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An integral representation, complete monotonicity, and inequalities of Cauchy numbers of the second kind

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ABSTRACT

In the paper, the author establishes an integral representation, finds the complete monotonicity, minimality, and logarithmic convexity, and presents some inequalities of Cauchy numbers of the second kind.

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1. Introduction

According to [1, pp. 293–294], there are two kinds of Cauchy numbers which may be defined respectively by

$$C_n = \int_0^1 \langle x\rangle_n \mathrm{d}x \quad \text{and} \quad c_n = \int_0^1 (x)_n \mathrm{d}x, \tag{1.1}$$

where

$$\langle x\rangle_n = \begin{cases} x(x-1)(x-2)\cdots(x-n+1), & n \ge 1 \\ 1, & n = 0 \end{cases} \tag{1.2}$$

and

$$(x)_n = \begin{cases} x(x+1)(x+2)\cdots(x+n-1), & n \ge 1 \\ 1, & n = 0 \end{cases} \tag{1.3}$$

are respectively called the falling and rising factorials. The coefficients expressing rising factorials $(x)_n$ in terms of falling factorials $\langle x\rangle_n$ are called Lah numbers. Lah numbers have an interesting meaning in combinatorics: they count the number of ways a set of n elements can be partitioned into k nonempty linearly ordered subsets. Shortly speaking, Cauchy numbers play important roles in some fields, such as approximate integrals, Laplace summation formula, and difference-differential equations, and are also related to some famous numbers such as Stirling numbers, Bernoulli numbers, and harmonic numbers. Therefore, Cauchy numbers deserve to be studied.

It is known [1, p. 294] that Cauchy numbers of the second kind c_k may be generated by

$$\frac{-t}{(1-t)\ln(1-t)} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!} \tag{1.4}$$

which is equivalent to

$$\frac{t}{(1+t)\ln(1+t)} = \sum_{n=0}^{\infty} (-1)^n c_n \frac{t^n}{n!}. \tag{1.5}$$

The first few Cauchy numbers of the second kind c_k are

$$c_0 = 1, \quad c_1 = \frac{1}{2}, \quad c_2 = \frac{5}{6}, \quad c_3 = \frac{9}{4}, \quad c_4 = \frac{251}{30},$$
$$c_5 = \frac{475}{12}, \quad c_6 = \frac{19087}{84}. \tag{1.6}$$

In this paper, we will establish an integral representation, find the complete monotonicity, minimality, and logarithmic convexity, and present some inequalities of Cauchy numbers of the second kind c_n.

2. An integral representation of Cauchy numbers

We first establish an integral representation of Cauchy numbers of the second kind c_n.

Theorem 2.1. *For $n \in \{0\} \cup \mathbb{N}$, Cauchy numbers of the second kind c_n have an integral representation*

$$c_n = n! \int_0^\infty \frac{\mathrm{d}u}{u[\pi^2 + (\ln u)^2](1+u)^n}. \tag{2.1}$$

Proof. Recall from [8] that the function

$$F(z) = \begin{cases} \dfrac{z}{(1+z)\ln(1+z)}, & z \in \mathbb{C} \setminus (-\infty, -1] \setminus \{0\} \\ 1, & z = 0 \end{cases} \tag{2.2}$$

has the integral representation

$$F(z) = \int_0^\infty \frac{u+1}{u[(\ln u)^2 + \pi^2]} \frac{\mathrm{d}u}{u+1+z}, \quad z \in \mathbb{C} \setminus (-\infty, -1], \tag{2.3}$$

where $\mathbb{C}$ is the set of all complex numbers. Differentiating n times on both sides of (1.5) and (2.3) yields

$$F^{(n)}(t) = \sum_{k=n}^\infty (-1)^k c_k \frac{t^{k-n}}{(k-n)!} = \sum_{k=0}^\infty (-1)^{k+n} c_{k+n} \frac{t^k}{k!}$$

and

$$F^{(n)}(t) = (-1)^n n! \int_0^\infty \frac{u+1}{u[(\ln u)^2 + \pi^2]} \frac{\mathrm{d}u}{(u+1+t)^{n+1}}.$$

Hence,

$$\sum_{k=0}^\infty (-1)^{k+n} c_{k+n} \frac{t^k}{k!} = (-1)^n n! \int_0^\infty \frac{u+1}{u[(\ln u)^2 + \pi^2]} \frac{\mathrm{d}u}{(u+1+t)^{n+1}}.$$

Further letting $t \to 0$ on both sides of the above equation gives the integral representation (2.1). The proof of Theorem 2.1 is complete. □

3. Complete monotonicity and minimality of Cauchy numbers

Basing on the integral representation (2.1), we now find complete monotonicity and minimality of Cauchy numbers of the second kind c_n.

Recall from monographs [5, pp. 372–373] and [11, p. 108, Definition 4] that a sequence $\{\mu_n\}_{0\le n\le\infty}$ is said to be completely monotonic if its elements are non-negative and its successive differences are alternatively non-negative, that is

$$(-1)^k\Delta^k\mu_n\ge 0 \tag{3.1}$$

for $n,k\ge 0$, where

$$\Delta^k\mu_n=\sum_{m=0}^{k}(-1)^m\binom{k}{m}\mu_{n+k-m}. \tag{3.2}$$

Recall from [11, p. 163, Definition 14a] that a completely monotonic sequence $\{a_n\}_{n\ge0}$ is minimal if it ceases to be completely monotonic when a_0 is decreased.

Theorem 3.1. *The infinite sequence of Cauchy numbers of the second kind*

$$\left\{\frac{c_n}{n!}\right\}_{n\ge0} \tag{3.3}$$

is completely monotonic and minimal.

Proof. It was stated in [5, pp. 372–373] and [11, p. 108, Theorem 4a] that a necessary and sufficient condition that the sequence $\{\mu_n\}_0^\infty$ should have the expression

$$\mu_n=\int_0^1 t^n\,\mathrm{d}\alpha(t) \tag{3.4}$$

for $n\ge0$, where $\alpha(t)$ is non-decreasing and bounded for $0\le t\le 1$, is that it should be completely monotonic. Theorem 14a in [11, p. 164] states that a completely monotonic sequence $\{\mu_n\}_{n\ge0}$ is minimal if and only if the integral representation (3.4) is valid for $n\ge0$ and $\alpha(t)$ is a non-decreasing bounded function continuous at $t=0$.

From (2.1), it follows that for $n\in\mathbb{N}$

$$\begin{aligned}\frac{c_n}{n!}&=\int_0^\infty\frac{1}{u[(\ln u)^2+\pi^2]}\frac{\mathrm{d}u}{(u+1)^n}\\&=\int_1^0\frac{1}{(-\ln t)\{[\ln(-\ln t)]^2+\pi^2\}}\frac{\mathrm{d}(-\ln t)}{(1-\ln t)^n}\end{aligned}$$

$$= \int_0^1 \frac{\mathrm{d}t}{t(-\ln t)\{[\ln(-\ln t)]^2 + \pi^2\}(1-\ln t)^n}$$
$$= \int_0^1 t^n \frac{\mathrm{d}t}{t^{n+1}(-\ln t)\{[\ln(-\ln t)]^2 + \pi^2\}(1-\ln t)^n}$$
$$= \int_0^1 t^n \mathrm{d}\left[\int_0^t \frac{1}{u^{n+1}(-\ln u)\{[\ln(-\ln u)]^2 + \pi^2\}(1-\ln u)^n}\mathrm{d}u\right].$$

This implies the complete monotonicity and minimality of the sequence (3.3).

The complete monotonicity may be alternatively proved as follows. In [5, p. 373], it was stated that if a function $f(t)$ is completely monotonic on $[0,\infty)$, that is, $(-1)^k f^{(k)}(t) \ge 0$ for $k \ge 0$, then the sequence $\{(-1)^n f^{(n)}(n)\}$ is completely monotonic. It is clear that the function of x

$$\int_0^\infty \frac{1}{u[(\ln u)^2 + \pi^2]} \frac{\mathrm{d}u}{(u+1)^x}$$

is completely monotonic on $[0,\infty)$. Hence, by the integral representation (2.1), the sequence (3.3) is completely monotonic. The proof of Theorem 3.1 is complete. □

4. Positivity of determinants for Cauchy numbers

With the help of the integral representation (2.1), we now present the positivity of two determinants of Cauchy numbers of the second kind c_n.

Theorem 4.1. *Let $m \in \mathbb{N}$ and let n and a_k for $1 \le k \le m$ be nonnegative integers. Then*

$$\left|(-1)^{a_i+a_j} c_{n+a_i+a_j}\right|_m \ge 0 \tag{4.1}$$

and

$$(-1)^{mn}|c_{n+a_i+a_j}|_m \ge 0, \tag{4.2}$$

where $|a_{kj}|_m$ denotes a determinant of order m with elements a_{kj}.

Proof. From the proof of Theorem 2.1, we observe that

$$\frac{c_n}{n!} = \lim_{t\to 0^+} h_n(t), \tag{4.3}$$

where

$$h_n(t) = \int_0^\infty \frac{u+1}{u[(\ln u)^2 + \pi^2]} \frac{\mathrm{d}u}{(u+1+t)^{n+1}} \tag{4.4}$$

is completely monotonic on $[0, \infty)$ and

$$h_n^{(k)}(t) = (-1)^k \frac{(n+k)!}{n!} h_{n+k}(t) \to (-1)^k \frac{(n+k)!}{n!} \frac{c_{n+k}}{(n+k)!} = (-1)^k \frac{c_{n+k}}{n!} \tag{4.5}$$

as $t \to 0^+$.

In [3], or see [5, p. 367], it was obtained that if f is a completely monotonic function on $[0, \infty)$, then

$$\left| f^{(a_i+a_j)}(x) \right|_m \geq 0 \tag{4.6}$$

and

$$\left| (-1)^{a_i+a_j} f^{(a_i+a_j)}(x) \right|_m \geq 0. \tag{4.7}$$

Applying f in (4.6) and (4.7) to the function $h_n(x)$ yields

$$\left| h_n^{(a_i+a_j)}(x) \right|_m \geq 0 \tag{4.8}$$

and

$$\left| (-1)^{a_i+a_j} h_n^{(a_i+a_j)}(x) \right|_m \geq 0. \tag{4.9}$$

Letting $x \to 0^+$ in (4.8) and (4.9) and making use of (4.5) produce

$$\left| (-1)^{a_i+a_j} \frac{c_{n+a_i+a_j}}{n!} \right|_m \geq 0 \tag{4.10}$$

and

$$\left| (-1)^n \frac{c_{n+a_i+a_j}}{n!} \right|_m \geq 0. \tag{4.11}$$

Further simplifying (4.10) and (4.11) leads to (4.1) and (4.2). The proof of Theorem 4.1 is complete. □

5. Inequalities for products of Cauchy numbers

In this final section, by virtue of the integral representation (2.1), we discover some inequalities and, as a consequence, the logarithmic convexity of Cauchy numbers of the second kind c_n.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$ and $\mu = (\mu_1, \mu_2, \dots, \mu_n) \in \mathbb{R}^n$. The sequence λ is said to be majorized by μ (in symbols $\lambda \preceq \mu$) if

$$\sum_{\ell=1}^{k} \lambda_{[\ell]} \le \sum_{\ell=1}^{k} \mu_{[\ell]}$$

for $k = 1, 2, \dots, n-1$ and

$$\sum_{\ell=1}^{n} \lambda_\ell = \sum_{\ell=1}^{n} \mu_\ell,$$

where $\lambda_{[1]} \ge \lambda_{[2]} \ge \dots \ge \lambda_{[n]}$ and $\mu_{[1]} \ge \mu_{[2]} \ge \dots \ge \mu_{[n]}$ are rearrangements of λ and μ in a descending order. A sequence λ is said to be strictly majorized by μ (in symbols $\lambda \prec \mu$) if λ is not a permutation of μ.

Theorem 5.1. *Let $m \in \mathbb{N}$ and let λ and μ be two m-tuples of nonnegative integers such that $\lambda \preceq \mu$. Then*

$$\prod_{i=1}^{m} c_{\lambda_i} \le \prod_{i=1}^{m} c_{\mu_i}. \tag{5.1}$$

Proof. In [10, p. 106, Theorem A] and [5, p. 367, Theorem 2], a minor correction of [2, Theorem 1], it was obtained that if f is a completely monotonic function on $(0, \infty)$ and $\lambda \preceq \mu$, then

$$\left|\prod_{i=1}^{n} f^{(\lambda_i)}(x)\right| \le \left|\prod_{i=1}^{n} f^{(\mu_i)}(x)\right|. \tag{5.2}$$

The equality in (5.2) is valid only when λ and μ are identical or when $f(x) = e^{-cx}$ for $c \ge 0$. Applying the inequality (5.2) to $h_n(x)$ creates

$$\left|\prod_{i=1}^{m} h_n^{(\lambda_i)}(t)\right| \le \left|\prod_{i=1}^{m} h_n^{(\mu_i)}(t)\right|.$$

Taking the limit $t \to 0^+$ on both sides of the above inequality and making use of (4.5) reveal

$$\left|\prod_{i=1}^{m}(-1)^{\lambda_i}\frac{c_{n+\lambda_i}}{n!}\right| \le \left|\prod_{i=1}^{m}(-1)^{\mu_i}\frac{c_{n+\mu_i}}{n!}\right| \tag{5.3}$$

which is equivalent to (5.1). The proof of Theorem 5.1 is complete. □

Corollary 5.2. *The infinite sequence $\{c_n\}_{n\ge0}$ is logarithmically convex.*

Proof. This follows from the majorization relation $(i+2,i)\succeq(i+1,i+1)$ for $i\ge0$ and Theorem 5.1.

This may also be verified as follows. In [5, p. 369] and [6, p. 429, Remark], it was stated that if $f(t)$ is a completely monotonic function such that $f^{(k)}(t)\ne0$ for $k\ge0$, then the sequence

$$s_i(t)=\ln\bigl[(-1)^{i-1}f^{(i-1)}(t)\bigr],\quad i\ge1 \tag{5.4}$$

is convex. Applying this result to the function $h_n(t)$ and making use of (4.5) figures out that the sequence

$$s_i(t)=\ln\bigl[(-1)^{i-1}h_n^{(i-1)}(t)\bigr]\to\ln\frac{c_{n+i-1}}{n!},\quad t\to0^+ \tag{5.5}$$

for $i\ge1$ is convex. Hence, the sequence $\{c_n\}_{n\ge0}$ is logarithmically convex. □

Corollary 5.3. *For $\ell\ge0$ and $n>k>0$, we have*

$$(c_{\ell+k})^n\le(c_{\ell+n})^k(c_\ell)^{n-k}. \tag{5.6}$$

Proof. As done in [2], considering the majorization relation

$$(\overbrace{k,k,\dots,k}^{n})\prec(\overbrace{n,\dots,n}^{k},\overbrace{0,\dots,0}^{n-k})$$

for $n>k$, the inequality (5.2) becomes

$$(-1)^{nk}\bigl[f^{(k)}(t)\bigr]^n\le(-1)^{nk}\bigl[f^{(n)}(t)\bigr]^k\bigl[f(t)\bigr]^{n-k},\quad n>k>0.$$

Substituting $h_\ell(t)$ for f in the above inequality, letting $t\to0^+$, and utilizing (4.5) procure

$$(-1)^{nk}\bigl[h_\ell^{(k)}(t)\bigr]^n\le(-1)^{nk}\bigl[h_\ell^{(n)}(t)\bigr]^k\bigl[h_\ell(t)\bigr]^{n-k},$$
$$(-1)^{nk}\left[(-1)^k\frac{c_{\ell+k}}{\ell!}\right]^n\le(-1)^{nk}\left[(-1)^n\frac{c_{\ell+n}}{\ell!}\right]^k\left(\frac{c_\ell}{\ell!}\right)^{n-k}$$

for $n>k>0$ and $\ell\ge0$. This may be simplified as (5.6). The required proof is complete. □

Theorem 5.4. *If $\ell \ge 0$, $n \ge k \ge m$, $k \ge n-k$, and $m \ge n-m$, then*

$$c_{\ell+k}c_{\ell+n-k} \ge c_{\ell+m}c_{\ell+n-m}. \tag{5.7}$$

Proof. In [9, p. 397, Theorem D], it was recovered that if $f(x)$ is completely monotonic on $(0,\infty)$ and if $n \ge k \ge m$, $k \ge n-k$, and $m \ge n-m$, then

$$(-1)^n f^{(k)}(x)f^{(n-k)}(x) \ge (-1)^n f^{(m)}(x)f^{(n-m)}(x). \tag{5.8}$$

Replacing $f(x)$ by the function $h_\ell(t)$ in the above inequality leads to

$$(-1)^n h_\ell^{(k)}(t)h_\ell^{(n-k)}(t) \ge (-1)^n h_\ell^{(m)}(t)h_\ell^{(n-m)}(t).$$

Further taking $t \to 0^+$ and employing (4.5) find

$$(-1)^n(-1)^k\frac{c_{\ell+k}}{\ell!}(-1)^{n-k}\frac{c_{\ell+n-k}}{\ell!} \ge (-1)^n(-1)^m\frac{c_{\ell+m}}{\ell!}(-1)^{n-m}\frac{c_{\ell+n-m}}{\ell!}.$$

Simplifying this inequality leads to (5.7). The proof of Theorem 5.4 is complete. □

Theorem 5.5. *For $n,m \in \mathbb{N}$ and $\ell \ge 0$, let*

$$\begin{aligned}
\mathcal{G}_{n,m,\ell} &= c_{\ell+n+2m}(c_\ell)^2 - c_{\ell+n+m}c_{\ell+m}c_\ell - c_{\ell+n}c_{\ell+2m}c_\ell + c_{\ell+n}(c_{\ell+m})^2,\\
\mathcal{H}_{n,m,\ell} &= c_{\ell+n+2m}(c_\ell)^2 - 2c_{\ell+n+m}c_{\ell+m}c_\ell + c_{\ell+n}(c_{\ell+m})^2,\\
\mathcal{I}_{n,m,\ell} &= c_{\ell+n+2m}(c_\ell)^2 - 2c_{\ell+n}c_{\ell+2m}c_\ell + c_{\ell+n}(c_{\ell+m})^2.
\end{aligned}$$

Then

$$\mathcal{G}_{n,m,\ell} \ge 0, \qquad \mathcal{H}_{n,m,\ell} \ge 0, \tag{5.9}$$

$$\mathcal{H}_{n,m,\ell} \lesseqqgtr \mathcal{G}_{n,m,\ell} \quad \text{when } m \lesseqqgtr n, \tag{5.10}$$

and

$$\mathcal{I}_{n,m,\ell} \ge \mathcal{G}_{n,m,\ell} \ge 0 \quad \text{when } n \ge m. \tag{5.11}$$

Proof. In [10, Theorem 1 and Remark 2], it was obtained that if f is completely monotonic on $(0,\infty)$ and

$$G_{n,m} = (-1)^n\{f^{(n+2m)}f^2 - f^{(n+m)}f^{(m)}f - f^{(n)}f^{(2m)}f + f^{(n)}[f^{(m)}]^2\}, \tag{5.12}$$

$$H_{n,m} = (-1)^n\{f^{(n+2m)}f^2 - 2f^{(n+m)}f^{(m)}f + f^{(n)}[f^{(m)}]^2\}, \tag{5.13}$$

$$I_{n,m} = (-1)^n\{f^{(n+2m)}f^2 - 2f^{(n)}f^{(2m)}f + f^{(n)}[f^{(m)}]^2\} \tag{5.14}$$

for $n,m \in \mathbb{N}$, then $G_{n,m} \ge 0$, $H_{n,m} \ge 0$, and

$$H_{n,m} \lesseqgtr G_{n,m} \quad \text{when } m \lesseqgtr n, \tag{5.15}$$

$$I_{n,m} \ge G_{n,m} \ge 0 \quad \text{when } n \ge m. \tag{5.16}$$

Replacing $f(t)$ by $h_\ell(t)$ in $G_{n,m}$, $H_{n,m}$, and $I_{n,m}$ and simplifying produce

$$\begin{aligned} G_{n,m} &= (-1)^n \bigl\{h_\ell^{(n+2m)} h_\ell^2 - h_\ell^{(n+m)} h_\ell^{(m)} h_\ell - h_\ell^{(n)} h_\ell^{(2m)} h_\ell + h_\ell^{(n)} \bigl[h_\ell^{(m)}\bigr]^2\bigr\}, \\ H_{n,m} &= (-1)^n \bigl\{h_\ell^{(n+2m)} h_\ell^2 - 2h_\ell^{(n+m)} h_\ell^{(m)} h_\ell + h_\ell^{(n)} \bigl[h_\ell^{(m)}\bigr]^2\bigr\}, \\ I_{n,m} &= (-1)^n \bigl\{h_\ell^{(n+2m)} h_\ell^2 - 2h_\ell^{(n)} h_\ell^{(2m)} h_\ell + h_\ell^{(n)} \bigl[h_\ell^{(m)}\bigr]^2\bigr\}. \end{aligned}$$

Further taking $t \to 0^+$ and employing (4.5) discover

$$(\ell!)^3 G_{n,m} = \mathcal{G}_{n,m,\ell}, \qquad (\ell!)^3 H_{n,m} = \mathcal{H}_{n,m,\ell}, \qquad (\ell!)^3 I_{n,m} = \mathcal{I}_{n,m,\ell}.$$

The proof of Theorem 5.5 is complete. $\square$

Theorem 5.6. *If $m \ge 1$ and $a_0, a_1, \dots, a_m$ be nonnegative integers, then*

$$\left(\frac{c_{a_0}}{a_0!}\right)^{m-1} \frac{c_{\sum_{k=0}^m a_k}}{(\sum_{k=0}^m a_k)!} \ge \prod_{k=1}^m \frac{c_{a_0+a_k}}{(a_0+a_k)!} \tag{5.17}$$

and

$$\left|\frac{c_{a_i+a_j}}{(a_i+a_j)!}\right|_m \ge 0. \tag{5.18}$$

Proof. In [4] and [5, pp. 369 and 374], it was obtained that if f is completely monotonic on $[0,\infty)$ and $m \ge 1$, then

$$\bigl[f(x_0)\bigr]^{m-1} f\left(\sum_{k=0}^m x_k\right) \ge \prod_{k=1}^m f(x_0+x_k) \tag{5.19}$$

and

$$\bigl|f(x_i+x_j)\bigr|_m \ge 0. \tag{5.20}$$

We consider the function (4.4) from an alternative viewpoint

$$\mathfrak{h}(t,s) = \int_0^\infty \frac{u+1}{u[(\ln u)^2+\pi^2]} \frac{du}{(u+1+t)^{s+1}} \tag{5.21}$$

and find that $\mathfrak{h}(t,s)$ is a completely monotonic function of $s \in [0,\infty)$. Replacing the function f and nonnegative numbers $x_0, x_1, \dots, x_m$ in (5.19) and (5.20) by the function $\mathfrak{h}(t,s)$ and nonnegative integers $a_0, a_1, \dots, a_m$ respectively yields

$$\left[\mathfrak{h}(t,a_0)\right]^{m-1}\mathfrak{h}\left(t,\sum_{k=0}^{m}a_k\right)\ge\prod_{k=1}^{m}\mathfrak{h}(t,a_0+a_k) \tag{5.22}$$

and

$$\left|\mathfrak{h}(t,a_i+a_j)\right|_m\ge 0. \tag{5.23}$$

By virtue of (4.3), we obtain

$$\lim_{t\to 0}\mathfrak{h}(t,a_i)=\frac{c_{a_i}}{a_i!}. \tag{5.24}$$

Therefore, taking $t\to 0$ in (5.22) and (5.23) leads to (5.17) and (5.18). The proof of Theorem 5.6 is complete. □

Remark 5.7. This paper is a slightly revised and corrected version of the preprint [7].

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