Journal Pre-proof

The multiplication table for smooth integers

Marzieh Mehdizadeh

PII: S0022-314X(20)30299-7
DOI: https://doi.org/10.1016/j.jnt.2020.09.019
Reference: YJNTH 6622

To appear in: *Journal of Number Theory*

Received date: 11 September 2017
Revised date: 9 August 2020
Accepted date: 17 September 2020

Please cite this article as: M. Mehdizadeh, The multiplication table for smooth integers, *J. Number Theory* (2021), doi: https://doi.org/10.1016/j.jnt.2020.09.019.

The Multiplication Table for Smooth Integers

Marzieh Mehdizadeh[1]

Départment de Mathématiques et Statistique, Université de Montréal. Montréal, QC, Canada.

Abstract

The Erdős multiplication table problem asks how many distinct integers appear in the $N \times N$ multiplication table. The order of magnitude of this quantity was determined by Ford [1]. In this paper we study the number of y-smooth entries of the $N \times N$ multiplication table, that is to say entries with no prime factors greater than y.

Key words: Probailistic and Multiplicative number thorey, Smooth integers.

1. Introduction

The *multiplication table problem* involves estimating

$$A(x) := \#\{ab : a, b \le \sqrt{x}, \quad \text{and} \quad a, b \in \mathbb{N}\}.$$

This interesting question, posed by Erdős, has been studied by many authors. Erdős in [2], showed that for all $\varepsilon > 0$, we have

$$\frac{x}{(\log x)^{\delta+\epsilon}} \le A(x) \le \frac{x}{(\log x)^{\delta-\varepsilon}} \quad (x \to \infty), \tag{1}$$

where

$$\delta = 1 - \frac{1 + \log\log 2}{\log 2} = 0.0860\ldots. \tag{2}$$

The best estimate of $A(x)$ is a result due to Kevin Ford [1], who significantly improved (1) by showing that

$$A(x) \asymp \frac{x}{(\log x)^{\delta}(\log\log x)^{3/2}}. \tag{3}$$

Notation: In this paper, we use the notation $f(x) \asymp g(x)$ if both $f(x) \ll g(x)$ and $g(x) \ll f(x)$ hold, where we write $f(x) \ll g(x)$ or $f(x) = O(g(x))$ interchangeably to mean that $|f(x)| \le cg(x)$ holds with some constant c for all x in a range which will normally be clear from the context. Also, the notation $f(x) \sim g(x)$ means that $\frac{f(x)}{g(x)} \to 1$ as $x \to \infty$, and $f(x) = o(g(x))$ means that $\frac{f(x)}{g(x)} \to 0$ as $x \to \infty$.
Also, u is defined as

$$u := \frac{\log x}{\log y} \qquad x \ge y \ge 2,$$

and we let $\log_k x$ denote the k-fold iterated logarithm, defined by $\log_1 x := \log x$ and $\log_k x = \log\log_{k-1} x$, for $k > 1$. Motivated by this background, in this paper we investigate the multiplication table problem for

[1] marzieh.mehdizadeh@gmail.com

smooth integers. The set of y-smooth numbers, is defined by

$$S(x,y) := \{n \le x : P(n) \le y\},$$

where $P(n)$ denotes the largest prime factor of an integer $n \ge 2$, with the convention $P(1) = 1$. Set

$$\Psi(x,y) := \big|S(x,y)\big|.$$

Our main aim in this work is to study

$$A(x,y) := \#\{ab : a, b \in S(\sqrt{x}, y)\}.$$

Hence computing $A(x,y)$ is equivalent to estimating the size of $S(\sqrt{x},y) \cdot S(\sqrt{x},y)$, where dot sign stands for the set multiplication. A simple approximation of $\Psi(x,y)$ due to Canfield, Erdős and Pomerance [5] states that for a fixed $\epsilon > 0$ we have

$$\Psi(x,y) = xu^{-u(1+o(1))} \quad \text{as} \quad u \to \infty, \tag{4}$$

for $u \le y^{1-\epsilon}$, that is for $y \ge (\log x)^{1+\epsilon}$.

By estimate (4), one can see that for u large (or y small), the value of $\Psi(x,y)$ is small. Moreover, most integers counted by $\Psi(x,y)$ have a lot of prime factors.

If u is small (which means that y is large), then from (4) it follows that the value of $\Psi(x,y)$ is comparable in size to x. In this case, $S(x,y)$ contains integers with large prime factors and we expect the size of $S(\sqrt{x},y) \cdot S(\sqrt{x},y)$ to be small compared to $\big|S(x,y)\big|$.

The behaviour of $A(x^2,y) = \big|S(x,y) \cdot S(x,y)\big|$ in different ranges of y, particularly in the range where y is relatively large or small, was considered by Banks and Covert [6] in the context of a sum-product problem. They used combinatorial tools.

Here we present a simple idea to prove that $A(x,y)$ has the same size as $\Psi(x,y)$ when y is small compared to $\log x$. Let $n \le \frac{x}{y}$ be a y-smooth number. If $n \le \sqrt{x}$ then trivially we have $n \in A(x,y)$. Thus, we may assume that $\sqrt{x} \le n$. Let $p_1 \le p_2 \le \cdots \le p_k$ be prime factors of n. Consider the following sequence obtained from prime factorization of n:

$$n_0 = 1, \qquad n_j = \prod_{i=1}^{j} p_i, \qquad 1 \le j \le k.$$

Since $n \ge \sqrt{x}$ then there exists a unique integer s, with $0 \le s < k$ such that $n_s < \sqrt{x} \le n_{s+1}$. Each prime factor of n is less than y, therefore

$$n_s \le \sqrt{x} \le n_{s+1} \le n_s y.$$

Set $d = n_s$, then

$$\frac{\sqrt{x}}{y} \le d \le \sqrt{x}.$$

Since $n \le x/y$, then we easily conclude that

$$\frac{n}{d} \le \sqrt{x}.$$

Therefore,

$$\Psi(x/y, y) \le A(x,y) \le \Psi(x,y),$$

and by a simple argument we deduce that $\Psi(x/y, y) \sim \Psi(x, y)$ when $y = o(\log x)$ as both x and y tends to infinity (see Corollary 2.3). This argument leads us to state the following theorem.

Theorem 1.1. *If $y = o(\log x)$ then we have*

$$A(x, y) \sim \Psi(x, y) \quad \textit{as} \quad x, y \to \infty.$$

The problem gets harder and hence more interesting, when y takes larger values compared to $\log x$. We shall prove the following theorem for small values of y compared to x.

Theorem 1.2. *Defining the constant $L := \frac{1-\log 2}{\log 2}$, we have*

$$A(x, y) \sim \Psi(x, y) \quad \textit{as} \quad x, y \to \infty,$$

when u and y satisfy

$$\frac{u \log u}{(\log y \log_2 y \log_3 y)^2} \to \infty. \tag{5}$$

In particular this holds when

$$y \le \exp\left\{ \frac{(\log x)^{\frac{1}{3}}}{(\log_2 x)^{\frac{1}{3}+\epsilon}} \right\}, \tag{6}$$

for $\epsilon > 0$ arbitrarily small.

Theorem 1.2 is proved in Section 3. The proof relies on some probabilistic arguments and recent estimates for $\Psi(x, y)$.

If y takes values very close to x, which implies u is small compared to $\log\log y$, then we will show the following theorem.

Theorem 1.3. *Let $\epsilon > 0$ is arbitrarily small, then we have*

$$A(x, y) = o(\Psi(x, y)) \quad \textit{as} \quad x, y \to \infty,$$

where u and y satisfy

$$u < (L - \epsilon) \log_2 y. \tag{7}$$

In particular this holds when

$$y \ge \exp\left\{ \frac{\log x}{(L-\epsilon)\log_2 x} \right\}, \tag{8}$$

Theorem 1.3 is proved in Section 4, by applying an idea of Erdős [7], suitably modified for y-smooth integers.

In what follows, we will give a heuristic argument that predicts the behaviour of $A(x, y)$ in the ranges (5) and (7).

We define the function $\tau(n; A, B)$ to be the number of all divisors of n in the interval $(A, B]$. In other words.

$$\tau(n; A, B) := \#\{d : d|n \Rightarrow A < d \le B\}.$$

Let $n \in S((1-\eta)x, y)$ be a square-free number with k prime factors where η is a function of x tending sufficiently slowly to 0, as $x \to \infty$. Assume that the set

$$D(n) := \{\log d : d|n\}$$

is uniformly distributed in the interval $[0, \log n]$. So

$$P\left(d \in (A, B)\right) := \tau(n)\frac{\log B - \log A}{\log n}, \quad \text{with } \tau(n) = \tau(n; 0, n) \tag{9}$$

where the sample space is defined by

$$\Pi_k(x) := \{n \leq x : \omega(n) = k\},$$

and n is chosen uniformly at random. By this assumption, the expected value of the function $\tau(n, (1-\eta)\sqrt{x}, \sqrt{x})$ is as follows

$$\mathbb{E}\left[\tau(n, (1-\eta)\sqrt{x}, \sqrt{x})\right] \approx \frac{2^k \log \frac{1}{1-\eta}}{\log x} \approx \frac{2^k}{u \log y}. \tag{10}$$

Alladi and Hildebrand in [8], respectively [9], showed that the normal number of prime factors of y-smooth integers is very close to its expected value $u + \log_2 y$ in different ranges of y. Hence, from (10), we deduce that

$$\mathbb{E}\left[\tau(n, (1-\eta)\sqrt{x}, \sqrt{x})\right] \asymp \frac{2^{u+\log_2 y}}{u \log y} = \frac{2^{u(1+o(1))+\log_2 y}}{\log y}.$$

If $2^{u(1+o(1))+\log_2 y}/\log y \to \infty$, then we expect that for almost all $n \in S((1-\eta)x, y)$ will have a divisor d in the interval $((1-\eta)\sqrt{x}, \sqrt{x}]$.

Since $n \leq (1-\eta)x$, then we have $n/d \leq \sqrt{x}$, and we deduce that $n \in A(x, y)$, this means that

$$A(x, y) \geq (1 + o(1))\Psi((1-\eta)x, y) \geq (1 + o(1))\,\Psi(x, y),$$

where the last inequality is obtained by using Theorem 2.2. Trivially $A(x, y) \leq \Psi(x, y)$. So by this argument, we expect that

$$A(x, y) \sim \Psi(x, y).$$

On the other hand, if

$$\frac{2^{u(1+o(1))+\log_2 y}}{\log y} \to 0, \tag{11}$$

then we expect that none of integers in $S((1-\eta)x, y)$ have a divisor in $((1-\eta)\sqrt{x}, \sqrt{x}]$ (except a set with density 0). So for almost all integers $n \leq (1-\eta)x$ if $d|n$ then either $d < (1-\eta)\sqrt{x}$ or $d > \sqrt{x}$. If d is the biggest divisor of n less than $(1-\eta)\sqrt{x}$ then for any prime divisor p of n we have $d \leq (1-\eta)\sqrt{x} \leq dp \leq dy$. Therefore,

$$d' = \frac{n}{d} \geq \frac{ny}{(1-\eta)\sqrt{x}} \geq \sqrt{x},$$

where the last inequality is a deduction of (11) as $x \to \infty$. So we conclude that

$$A(x, y) = o(\Psi(x, y)) \quad \text{as} \quad x \to \infty$$

This heuristic gives some evidence for the following conjecture:

Conjecture 1. Put $L := \frac{1-\log 2}{\log 2}$. We have the following dichotomy

1. If $u - \log u - L\log_2 y \to +\infty$, then we have

$$A(x,y) \sim \Psi(x,y) \quad \text{as} \quad x, y \to \infty.$$

In particular this holds when

$$y \le \exp\left\{\frac{\log x}{L\log_2 x}\right\}.$$

2. If $u - \log u - L\log_2 y \to -\infty$, then we have

$$A(x,y) = o(\Psi(x,y)) \quad \text{as} \quad x, y \to \infty.$$

In particular, this holds when

$$y \ge \exp\left\{\frac{\log x}{(L-\epsilon)\log_2 x}\right\},$$

where $\epsilon > 0$ is small.

Theorem 1.2 and Theorem 1.3 are in the direction of the first case and the second case of Conjecture 1 respectively, but the claimed range in the first case of conjecture is larger than the claimed range in Theorem 1.2, this is a consequence of our uniformly assumption regarding $D(n)$.

2. Preliminaries

In this section, we review some results used in the proof of our main theorems. We first fix some notation. The Dickman–de Bruijn function $\rho(u) isacontinuousfunctionthatsatisfiesthedelaydifferentialequationu\rho'(u)+(u-1)=0$,with initial conditions $\rho(u)=1$ for $0\le u \le 1$. By [10, 3.9] we have the estimate for $\rho(u)$

$$\rho(u) = \left(\frac{e+o(1)}{u\log u}\right)^u \quad \text{as} \quad u \to \infty. \tag{12}$$

Theorem 2.1 (Hildebrand [11])**.** *The estimate*

$$\Psi(x,y) = x\rho(u)\left(1 + O_\epsilon\left(\frac{\log(u+1)}{\log y}\right)\right) \tag{13}$$

holds uniformly in the range

$$x \ge 3, \qquad 1 \le u \le \frac{\log x}{(\log_2 x)^{\frac{5}{3}+\epsilon}}, \quad \textit{that is,} \quad y \ge \exp\left((\log_2 x)^{\frac{5}{3}+\epsilon}\right), \tag{14}$$

where ϵ is any fixed positive number.

It is good to mention that the estimate in (4) can be deduced by combining (13) with the asymptotic formula (12). We will apply this estimate in the proof of Theorem 1.3. However this estimate of $\Psi(x,y)$ is not very sharp for large values of u, for which the saddle point method is more effective.

Let $\alpha := \alpha(x, y)$ be any real number satisfying

$$\sum_{p \le y} \frac{\log p}{p^\alpha - 1} = \log x. \tag{15}$$

It can be shown that α is unique. This function will play an essential role in this work, so we briefly recall some fundamental properties of this function that are used frequently. By [12, Lemma 3.1] we have the following estimates for α.

$$\alpha(x, y) = \frac{\log\left(1 + \frac{y}{\log x}\right)}{\log y} \left\{1 + O\left(\frac{\log_2 y}{\log y}\right)\right\} \qquad x \ge y \ge 2. \tag{16}$$

For any $\epsilon > 0$, we have the particular cases

$$\alpha(x, y) = 1 - \frac{\xi(u)}{\log y} + O\left(\frac{1}{L_\epsilon(y)} + \frac{1}{u(\log y)^2}\right) \qquad \text{if } \ y \ge (\log x)^{1+\epsilon}, \tag{17}$$

where

$$L_\epsilon(y) = \exp\left\{(\log y)^{3/5-\epsilon}\right\}, \tag{18}$$

and $\xi(t)$ is the unique real non-zero root of the equation:

$$e^{\xi(t)} = 1 + t\xi(t), \tag{19}$$

and we have the following useful estimate for ξ

$$\xi(t) = \log(t \log t) + O\left(\frac{\log_2 t}{\log t}\right) \quad t \ge 3. \tag{20}$$

Also for small values of y, we have

$$\alpha(x, y) = \frac{\log(1 + \frac{y}{\log x})}{\log y} \left\{1 + O\left(\frac{1}{\log y}\right)\right\} \ \text{ if } \ 2 \le y \le (\log x)^2. \tag{21}$$

We now turn to another ingredient related to the behaviour of $\Psi(x, y)$. The following estimate is a special case of a general result of de La Bretéche and Tenenbaum [12, Theorem 2.4].

Theorem 2.2. *If $d \le x/y$, then uniformly for $x \ge y \ge 2$ we have*

$$\Psi(\frac{x}{d}, y) = \left\{1 + O\left(\frac{1}{u} + \frac{\log y}{y}\right)\right\} \frac{\Psi(x, y)}{d^\alpha}. \tag{22}$$

We deduce the following corollary by Theorem 2.2.

Corollary 2.3. *If $y \ge 2$ and $y = o(\log x)$, then we have*

$$\Psi(x/y, y) \sim \Psi(x, y) \quad \textit{as} \quad x \to \infty. \tag{23}$$

Proof. (i): Let $y \ge (\log_2 x)^2$ and $y = o(\log x)$. By applying (22), if $d = y$, we obtain

$$\Psi(x/y, y) = \frac{\Psi(x, y)}{y^\alpha} \left\{1 + O\left(\frac{\log y}{y}\right)\right\}. \tag{24}$$

By combining the above estimate along with estimate (21), we get

$$\Psi(x/y,y) = \frac{\Psi(x,y)}{\left(1+\frac{y}{\log x}\right)^{1+O\left(\frac{1}{\log y}\right)}}\left\{1+O\left(\frac{\log y}{y}\right)\right\}. \tag{25}$$

We remark again that $y = o(\log x)$, so we obtain

$$\frac{1}{(1+y/\log x)^{1+O\left(\frac{1}{\log y}\right)}} \to 1 \qquad \text{when} \qquad x\to\infty.$$

Also, we have

$$\frac{\log y}{y} \to 0 \quad \text{when} \quad x \to \infty,$$

since $y \geq (\log_2 x)^2$. Thus, by (25), we conclude

$$\frac{\Psi(\frac{x}{y},y)}{\Psi(x,y)} \to 1 \quad \text{when} \quad x\to\infty.$$

(ii) : Let $2 \leq y \leq (\log_2 x)^2$, then by invoking Ennola's theorem [13], we get

$$\begin{aligned}
\Psi(x/y,y) &= \frac{1}{\pi(y)!}\prod_{p\leq y}\frac{\log x/y}{\log p}\left\{1+O\left(\frac{y^2}{\log x\log y}\right)\right\}\\
&= \frac{1}{\pi(y)!}\prod_{p\leq y}\frac{\log x}{\log p}\prod_{p\leq y}\left(1-\frac{\log y}{\log x}\right)\left\{1+O\left(\frac{y^2}{\log x\log y}\right)\right\}\\
&= \Psi(x,y)\left(1+O\left(\pi(y)\frac{\log y}{\log x}\right)\right)\\
&= \Psi(x,y)\left(1+O\left(\frac{y}{\log x}\right)\right),
\end{aligned} \tag{26}$$

which gives that

$$\Psi(x/y,y) \sim \Psi(x,y) \quad \text{as} \quad x\to\infty,$$

and this completes the proof. □

Finally, we define

$$\theta(x,y,z) := \#\{n \leq x : p|n \Rightarrow z < p \leq y\}.$$

This function has been studied in the literature. Namely Friedlander [14] and Saias [15, 16] gave several estimates for $\theta(x,y,z)$ in different ranges. The following theorem is due to Saias [16, Theorem 5] and [17, Theorem B] and is used in Section 4.

Theorem 2.4. *There exists a constant $c > 0$ such that for $x \geq y \geq z \geq 2$ we have*

$$\theta(x,y,z) \leq c\frac{\Psi(x,y)}{\log z}. \tag{27}$$

In the more restricted range $z^{(\log 2u)} \leq y \leq x$ we have

$$\theta(x,y,z) = \Psi(x,y)\prod_{p\leq z}(1-\frac{1}{p})\left(1+O\left(\frac{(\log 2u)\log z}{\log y}\right)\right). \tag{28}$$

3. Proof of Theorem 1.2

We begin this section by setting some notation. Let η be defined by

$$\eta := \frac{1}{\log_3 y},$$

and set

$$N := \left\lfloor \frac{\log_2 y - \log \eta}{\log 2} + 2 \right\rfloor, \tag{29}$$

These two quantities play an essential role in our proof.

The proof of Theorem 1.2 combines both probabilistic and combinatorial techniques. Before working out the details, we give a sketch of proof.

The first step in proving Theorem 1.2 is to study the number of prime factors of $n \in S(x, y)$ in the *narrow intervals*

$$J_i := \left[(1-\kappa)y^{1-\frac{1}{2^i}}, y^{1-\frac{1}{2^i}}\right], \qquad 1 \leq i \leq N,$$

of *multiplicative length* $(1-\kappa)^{-1}$, where κ is defined as

$$\kappa := \frac{\eta}{2N}. \tag{30}$$

Also, we define the *tail interval*

$$J_\infty := [(1-\kappa)y, y].$$

Let $\omega_i(n)$ be the number of prime factors of n in J_i for each $i \in \{1, 2, \ldots, N, \infty\}$, more formally

$$\omega_i(n) := \#\{p|n : p \in J_i\}. \tag{31}$$

We define

$$\mu_i(x, y) := \frac{1}{\Psi(x, y)} \sum_{n \in S(x,y)} \omega_i(n), \tag{32}$$

to be the expected values of $\omega_i(n)$. In Proposition 3.4, we will prove that for almost all y-smooth integers the value of $\omega_i(n)$ exceeds $\mu_i(x, y)/2$. We establish this by applying the Chebyshev inequality

$$\frac{\#\{n \in S(x, y) : \omega_i(n) \leq \frac{\mu_i(x,y)}{2}\}}{\Psi(x, y)} \leq \frac{4\sigma_i^2(x, y)}{\mu_i^2(x, y)}, \tag{33}$$

where

$$\sigma_i^2(x, y) := \frac{1}{\Psi(x, y)} \sum_{n \in S(x,y)} \left(\omega_i(n) - \mu_i(x, y)\right)^2, \tag{34}$$

is the variance of $\omega_i(n)$ and $i \in \{1, 2, \ldots, N, \infty\}$. By using th above information, we will conclude that there is at least one prime factor p_i in each J_i for $1 \leq i \leq N$ and N prime factors $q_1, \ldots, q_N$ in J_∞. Then by using the product of these prime factors in Corollary 3.5, we will find a divisor of n in the interval $\left[(1-\kappa)^N y^{N-j/2^N}, y^{N-j/2^N}\right]$, where j is an integer in $\{0, 1, \ldots 2^N - 1\}$.

Also, we find another divisor of n in the interval $\left(\frac{\sqrt{n}}{y^N} y^{j/2^N}, \frac{\sqrt{n}}{y^N} y^{(j+1)/2^N}\right)$. Multiplying these two divisors will give a new divisor of n such that we can write n as the product of two divisors less than $\sqrt{x}$.

Before stating some technical lemmas, we obtain an estimate for the expected value of $\omega_i(n)$ for all $1 \le i \le N$ and $i = \infty$. By changing the order of summation in (32), we easily see that

$$\mu_i(x,y) = \sum_{p \in J_i} \frac{\Psi(\frac{x}{p}, y)}{\Psi(x,y)}. \tag{35}$$

By (22), we have the following estimate

$$\mu_i(x,y) = \sum_{p \in J_i} \frac{1}{p^{\alpha}} \left(1 + O\left(\frac{1}{u} + \frac{\log y}{y}\right)\right), \tag{36}$$

for all $1 \le i \le N$ and $x \ge y \ge 2$. Also, we obtain the following estimate for $\mu_i(x/q, y)$, where q is a prime divisor of $n \in S(x,y)$.

$$\mu_i(x/q,y) = \sum_{p \in J_i} \frac{1}{p^{\alpha_q}} \left\{1 + O\left(\frac{1}{u_q} + \frac{\log y}{y}\right)\right\}, \tag{37}$$

where $u_q := u - \log q / \log y$. By substitution we obtain $x/q = y^{u_q}$. We define the saddle point $\alpha_q := \alpha(x/q, y)$ as the unique real number satisfying

$$\sum_{p \le y} \frac{\log p}{p^{\alpha_q} - 1} = \log \frac{x}{q}. \tag{38}$$

We are ready to prove the following lemma showing the difference between $\mu_i(x/q, y)$ and $\mu_i(x,y)$ is small.

Lemma 3.1. *Let q be a prime divisor of $n \in S(x,y)$, then we have*

$$\left|\mu_i(\frac{x}{q}, y) - \mu_i(x,y)\right| \ll \frac{\mu_i(x,y)}{u}.$$

Proof. First we represent the saddle point α as a function of u, so we have

$$\alpha(u) = \alpha(y^u, y).$$

We use the estimate

$$0 < -\alpha'(u) := -\frac{d\alpha(u)}{du} \asymp \frac{\bar{u}}{u^2 \log y}, \tag{39}$$

established in [18, formula 6.6], where $\bar{u} := \min\{u, \frac{y}{\log y}\}$. By (39), we deduce

$$\left|\alpha'(u)\right| \ll \frac{1}{u \log y}. \tag{40}$$

Then applying (40), gives that

$$\begin{aligned} \alpha - \alpha_q \le \int_{u_q}^{u} \left|\alpha'(v)\right| dv &\ll \int_{u_q}^{u} \frac{dv}{v \log y} \\ &= \frac{1}{\log y} \log\left(\frac{u}{u_q}\right) \asymp \frac{\log q}{\log y \log x}. \end{aligned} \tag{41}$$

By combining (36) and (37) and using $O(1/u) = O(1/u_q)$, we get

$$\left|\mu_i(\frac{x}{q}, y) - \mu_i(x,y)\right| \le \sum_{p \in J_i} \frac{1}{p^{\alpha}} \left\{\left|p^{\alpha - \alpha_q} - 1\right| + O\left(\frac{1}{u} + \frac{\log y}{y}\right)\right\}. \tag{42}$$

By the Taylor expansion of the exponential function and invoking (41) we obtain

$$\exp\{(\alpha-\alpha_q)\log p\}-1 \ll \frac{\log p \log q}{\log y \log x}. \tag{43}$$

We recall that $p, q \le y$ for $1 \le i \le N$ and $i=\infty$. From this we infer that

$$\left|p^{\alpha-\alpha_q}-1\right| \ll \frac{1}{u},$$

this finishing the proof. □

The following lemma provides an upper bound for $\sigma_i^2(x,y)$ (defined in (34)) for each $i \in \{1,2,\ldots,N,\infty\}$.

Lemma 3.2. *For $i \in \{1,2,\ldots,N,\infty\}$, we have*

$$\sigma_i^2(x,y) \ll \mu_i(x,y) + \frac{\mu_i^2(x,y)}{u}.$$

Proof. By the definition of $\sigma_i^2(x,y)$ given in (34), we have

$$\sigma_i^2(x,y) = \frac{1}{\Psi(x,y)} \sum_{n\in S(x,y)} \left[\omega_i^2(n) - 2\mu_i(x,y)\omega_i(n) + \mu_i^2(x,y)\right].$$

Using the definition of $\omega_i(n)$ in (31), gives

$$\sum_{n\in S(x,y)} \omega_i(n) = \sum_{n\in S(x,y)} \sum_{p\in J_i} \mathbb{1}_{p|n} = \sum_{p\in J_i} \Psi(\frac{x}{p}, y),$$

where the indicator function $\mathbb{1}_{p|n}$ is 1 or 0 according to the prime p divides n or not. By the definition of $\mu_i(x,y)$ in (35), we deduce that

$$\sum_{n\in S(x,y)} \omega_i(n) = \Psi(x,y)\mu_i(x,y).$$

By (32), we have

$$\begin{aligned}
\Psi(x,y)\sigma_i^2(x,y) &= \sum_{n\in S(x,y)} \left[\omega_i^2(n) - 2\mu_i(x,y)\omega_i(n) + \mu_i^2(x,y)\right] \\
&= \sum_{n\in S(x,y)} \omega_i^2(n) - 2\Psi(x,y)\mu_i^2(x,y) + \Psi(x,y)\mu_i^2(x,y) \\
&= \left(\sum_{\substack{p,q\in J_i \\ p\neq q}} \Psi(x/pq, y) \right) - \Psi(x,y)\mu_i^2(x,y) + \sum_{p\in J_i} \Psi(x/p, y) \\
&:= S_1 + S_2,
\end{aligned}$$

where $S_1 := \sum_{\substack{p,q\in J_i \\ p\neq q}} \Psi(\frac{x}{pq}, y) - \Psi(x,y)\mu_i^2(x,y)$ and $S_2 := \sum_{p\in J_i} \Psi(x/p, y)$. We next find an upper bound for each S_i. We first consider S_1, by using (35) we get

$$\sum_{\substack{p,q\in J_i \\ p\neq q}} \Psi(\frac{x}{pq}, y) - \Psi(x,y)\mu_i^2(x,y) \le \sum_{p\in J_i} \Psi(\frac{x}{p}, y) \left(\mu_i(\frac{x}{p}, y) - \mu_i(x,y)\right). \tag{44}$$

By Lemma 3.1 and using (44), we obtain the following upper bound for S_1

$$S_1 \le C\frac{\Psi(x,y)\mu_i^2(x,y)}{u}, \tag{45}$$

where C is a positive constant. It remains to estimate S_2. From (35) we have

$$S_2 = \Psi(x,y)\mu_i(x,y).$$

By substituting the upper bounds for S_1 and S_2, we get

$$\sigma_i^2(x,y) = \frac{S_1+S_2}{\Psi(x,y)} \ll \left(\mu_i(x,y) + \frac{\mu_i^2(x,y)}{u}\right),$$

60 and the proof is complete. □

By recalling that α and κ defined in (15) and (30) respectively, we give the order of magnitude for $\mu_i(x,y)$, where $i \in \{1,2,\ldots,N,\infty\}$

Lemma 3.3. *We have*

$$\mu_i(x,y) \asymp \kappa \frac{Y^{1-\frac{1}{2^i}}}{\log y},$$

where $i \in \{1,2,\ldots,N,\infty\}$*, and*

$$Y := y^{1-\alpha}.$$

Proof. By the definition of each J_i, we obtain the simple inequalities

$$\frac{1}{y^{\alpha(1-1/2^i)}} \# \{p \in J_i\} \leq \sum_{p\in J_i} \frac{1}{p^\alpha} \leq \frac{1}{(1-\kappa)y^{\alpha(1-1/2^i)}} \# \{p \in J_i\}. \tag{46}$$

By applying the prime number theorem, we obtain

$$\begin{aligned}
\#\{p \in J_i\} &= \pi(y^{1-1/2^i}) - \pi((1-\kappa)y^{1-1/2^i}) \\
&= \frac{y^{1-1/2^i}}{\log\left(y^{1-1/2^i}\right)} - \frac{(1-\kappa)y^{1-1/2^i}}{\log\left((1-\kappa)y^{1-1/2^i}\right)} + O\left(\frac{y^{1-1/2^i}}{\log^2 y}\right) \\
&= \frac{y^{1-1/2^i}}{(1-1/2^i)\log y} - \frac{(1-\kappa)y^{1-1/2^i}}{(1-1/2^i)\log y}\left(1+O\left(\frac{1}{\log y}\right)\right) \qquad (47)\\
&= \frac{\kappa y^{1-1/2^i}}{(1-1/2^i)\log y}(1+o(1)),
\end{aligned}$$

The last equality is true, since the given values of κ and N in (30) and (29) imply

$$\kappa \asymp \frac{1}{\log_2 y \log_3 y}. \tag{48}$$

By substituting in (46) the estimate for $\#\{p \in J_i\}$ given in (47), we obtain

$$\mu_i(x,y) \asymp \kappa \frac{Y^{1-1/2^i}}{\log y}, \tag{49}$$

and we get our desired result. □

Having the above lemmas at our disposal, we are now ready for proving the following proposition.

Proposition 3.4. *If u and y are in range given in* (5)*, we have*

$$\# \left\{ n \in S(x,y) : \omega_i(n) > \frac{\mu_i(x,y)}{2} \quad \forall i \in \{1,\ldots,N,\infty\} \right\} \sim \Psi(x,y) \quad as \quad x,y \to \infty,$$

Proof. By Chebyshev's inequality in (33) and using the upper bound for $\sigma_i^2(x,y)$ in Lemma 3.2, we get

$$\#\left\{n \in S(x,y) : \omega_i(n) \leq \frac{\mu_i(x,y)}{2}\right\} \ll \Psi(x,y)\left(\frac{1}{\mu_i(x,y)} + \frac{1}{u}\right).$$

Define

$$M := \#\left\{n \in S(x,y) : \exists i \in \{1,\ldots,N,\infty\} \quad \text{such that} \quad \omega_i(n) \leq \frac{\mu_i(x,y)}{2}\right\}.$$

By the above inequality, we obtain an upper bound of M as follows:

$$M \ll \Psi(x,y)\left[\frac{1}{\mu_\infty(x,y)} + \sum_{i=1}^{N}\frac{1}{\mu_i(x,y)} + \frac{N+1}{u}\right]. \tag{50}$$

Our main task is to show that in the range (5) we have

$$\frac{M}{\Psi(x,y)} \to 0.$$

By using Lemma 3.3 and substituting the order of magnitude of $\mu_i(x,y)$ in (50), we get

$$M \ll \Psi(x,y)\left[\frac{\log y}{\kappa Y} + \frac{\log y}{\kappa}\sum_{i=1}^{N}\frac{1}{Y^{1-1/2^i}} + \frac{N+1}{u}\right]. \tag{51}$$

65 In what follows, we find a lower bound for Y in two different ranges of y.

(**i**) If $y \leq (\log x)^2$, then by (21) $\alpha \leq 1/2 + o(1)$ as $y \to \infty$. Therefore,

$$Y \geq y^{1/2-o(1)} \geq y^{1/3}.$$

By substituting this lower bound in (51) and using the precise value of N in (29), we have

$$\begin{aligned} M &\ll \Psi(x,y)\left[\frac{\log y}{\kappa y^{1/3}} + \frac{N}{u} + \frac{\log y}{\kappa y^{1/3}}\sum_{i=1}^{N} y^{1/3(2^i)}\right] \\ &\ll \Psi(x,y)\left[\frac{\log_2 y}{u} + \frac{y^{1/6}\log y}{\kappa y^{1/3}}\left(1 + O\left(\frac{N}{y^{1/12}}\right)\right)\right] \\ &\ll \Psi(x,y)\frac{\log y}{\kappa y^{1/6}}. \end{aligned} \tag{52}$$

By using the asymptotic value of κ in (48), we obtain

$$M \ll \Psi(x,y)\frac{\log y \log_2 y \log_3 y}{y^{1/6}},$$

and clearly we have

$$M = o(\Psi(x,y)) \quad \text{as} \quad x,y \to \infty,$$

this finishing the proof for the case $y \leq (\log x)^2$.

(**ii**) If $y \geq (\log x)^2$, by applying (17), we have

$$1 - \alpha = \frac{\xi(u)}{\log y} + O\left(\frac{1}{L_\epsilon(y)} + \frac{1}{u(\log y)^2}\right). \tag{53}$$

Using (20), we have the following estimate of ξ

$$\xi(t) = \log(t \log t) + O\left(\frac{\log_2 t}{\log t}\right) \quad \text{if} \quad t > 3.$$

Therefore,

$$1 - \alpha = \frac{\log(u \log u)}{\log y} + O\left(\frac{\log_2 u}{\log y \log u}\right),$$

Thus, we get

$$\begin{aligned} Y &= u \log u \left[1 + O\left(\frac{\log_2 u}{\log u}\right)\right] \\ &\asymp u \log u. \end{aligned} \tag{54}$$

By combining the above with the estimate in (54), and using the value of N in (29), we get

$$\begin{aligned} M &\ll \Psi(x,y)\left[\frac{\log y}{\kappa u \log u} + \frac{\log y}{\kappa u \log u}\sum_{i=1}^{N}(u \log u)^{1/2^i} + \frac{N+1}{u}\right] \\ &\ll \Psi(x,y)\left[\frac{\log y}{\kappa u \log u}\left((u \log u)^{1/2} + (u \log u)^{1/2^2} + ... + (u \log u)^{1/2^N}\right) + \frac{N+1}{u}\right] \\ &\ll \Psi(x,y)\left[\frac{\log y}{\kappa (u \log u)^{1/2}}\left(1 + O\left(N(u \log u)^{-1/4}\right)\right) + \frac{N+1}{u}\right] \\ &\ll \Psi(x,y)\left[\frac{\log_2 y}{u} + \frac{\log y}{\kappa (u \log u)^{1/2}}\right], \end{aligned} \tag{55}$$

Using (48) to estimate κ, one arrive at the following upper bound of M:

$$M \ll \Psi(x,y)\frac{\log y \log_2 y \log_3 y}{(u \log u)^{1/2}}. \tag{56}$$

So there exists a constant c such that for all $i \in \{1, \ldots, N, \infty\}$, we have

$$\#\left\{n \in S(x,y) : \omega_i(n) > \frac{\mu_i(x,y)}{2} \quad \forall i\right\} \geq \Psi(x,y)\left(1 - c\frac{\log y \log_2 y \log_3 y}{(u \log u)^{1/2}}\right), \tag{57}$$

and this finishes the proof, since by assumption $u \log u/(\log y \log_2 y \log_3 y)^2 \to \infty$. $\square$

Corollary 3.5. *If x and y are in the range* (5)*, then almost all n in $S(x,y)$ are divisible by at least one prime factor p_i in J_i, and N prime factors $q_1, \ldots, q_N$ in J_∞. Moreover, the product $\prod_{i=1}^{N} p_i q_i$ has a divisor D_j in each of intervals $[(1-\kappa)^N y^{N-j/2^N}, y^{N-j/2^N}]$, where $j \in \{0, 1, \ldots, 2^N - 1\}$.*

Proof. The first assertion is a direct consequence of Proposition 3.4.
For the second part, let n be in $S(x,y)$ with x and y satisfy in (5). Let $p_i \in J_i$ for $1 \leq i \leq N$, $q_1, \ldots, q_N \in J_\infty$, and j be an arbitrary integer in $\{0, 1, \ldots, 2^N - 1\}$. Moreover, we define

$$a_0 := N - \sum_{i=1}^{N} a_i,$$

where a_i's get the values 0 or 1 such that

$$j = \sum_{i=1}^{N} a_i 2^{N-i}. \tag{58}$$

Journal Pre-proof

We consider the divisor of D_j of n having the following form:

$$D_j := \prod_{i=1}^{N} p_i^{a_i} \prod_{i=1}^{a_0} q_i,$$

By using the bounds of p_i's and q_i's, we deduce that

$$(1-\kappa)^N y^{N-\sum_{i=1}^{N} a_i/2^i} \leq D_j \leq y^{N-\sum_{i=1}^{N} a_i/2^i},$$

By using (58), we have

$$(1-\kappa)^N y^{N-j/2^N} \leq D_j \leq y^{N-j/2^N},$$

and this finishes our proof. $\square$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $n \leq (1-\eta)x$ be a y-smooth integer with at least one prime factor p_i in each J_i , where $i = 1, \ldots, N$, and N prime divisors $q_1, q_2, \ldots, q_N$ in J_∞. Set

$$m := \frac{n}{\prod_{i=1}^{N} p_i q_i}.$$

By this definition, we get

$$m = \frac{n}{\prod_{i=1}^{N} p_i q_i} \geq \frac{n}{y^{2N}} > \sqrt{n},$$

when $4N \leq u$. Let $\{r_v\}$ be the increasing sequence of prime factors of m taken with multiplicity and set $d_v = r_1 \ldots r_v$.

Clearly, m has at least one divisor among the r_v bigger than $\frac{\sqrt{n}}{y^N}$. We suppose that l is the smallest integer such that $d_l \geq \frac{\sqrt{n}}{y^N}$, and evidently we have

$$d_{l-1} \leq \frac{\sqrt{n}}{y^N}.$$

So, we arrive at the following bounds for d_l

$$\frac{\sqrt{n}}{y^N} \leq d_l \leq y d_{l-1} \leq \frac{\sqrt{n}}{y^{N-1}}. \tag{59}$$

We pick $k \in \{0, 1, 2..., 2^N - 1\}$ such that

$$\frac{\sqrt{n}}{y^N} y^{k/2^N} \leq d_l \leq \frac{\sqrt{n}}{y^N} y^{(k+1)/2^N}. \tag{60}$$

By the second part of Corollary 3.5, for every k in $\{0, 1, \ldots, 2^N - 1\}$ there exists a divisor D_k of n such that

$$(1-\kappa)^N y^{N-k/2^N} \leq D_k \leq y^{N-k/2^N}.$$

We define $d := d_l D_k$. Then we have

$$(1-\kappa)^N \sqrt{n} \leq d \leq y^{1/2^N} \sqrt{n}.$$

By using the values of N in (29) and κ in (30), we have

$$y^{\frac{1}{2^N}} \leq e^{\eta/2},$$

and

$$(1-\kappa)^N = \exp\left(N(-\kappa + O(\kappa^2))\right) = \exp\left(-\frac{\eta}{2} + O(\frac{\eta^2}{N})\right) = \exp\left(-\frac{\eta}{2} + O(\eta^3)\right).$$

Now by applying the Taylor expansion for exponential function we obtain

$$\left(1 - \frac{\eta}{2} + \frac{\eta^2}{8} + O(\eta^3)\right)^{1/2} \sqrt{n} \le d \le \left(1 + \frac{\eta}{2} + \frac{\eta^2}{8} + O(\eta^3)\right)^{1/2} \sqrt{n}. \tag{61}$$

By using the assumption $n \le (1-\eta)x$ in the upper bound and lower bound above, we obtain

$$d \le \left(1 - \frac{\eta}{2} + O(\eta^2)\right)^{1/2} \sqrt{x} \le \sqrt{x},$$

and

$$\frac{n}{d} \le \left(1 - \frac{\eta}{2} + O(\eta^2)\right)^{1/2} \sqrt{x} \le \sqrt{x}.$$

Thus, we write $n \in S((1-\eta)x, y)$ as the product of two divisors less than $\sqrt{x}$, and we deduce that

$$\Psi\left((1-\eta)x, y\right) - o\left(\Psi(x,y)\right) \le A(x,y) \le \Psi(x,y),$$

By using (22), we have

$$\frac{\Psi\left((1-\eta)x, y\right)}{\Psi(x,y)} = (1-\eta)^{\alpha}\left\{1 + O\left(\frac{1}{u} + \frac{\log y}{y}\right)\right\} \to 1 \quad \text{as} \quad x, y \to \infty,$$

this finishing the proof. □

4. Proof of Theorem 1.3

In this section, we shall study the behaviour of $A(x,y)$ for large values of y. When y takes values very close to x, then the set of y-smooth integers contains integers having large prime factors. As we explained in the heuristic argument, we expect that $A(x,y) = o(\Psi(x,y))$. To show this assertion, we recall the idea of Erdős used to prove the multiplication table problem for integers up to x.

We start our argument by giving an upper bound for $A^*(x)$, defined by

$$A^*(x) := \#\left\{ab : a, b \le \sqrt{x} \quad \text{and} \quad (a,b) = 1\right\}. \tag{62}$$

We shall find an upper bound of $A^*(x)$ by considering the number of prime factors of a and b. We first define

$$\pi_k(x) := \#\{n \le x : \omega(n) = k\}$$

Therefore,

$$\begin{aligned} A^*(x) &\le \sum_k \min\left\{\pi_k(x), \sum_{j=1}^{k-1} \pi_j(\sqrt{x})\pi_{k-j}(\sqrt{x})\right\} \\ &\le \sum_k \min\left\{\frac{cx}{\log x}\frac{(\log_2 x + C)^{k-1}}{(k-1)!}, \sum_{j=1}^{k-1} \frac{c\sqrt{x}}{\log\sqrt{x}}\frac{(\log_2\sqrt{x} + C)^{j-1}}{(j-1)!}\frac{c\sqrt{x}}{\log\sqrt{x}}\frac{(\log_2\sqrt{x} + C)^{k-j-1}}{(k-j-1)!}\right\}, \end{aligned} \tag{63}$$

where in the last inequality, we used the well-known result of Hardy and Ramanujan stating there are absolute constants C and c such that

$$\pi_k(x) \leq \frac{cx}{\log x}\frac{(\log_2 x + C)^{k-1}}{(k-1)!} \quad \text{for} \quad k = 0, 1, 2, .. \quad \text{and} \quad x \geq 2. \tag{64}$$

By simplifying the upper bound in (63) and using Stirling's formula

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$$

we obtain

$$\begin{aligned} A^*(x) &\leq \sum_k \min\left\{\frac{cx}{\log x}\frac{(\log_2 x + C)^{k-1}}{(k-1)!}, \frac{4c^2Cx}{(\log x)^2}\sum_{j=0}^{k-2}\frac{1}{(k-2)!}\binom{k-2}{j}(\log_2\sqrt{x}+C)^{k-2}\right\} \\ &= \sum_k \min\left\{\frac{cx}{\log x}\frac{(\log_2 x + C)^{k-1}}{(k-1)!}, \frac{4c^2Cx}{(\log x)^2}\frac{(2\log_2\sqrt{x}+C)^{k-2}}{(k-2)!}\right\} \\ &\leq \sum_{k\leq\frac{\log_2 x}{\log 2}}\frac{4c^2Cx}{(\log x)^2}\frac{(2\log_2\sqrt{x}+C)^{k-2}}{(k-2)!} + \sum_{k>\frac{\log_2 x + C}{\log 2}}\frac{cx}{\log x}\frac{(\log_2 x + C)^{k-1}}{(k-1)!} \\ &\ll \frac{x}{(\log x)^{1-\frac{1+\log\log 2}{\log 2}}(\log_2 x)^{1/2}} \to 0 \quad \text{as} \quad x \to \infty. \end{aligned} \tag{65}$$

We shall get the same upper bound for $A(x)$. Let $n \leq x$ and there are a and b less than $\sqrt{x}$ such that $n = ab$. If $(a, b) = 1$ then n is counted by $A(x)$, and if $(a, b) = d > 1$ then we write n as $n = a'b'd^2$ such that $(a', b') = 1$. So $n/d^2 \leq x/d^2$, and $\frac{n}{d^2}$ will be counted by $A(x/d^2)$. Therefore,

$$A(x) \leq \sum_{d\leq\sqrt{x}} A^*(\frac{x}{d^2}) \ll A^*(x)$$

By (65), we get

$$A(x) \ll \frac{x}{(\log x)^{1-\frac{1+\log\log 2}{\log 2}}(\log_2 x)^{1/2}}.$$

Thus,

$$A(x) = o(x) \quad \text{as} \quad x \to \infty.$$

Motivated by Erdős' idea for the multiplication table of integers up to x, we apply a similar method to find an upper bound for $A(x, y)$.

The first step of proof is to study the following function which plays a crucial role in this section. Let

$$N_k(x, y, z) := \#\{n \in S(x, y) : \omega_z(n) = k\},$$

where $\omega_z(n)$ is the truncated version of $\omega(n)$, only counting divisibility by distinct primes not exceeding z with their multiplicities. In other words

$$\omega_z(n) := \sum_{\substack{p^v||n \\ p\leq z}} 1.$$

In the following lemma, by using induction on k, we shall find an upper bound of type (64) for $N_k(x, y, z)$. The reason of applying truncation is to sieve out prime factors exceeding some power of y which are the cause of big error terms as k increases in each step of induction. The upper bound of $N_k(x, y, z)$ leads us to generalize Erdős' idea for y-smooth integers in a certain range of y.

Lemma 4.1. *Let $u \le C \log\log y$, for a fixed constant $C < 1$, and choose the parameter z such that*

$$\log\log z \le Cu.$$

Then, there are constants A and B depending at most on C such that the inequality

$$N_k(x, y, z) \le \frac{A\Psi(x, y)}{\log z} \frac{(\log\log z + B)^k}{k!} \tag{66}$$

holds for every integer $k \ge 0$.

Proof. When $k = 0$, by (27), evidently we have

$$N_0(x, y, z) = \theta(x, y, z) \le c_0 \frac{\Psi(x, y)}{\log z},$$

where $c_0 > 0$ is a constant. When $k = 1$, we can write n as $n = p^a m$, where $p \le z$ and every prime factor q of m is between z and y, then using the definition of $\theta(x, y, z)$ and applying (22) we have

$$\begin{aligned} N_1(x, y, z) &= \sum_{p \le z} \sum_{\substack{m \le x/p^a \\ q|m \Rightarrow z < q \le y}} 1 \le \sum_{p \le z} \theta(\frac{x}{p}, y, z) \\ &= \sum_{p \le z} \left[\Psi(\frac{x}{p}, y) - \Psi(\frac{x}{p}, z) \right] \\ &= \sum_{p \le z} \frac{1}{p^\alpha} \left[\Psi(x, y) - \Psi(x, z) \right] \left(1 + O(\frac{1}{u}) \right) \\ &= \sum_{p \le z} \frac{1}{p^\alpha} \theta(x, y, z) \left(1 + O(\frac{1}{u}) \right) \end{aligned} \tag{67}$$

By applying the inequality in (27), there is constant c such that

$$N_1(x, y, z) \le c \frac{\Psi(x, y)}{\log z} \sum_{p \le z} \frac{1}{p^\alpha} \left\{ 1 + O\left(\frac{1}{u} \right) \right\}.$$

In order to estimate the later sum we note that

$$\begin{aligned} \sum_{p \le z} \frac{1}{p^\alpha} &= \sum_{p \le z} \frac{1}{p} \left(p^{1-\alpha} \right) \\ &= \sum_{p \le z} \frac{1}{p} \left\{ 1 + O\left((1 - \alpha) \log p \right) \right\}. \end{aligned} \tag{68}$$

Since $(1 - \alpha) \log z$ is bounded in our range (see (70)). Therefore,

$$\sum_{p \le z} \frac{1}{p^\alpha} = \log_2 z + O\left(1 + (1 - \alpha) \log z \right), \tag{69}$$

By using the estimates of α in (17), ξ in (20) and the upper bound of z, we get

$$(1-\alpha)\log z \ll \frac{\log u}{\log y}\log z \ll \frac{\log_3 y(\log y)^{C^2}}{\log y} \ll 1, \tag{70}$$

and we obtain

$$\sum_{p\le z}\frac{1}{p^\alpha} = \log_2 z + O\left(1\right). \tag{71}$$

Thus,

$$\sum_{p\le z}\frac{1}{p^\alpha}\left\{1+O\left(\frac{1}{u}\right)\right\} = \log_2 z + O(1), \tag{72}$$

since we have $\log\log z \le Cu$.

Substituting (72) in the upper bound of $N_1(x,y,z)$ gives

$$N_1(x,y,z) \le \frac{c\Psi(x,y)}{\log z}\left(\log_2 z + O(1)\right).$$

Now we argue by induction: we assume that the estimate in (66) is true for any positive integer k, we now prove it for $n\in S(x,y)$ with $\omega_z(n)=k+1$. We write n as $n=p_1^{a_1}\cdots p_{k+1}^{a_{k+1}}m$ such that $p_1,\ldots,p_{k+1}\le z$ and and m having only prime factors greater than z and less than y. Then by definition of $\theta(x,y,z)$ we have

$$\begin{aligned}
N_{k+1}(x,y,z) &= \frac{1}{(k+1)!}\sum_{p_1,\ldots,p_{k+1}\le z}\theta(\frac{x}{p_1^{a_1}\cdots p_{k+1}^{a_{k+1}}},y,z)\\
&\le \frac{1}{(k+1)!}\sum_{p_1,\ldots,p_{k+1}\le z}\theta(\frac{x}{p_1\cdots p_{k+1}},y,z)\\
&= \frac{1}{(k+1)!}\sum_{p_1,\ldots,p_{k+1}\le z}\left[\Psi(\frac{x}{p_1\cdots p_{k+1}},y)-\Psi(\frac{x}{p_1\cdots p_{k+1}},z)\right]\\
&= \frac{1}{(k+1)!}\sum_{p_1\cdots p_k\le z}\sum_{p_{k+1}\le z}\frac{1}{p_{k+1}^\alpha}\left[\Psi(\frac{x}{p_1\cdots p_k},y)-\Psi(\frac{x}{p_1\cdots p_k},z)\right]\left(1+O(\frac{1}{u})\right) \qquad (73)\\
&= \frac{1}{k+1}\left(\log_2 z+O(1)\right)\frac{1}{k!}\sum_{p_1\cdots p_k\le z}\left[\Psi(\frac{x}{p_1,\cdots p_k},y)-\Psi(\frac{x}{p_1\cdots p_k},z)\right]\\
&= \frac{1}{k+1}\left(\log_2 z+O(1)\right)\frac{1}{k!}\sum_{p_1,\ldots,p_k\le z}\theta(\frac{x}{p_1\cdots p_k},y,z)\\
&= \frac{1}{k+1}\left(\log_2 z+O(1)\right)N_k(x,y,z).
\end{aligned}$$

So by putting $A=c$ and B an upper bound on the constant implied by $O(1)$ we get our desired result. $\square$

Proof of Theorem 1.3. For $\epsilon>0$ small enough, we set $u<\left(\frac{\lambda}{\log 2}-\epsilon\right)\log_2 y$, where λ is a fixed real number in the interval $(0,1-\log 2)$.

We now choose z such that

$$\log\log z = \frac{\log 2}{\lambda}u, \tag{74}$$

so the given ranges of u and z satisfy Lemma 4.1.

By the definition of $A(x,y)$ the following evident upper bound holds:

$$A(x,y) \leq \sum_{k} \min\left\{ \sum_{\substack{n\in S(x,y)\\ \omega_z(n)=k}} 1, \sum_{j=1}^{k-1} \sum_{\substack{a\in S(\sqrt{x},y)\\ \omega_z(a)=j}} 1 \sum_{\substack{b\in S(\sqrt{x},y)\\ \omega_z(b)=k-j}} 1 \right\}. \tag{75}$$

We set

$$L = \lfloor H \log_2 z \rfloor,$$

where

$$H := \frac{1-\lambda}{\log 2}.$$

By using (75),we obtain the following upper bound for $A(x,y)$

$$\begin{aligned} A(x,y) &\leq \#\left\{n \in S(x,y) : \omega_z(n) > L\right\} + \#\left\{ab : a, b \in S(\sqrt{x},y), \omega_z(a) + \omega_z(b) \leq L\right\} \\ &\leq \sum_{k>L} N_k(x,y,z) + \sum_{k\leq L} \sum_{j=0}^{k} N_j(\sqrt{x},y,z) N_{k-j}(\sqrt{x},y,z). \end{aligned} \tag{76}$$

By applying Lemma 4.1, we have

$$\begin{aligned} A(x,y) &\ll \sum_{k>L} \frac{\Psi(x,y)}{\log z} \frac{(\log_2 z + c)^k}{k!} + \sum_{k\leq L} \sum_{j=0}^{k} \frac{\Psi^2(\sqrt{x},y)}{\log^2 z} \frac{(\log_2 z + c)^j}{j!} \frac{(\log_2 z + c)^{k-j}}{(k-j)!} \\ &= \sum_{k>L} \frac{\Psi(x,y)}{\log z} \frac{(\log_2 z + c)^k}{k!} + \sum_{k\leq L} \frac{\Psi^2(\sqrt{x},y)}{\log^2 z} \sum_{j=0}^{k} \frac{1}{k!} \binom{k}{j} (\log_2 z + c)^k \\ &= \sum_{k>L} \frac{\Psi(x,y)}{\log z} \frac{(\log_2 z + c)^k}{k!} + \sum_{k\leq L} \frac{\Psi^2(\sqrt{x},y)}{\log^2 z} \frac{(2\log_2 z + 2c)^k}{k!}. \end{aligned} \tag{77}$$

By applying the estimate (12), Theorem 13 and using the assumption (74), we get

$$\frac{\Psi^2(\sqrt{x},y)}{\Psi(x,y)} \asymp (\log z)^{\lambda} \quad \text{as} \quad u, y \to \infty. \tag{78}$$

Thus,

$$A(x,y) \ll \frac{\Psi(x,y)}{\log z} \sum_{k>L} \frac{(\log_2 z + c)^k}{k!} + \frac{(\log z)^{\lambda}\Psi(x,y)}{\log^2 z} \sum_{k\leq L} \frac{(2\log_2 z + 2c)^k}{k!}. \tag{79}$$

The maximum values of the functions in the latter two sums (with respect to k) are attained at $k = \log_2 z + O(1)$ and $k = 2\log_2 z + O(1)$ respectively. By using Stirling's formula $k! \sim \sqrt{2\pi} k^{k+\frac{1}{2}} e^{-k}$, for each summation we have

$$\begin{aligned} \sum_{k>L} \frac{(\log_2 z)^k}{k!} &= \sum_{H\log_2 z < k \leq e\log_2 z} \frac{(\log_2 z)^k}{k!} + \sum_{k > e\log_2 z} \frac{(\log_2 z)^k}{k!} \\ &\ll \left(\frac{e}{H}\right)^{H\log_2 z} \ll \frac{1}{(\log z)^{H\log H - H}}. \end{aligned} \tag{80}$$

The function in the second summation in (79) is increasing for $k \leq L$, and we have

$$\sum_{k\leq L} \frac{(2\log_2 z + 2c)^k}{k!} \ll \left(\frac{2e}{H}\right)^{H\log_2 z} = \frac{1}{(\log z)^{H\log H - H - H\log 2}}. \tag{81}$$

Substituting the upper bounds obtained in (80) and (81) in (79), and using the definition of H, gives

$$A(x,y) \ll \frac{\Psi(x,y)}{(\log z)^{G(H)}},$$

where

$$G(H) := 1 + H\log H - H.$$

The function $G(H)$ is increasing in the interval $(1,2)$ with a zero at $H = 1$. Thus, for any arbitrary $0 < \lambda < 1 - \log 2$, we have

$$A(x,y) = o(\Psi(x,y)) \quad \text{as} \quad x, y \to \infty,$$

and so we obtained our desired result. $\square$

5. Acknowledgement

I would like to thank Andrew Granville and Dimitris Koukoulopoulos for all their advice and encouragement as well as their valuable comments on an earlier version of the present paper. I am also grateful to Sary Drappeau, Farzad Aryan and Oleksiy Klurman for helpful conversations.

References

[1] K. Ford, Integers with a divisor in $(y, 2y]$, in: Anatomy of integers, Vol. 46 of CRM Proc. Lecture Notes, Amer. Math. Soc., Providence, RI, 2008, pp. 65–80.

[2] P. Erdős, Some remarks on number theory, Riveon Lematematika 9 (1955) 45–48.

[3] E. R. Canfield, P. Erdős, C. Pomerance, On a problem of Oppenheim concerning "factorisatio numerorum", J. Number Theory 17 (1) (1983) 1–28. doi:10.1016/0022-314X(83)90002-1.
URL http://dx.doi.org/10.1016/0022-314X(83)90002-1

[4] W. D. Banks, D. J. Covert, Sums and products with smooth numbers, J. Number Theory 131 (6) (2011) 985–993. doi:10.1016/j.jnt.2010.11.001.
URL http://dx.doi.org/10.1016/j.jnt.2010.11.001

[5] P. Erdős, An asymptotic in the theory of numbers, (Russian), Vestnik Leningrad. Univ. 15 (13) (1960) 41–49.

[6] K. Alladi, An Erdős-Kac theorem for integers without large prime factors, Acta Arith. 49 (1) (1987) 81–105.

[7] A. Hildebrand, On the number of prime factors of integers without large prime divisors, J. Number Theory 25 (1) (1987) 81–106. doi:10.1016/0022-314X(87)90017-5.
URL http://dx.doi.org/10.1016/0022-314X(87)90017-5

[8] A. Granville, Smooth numbers: computational number theory and beyond, in: Algorithmic number theory: lattices, number fields, curves and cryptography, Vol. 44 of Math. Sci. Res. Inst. Publ., Cambridge Univ. Press, Cambridge, 2008, pp. 267–323.

[9] A. Hildebrand, On the number of positive integers $\leq x$ and free of prime factors $> y$, J. Number Theory 22 (3) (1986) 289–307. `doi:10.1016/0022-314X(86)90013-2`.
URL `http://dx.doi.org/10.1016/0022-314X(86)90013-2`

[10] R. de la Bretèche, G. Tenenbaum, Propriétés statistiques des entiers friables, Ramanujan J. 9 (1-2) (2005) 139–202. `doi:10.1007/s11139-005-0832-6`.
URL `http://dx.doi.org/10.1007/s11139-005-0832-6`

[11] V. Ennola, On numbers with small prime divisors, Ann. Acad. Sci. Fenn. Ser. A I No. 440 (1969) 16.

[12] J. B. Friedlander, Integers free from large and small primes, Proc. London Math. Soc. (3) 33 (3) (1976) 565–576.

[13] É. Saias, Entiers sans grand ni petit facteur premier. II, Acta Arith. 63 (4) (1993) 287–312.

[14] É. Saias, Entiers sans grand ni petit facteur premier. III, Acta Arith. 71 (4) (1995) 351–379.

[15] É. Saias, Sur le nombre des entiers sans grand facteur premier, J. Number Theory 32 (1) (1989) 78–99. `doi:10.1016/0022-314X(89)90099-1`.
URL `http://dx.doi.org/10.1016/0022-314X(89)90099-1`

[16] A. Hildebrand, G. Tenenbaum, On integers free of large prime factors, Trans. Amer. Math. Soc. 296 (1) (1986) 265–290.