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An explicit bound for the number of partitions into roots

Florian Luca [a,*], Dimbinaina Ralaivaosaona [b]

[a] *School of Mathematics, University of the Witwatersrand, Private Bag X3, Wits 2050, South Africa*
[b] *Department of Mathematical Sciences, Stellenbosch University, Private Bag X1, Matieland 7602, South Africa*

A R T I C L E I N F O

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A B S T R A C T

We use the saddle point method to prove an explicit upper bound for the number of representations of a positive integer n into the form $\lfloor\sqrt{a_1}\rfloor + \lfloor\sqrt{a_2}\rfloor + \ldots + \lfloor\sqrt{a_k}\rfloor$, where k and $a_1, a_2, \ldots, a_k$ are positive integers. We also give an asymptotic formula for this number as $n \to \infty$.

1. Introduction

The number of representations of a positive integer n of the form $\lfloor\sqrt{a_1}\rfloor + \lfloor\sqrt{a_2}\rfloor + \cdots + \lfloor\sqrt{a_k}\rfloor$ with positive integers k and $a_1 \le a_2 \le \cdots \le a_k$ was introduced by Balasubramanian and Luca in [1]. In their paper, they investigated the size of the set

* Corresponding author.
E-mail addresses: Florian.Luca@wits.ac.za (F. Luca), naina@sun.ac.za (D. Ralaivaosaona).

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$$\mathcal{F}(x) = \{\Delta(n) \leq x\},$$

where $\Delta(n)$ is the number of unordered factorizations of the positive integer n into factors > 1. They proved that for all $x \geq 1$,

$$|\mathcal{F}(x)| \leq \exp\left(9(\log x)^{2/3}\right). \tag{1}$$

Their proof of inequality (1) however depended on the following bound, which was also proved in the same paper:

$$q(n) \leq \exp\left(5n^{2/3}\right), \tag{2}$$

where $q(n)$ denotes the number of representations of n of the form

$$n = \lfloor\sqrt{a_1}\rfloor + \lfloor\sqrt{a_2}\rfloor + \ldots + \lfloor\sqrt{a_k}\rfloor,$$

where k and $a_1 \leq a_2 \leq \cdots \leq a_k$ are positive integers.

Chen and Li [2] found a gap in the proof of (2). They provided an alternative proof that gives a weaker upper bound as well as a lower bound for $q(n)$. Their result is the following:

$$\exp\left(c_1 n^{2/3}\right) \leq q(n) \leq \exp\left(c_2 n^{2/3}\right),$$

where $c_1 \approx 5.385 \times 10^{-24}$ and $c_2 \approx 22.962$. In this paper, we use the saddle point method to obtain the following explicit upper bound on $q(n)$:

Theorem 1. *For $n \geq 50$ we have*

$$\log q(n) \leq \frac{6\zeta(3)^{1/3}}{4^{2/3}} n^{2/3} \left(1 + \frac{3.2}{\sqrt[3]{n+2}}\right), \tag{3}$$

with

$$\frac{6\zeta(3)^{1/3}}{4^{2/3}} \approx 2.532.$$

The condition $n \geq 50$ is only a technicality that we assume to obtain a relatively small constant in the second term of the bound in (3). For $n \leq 50$, Fig. 1 shows that an even better upper bound holds (in Fig. 1 the curve represents the graph of the function $x \mapsto \frac{6\zeta(3)^{1/3}}{4^{2/3}} x^{2/3}$ and the dotted line represents the graph of the sequence of general term $\log q(n)$).

Note that if $n \geq 50$ then

$$\frac{6\zeta(3)^{1/3}}{4^{2/3}} \left(1 + \frac{3.2}{\sqrt[3]{n+2}}\right) \geq \frac{6\zeta(3)^{1/3}}{4^{2/3}} \left(1 + \frac{3.2}{\sqrt[3]{52}}\right) \approx 4.7,$$

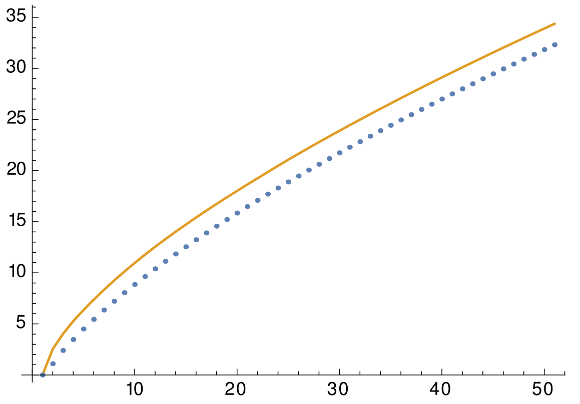

Fig. 1. Graphs of $\log q(n)$ and $x \mapsto \frac{6\zeta(3)^{1/3}}{4^{2/3}} x^{2/3}$.

so Theorem 1 and Fig. 1 show that the Balasubramanian–Luca bound (2) is true for all $n \geq 1$.

Fig. 1 seems to suggest that $\frac{6\zeta(3)^{1/3}}{4^{2/3}} n^{2/3}$ is an upper bound for $\log q(n)$. Unfortunately, this is not the case for large n as we can see in the following asymptotic formula:

Theorem 2. *As $n \to \infty$, we have*

$$q(n) \sim K n^{-8/9} \exp\left(\frac{6\zeta(3)^{1/3}}{4^{2/3}} n^{2/3} + \frac{\zeta(2)}{(4\zeta(3))^{1/3}} n^{1/3}\right)$$

where

$$K = \frac{(4\zeta(3))^{7/18}}{\pi \mathrm{A}^2 \sqrt{12}} \exp\left(\frac{4\zeta(3) - \zeta(2)^2}{24\zeta(3)}\right) \approx 0.110678,$$

and A *is the Glaisher–Kinkelin constant [3, Section 2.15].*

The saddle-point method is a standard technique to obtain asymptotic results for the number of partitions of positive integers. Here, we are interested in counting the number of partitions of n into members of the sequence $\lfloor\sqrt{1}\rfloor, \lfloor\sqrt{2}\rfloor, \lfloor\sqrt{3}\rfloor, \ldots$. There is a general scheme due to Meinardus [6] that gives an asymptotic formula for the number of partitions of n into members of a given sequence of positive integers Λ, provided that Λ satisfies certain properties known as the Meinardus' scheme. However, this scheme does not apply to our case since the Dirichlet series associated to our sequence has multiple poles on the positive real axis: one can show that the Dirichlet series associated to the integer part of the square-root sequence is

$$D(s) = \sum_{k=1}^{\infty} \frac{2k+1}{k^s}.$$

The function $D(s)$ is analytic in the half-plane $\mathrm{Re}(s) > 2$. Moreover, we have $D(s) = 2\zeta(s-1)+\zeta(s)$, where $\zeta(s)$ is the Riemann zeta function. Hence, $D(s)$ admits an analytic continuation to the entire complex plane with simple poles at $s = 2$ and $s = 1$. In order to apply the Meinardus' result directly, it is required that the Dirichlet series admits only one simple pole in the half-plane $\mathrm{Re}(s) > 0$. There is a generalization of the Meinardus' result that can be applied to our case which has been given in [5]. However, the coefficients in the asymptotic formula of [5] were only given implicitly and have never been computed. Since, we are also interested in obtaining an explicit bound, we will present full proofs of our theorems in this paper.

2. Estimates of sums via Mellin transform

Consider the following function

$$f(x) = -\sum_{k=1}^{\infty}(2k+1)\log(1-e^{-kx}).$$

We use Mellin transform to estimate $f^{(l)}(x)$ for $l = 0, 1, 2, 3$, when x is a small positive number, where $f^{(l)}$ is the l-th derivative of f. For a more elaborate account of the Mellin transform technique, the reader can consult [4]. We begin with $f(x)$. Its Mellin transform is

$$\Phi_0(s) = \zeta(s+1)\Gamma(s)(2\zeta(s-1)+\zeta(s)).$$

The Mellin inversion formula yields

$$f(x) = \frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty}\Phi_0(s)x^{-s}ds.$$

By shifting the path of integration in the Mellin inversion formula to $\mathrm{Re}(s) = 0.5$, we obtain

$$f(x) = 2\zeta(3)x^{-2} + \zeta(2)x^{-1} + E_0(x),$$

where

$$E_0(x) = \frac{1}{2\pi i}\int_{0.5-i\infty}^{0.5+i\infty}\Phi_0(s)x^{-s}ds.$$

The first two terms in the above estimate are the contributions of the poles at $s = 2$ and $s = 1$ of $\Phi_0(s)$. Note that we are allowed to shift the path of integration since

$|\Gamma(s)|$ decreases exponentially along any fixed vertical line while $|\zeta(s)|$ can grow at most polynomially on such lines. Now we estimate the error term $E_0(x)$. We have

$$|E_0(x)| \leq x^{-1/2}\frac{\zeta(1.5)}{2\pi}\Big(2\int_{-\infty}^{\infty}\Big|\Gamma(0.5+it)\zeta(-0.5+it)\Big|dt + \int_{-\infty}^{\infty}\Big|\Gamma(0.5+it)\zeta(0.5+it)\Big|dt\Big).$$

The integrals on the right hand side are both convergent, so $|E_0(x)|$ is bounded by $Cx^{-1/2}$, where C is a positive constant. To estimate C, we make use of the following bounds:

$$|\Gamma(0.5+it)| = \sqrt{\frac{\pi}{\cosh(\pi t)}}; \tag{4}$$

$$|\zeta(0.5+it)| \leq 0.732|4.678+it|^{1/6}\log|4.678+it|; \tag{5}$$

$$|\zeta(-0.5+it)| \leq \frac{\zeta(1.5)}{\sqrt{2\pi^3}}\Big|\sin\Big(\frac{-\pi+2i\pi t}{4}\Big)\Gamma(1.5+it)\Big|. \tag{6}$$

Equation (4) is a well-known property of $\Gamma(s)$ (a consequence of the Euler's reflection formula), inequality (5) can be found in [7], and inequality (6) follows from the functional equation of $\zeta(s)$. Using these estimates, one can numerically estimate a bound for the constant C. We get

$$|E_0(x)| \leq 2.8x^{-1/2}.$$

For $l = 1, 2, 3$, the Mellin transform of $f^{(l)}(x)$ is

$$\Phi_l(s) := (-1)^l\zeta(s-l+1)\Gamma(s)\Big(2\zeta(s-l-1)+\zeta(s-l)\Big).$$

We apply the same technique as before: the Mellin inversion formula gives us

$$f^{(l)}(x) = \frac{1}{2\pi i}\int_{l+3-i\infty}^{l+3+i\infty}\Phi_l(s)x^{-s}ds.$$

We only need to shift the path of integration to $\mathrm{Re}(s) = 0.5+l$. Hence, for any $x > 0$ we obtain

$$f'(x) = -4\zeta(3)x^{-3} - \zeta(2)x^{-2} + E_1(x) \text{ with } |E_1(x)| \leq 2.9\ x^{-3/2}, \tag{7}$$

$$f''(x) = 12\zeta(3)x^{-4} + 2\zeta(2)x^{-3} + E_2(x) \text{ with } |E_2(x)| \leq 6.1\ x^{-5/2}, \tag{8}$$

$$f'''(x) = -48\zeta(3)x^{-5} - 6\zeta(2)x^{-4} + E_3(x) \text{ with } |E_3(x)| \leq 19.1\ x^{-7/3}. \tag{9}$$

The upper bounds given for the above error terms are not best possible but they are good enough for our purpose.

3. Saddle point method

We consider the generating function

$$F(z) = \prod_{k=1}^{\infty} \left(1 - z^k\right)^{-2k-1}. \tag{10}$$

One can easily show that for a positive integer n, $q(n)$ is the coefficient of z^n in $F(z)$. In addition, $F(z)$ is analytic in the disk $|z| < 1$. Therefore, Cauchy's integral formula yields

$$q(n) = \frac{1}{2\pi i} \oint_{|z|=\rho} F(z) \frac{dz}{z^{n+1}}, \tag{11}$$

for any positive number $\rho < 1$. Setting $z = e^{-\tau}$ where $\tau = r + it$ with $r > 0$, and $f(\tau) = \log F(e^{-\tau})$, we deduce that

$$q(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\Big(f(r+it) + int\Big) dt. \tag{12}$$

We now estimate the integral on the right hand side of (12): we first choose r to be the unique positive solution of the equation

$$n = -f'(r) = \sum_{k=1}^{\infty} \frac{k(2k+1)}{e^{kr} - 1}. \tag{13}$$

The existence and uniqueness of the positive solution r can be verified by noting that the series on the right hand side above is a monotone decreasing function of r. In particular, we have $r = r(n) \to 0$ as $n \to \infty$. Furthermore, since $-f'(1) = 5.82956\ldots$, from now on, we will assume that $n \geq 6$ so that $r < 1$ (we need this fact later).

Next, we split the integral in (12) into two parts, namely

$$I_1 = \frac{1}{2\pi} \int_{-t_0}^{t_0} \exp\Big(f(r+it) + int\Big) dt,$$

where $t_0 = r^{11/6}$, and $I_2 = q(n) - I_1$ so that $q(n) = I_1 + I_2$. We estimate I_1 and I_2 separately.

3.1. Estimate of I_1

We can approximate $f(r+it)$ by means of its Taylor expansion. For $|t| \le t_0$, we have

$$f(r+it) = f(r) + f'(r)it - f''(r)\frac{t^2}{2} + R(t,r),$$

where the error term $R(t,r)$ satisfies

$$|R(t,r)| \le \frac{r^{11/2}}{6} \max_{|\eta|\le t_0} |f'''(r+i\eta)|.$$

The following lemma provides an explicit upper bound for the error term $R(t,r)$.

Lemma 3. *For $0 < r \le 1$ and $|t| \le r^{11/6}$ we have*

$$|R(t,r)| \le 15\sqrt{r}.$$

Proof. By definition

$$f'''(\tau) = -\sum_{k=1}^{\infty} \frac{k^3(2k+1)e^{k\tau}(1+e^{k\tau})}{(e^{k\tau}-1)^3}.$$

Hence,

$$\begin{aligned} |f'''(r+i\eta)| &\le \sum_{k=1}^{\infty} \frac{k^3(2k+1)e^{kr}(1+e^{kr})}{|e^{k(r+i\eta)}-1|^3} \\ &\le \sum_{k=1}^{\infty} \frac{k^3(2k+1)e^{kr}(1+e^{kr})}{(e^{kr}-1)^3}, \end{aligned}$$

since $|e^{k(r+i\eta)} - 1\Big| \ge e^{kr} - 1$ for any real number η. Thus,

$$|f'''(r+i\eta)| \le |f'''(r)|.$$

Now by the estimate of $f'''(r)$ in (9) we have

$$\begin{aligned} |f'''(r+i\eta)| &\le 48\zeta(3)r^{-5} + 6\zeta(2)r^{-4} + 19.1r^{-7/3} \\ &\le \Big(48\zeta(3) + 6\zeta(2) + 19.1\Big)r^{-5} \qquad (r \le 1). \end{aligned}$$

Therefore,

$$|R(t,r)| \le \frac{48\zeta(3)+6\zeta(2)+19.1}{6}\sqrt{r} \le 15\sqrt{r},$$

which completes the proof of the lemma. $\square$

Lemma 4. *We have*

$$\int_{-t_0}^{t_0} \exp\Big(-f''(r)\frac{t^2}{2}\Big)dt = \sqrt{\frac{2\pi}{f''(r)}} + J_1(r),$$

with

$$|J_1(r)| \le 0.4\ r^{13/6}\exp(-5\ r^{-1/3}).$$

Proof. Note that $f''(r)$ is positive when r is positive. So we have

$$\int_{-\infty}^{\infty} \exp\Big(-f''(r)\frac{t^2}{2}\Big)dt = \sqrt{\frac{2\pi}{f''(r)}}.$$

Hence,

$$\begin{aligned}\left|\int_{-t_0}^{t_0} \exp\Big(-f''(r)\frac{t^2}{2}\Big)dt - \sqrt{\frac{2\pi}{f''(r)}}\right| &\le 2\int_{t_0}^{\infty} \exp\Big(-f''(r)\frac{t^2}{2}\Big)dt \\ &\le 2\int_{t_0}^{\infty} \exp\Big(-t_0 f''(r)\frac{t}{2}\Big)dt \\ &\le \frac{4\exp\Big(-t_0^2 f''(r)/2\Big)}{t_0 f''(r)}. \qquad (14)\end{aligned}$$

Since $r<1$, using (8) we obtain

$$\begin{aligned} f''(r) &\ge 12\zeta(3)r^{-4} + 2\zeta(2)r^{-3} - 6.1r^{-5/2} \\ &\ge \Big(12\zeta(3)+2\zeta(2)-6.1\Big)r^{-4} \\ &\ge 10\ r^{-4}. \qquad (15)\end{aligned}$$

Combining the last inequality (15) with the above upper bound (14) proves the result. $\square$

We are now ready to estimate I_1. Recall that

$$I_1 = \frac{e^{nr+f(r)}}{2\pi}\int_{-t_0}^{t_0} \exp\Big(-f''(r)\frac{t^2}{2} + R(t,r)\Big)dt.$$

Applying the bounds in Lemma 3 and Lemma 4, we have

$$|I_1| \leq \exp\left(nr + f(r) + 15\sqrt{r}\right)\left(\frac{1}{\sqrt{2\pi f''(r)}} + \frac{|J_1(r)|}{2\pi}\right)$$

$$\leq \exp\left(nr + f(r) + 15\sqrt{r}\right)\left(\frac{r^2}{\sqrt{20\pi}} + \frac{|J_1(r)|}{2\pi}\right).$$

Here, we used the lower bound (15) that is only valid for $r < 1$. By the upper bound on $|J_1|$ from Lemma 4, and assuming still that $r < 1$, we finally deduce the following estimate for $|I_1|$:

$$|I_1| \leq 0.13\ \exp\left(nr + f(r) + 15\sqrt{r}\right). \tag{16}$$

For $r \to 0^+$, one can easily deduce the following asymptotic estimate from Lemma 3 and Lemma 4:

$$I_1 = \frac{e^{nr+f(r)}}{\sqrt{2\pi f''(r)}}\left(1 + \mathcal{O}(\sqrt{r})\right). \tag{17}$$

3.2. Estimate of I_2

We continue to assume that $n \geq 6$ so $r < 1$. We start with the following lemma.

Lemma 5. *For real t with $t_0 \leq |t| \leq \pi$, we have*

$$\frac{|F(e^{-r-it})|}{F(e^{-r})} \leq \exp\left(-\frac{r^{-1/3}}{64}\right).$$

Proof. In view of (10), we have

$$\frac{|F(e^{-r-it})|}{F(e^{-r})} = \exp \operatorname{Re}\left(\sum_{k=1}^{\infty}(2k+1)\left(\log\left(1 - e^{-kr}\right) - \log\left(1 - e^{-k(r+it)}\right)\right)\right)$$

$$= \exp\left(-\sum_{k=1}^{\infty}(2k+1)\left(\sum_{l=1}^{\infty}\frac{e^{-klr}}{l}\left(1 - \cos(klt)\right)\right)\right)$$

$$\leq \exp\left(-\sum_{k=1}^{\infty}(2k+1)e^{-kr}\left(1 - \cos(kt)\right)\right)$$

$$\leq \exp\left(-2\sum_{k=1}^{\infty}ke^{-kr}\left(1 - \cos(kt)\right)\right).$$

Furthermore, we have

$$\begin{aligned}\sum_{k=1}^{\infty} ke^{-kr}\Big(1-\cos(kt)\Big) &= \frac{e^{-r}}{(1-e^{-r})^2} - \mathrm{Re}\left(\frac{e^{-r-it}}{(1-e^{-r-it})^2}\right)\\ &\geq \frac{e^{-r}}{(1-e^{-r})^2} - \frac{e^{-r}}{|1-e^{-r-it}|^2}\\ &\geq \frac{2e^{-2r}(1-\cos t)}{(1-e^{-r})^2(1-2e^{-r}\cos t+e^{-2r})}.\end{aligned}$$

Since the function $r \mapsto r^2e^{-2r}/(1-e^{-r})^2$ is decreasing for $r \in (0,1]$, we get

$$\frac{e^{-2r}}{(1-e^{-r})^2} \geq \frac{e^{-2}}{(1-e^{-1})^2}r^{-2}.$$

Moreover, we have

$$\frac{1-\cos t}{1-2e^{-r}\cos t+e^{-2r}} = \frac{1}{2}\left(e^{r} - \frac{e^{r}(1-e^{-r})^2}{1-2e^{-r}\cos t+e^{-2r}}\right).$$

So for fixed r, the above term is smallest when $\cos t$ is nearest to 1, that is when $t = t_0$. In this case (i.e., when $t = t_0$), we have the following bounds:

$$1-\cos t \geq \min\left\{1, \frac{4}{\pi^2}t^2\right\} = \frac{4}{\pi^2}r^{11/3},$$

and

$$\begin{aligned}1-2e^{-r}\cos t+e^{-2r} &= |1-e^{-r-it}|^2\\ &\leq \left(|r+it|\sum_{k=1}^{\infty}\frac{|r+it|^{k-1}}{k!}\right)^2\\ &\leq |r+it|^2e^{2|r+it|}\\ &\leq 2e^{2\sqrt{2}}r^2.\end{aligned}$$

Putting these estimates together, we obtain

$$\sum_{k=1}^{\infty} ke^{-kr}\Big(1-\cos(kt)\Big) \geq \frac{4e^{-2\sqrt{2}}}{(e-1)^2\pi^2}r^{-1/3},$$

which implies

$$\frac{|F(e^{-r-it})|}{F(e^{-r})} \leq \exp\left(-\frac{8e^{-2\sqrt{2}}}{(e-1)^2\pi^2} r^{-1/3}\right).$$

This completes the proof since

$$\frac{1}{64} \leq \frac{8e^{-2\sqrt{2}}}{(e-1)^2\pi^2} \approx 0.0162. \quad \square$$

We notice from Lemma 5 that for $|t| \geq t_0$, $|F(e^{-r-it})|$ is small compared to $F(r)$ as $r \to 0^+$, so the contribution of I_2 to $q(n)$ should also be small. We show that this is indeed the case by bounding the ratio $|I_2|/|I_1|$. We deduce from the definition of I_2 that

$$|I_2| \leq \frac{e^{nr}}{\pi} \int_{t_0}^{\pi} |F(e^{-r-it})| dt = \frac{e^{nr+f(r)}}{\pi} \int_{t_0}^{\pi} \frac{|F(e^{-r-it})|}{F(e^{-r})} dt.$$

Hence, Lemma 5 yields

$$|I_2| \leq \exp\left(nr + f(r) - \frac{r^{-1/3}}{64}\right). \tag{18}$$

Furthermore, by (17) and (15) we deduce the following bound:

$$\frac{|I_2|}{|I_1|} = \mathcal{O}\left(r^{-2} \exp\left(-\frac{r^{-1/3}}{64}\right)\right) \qquad (r \to 0^+). \tag{19}$$

3.3. Estimates of $q(n)$

For $n \geq 6$ (to guarantee that $r < 1$), by (16) and (18), we have the following bound:

$$q(n) \leq \exp\left(nr + f(r) + 15\sqrt{r}\right)\left(0.13 + \exp\left(-15\sqrt{r} - \frac{r^{-1/3}}{64}\right)\right).$$

The minimum of the function $r \mapsto 15\sqrt{r} + \dfrac{r^{-1/3}}{64}$ is obtained in $r = 1440^{-6/5}$ and equals $0.477544\ldots > 0.47$. Thus,

$$0.13 + \exp\left(-15\sqrt{r} - \frac{r^{-1/3}}{64}\right) \leq 0.13 + e^{-0.47} < 1.$$

Therefore,

$$q(n) \leq \exp\left(nr + f(r) + 15\sqrt{r}\right). \tag{20}$$

Furthermore, if $n \to \infty$ (so, $r \to 0^+$) then by (19), the ratio $|I_2|/|I_1|$ tends to zero faster than any power of r. Thus, I_1 is the main term of $q(n)$ and from (17) we deduce the asymptotic formula

$$q(n) = \frac{e^{nr+f(r)}}{\sqrt{2\pi f''(r)}}\Big(1 + \mathcal{O}(\sqrt{r})\Big), \tag{21}$$

as $n \to \infty$.

We are now ready to prove our main theorems. In fact, they follow by expressing the estimates (20) and (21) in terms of n.

4. Proof of Theorem 1

We make use of the following estimates proved in Section 2:

$$\Big|f(r) - 2\zeta(3)r^{-2} - \zeta(2)r^{-1}\Big| \leq 2.8\ r^{-1/2}; \tag{22}$$

$$\Big|f'(r) + 4\zeta(3)r^{-3} + \zeta(2)r^{-2}\Big| \leq 2.9\ r^{-3/2}. \tag{23}$$

One can easily check from (13) that $r < 0.5$ if $n \geq 50$. We assume from now on that $n \geq 50$, therefore that $r < 0.5$. Using (22), we have

$$\begin{aligned} nr + f(r) + 15\sqrt{r} &\leq 6\zeta(3)r^{-2} + 2\zeta(2)r^{-1} + 6r^{-1/2} + 15\sqrt{r} \\ &\leq 6\zeta(3)r^{-2} + 2(3\sqrt{r} + \zeta(2))r^{-1} + 15\sqrt{r} \\ &\leq 6\zeta(3)r^{-2} + 2(2.2 + \zeta(2))r^{-1} + 11. \end{aligned} \tag{24}$$

On the other hand, we also have the lower bound

$$n \geq 4\zeta(3)r^{-3} + \zeta(2)r^{-2} - 3r^{-3/2} \geq 4\zeta(3)r^{-3} - 2. \tag{25}$$

In the last inequality above we used the fact that the minimum of the function $x \mapsto \zeta(2)x^2 - 3x^{3/2}$ is at $x = 64/(9\pi^4)$ and its value there is $-1.42017\ldots > -2$. Thus, inequality (25) gives

$$r^{-1} \leq \Big(\frac{n+2}{4\zeta(3)}\Big)^{1/3}.$$

Finally, from inequalities (24) and (25), we get

$$nr + f(r) + 15\sqrt{r} \leq 6\zeta(3)\Big(\frac{n+2}{4\zeta(3)}\Big)^{2/3} + 2(2.2 + \zeta(2))\Big(\frac{n+2}{4\zeta(3)}\Big)^{1/3} + 11.$$

The last expression above is smaller than

$$\frac{6\zeta(3)^{1/3}}{4^{2/3}} n^{2/3}\left(1+\frac{3.2}{(n+2)^{1/3}}\right),$$

for $n \geq 50$, which via (20) finishes the proof.

5. Proof of Theorem 2

To obtain an asymptotic formula for $q(n)$ as $n \to \infty$, we need more terms in the estimates of $f(r)$ and $f'(r)$ in Section 2. In the integral expressions of $f(r)$ and $f'(r)$ via Mellin inversion formula, we can shift their paths of integration further to the left to obtain more terms in their asymptotic expansions as $r \to 0^+$. Aside from the poles that we considered in Section 2, $\Phi_0(s)$ has a double pole at $s = 0$, and $s = 2, 1, 0$ are the only singularities of $\Phi_0(s)$ in the right half-plane $\mathrm{Re}(s) > -1$. In addition, the Laurent expansion of $\Phi_0(s)$ about $s = 0$ is

$$\frac{D(0)}{s^2} + \frac{D'(0)}{s} + \mathcal{O}(1) = \frac{-2}{3s^2} + \frac{D'(0)}{s} + \mathcal{O}(1),$$

where $D(s) = 2\zeta(s-1) + \zeta(s)$. Therefore, as $r \to 0^+$, we have

$$f(r) = 2\zeta(3)r^{-2} + \zeta(2)r^{-1} + \frac{2}{3}\log r + D'(0) + \mathcal{O}(r). \tag{26}$$

Similarly, by looking at the contribution of the simple pole $s = 1$ of $\Phi_1(s)$ (which is the only singularity of $\Phi_1(s)$, aside from $s = 3$ and 2, that lies in the right half-plane $\mathrm{Re}(s) > 0$), we have the asymptotic estimate for $f'(r)$ as $r \to 0^+$:

$$f'(r) = -4\zeta(3)r^{-3} - \zeta(2)r^{-2} + \frac{2}{3}r^{-1} + \mathcal{O}(1). \tag{27}$$

From (13), we immediately obtain an estimate of n in terms of r

$$n = 4\zeta(3)r^{-3} + \zeta(2)r^{-2} - \frac{2}{3}r^{-1} + \mathcal{O}(1).$$

Since we need r in terms of n instead, we have to invert the above estimate. This is done in the next lemma.

Lemma 6. *As $n \to \infty$, we have*

$$r^{-1} = A_1 n^{1/3} + A_2 + A_3 n^{-1/3} + \mathcal{O}(n^{-2/3}), \tag{28}$$

where

$$A_1 = (4\zeta(3))^{-1/3}, A_2 = -\frac{\zeta(2)}{12\zeta(3)}, \quad \textit{and} \quad A_3 = \frac{\zeta(2)^2 + 8\zeta(3)}{9(4\zeta(3))^{5/3}}.$$

Proof. We adopt the following abbreviations in this proof:

$$x = r^{-1}, \ a = 4\zeta(3), \ b = \zeta(2), \ \text{and } c = -\frac{2}{3}.$$

Thus, we have

$$n = ax^3 + bx^2 + cx + \mathcal{O}(1). \tag{29}$$

We write

$$x = A_1 n^{1/3} + A_2 + A_3 n^{-1/3} + \mathcal{O}(1),$$

and compute

$$\begin{aligned} x^2 &= A_1^2 n^{2/3} + (2A_1A_2)n^{1/3} + (A_2^2 + 2A_1A_3) + \mathcal{O}(n^{-1/3}); \\ x^3 &= A_1^3 n + (3A_1^2A_2)n^{2/3} + (3A_1^2A_3 + 3A_1A_2^2)n^{1/3} + \mathcal{O}(1). \end{aligned}$$

We get

$$\begin{aligned} n &= ax^3 + bx^2 + cx + \mathcal{O}(1) \\ &= aA_1^3 n + (3aA_1^2A_2 + bA_1^2)n^{2/3} \\ &\quad + (3aA_1^2A_3 + 3aA_1A_2^2 + 2bA_1A_2 + cA_3)n^{1/3} + \mathcal{O}(1). \end{aligned}$$

Identifying coefficients of n, $n^{2/3}$ and $n^{1/3}$, we get

$$\begin{aligned} aA_1^3 &= 1 \\ 3aA_1^2A_2 + bA_1^2 &= 0 \\ 3aA_1^2A_3 + 3aA_1A_2^2 + 2bA_1A_2 + cA_1 &= 0. \end{aligned}$$

Solving we get

$$A_1 = a^{-1/3}, \qquad A_2 = -\frac{b}{3a}, \qquad A_3 = \frac{b^2 - 3ac}{9a^{5/3}},$$

which is what we wanted. $\square$

Recall the asymptotic formula (21) for $q(n)$ that we proved in Section 3:

$$q(n) = \frac{e^{nr+f(r)}}{\sqrt{2\pi f''(r)}}\Big(1 + \mathcal{O}(\sqrt{r})\Big).$$

Next, we recall from (8) that $f''(r) = 12\zeta(3)r^{-4} + \mathcal{O}(r^{-3})$ as $r \to 0^+$. Hence,

$$q(n) = \frac{r^2 e^{nr+f(r)}}{\sqrt{24\pi\zeta(3)}} \Big(1 + \mathcal{O}(\sqrt{r})\Big). \tag{30}$$

Similarly, from (13) together with (26) and (27), we have

$$nr + f(r) = 6\zeta(3)r^{-2} + 2\zeta(2)r^{-1} + \frac{2}{3}\log r + D'(0) - \frac{2}{3} + \mathcal{O}(r). \tag{31}$$

Applying the estimate (28) for r^{-1} from Lemma 6, we obtain

$$\begin{aligned} r^{-2} &= A_1^2 n^{2/3} + 2A_1A_2n^{1/3} + 2A_1A_3 + A_2^2 + \mathcal{O}(n^{-1/3}); \\ r^{-1} &= A_1 n^{1/3} + A_2 + \mathcal{O}(n^{-1/3}); \\ \log r &= -\frac{1}{3}\log n - \log A_1 + \mathcal{O}(n^{-1/3}). \end{aligned}$$

Finally, (30) together with (31) and the three last estimates above yield

$$q(n) = Kn^{-8/9} \exp\Big(6\zeta(3)A_1^2 n^{2/3} + 2A_1(\zeta(2) + 6\zeta(3)A_2)n^{1/3}\Big)\Big(1 + \mathcal{O}(n^{-1/6})\Big),$$

where

$$K = \frac{\exp\Big(6\zeta(3)(2A_1A_3 + A_2^2) + 2\zeta(2)A_2 - \frac{2}{3}\log A_1 + D'(0) - 2/3\Big)}{A_1^2\sqrt{24\pi\zeta(3)}}.$$

Replacing A_1, A_2 and A_3 by their definitions in Lemma 6, and noting that

$$D'(0) = 2\zeta'(-1) + \zeta'(0) = \frac{1}{6} - 2\log \mathrm{A} - \log\sqrt{2\pi},$$

where A is the Glaisher–Kinkelin constant, we complete the proof of Theorem 2.

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