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Densities for some real quadratic fields with infinite Hilbert 2-class field towers

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Abstract

Let K be a real quadratic field with 2-class rank equal to 4 or 5 and 4-class rank equal to 3. This paper computes density information for such fields to have infinite Hilbert 2-class field towers.

MSC: 11R11; 11R29

Keywords: Densities; Real quadratic fields; Hilbert 2-class field towers

1. Introduction

Let K be a finite algebraic extension of the rational numbers $\mathbb{Q}$. Let C_K denote the 2-class group of K (i.e., the Sylow 2-subgroup of the ideal class group of K), and let $C_K^i = \{c^i\colon c \in C_K\}$, where i is a positive integer. Let K_1 be the Hilbert 2-class field of K (i.e., the maximal abelian unramified extension of K whose Galois group is a 2-group), and let K_i be the Hilbert 2-class field of K_{i-1} for $i \geqslant 2$. Then

$$K \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_i \subseteq \cdots$$

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is the Hilbert 2-class field tower of K. If $K_i \neq K_{i-1}$ for all i, then the Hilbert 2-class field tower of K is said to be infinite.

Next, let r_K denote the 2-class rank of K; i.e.,

$$r_K = \dim_{\mathbb{F}_2}\left(C_K/C_K^2\right), \tag{1}$$

where $\mathbb{F}_2$ is the finite field with two elements, and the elementary abelian 2-group C_K/C_K^2 is viewed as a vector space over $\mathbb{F}_2$. Similarly, let s_K denote the 4-class rank of K; i.e.,

$$s_K = \dim_{\mathbb{F}_2}\left(C_K^2/C_K^4\right). \tag{2}$$

From [3, p. 233], we know that the Hilbert 2-class field tower of K is infinite if

$$r_K \geqslant 2 + 2\sqrt{z_K + 1}, \tag{3}$$

where z_K is the number of infinite primes of K.

We shall now summarize some results that are known for quadratic fields K. Suppose first that K is an imaginary quadratic extension of $\mathbb{Q}$. Then $z_K = 1$. So if $r_K \geqslant 5$, then (3) is satisfied and K has infinite Hilbert 2-class field tower. If $r_K = 3$ or 4 and if $s_K \geqslant 3$, then K has infinite Hilbert 2-class field tower (cf. [11,12]). If $r_K = 4$ and $s_K = 2$, then most K have infinite Hilbert 2-class field towers, and if $r_K = 4$ and $s_K \leqslant 1$, then many K have infinite Hilbert 2-class field towers (cf. [1,2,9]). Thus there is considerable support for the conjecture that all imaginary quadratic fields with 2-class rank equal to 4 have infinite Hilbert 2-class field towers. Also, a positive proportion of the K with $r_K = 3$ and $s_K = j$ (with $j = 0$, 1, or 2) have infinite Hilbert 2-class field towers (cf. [7,8]).

Now suppose K is a real quadratic extension of $\mathbb{Q}$. Then $z_K = 2$. So if $r_K \geqslant 6$, (3) is satisfied and K has infinite Hilbert 2-class field tower. If $r_K = 4$ or 5 and $s_K \geqslant 4$, then K has infinite Hilbert 2-class field tower (cf. [14]). Also, a positive proportion of the K with $r_K = 5$ and $s_K = j$ (with $j = 0, 1, 2$, or 3) have infinite Hilbert 2-class field towers, and a positive proportion of the K with $r_K = 4$ and $s_K = j$ (with $j = 0, 1, 2$, or 3) have infinite Hilbert 2-class field towers (cf. [7]). In this paper we shall show that when $r_K = 4$ or 5 and $s_K = 3$, then most such K have infinite Hilbert 2-class field towers. More precisely, for square-free integers $m > 1$, $r = 4$ or 5, and positive real numbers x, we let

$$A_{r;x} = \left\{K = \mathbb{Q}(\sqrt{m})\colon\ r_K = r \text{ and } m \leqslant x\right\}, \tag{4}$$

$$A_{r,3;x} = \{K \in A_{r;x}\colon\ s_K = 3\}, \tag{5}$$

$$A^*_{r,3;x} = \{K \in A_{r,3;x}\colon \text{ the Hilbert 2-class field tower of } K \text{ is infinite}\}. \tag{6}$$

We then define a density $\alpha^*_{r,3}$ by

$$\alpha^*_{r,3} = \liminf_{x\to\infty} \frac{|A^*_{r,3;x}|}{|A_{r,3;x}|}, \tag{7}$$

where $|A|$ denotes the cardinality of a set A.

Theorem 1. *For real quadratic fields with* 2*-class rank* $r = 4$ *or* 5*, let* $\alpha^*_{r,3}$ *be the density defined by* (7)*. Then* $\alpha^*_{5,3} = 1$ *and* $\alpha^*_{4,3} \geqslant \frac{1650}{1710} > 0.9649$.

This theorem provides some support for a conjecture that all real quadratic fields with 2-class rank greater than or equal to 4 have infinite Hilbert 2-class field towers.

2. Proof of Theorem 1

Let $K = \mathbb{Q}(\sqrt{m}\,)$, where $m > 1$ is a square-free integer. Let r_K be the 2-class rank of K, and let t be the number of primes that ramify in $K/\mathbb{Q}$. It is well known that

$$r_K = \begin{cases} t-1 & \text{if no prime dividing } m \text{ is congruent to 3 (mod 4)}, \\ t-2 & \text{if some prime dividing } m \text{ is congruent to 3 (mod 4)}. \end{cases} \tag{8}$$

Let r be a fixed positive integer. If $r_K = r$, then (8) implies that m has one of the following forms:

(i) $m = p_1 \cdots p_{r+1}$ with distinct primes $p_1, \dots, p_{r+1}$ and no $p_i \equiv 3 \pmod 4$;
(ii) $m = p_1 \cdots p_{r+1}$ with distinct odd primes $p_1, \dots, p_{r+1}$ and an odd number of $p_i \equiv 3 \pmod 4$;
(iii) $m = 2p_1 \cdots p_{r+1}$ with distinct odd primes $p_1, \dots, p_{r+1}$ and at least one $p_i \equiv 3 \pmod 4$;
(iv) $m = p_1 \cdots p_{r+2}$ with distinct odd primes $p_1, \dots, p_{r+2}$ and a positive even number of $p_i \equiv 3 \pmod 4$.

Let x be a positive real number and let N_x be the number of square-free positive integers $m \leqslant x$ with $r+2$ prime factors. Then

$$N_x \sim \frac{1}{(r+1)!} \cdot \frac{x(\log\log x)^{r+1}}{\log x} \quad (\text{as } x \to \infty)$$

(cf. [13, Theorem 437]). Now using the m from case (iv) above, we define

$$Y_{r;x} = \bigl\{K = \mathbb{Q}(\sqrt{m}\,)\colon\ m = p_1 \cdots p_{r+2} \leqslant x \text{ with odd primes } p_1 < \cdots < p_{r+2} \\ \text{and with a positive even number of } p_i \equiv 3 \pmod 4\bigr\}. \tag{9}$$

Then

$$|Y_{r;x}| \sim \left(\frac{1}{2} - \frac{1}{2^{r+2}}\right) \cdot \frac{1}{(r+1)!} \cdot \frac{x(\log\log x)^{r+1}}{\log x} \quad (\text{as } x \to \infty), \tag{10}$$

since

$$\bigl|\bigl\{m = p_1 \cdots p_{r+2} \leqslant x \text{ with primes } p_1 < \cdots < p_{r+2} \text{ and } m \equiv 1 \pmod 4\bigr\}\bigr| \sim \frac{1}{2} N_x$$

and

$$\left|\left\{m = p_1 \cdots p_{r+2} \leqslant x \text{ with primes } p_1 < \cdots < p_{r+2} \text{ and each } p_i \equiv 1 \pmod 4\right\}\right| \\ \sim \frac{1}{2^{r+2}} N_x.$$

Next we note that the m in (4) come from cases (i)–(iv) above, but m in cases (i)–(iii) contribute only $o(x(\log\log x)^{r+1}/\log x)$ to $|A_{r;x}|$. So

$$|A_{r;x}| \sim |Y_{r;x}| \quad (\text{as } x \to \infty). \tag{11}$$

Then for $r = 4$ or 5, we define

$$Y_{r,3;x} = \{K \in Y_{r;x}\colon \text{ the 4-class rank } s_K = 3\}, \tag{12}$$

$$Y^*_{r,3;x} = \{K \in Y_{r,3;x}\colon \text{ the Hilbert 2-class field tower of } K \text{ is infinite}\}. \tag{13}$$

From [5, Proposition 5.1],

$$|Y_{r,3;x}| \gg \frac{x(\log\log x)^{r+1}}{\log x}$$

(in the formula for $|B_{t,e;x}|$ in [5, Proposition 5.1], use $t = r + 2$, $e = 3$, but sum over $\ell \geqslant 2$ rather than $\ell \geqslant 0$ to get $|Y_{r,3;x}|$). From (10) and from [7, Theorem 2 and Eq. 13],

$$\left|Y^*_{r,3;x}\right| \gg \frac{x(\log\log x)^{r+1}}{\log x}.$$

So we may disregard the m in cases (i)–(iii) when calculating asymptotic formulas for $|A_{r,3;x}|$ and $|A^*_{r,3;x}|$ for $r = 4$ or 5. In other words, for $r = 4$ or 5,

$$|A_{r,3;x}| \sim |Y_{r,3;x}| \quad (\text{as } x \to \infty),$$

$$\left|A^*_{r,3;x}\right| \sim \left|Y^*_{r,3;x}\right| \quad (\text{as } x \to \infty).$$

Then from (7) we get

$$\alpha^*_{r,3} = \liminf_{x\to\infty} \frac{|Y^*_{r,3;x}|}{|Y_{r,3;x}|} \tag{14}$$

for $r = 4$ or 5.

We first consider $r = 5$. Since only the m in case (iv) are used in $Y_{5,3;x}$, then the number of ramified primes in $\mathbb{Q}(\sqrt{m})$ is seven. From [10, Theorem 1], $\mathbb{Q}(\sqrt{m})$ has infinite Hilbert 2-class field tower if seven primes ramify in $\mathbb{Q}(\sqrt{m})$ and the 4-class rank of $\mathbb{Q}(\sqrt{m})$ is 3. So $Y^*_{5,3;x} = Y_{5,3;x}$ and $\alpha^*_{5,3} = 1$.

So we consider $r = 4$. For $K = \mathbb{Q}(\sqrt{m}) \in Y_{4;x}$, we see that $m = p_1 \cdots p_6$ with odd primes $p_1 < p_2 < \cdots < p_6$ and with a positive even number of $p_i \equiv 3 \pmod 4$. From [5, Eqs. 5.5 and 5.6],

$$s_K = 5 - \operatorname{rank} M_K, \tag{15}$$

where M_K is a 6×6 Rédei matrix over $\mathbb{F}_2$ whose entries a_{ij} are defined by Legendre symbols as follows:

$$(-1)^{a_{ij}} = \begin{cases} \left(\frac{P_i}{p_j}\right) & \text{if } i \neq j, \\ \left(\frac{\bar{P}_i}{p_j}\right) & \text{if } i = j, \end{cases} \tag{16}$$

where $P_i = p_i$ if $p_i \equiv 1 \pmod 4$, $P_i = -p_i$ if $p_i \equiv 3 \pmod 4$, and $\bar{P}_i = m/P_i$. Actually the matrix M_K whose entries satisfy (16) is the transpose of the matrix M_K in [5], but the transpose of M_K in [5] is used to derive other formulas in [5]. Also, the 4-class rank in [5] is the 4-rank of the narrow ideal class group of K, but since some primes congruent to $3 \pmod 4$ divide m, then the 4-class rank of K in the usual sense is the same as the 4-class rank of K in the narrow sense. We also note that from properties of Legendre symbols, the sum of the entries in each column of M_K is zero.

Now since $p_i \equiv 1$ or $3 \pmod 4$ for $1 \leqslant i \leqslant 6$ and since a positive even number of p_i satisfy $p_i \equiv 3 \pmod 4$, there are $\binom{6}{2} + \binom{6}{4} + \binom{6}{6} = 31$ possible sets of these congruence conditions: 15 with exactly two $p_i \equiv 3 \pmod 4$, 15 with exactly four $p_i \equiv 3 \pmod 4$, and one with all six $p_i \equiv 3 \pmod 4$. For each of the 31 possible sets of congruence conditions, there are 32 768 possible matrices M_K whose entries satisfy (16). (These matrices are determined by the $2^{15} = 32\,768$ possible values of the 15 Legendre symbols $\left(\frac{P_i}{p_j}\right)$ for $1 \leqslant i < j \leqslant 6$, given a set of congruence conditions (mod 4) for $p_1, \ldots, p_6$.) Then combining the possible sets of congruence conditions (mod 4) and the number of matrices for each set of congruence conditions (mod 4), there are $31 \cdot 32\,768 = 1\,015\,808$ possibilities to consider, and each is asymptotically equally likely to occur as $x \to \infty$. (This follows from [5, Eq. 2.11 and Formula 2.12], which depend on character sum estimates similar to those in Section 4 of [4] and Section 5 of [6]. Actually Eq. 2.11 and Formula 2.12 in [5] are derived for imaginary quadratic fields, but the same arguments work for real quadratic fields.)

Although there are 1 015 808 possibilities to consider when analyzing $Y_{4;x}$, we are actually interested in $K \in Y_{4,3;x}$. So we want the 4-class rank $s_K = 3$. From (15), we want $\operatorname{rank} M_K = 2$. The condition $\operatorname{rank} M_K = 2$ will substantially reduce the number of possibilities to consider.

Case (a). Suppose that $K = \mathbb{Q}(\sqrt{m})$, where $m = p_1 \cdots p_6$ with odd primes $p_1 < p_2 < \cdots < p_6$ and exactly two $p_i \equiv 3 \pmod 4$, and $s_K = 3$.

For convenience we consider $p_1 \equiv p_2 \equiv 3 \pmod 4$ and $p_i \equiv 1 \pmod 4$ for $3 \leqslant i \leqslant 6$. (Similar arguments work for other arrangements of the six primes with two of them congruent to $3 \pmod 4$.) Since the sum of the entries in each column of M_K is zero, we may delete the first row of M_K without changing the rank. Then by adding columns 2 through 6 to column 1, we get a matrix of the form

$$\bar{M} = \left[\begin{array}{c:c:c} 1 & A & B \\ \hdashline 0 & & \\ 0 & B^T & C \\ 0 & & \\ 0 & & \end{array}\right], \tag{17}$$

where A is a 1×1 matrix over $\mathbb{F}_2$, B is a 1×4 matrix over $\mathbb{F}_2$, B^T is the transpose of B, and C is a 4×4 symmetric matrix over $\mathbb{F}_2$ (cf. [5, Eq. 5.12]). We can calculate the number of possible $\bar{M}$ in (17) with $\operatorname{rank} \bar{M} = 2$ by using the algorithm in the remark following Proposition 5.10 in [5]. That calculation gives 90 matrices $\bar{M}$ in (17) with $\operatorname{rank} \bar{M} = 2$. Since there are $\binom{6}{2} = 15$ arrangements of the six primes dividing m with exactly two of them congruent to $3 \pmod 4$, there are $90 \cdot 15 = 1350$ possibilities to consider in Case (a).

We shall call a matrix M_K "good" if it corresponds to a field K that we know has infinite Hilbert 2-class field tower. We shall need two preliminary lemmas. Although these results are known to specialists, we shall sketch their proofs.

Lemma 1. *Suppose L is a real quadratic field and suppose F is a quadratic extension of L with at least ten primes ramified in F/L. Then F has infinite Hilbert 2-class field tower.*

Proof. From genus theory the 2-class rank

$$r_F \geqslant t - 1 - \dim_{\mathbb{F}_2}\bigl(E_L/\bigl(E_L \cap N_{F/L}\bigl(F^\times\bigr)\bigr)\bigr),$$

where t is the number of primes that ramify in F/L, E_L is the group of units in the ring of integers of L, and $N_{F/L} : F^\times \to L^\times$ is the norm map. Since $t \geqslant 10$ and $\dim_{\mathbb{F}_2}(E_L/(E_L \cap N_{F/L}(F^\times))) \leqslant 2$, then $r_F \geqslant 7$. So r_F satisfies (3) and hence F has infinite Hilbert 2-class field tower. $\square$

Lemma 2. *Suppose L is a totally real field with $[L : \mathbb{Q}] = 4$. Suppose q_1, q_2, q_3 are rational primes, each congruent to $1 \pmod 4$, that split completely in L. Let $F = L(\sqrt{q_1 q_2 q_3})$. Then F has infinite Hilbert 2-class field tower.*

Proof. Since q_1, q_2, q_3 split completely in L, then twelve primes ramify in F/L. Since each $q_i \equiv 1 \pmod 4$, then $-1 \in (E_L \cap N_{F/L}(F^\times))$. So $\dim_{\mathbb{F}_2}(E_L/(E_L \cap N_{F/L}(F^\times))) \leqslant 3$. So the 2-class rank $r_F \geqslant 12 - 1 - 3 = 8$. Then r_F satisfies (3), and hence F has infinite Hilbert 2-class field tower. $\square$

Now if $\operatorname{rank} \bar{M} = 2$ in (17), then the rank of the submatrix $[B^T\ C]$ must be 1. Suppose the first row of $[B^T\ C]$ consists entirely of zeros. From the way $\bar{M}$ was created from the entries in (16), this means $\left(\frac{P_3}{p_j}\right) = 1$ for all $j \neq 3$. Then p_1, p_2, p_4, p_5, p_6 split in $L = \mathbb{Q}(\sqrt{P_3})$, and then ten primes ramify in F/L, where $F = L(\sqrt{m})$. By Lemma 1,

F has infinite Hilbert 2-class field tower. Since F is contained in the Hilbert 2-class field of $K = \mathbb{Q}(\sqrt{m}\,)$, then K has infinite Hilbert 2-class field tower. A similar argument can be used if any row of $[B^T\ C]$ consists entirely of zeros. So the matrix M_K is good if any row of $[B^T\ C]$ consists entirely of zeros.

So suppose there are no rows of $[B^T\ C]$ consisting entirely of zeros. Since $\operatorname{rank}[B^T\ C] = 1$, then every row of $[B^T\ C]$ is the same. One possibility is when $B = [1\ 1\ 1\ 1]$ and every entry in C is zero. From the way $\bar{M}$ was created from the entries in (16), we see that $\left(\frac{P_i}{p_j}\right) = -1$ for $1 \leqslant i \leqslant 2$ and $3 \leqslant j \leqslant 6$. Also $\left(\frac{P_3}{p_j}\right) = 1$ for $4 \leqslant j \leqslant 6$. Then p_4, p_5, p_6 split completely in $L = \mathbb{Q}(\sqrt{P_1P_2}, \sqrt{P_3}\,)$. Then we can apply Lemma 2 to conclude that $F = L(\sqrt{P_4P_5P_6}\,)$ has infinite Hilbert 2-class field tower. Since $L(\sqrt{m}\,) = L(\sqrt{P_4P_5P_6}\,) = F$, then F is contained in the Hilbert 2-class field of $K = \mathbb{Q}(\sqrt{m}\,)$, and hence K has infinite Hilbert 2-class field tower. So M_K is a good matrix in this situation.

The other possibilities with $\operatorname{rank}[B^T\ C] = 1$ and no row of zeros in $[B^T\ C]$ are $B = [0\ 0\ 0\ 0]$ or $[1\ 1\ 1\ 1]$, and every entry in C is 1. In these situations we are not able to determine whether K has infinite Hilbert 2-class field tower. Since $A = [0]$ or $[1]$ in (17), there are 4 possible matrices $\bar{M}$ in (17) for which we are unable to determine whether the field has infinite Hilbert 2-class field tower. Since there are $\binom{6}{2} = 15$ arrangements of the six primes dividing m with exactly two of them congruent to 3 (mod 4), there are $4 \cdot 15 = 60$ of the 1350 possibilities in Case (a) in which we are unable to determine whether K has infinite Hilbert 2-class field tower. For the other 1290 possibilities, K has infinite Hilbert 2-class field tower.

Case (b). Suppose that $K = \mathbb{Q}(\sqrt{m}\,)$, where $m = p_1 \cdots p_6$ with odd primes $p_1 < p_2 < \cdots < p_6$ and exactly four $p_i \equiv 3 \pmod 4$, and $s_K = 3$.

For convenience we suppose $p_i \equiv 3 \pmod 4$ for $1 \leqslant i \leqslant 4$ and $p_5 \equiv p_6 \equiv 1 \pmod 4$. (Similar arguments work for other arrangements.) Analogous to (17), we get a matrix

$$\bar{M} = \left[\begin{array}{c:c:c} 1 & & \\ 1 & A & B \\ 1 & & \\ \hdashline 0 & & \\ 0 & B^T & C \end{array}\right], \tag{18}$$

where $A = [A_{ij}]$ is a 3×3 antisymmetric matrix over $\mathbb{F}_2$ (i.e., $A_{ij} \neq A_{ji}$ if $i \neq j$), B is a 3×2 matrix over $\mathbb{F}_2$, B^T is the transpose of B, and C is a 2×2 symmetric matrix over $\mathbb{F}_2$. If we use the algorithm in the remark following Proposition 5.10 in [5], we get 24 possible matrices $\bar{M}$ in (18) with $\operatorname{rank} \bar{M} = 2$, and in each of them, $\operatorname{rank}[B^T\ C] = 0$. Then $\left(\frac{P_6}{p_j}\right) = 1$ for $1 \leqslant j \leqslant 5$. So p_1, p_2, p_3, p_4, p_5 split is $L = \mathbb{Q}(\sqrt{P_6}\,)$, and ten primes ramify in F/L, where $F = L(\sqrt{m}\,)$. By Lemma 1, F has infinite Hilbert 2-class field tower. Since F is contained in the Hilbert 2-class field of $K = \mathbb{Q}(\sqrt{m}\,)$, then K has infinite Hilbert 2-class field tower. Since there are $\binom{6}{4} = 15$ arrangements of the six primes dividing m with exactly four of them congruent to 3 (mod 4), there are $24 \cdot 15 = 360$ possibilities in Case (b), and all of them correspond to fields K which have infinite Hilbert 2-class field towers.

Case (c). Suppose $K = \mathbb{Q}(\sqrt{m})$, where $m = p_1 \cdots p_6$ and each $p_i \equiv 3 \pmod 4$, and $s_K = 3$.

In this case the 6×6 matrix M_K is antisymmetric (i.e., $a_{ij} \neq a_{ji}$ for all $i \neq j$ in (16)). From [5, Corollary 3.3], $\operatorname{rank} M_K \geqslant 3$. Hence there are no fields K in Case (c) with $\operatorname{rank} M_K = 2$, and we may disregard this case.

Now tabulating the results from Cases (a) and (b), we see that there are $1350 + 360 = 1710$ possibilities to consider, and for all but 60 of the possibilities, we know that K has infinite Hilbert 2-class field tower. This implies that the density $\alpha^*_{4,3}$ defined by (7) satisfies $\alpha^*_{4,3} \geqslant \frac{1650}{1710}$, which completes the proof of Theorem 1.

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