{Kato}} \in \mathbb{H}^1(T)$ in Conjecture 1.4 by the image of $\mathbf{z}_{\gamma_0}^{(p)}$ under the map

$$\mathbf{H}^1(V_{\mathcal{O}_\lambda}(f)) \xrightarrow{\otimes (\zeta_{p^n})_n} \mathbf{H}^1(V_{\mathcal{O}_\lambda}(f)(1)) \xrightarrow{\text{cores}} \mathbb{H}^1(T)$$

where the corestriction map is from $\mathbb{Q}(\mu_{p^\infty})$ to \mathbb{Q}_∞ .

Remark A.14. Note that γ_0^\pm corresponds to the Néron periods of the elliptic curve since T comes from the p -adic Tate module of E .

A.3. The Euler system: an extension to the tame direction

The goal of this section is to extend the interpolation property of Kato’s modification $\mathbf{z}_\gamma^{(p)}$ obtained from

$$\left(c, d \mathbf{z}_{p^n}^{(p)}(f, k, j_1, \alpha_1, \text{prime}(pN)) \right)_{n \geq 1} \text{ and } \left(c, d \mathbf{z}_{p^n}^{(p)}(f, k, j_2, \alpha_2, \text{prime}(pN)) \right)_{n \geq 1}$$

to the tame direction *very slightly*. The motivation of the extension is the following question.

Question A.15. Do we have an *integral* Euler system for the triple $(V_{\mathcal{O}_\lambda}(f)(k - r), F_\lambda, \text{prime}(cdpN))$ modifying $c, d \mathbf{z}_m^{(p)}(f, r, j, \xi, \text{prime}(mN)) \in \mathbb{H}^1(\mathbb{Z}[1/p, \zeta_m], V_{\mathcal{O}_\lambda}(f)(k - r))$ such that

- (1) it is *independent* of the choices of (c, d) and (j, ξ) (Proposition A.9) and

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(2) it satisfies the interpolation property for *all finite order characters on* $\text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$ (Theorem A.4)?

We do not have the answer to this question yet. In [Rub98, Corollary 7.2], when $k = 2$, $r = 1$ and $j_1 = j_2 = 1$, Rubin could modify the Euler system

$${}_{c,d}z_m^{(p)}(f, 1, 1, \xi, \text{prime}(mpN))$$

by choosing c and d more carefully, i.e. $c \not\equiv 1 \pmod{p}$ and $d \not\equiv 1 \pmod{p}$ in order to have

$$c \cdot d \cdot (c - \sigma_c) \cdot (d - \sigma_d) \in \Lambda^\times.$$

Thus, Rubin’s modification yields both the integrality and the interpolation property for finite order characters on $\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. We extend Rubin’s strategy to the case we need for Theorem 3.1 using the following lemma.

Lemma A.16. *Assume that*

(**) $j = k - 1$, $k < p$, and $r \neq 2$.

Then there exist infinitely many integers c and d such that

$$(c^2 - c^{r+1-j} \cdot \sigma_c) \cdot (d^2 - d^{j+1+r-k} \cdot \sigma_d) \in \mathbb{Z}_p[\text{Gal}(F/\mathbb{Q})]^\times,$$

where F is a cyclic extension of \mathbb{Q} of p -power degree ramified only at a prime ℓ with $\ell \equiv 1 \pmod{p}$ and $(\ell, cdp) = 1$.

Proof. Since

$$\begin{aligned} & (c^2 - c^{r+1-j} \cdot \sigma_c) \cdot (d^2 - d^{j+1+r-k} \cdot \sigma_d) \\ &= c^{r+1-j} \cdot (c^{1-r+j} - 1 + 1 - \sigma_c) \cdot d^{j+1+r-k} \cdot (d^{1-j-r+k} - 1 + 1 - \sigma_d) \\ &= c^{r+1-k+1} \cdot (c^{1-r+k-1} - 1 + 1 - \sigma_c) \cdot d^{k-1+1+r-k} \cdot (d^{1-k+1-r+k} - 1 + 1 - \sigma_d) \\ &= c^{r-k+2} \cdot (c^{k-r} - 1 + 1 - \sigma_c) \cdot d^r \cdot (d^{2-r} - 1 + 1 - \sigma_d), \end{aligned}$$

under Assumption (**), it suffices to choose c and d such that their reductions mod p are primitive roots mod p . \square

Remark A.17. Assumption (**) is already observed in [Kat04, (5.2.3)]. If $k = 2$, Assumption (**) in Lemma A.16 is automatic.

Following [Kat04, §13.3], we consider the following Euler system

$${}_{c,d}z_m^{(p)}(f, r, k - 1, \xi, \text{prime}(mpN))$$

in $H^1(\mathbb{Z}[1/p, \zeta_m], V_{\mathcal{O}_\lambda}(f)(k - r))$ for the triple $(V_{\mathcal{O}_\lambda}(f)(k - r), F_\lambda, \text{prime}(cdpN))$ where $\xi \in \text{SL}_2(\mathbb{Z})$.

Remark A.18.

(1) Note that we have the following equality at the level of Iwasawa cohomology

$$\begin{aligned} & \left({}_{c,d}z_{mp^n}^{(p)}(f, r, k - 1, \xi, \text{prime}(mpN)) \right)_n \\ &= \left({}_{c,d}z_{mp^n}^{(p)}(f, k, k - 1, \xi, \text{prime}(mpN)) \right)_n \otimes (\zeta_{p^n}^{\otimes k-r})_n \end{aligned}$$

in $\varprojlim_n H^1(\mathbb{Z}[1/p, \zeta_{mp^n}], V_{\mathcal{O}_\lambda}(f)(k - r))$.

(2) Since we do not expect

$$\prod_{\ell|N} (1 - \overline{a_\ell(f)} \ell^{-r} \sigma_\ell^{-1})$$

is invertible in general, we do not invert these Euler factors at bad primes.

Under Assumption (**) in Lemma A.16, let $\nu(c, d, k - 1, k - r)$ be any element of $\mathbb{Z}_p[[\text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})]]$ which restricts to

$$((c^2 - c^{r+1-j} \cdot \sigma_c) \cdot (d^2 - d^{j+1+r-k} \cdot \sigma_d))^{-1} \in \mathbb{Z}_p[[\text{Gal}(\tilde{F}/\mathbb{Q})]]$$

where \tilde{F} is the compositum of all cyclic extensions of \mathbb{Q} of p -power degree ramified only at a prime ℓ with $(\ell, pcdN) = 1$ and \mathbb{Q}_∞ . Then we define

$$\begin{aligned} z_{\gamma,m}^{(Np)}(k - r) &:= \left\{ \nu(c, d, k - 1, k - r) \cdot b_1 \cdot {}_{c,d}z_m^{(p)}(f, r, k - 1, \alpha_1, \text{prime}(pNm)) \right\}^- \\ &+ \left\{ \nu(c, d, k - 1, k - r) \cdot b_2 \cdot {}_{c,d}z_m^{(p)}(f, r, k - 1, \alpha_2, \text{prime}(pNm)) \right\}^+ \end{aligned}$$

in $H^1(\mathbb{Z}[1/p, \zeta_m], V_{\mathcal{O}_\lambda}(f)(k - r))$ and then $z_{\gamma,m}^{(Np)}(k - r)$ obviously forms an integral Euler system and interpolates Np -imprimitive L -values of f at $s = r$ twisted by finite order characters on $\text{Gal}(\tilde{F}/\mathbb{Q})$. Since the choice of $\nu(c, d, k - 1, k - r)$ is arbitrary, we do not know the precise interpolation formula for general finite order characters on $\text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$.

A.4. Optimization of periods

The goal here is to make a *good* choice of γ in $z_{\gamma,m}^{(Np)}(k-r)$.

We recall that the period map appeared in §A.1 induces the Shimura isomorphism

$$\text{per}_f : (S(f) \otimes \mathbb{C})^2 \simeq V_F(f) \otimes \mathbb{C} \tag{A.2}$$

defined by $(f, f) \mapsto \text{per}_f(f) + \iota' \text{per}_f(f)$ where $\iota'(x \otimes y) = x \otimes \bar{y}$ for $x \in V_F(f)$, $y \in \mathbb{C}$. See [Kat04, (7.13.2)] for detail.

A.4.1. Néron periods

Assume that f corresponds to an elliptic curve E over \mathbb{Q} . Let $\omega_f \in S(f)$ be the element corresponding to the Néron differential ω_E of E under the modular parametrization. Then we have

$$\text{per}_f(\omega_f) = \Omega_E^+ \cdot \gamma^+ + \Omega_E^- \cdot \gamma^-$$

for some non-zero $\gamma^\pm \in V_{\mathbb{Z}}(f)$. Here, $\gamma^\pm \in V_{\mathbb{Z}}(f) \subseteq T(-1)$ is due to Theorem 1.2 and [Kat04, Theorem 13.6] (cf. [AS86]).

Choice A.19. We choose $b_1, b_2 \in \mathbb{Q}(=\mathbb{Q}_{f,\lambda})$ such that

$$\gamma^+ = b_1 \cdot \delta(f, 1, \alpha_1)^+, \quad \gamma^- = b_2 \cdot \delta(f, 1, \alpha_2)^-$$

in Equation (A.1).

Definition A.20. Due to Lemma A.16 with Remark A.17, the Euler system in §3.1 can be chosen by

$$c_{\mathbb{Q}(\mu_m)}(1) := z_{\gamma,m}^{(Np)}(1)$$

where f is the modular form corresponding to E , $k = 2$, $j_1 = j_2 = 1$, and α_1, α_2 and γ^\pm follows Choice A.19.

We also consider the Iwasawa cohomology version. Let

$$\begin{aligned} \mathbf{z}_{\gamma}^{(Np)} := & \left\{ \nu(c, d, 1, 1) \cdot b_1 \cdot \left(c, d \mathbf{z}_{p^n}^{(Np)}(f, 1, 1, \alpha_1, \text{prime}(pNm)) \right)_{n \geq 1} \right\}^- \\ & + \left\{ \nu(c, d, 1, 1) \cdot b_2 \cdot \left(c, d \mathbf{z}_{p^n}^{(Np)}(f, 1, 1, \alpha_2, \text{prime}(pNm)) \right)_{n \geq 1} \right\}^+ . \end{aligned}$$

Then we define an N -imprimitive analogue of \mathbf{z}_{Kato} (Definition A.13) for elliptic curves by

$$\mathbf{z}_{\text{Kato}}^{(N)} := \text{cores}_{\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}_\infty} \left(\mathbf{z}_{\gamma_0}^{(Np)}(1) \right). \tag{A.3}$$

Lemma A.21. *Under Assumption 1.1.(1), we have*

$$\text{char}_\Lambda \left(\frac{\mathbb{H}^1(T)}{\Lambda \mathbf{z}_{\text{Kato}}^{(N)}} \right) = \text{char}_\Lambda \left(\frac{\mathbb{H}^1(T)}{\Lambda \mathbf{z}_{\text{Kato}}} \right).$$

Proof. By definition, we have

$$\mathbf{z}_{\text{Kato}}^{(N)} = \prod_{\ell|N_{\text{st}}} (1 - \ell^{-1} \sigma_\ell^{-1}) \cdot \prod_{\ell|N_{\text{ns}}} (1 + \ell^{-1} \sigma_\ell^{-1}) \cdot \mathbf{z}_{\text{Kato}},$$

and, indeed,

$$\prod_{\ell|N_{\text{st}}} (1 - \ell^{-1} \sigma_\ell^{-1}) \cdot \prod_{\ell|N_{\text{ns}}} (1 + \ell^{-1} \sigma_\ell^{-1})$$

is invertible in Λ under Assumption 1.1.(1). \square

A.4.2. Canonical periods

We fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \simeq \mathbb{C}$. Let $f \in S_k(\Gamma_1(N), \psi)$ and $\bar{\rho} = \bar{\rho}_f$ be the mod p representation associated to f . We assume the following property.

Assumption A.22. The localized Hecke algebra at the maximal ideal corresponding to $\bar{\rho}$ is Gorenstein.

Theorem A.23. *Assume that $2 \leq k < p$. If*

- (1) p does not divide N , or
- (2) p exactly divides N and $\bar{\rho}$ is p -distinguished,

then Assumption A.22 holds.

Proof. See [Vat99, Theorem 1.13]. \square

Under Assumption A.22, we can add a canonical p -integral structure on the Shimura isomorphism (A.2) following [Vat13, §3]

$$\begin{array}{ccc} S(f) \otimes \mathbb{Q}_p \otimes_\iota \mathbb{C} & \xrightarrow{\text{per}_f} & V_{\mathcal{O}_\lambda}(f) \otimes_{\mathcal{O}_\lambda} F_\lambda \otimes_\iota \mathbb{C} \\ \uparrow & & \uparrow \\ \mathcal{O}_\lambda \omega_f & \xrightarrow{\text{per}_f} & \Omega_f^+ \cdot V_{\mathcal{O}_\lambda}(f)^+ + \Omega_f^- \cdot V_{\mathcal{O}_\lambda}(f)^- \end{array}$$

where ω_f is an \mathcal{O}_λ -basis of $S(f) \otimes \mathbb{Q}_p$ and it corresponds to the *normalized* newform f . Here, Ω_f^\pm are the **integral canonical periods** of f . Under the period map, we have

$$\text{per}_f(\omega_f) = \Omega_f^+ \cdot \gamma^+ + \Omega_f^- \cdot \gamma^- \quad (\text{A.4})$$

for some non-zero $\gamma^\pm \in V_{\mathcal{O}_\lambda}(f)^\pm$. By making the same choice of γ as Choice A.19 with Equation (A.4), we obtain an “optimal” Euler system $z_{\gamma,m}^{(Np)}(k-r)$.

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