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Equivariant hom-Lie algebras and twisted derivations on (arithmetic) schemes

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ABSTRACT

In this paper we introduce the notion of global equivariant hom-Lie algebra. This is a Lie algebra-like structure associated with twisted derivations. We prove several results on the structure of modules of twisted derivations and how they form global equivariant hom-Lie algebras. Particular emphasis is put on examples and results in arithmetic geometry.

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1. Introduction

This paper is devoted to the study of difference operators in arithmetic settings. It turns out that difference operators are special cases of so-called twisted derivations. To recall, a (σ-)twisted derivation on a k-algebra A, where k is commutative ring with unity, is a pair $(\sigma, \partial_\sigma)$, where $\sigma, \partial_\sigma \in \mathrm{End}_k(A)$, such that

$$\partial_\sigma(ab) = \partial_\sigma(a)b + \sigma(a)\partial_\sigma(b), \quad a, b \in A.$$

See sections 2.2, 2.3 and 4.1.1for details. Normally we simply write ∂_σ instead of the more cumbersome $(\sigma, \partial_\sigma)$.

The main points of this paper is to give a global version (Theorem 2.8) of a result from [HLS06] as well as giving a global definition of hom-Lie algebras. In addition several examples indicating the possible uses of twisted derivations and hom-Lie algebras in algebraic geometry and number theory are provided in the last section.

1.1. Philosophy

Let me begin by spending a few moments commenting on the philosophy behind the above construction in the context of arithmetic.

Assume for simplicity that we are given an *abelian* group scheme $G_{/R}$ over a ring R. Then it can, as in Lie theory, be argued that the Lie algebra to G should be something like $\log G$ and this should give us derivations on the ring of functions on G. Now, we can pretend (with a somewhat clear conscience) that the Taylor expansion of $\log(\sigma)$ is

$$\log(\sigma) = \sum_{i=1} (-1)^{i+1} \frac{(\mathrm{id} - \sigma)^i}{i},$$

and we see that the first-order term $\big(\log(\sigma)\big)_1$ is $\mathrm{id} - \sigma$.

Operators on the form $a(\mathrm{id} - \sigma)$ are the most common type of twisted derivations, and in fact, it can be shown that on many rings *all* twisted derivations are of this type (again see sections 2.2 and 2.3). Therefore, as twisted derivations are the most natural source of hom-Lie algebras, we notice that it is reasonable to view hom-Lie algebras as *first-order Lie algebras*.

Pushing the analogy with Lie groups and Lie algebras and their relation $\mathfrak{g} = \log(G)$, it seems reasonable to view $\big(\log(G)\big)_1$ as the "true" hom-Lie algebra. This is what I refer to as *equivariant hom-Lie algebras* in this paper. This is because the original definition of hom-Lie algebra involved only one σ. An unfortunate result of this, and the main point where the analogy with Lie theory is flawed, is that the product in the equivariant structure is performed "one σ at a time".

As Lie algebras measure the "infinitesimal" action of the Lie group on some ring, hom-Lie algebras can be said to measure the "first-order infinitesimal" effect of the action as the following example hopefully illustrates.

Example 1.1. Let k be a complete field (for simplicity) and consider the field $k(t)$ of rational functions over k in the variable t. Put $\sigma(t) = \epsilon t$, $\epsilon \in k$. Then

$$D_\epsilon := (1-\epsilon)^{-1}(\mathrm{id} - \sigma)$$

is a twisted derivation on $k(t)$ as is easily seen. The (left) $k(t)$-module $k(t) \cdot D_\epsilon$ defines a hom-Lie algebra (as we will see). Now, as $\epsilon \to 0$ one can argue successfully that $D_\epsilon \to \frac{d}{dt}$, the ordinary derivation along t. Therefore, choosing ϵ small enough, σ becomes close to the identity and D_ϵ close to a derivation.

Of course, in general such a nice and clear-cut interpretation of something approaching zero, is not readily available but the intuition is still very much applicable. Note however, that in the p-adic world it is actually possible to make sense of such a limit (called "confluence") between differential operators and difference operators. See A. Pulita's paper [Pul08] for more details on this. In any case, intuitively, it is therefore natural to view the structure of hom-Lie algebras (at least the ones coming from twisted derivations) as measuring the relative effect of $\sigma \in G$, in a sense I hope to make sense of in the main text.

The subject of difference operators goes back centuries, but fell out of fashion during the past mid-century. Happily though, in recent time there has been a renewed interest in these kinds of operators, particularly in arithmetic. Let us briefly recall the essence.

Classically one was primarily interested in (algebraic) function fields over $\mathbb{C}$, so we will assume this set-up below.

Example 1.2. Of particular interest was (are) the following types of operators. Let R be a $\mathbb{C}$-algebra and consider a (not necessarily proper) subring of $R((t))$. Then

(a) $\sigma_{(h)}(f)(t) := f(t+h)$, for any $h \in R$,
  (i) $\partial(f)(t) := (\mathrm{id} - \sigma_{(h)})(f(t)) = f(t) - f(t+h)$,
  (ii) $\partial(f)(t) := h^{-1}\big(\mathrm{id} - \sigma_{(h)}\big)(f(t)) = h^{-1}\big(f(t) - f(t+h)\big)$,
  so-called *shifted difference operators*, and

(b) $\sigma_q(f)(t) := f(qt)$, for any $q \in R$,
  (i) $\partial(f)(t) := (\mathrm{id} - \sigma_q)(f(t)) = f(t) - f(qt)$,
  (ii) $\partial(f)(t) := \big((1-q)t\big)^{-1}\big(\mathrm{id} - \sigma_q\big)(f(t)) = \big((1-q)t\big)^{-1}\big(f(t) - f(qt)\big)$
  so-called *q-difference operators*,

all define σ-derivations.

However, in the 70's V. Drinfel'd used Frobenius difference operators on global function fields in connection with what he called elliptic modules (now called Drinfel'd modules). Since then a growing interest in difference operators and equations can be noted by a simple Google search.

For instance, recently q-difference operators has been studied in arithmetic contexts since the mid 90's, for instance by Y. André, L. Di Vizio [And01,DV02] and the already mentioned A. Pulita [Pul08], just to name a few. Also, K. Kedlaya and many others (see for instance the recent book [Ked10] by Kedlaya) study difference operators in the context of p-adic differential equations (Frobenius structures) and rigid cohomology.

As we indicated above, the underlying reason for the paper [HLS06] (and its antecedents [LS05,LS07]) is the study of the algebraic structure of q-difference operators. A standing assumption in these papers is that the ground field is $\mathbb{C}$ or a field of characteristic zero, but this is really an unnecessary assumption. More or less every result in those papers are true in any characteristic (with the possible exception of characteristic 2 or 3 at some places).

Therefore, it seems like a very good idea to have a "Lie algebra-like" structure in which to study these kind of operators.

1.2. Plan of the paper

Finally, let me briefly lay out the plan for the paper.

The paper starts out with a discussion of twisted derivations and a recollection of the main results from [HLS06]. Then in Section 2.3 comes the main theorem of the paper. In this section is the reader can also find a number of small examples. Then in Section 3 we define the main algebraic structure, equivariant hom-Lie algebras, as well as stating a number of simple base change results. Finally, in Section 4, a number of longer examples are provided. For instance, hom-Lie algebras associated with morphisms of schemes (in particular G-covers), and hom-Lie algebras of t-motives (where we indicate that these can be used in transcendence theory for t-motives).

Notation

The following notation will be adhered to throughout.

- k will denote a commutative, associative integral domain with unity.
- $\mathsf{Com}(k)$, $\mathsf{Com}(B)$ e.t.c, the category of, commutative, associative k-algebras (B-algebras, e.t.c) with unity. Morphisms of k-algebras (B-algebras, e.t.c) are always unital, i.e., $\phi(1) = 1$.

- $A^\times$ is the set of units in A (i.e., the set of invertible elements).
- $\mathsf{Mod}(A)$, the category of A-modules.
- $\mathrm{End}(A) := \mathrm{End}_k(A)$, the k-module of k-algebra morphisms on A.
- $\circlearrowleft_{a,b,c}(\,\cdot\,)$ will mean cyclic addition of the expression in bracket.
- Sch, denotes the category of schemes; Sch/S denotes the category of schemes over S (i.e., the category of S-schemes).
- We always assume that all schemes are Noetherian.
- When writing actions of group elements we will use the notations $\sigma(a)$ and a^σ, meaning the same thing: the action of σ on a.
- Sometimes we will use the notation $\underline{A} := \mathsf{Spec}(A)$.

The condition that k must be a domain can certainly be relaxed at several places in the presentation. But for simplicity we keep it as a standing assumption.

2. Twisted derivations

2.1. Generalities

Let $A \in \mathrm{ob}(\mathsf{Com}(k))$ and let $\sigma : A \to A$ be a k-linear map on A. Then a (classical) *twisted derivation* on A is a k-linear map $\partial : A \to A$ satisfying

$$\partial(ab) = \partial(a)b + \sigma(a)\partial(b).$$

We can generalize this as follows. Let $\phi \in \mathrm{End}(A)$, and let $M \in \mathrm{ob}(\mathsf{Mod}(A))$. The action of $a \in A$ on $m \in M$ will for now be denoted $a.m$. Then, a *twisted derivation on* M is k-linear map $\partial : M \to M$ such that

$$\partial(a.m) = \partial_A(a).m + \sigma(a).\partial(m), \tag{1}$$

where, by necessity, $\partial_A : A \to A$ is a twisted derivation on A (in the first sense). We call ∂_A the *restriction of* ∂ *to* A. Finally, a *twisted module derivation* is a k-linear map $\partial : A \to M$ such that

$$\partial(ab) = b.\partial(a) + \sigma(a).\partial(b),$$

for $\sigma \in \mathrm{End}(A)$. Normally we will not differentiate between left and right module structures, but there are times when such a distinction is necessary.

We will sometimes refer to the above as *σ-twisted (module) derivations* if we want to emphasize which σ we refer to.

Let $\sigma \in \mathrm{End}(A)$ and denote by $\sigma^*A := A \otimes_{A,\sigma} A$, the extension of scalars along σ. This means that we consider A as a left module over itself via σ, i.e., $a.b := \sigma(a)b$. The right module structure is left unchanged. This can also be viewed as the *right* A-module $\boldsymbol{e}A$, with left structure given by the commutation rule $a\boldsymbol{e} = \boldsymbol{e}\sigma(a)$.

If M is an A-module, we put

$$\sigma^* M := \sigma^* A \otimes_A M = A \otimes_{A,\sigma} M,$$

i.e., M is endowed with left module structure $a.m := \sigma(a)m$, and once more, the right structure is unaffected.

We note that a σ-derivation d_σ on A is actually a derivation $d : A \to \sigma^* A$ and conversely. Indeed,

$$d(ab) = d(a)b + a.d(b) = d(a)b + \sigma(a)d(b).$$

In the same manner, a σ-derivation $d_\sigma : A \to M$ is a derivation $d : A \to \sigma^* M$, and conversely. Therefore, there is a one-to-one correspondence between σ-derivations $d_\sigma : A \to M$ and derivations $d : A \to \sigma^* M$.

There is another way to connect derivations and σ-derivations as follows. Let J be the ideal generated by the set $(\mathrm{id} - \sigma)(A)$ and form the *blow-up algebra of* J:

$$\mathrm{Bl}_A(J) := \bigoplus_{i=0}^{\infty} J^i/J^{i+1}, \quad J^0 := A.$$

Then

$$\mathrm{id} - \sigma : \ \mathrm{Bl}_A(J) \to \mathrm{Bl}_A(J)$$

is in fact a graded derivation. Indeed, observe that $\sigma(J) \subseteq J$ since

$$\sigma(J) = \sigma\big(A(\mathrm{id} - \sigma)(A)\big) = \sigma(A)(\mathrm{id} - \sigma)\sigma(A) \subseteq A(\mathrm{id} - \sigma)(A) = J.$$

Similarly, we see that $\sigma(J^i) \subseteq J^i$. For $\bar{x} \in J^i/J^{i+1}$ and $\bar{y} \in J^l/J^{l+1}$, we have

$$(\mathrm{id} - \sigma)(xy) = (\mathrm{id} - \sigma)(x)y + x(\mathrm{id} - \sigma)(y) - (x - \sigma(x))(y - \sigma(y))$$

and $(x - \sigma(x))(y - \sigma(y))$ is in J^{j+l+1}. It is easily seen that this is independent on the lifts of $\bar{x}$ and $\bar{y}$.

This construction globalizes in the evident way.

2.1.1. Difference equations

Recall that a *difference equation* (or *difference module*) over A is an A-module M together with a σ-linear endomorphism $\boldsymbol{\sigma}$ on M, i.e., $\boldsymbol{\sigma}(am) = \sigma(a)\boldsymbol{\sigma}(m)$. Notice that this induces an A-linear homomorphism $M \to \sigma^* M$.

A σ-difference operator $\boldsymbol{\sigma}$ induces a σ*-connection*

$$\nabla^{(\sigma)} : M \to \sigma^* A \otimes M, \quad m \mapsto 1 \otimes (\mathrm{id} - \boldsymbol{\sigma})$$

and the solution space to the difference equations is $\ker \nabla^{(\sigma)}$. Notice that we have the Leibniz rule

$$\nabla^{(\sigma)}(am) = a\nabla^{(\sigma)}(m) + a(\mathrm{id} - \sigma)(a) \otimes m.$$

Conversely, a σ-connection

$$\nabla^{(\sigma)}(m) := 1 \otimes a(\mathrm{id} - \boldsymbol{\sigma}), \quad a \in A,$$

induces a difference equation on M as the kernel of $a(\mathrm{id} - \boldsymbol{\sigma})$.

We refer to [And01] for more on this, including a Tannakian formalism of σ-connections.

Example 2.1. The "universal" (this designation will be amply demonstrated in what follows) example of a σ-derivation is the following. Let $A \in \mathrm{ob}(\mathsf{Com}(k))$ and $M \in \mathrm{ob}(\mathsf{Mod}(A))$. Suppose $\boldsymbol{\sigma} : M \to M$ is σ-semilinear. Then a simple calculation shows that for all $b \in A$, $\partial := b(\mathrm{id} - \boldsymbol{\sigma}) : M \to M$, is a σ-twisted derivation on M. Notice that if $M = A$, we automatically get $\varphi = \sigma$.

Note 2.1. From now on we will usually not bother notationally separating σ as an endomorphism of A, and $\boldsymbol{\sigma}$ as an endomorphism of M. Both will in most instances be written as σ.

2.2. Modules of twisted derivations

Proposition 2.1. *Let M be an A-module. Then the k-modules of σ-twisted derivations,*

$$\begin{aligned} \mathrm{Der}_\sigma(M) &:= \{\partial \in \mathrm{End}_k(M) \mid \partial(a.m) = \partial_A(a).m + \sigma(a).\partial(m)\}, \quad \textit{and} \\ \mathrm{Der}_\sigma(A, M) &:= \{\partial \in \mathrm{Hom}_k(A, M) \mid \partial(ab) = \partial(a).b + \sigma(a).\partial(b)\} \end{aligned}$$

are left A-modules. Furthermore, we also have $\partial(1) = 0$.

Proof. The A-module structure is defined, in both cases, by $(a.\partial)(m) := a.\partial(m)$ (for m either in M or in A). Since A is commutative, we have

$$(b.\partial)(a.m) = b.\partial_A(a).m + b.\sigma(a).\partial(m) = b\partial_A(a).m + \sigma(a).(b.\partial)(m).$$

That $\partial(1) = 0$ follows easily, noting that $\sigma(1) = 1$, by the usual calculation. $\square$

Note that unlike the case of ordinary derivations, $\mathrm{Der}_\sigma(M)$ or $\mathrm{Der}_\sigma(A, M)$ are not Lie algebras.

Let, as before, $A \in \mathrm{ob}(\mathsf{Com}(k))$ and let $\sigma \in \mathrm{End}(A)$. Denote by Δ_σ a σ-twisted derivation on M whose restriction to A is ∂, i.e., $\Delta_\sigma \in \mathrm{Der}_\sigma(M)$ and $\partial \in \mathrm{Der}_\sigma(A)$. Assume that $\sigma(\mathrm{Ann}(\Delta_\sigma)) \subseteq \mathrm{Ann}(\Delta_\sigma)$, where

$$\mathrm{Ann}(\Delta_\sigma) := \{a \in A \mid a\Delta_\sigma(m) = 0, \quad \text{for all} \quad m \in M\},$$

and that

$$\partial \circ \sigma = q \cdot \sigma \circ \partial, \qquad \text{for some } q \in A. \tag{2}$$

Form the left A-module

$$A \cdot \Delta_\sigma := \{a \cdot \Delta_\sigma \mid a \in A\}.$$

Define

$$\langle\!\langle\, a \cdot \Delta_\sigma, b \cdot \Delta_\sigma \,\rangle\!\rangle := \sigma(a) \cdot \Delta_\sigma(b \cdot \Delta_\sigma) - \sigma(b) \cdot \Delta_\sigma(a \cdot \Delta_\sigma). \tag{3}$$

This should be interpreted as

$$\langle\!\langle\, a \cdot \Delta_\sigma, b \cdot \Delta_\sigma \,\rangle\!\rangle\,(m) := \sigma(a) \cdot \Delta_\sigma(b \cdot \Delta_\sigma(m)) - \sigma(b) \cdot \Delta_\sigma(a \cdot \Delta_\sigma(m)),$$

for $m \in M$. We now have the following fundamental theorem.

Theorem 2.2 *(Affine version). Under the above assumptions, equation (3) gives a well-defined k-linear product on $A \cdot \Delta_\sigma$ such that*

(i) $\langle\!\langle\, a \cdot \Delta_\sigma, b \cdot \Delta_\sigma \,\rangle\!\rangle = (\sigma(a)\partial(b) - \sigma(b)\partial(a)) \cdot \Delta_\sigma$;
(ii) $\langle\!\langle\, a \cdot \Delta_\sigma, a \cdot \Delta_\sigma \,\rangle\!\rangle = 0$;
(iii) $\circlearrowleft_{a,b,c} \big(\langle\!\langle\, \sigma(a) \cdot \Delta_\sigma, \langle\!\langle\, b \cdot \Delta_\sigma, c \cdot \Delta_\sigma \,\rangle\!\rangle \,\rangle\!\rangle + q \cdot \langle\!\langle\, a \cdot \Delta_\sigma, \langle\!\langle\, b \cdot \Delta_\sigma, c \cdot \Delta_\sigma \,\rangle\!\rangle \,\rangle\!\rangle\big) = 0$,

where, in (iii), q is the same as in (2).

Proof. Exactly the same proof as in [HLS06, Theorem 5]. □

Corollary 2.3. *In the case $\Delta_\sigma \in \mathrm{Der}_\sigma(A, M)$, defining the algebra structure directly by property (i) in the theorem gives (ii) and (iii) on $A \cdot \Delta_\sigma$.*

Proof. Obvious. □

We can extend σ to an algebra morphism on $A \cdot \Delta_\sigma$ by defining $\sigma(a \cdot \Delta_\sigma) := \sigma(a) \cdot \Delta_\sigma$.

Remark 2.2. Notice that for an ideal $I \subseteq A$ and $\Delta_\sigma \in \mathrm{Der}_\sigma(A, I)$, the module $A \cdot \Delta_\sigma$ and the product from the theorem, makes perfect sense. In particular, if I is σ-stable, $\Delta_\sigma(I) \subseteq I$ so Δ_σ induces a twisted derivation $\bar{\Delta}_\sigma$ on A/I and we can form $(A/I) \cdot \bar{\Delta}_\sigma$ with induced product.

Lemma 2.4. *Let $A \in \mathrm{ob}(\mathsf{Com}(k))$ and suppose there is an $x \in A$, such that*

$$x - \sigma(x) \in A^\times, \quad \mathrm{id} \neq \sigma \in \mathrm{End}(A),$$

then any σ-twisted derivation Δ_σ on M, with $M \in \mathrm{ob}(\mathsf{Mod}(A))$ and

$$\sigma \in \mathrm{End}(M), \quad \sigma(a.m) = \sigma(a).\sigma(m),$$

is of the form

$$\Delta_\sigma = (x - \sigma(x))^{-1}\partial_A(x)(\mathrm{id} - \sigma),$$

where ∂_A is the restriction of Δ_σ to A. If M is torsion-free over A, then $A \cdot \Delta_\sigma = \mathrm{Der}_\sigma(M)$ is free of rank one.

Proof. Let $m \in M$ be arbitrary. Then the first statement follows from

$$0 = \Delta_\sigma(m.x - x.m) = \Delta_\sigma(m)(x - \sigma(x)) + (\sigma(m) - m)\partial_A(x).$$

By assumption, $x - \sigma(x)$ is invertible, so

$$\Delta_\sigma(m) = (x - \sigma(x))^{-1}\partial_A(x)(\mathrm{id} - \sigma)(m), \quad \text{for all} \quad m \in M.$$

Clearly, when M is torsion-free over A, $a\Delta_\sigma(m) = 0 \Rightarrow a = 0$, so $\mathrm{Der}_\sigma(M)$ is free of rank one. □

Hence, "up to a localization" (at $x - \sigma(x)$), every σ-twisted derivation on $M \in \mathrm{ob}(\mathsf{Mod}(A))$ is of the form given in the lemma. This means that if there is an $x \in A$ such that $x - \sigma(x)$ is invertible, then giving a twisted derivation Δ_σ on M amounts to deciding what the restriction of Δ_σ to A is on x.

As an immediate consequence of the lemma we have:

Proposition 2.5. *Let A be a k-algebra and $\mathrm{id} \neq \sigma \in \mathrm{End}_k(A)$, $\sigma \in \mathrm{End}_k(M)$ such that $\sigma(a.m) = \sigma(a).\sigma(m)$. Suppose that for each $\mathfrak{p} \in \mathsf{Spec}(A)$ there is an $x \in A$ such that $x - \sigma(x) \notin \mathfrak{p}$. Then $\mathrm{Der}_\sigma(M)$ is locally free of rank one over A.*

Proof. For any $\mathfrak{p} \in \mathsf{Spec}(A)$, take $x \in A$ such that $x - \sigma(x) \notin \mathfrak{p}$. In the localization $A_\mathfrak{p}$ an element $x - \sigma(x)$ is a unit so we can apply the lemma. □

In case $M = A$ is a unique factorization domain (UFD), it is possible (see [HLS06, Theorem 4]) to prove a stronger version which does not assume the existence of $x \in A$ such that $x - \sigma(x) \in A^\times$:

Theorem 2.6. *If A is a UFD, and $\sigma \in \operatorname{End}(A)$, then*

$$\Delta_\sigma := \frac{\operatorname{id} - \sigma}{g}$$

generates $\operatorname{Der}_\sigma(A)$ *as a left A-module, where* $g := \gcd((\operatorname{id} - \sigma)(A))$.

Notice that the theorem and the proposition say slightly different things. The theorem states that g is a *factor* in $(\operatorname{id} - \sigma)(a)$ for all $a \in A$, and can be cancelled.

Example 2.2. When $A = K/k$ is a field (extension) the above theorem implies that every σ-twisted derivation is on the form given in the statement.

2.3. Global twisted derivations

We keep the notations from above.

The definition of twisted derivations can be globalized. Let $X \xrightarrow{f} S$ be an S-scheme and $\mathscr{A}$ a sheaf of coherent $\mathscr{O}_X$-algebras (notice the special case $\mathscr{A} = \mathscr{O}_X$). Let $G_{/S}$ be a finite flat group scheme acting on $\mathscr{A}$ (this induces an action on the global spectrum $\mathbf{Spec}_{\mathscr{O}_X}(\mathscr{A})$). In other words, we have a group homomorphism

$$G(R) \to \operatorname{Aut}(\mathscr{A})(R) = \operatorname{Aut}_R(\mathscr{A} \otimes_S R)$$

for every $\mathscr{O}_S$-algebra R. Let $\sigma \in G$. By this we mean the choice of a σ_R in every $G(R)$ for each S-scheme R. Then σ defines an automorphism of $\mathscr{A} \otimes R$ which we also denote σ.

Put $\mathscr{Z} = \mathbf{Spec}_{\mathscr{O}_X}(\mathscr{A})$ and let $z : \mathscr{Z} \to X$ be the corresponding affine morphism. A *σ-derivation* on $\mathscr{Z}$ is an endomorphism $\partial_U \in \mathscr{E}nd_S(\mathscr{A})(U)$ for each $U \subseteq \mathscr{Z}$ such that we have

$$\partial_U(xy) = \partial_U(x)y + \sigma|_U(x)\partial_U(y), \quad x, y \in \mathscr{A}(U). \tag{4}$$

We denote the sheaf of all σ-derivations on $\mathscr{Z}$ by $\mathscr{D}er_\sigma(\mathscr{A})$. The sheaf $\mathscr{D}er_\sigma(\mathscr{A})$ is coherent since $\mathscr{E}nd(\mathscr{A})$ is. Clearly, $\mathscr{D}er_\sigma(\mathscr{A})$ is a left $\mathscr{A}$-module.

Proposition 2.7. *Suppose X is regular. Then $\mathscr{D}er_\sigma(\mathscr{O}_X)$ is invertible.*

Proof. Since X is regular, each stalk is a regular local ring, hence a UFD. Now apply Theorem 2.6. □

Now, suppose that $\mathscr{E}$ is a coherent sheaf of $\mathscr{A}$-modules. We define $\mathscr{D}er_\sigma(\mathscr{E})$ in exactly the same way as $\mathscr{D}er_\sigma(\mathscr{O}_X)$, but now (4) becomes

$$\partial_U(x.e) = \partial_U|_{\mathscr{A}}(x).e + \sigma|_U(x).\partial_U(e), \quad x \in \mathscr{A}(U),\, e \in \mathscr{E}(U).$$

Let $\partial \in \mathscr{D}er_\sigma(\mathscr{A})$ and let $\Delta \in \mathscr{D}er_\sigma(\mathscr{E})$ such that $\Delta|_{\mathscr{A}} = \partial$. We define the $\mathscr{A}$-module $\mathscr{A}nn_{\mathscr{A}}(\Delta)$ as

$$\mathscr{A}nn_{\mathscr{A}}(\Delta)(U) := \big\{a \in \mathscr{A}(U) \mid a\Delta(e) = 0, \text{ for all } e \in \mathscr{E}(U)\big\}.$$

Assume that

$$\Delta_U \circ \sigma = q_U \cdot \sigma \circ \Delta_U, \quad q_U \in \mathscr{A}(U)$$

and form the left $\mathscr{A}$-module $\mathscr{A} \cdot \Delta$ by

$$(\mathscr{A} \cdot \Delta)(U) := \mathscr{A}(U) \cdot \Delta_U.$$

On $\mathscr{A} \cdot \Delta$ we introduce the product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ by

$$\langle\!\langle a \cdot \Delta_U, b \cdot \Delta_U \rangle\!\rangle_U := \sigma(a) \cdot \Delta_U(b \cdot \Delta_U) - \sigma(b) \cdot \Delta_U(a \cdot \Delta_U), \quad a, b \in \mathscr{A}(U).$$

We now have the following global version of Theorem 2.2.

Theorem 2.8 *(Global version). The above product is $\mathscr{O}_S$-linear and satisfies*

(i) $\langle\!\langle a \cdot \Delta_U, b \cdot \Delta_U \rangle\!\rangle_U = (\sigma(a)\partial_U(b) - \sigma(b)\partial_U(a)) \cdot \Delta_U$;
(ii) $\langle\!\langle a \cdot \Delta_U, a \cdot \Delta_U \rangle\!\rangle_U = 0$;
(iii) $\circlearrowleft_{a,b,c} \Big(\langle\!\langle \sigma(a) \cdot \Delta_U, \langle\!\langle b \cdot \Delta_U, c \cdot \Delta_U \rangle\!\rangle_U \rangle\!\rangle_U + q_U \cdot \langle\!\langle a \cdot \Delta_\sigma, \langle\!\langle b \cdot \Delta_U, c \cdot \Delta_U \rangle\!\rangle_U \rangle\!\rangle_U\Big) = 0$,

where $a, b, c \in \mathscr{A}(U)$.

The proof of this global version is simply a gluing of the affine version (i.e., a standard descent argument).

Assume now that $\mathfrak{p} \in X$. The set of all $\sigma \in G$ such that $\sigma(\mathfrak{p}) \subseteq \mathfrak{p}$ is called the *stabilizer group* or *decomposition group* at $\mathfrak{p}$, denoted $D_\mathfrak{p}$. Notice that this means that $\mathfrak{p}$ is a fixed point for all $\sigma \in D_\mathfrak{p}$. The subgroup of all $\sigma \in D_\mathfrak{p}$ that reduces to the identity modulo $\mathfrak{p}$, i.e., on the residue class field $k(\mathfrak{p})$, is called the *inertia group* at $\mathfrak{p}$, $I_\mathfrak{p}$. We let $I_\sigma(X)$ denote the set of all points $\mathfrak{p}$ in X such that $\sigma \in I_\mathfrak{p}$ and

$$I_G(X) := \bigcup_{\sigma \in G} I_\sigma(X),$$

the *inertia locus* on X. Notice that if X is defined over an algebraically closed field, $D_\mathfrak{p} = I_\mathfrak{p}$.

If X/G exists as an S-scheme then $\pi : X \to X/G$ defines a generic G-torsor. This means that π is étale over an open, dense subscheme, whose complement is a divisor. This divisor is called the *ramification divisor* and is the annihilator of the sheaf $\Omega_{X/(X/G)}$, denoted $\mathtt{ram}(\pi)$. The divisor $\pi(\mathtt{ram}(\pi))$ is called the *branch divisor*, denoted $\mathtt{branch}(\pi)$.

If $\mathfrak{p} \notin \mathtt{ram}(\pi)$, $\mathfrak{q} = \pi(\mathfrak{p})$, then the extension of residue fields $k(\mathfrak{p})/k(\mathfrak{q})$ is Galois and G acts transitively on the fibre $\pi^{-1}(\mathfrak{q})$. Furthermore, we have an isomorphism $D_{\mathfrak{p}} \simeq \mathrm{Gal}(k(\mathfrak{p})/k(\mathfrak{q}))$. On the other hand, if $\mathfrak{p} \in \mathtt{ram}(\pi)$ then the extension $k(\mathfrak{p})/k(\mathfrak{q})$ is normal (but not necessarily separable) and $D_{\mathfrak{p}}$ maps surjectively onto $\mathrm{Aut}(k(\mathfrak{p})/k(\mathfrak{q}))$. The kernel of this map is the inertia group $I_{\mathfrak{p}}$.

Theorem 2.9. *Let $X_{/S}$ be an integral S-scheme. Suppose G is a finite flat S-group acting on X such that X/G exists as an S-scheme. Let $\mathscr{E}$ be a torsion-free and coherent $\mathscr{O}_X$-module.*

(a) *The sheaf $\mathscr{D}er_\sigma(\mathscr{E})$ is invertible on the complement $Y_\sigma := X \setminus I_\sigma(X)$, for all $\sigma \in G \setminus \{\mathrm{id}\}$. Hence the image of the map*

$$\gamma: \quad G \setminus \{\mathrm{id}\} \to \mathrm{Pic}(Y_G), \quad \sigma \mapsto \mathscr{D}er_\sigma(\mathscr{E}|_{Y_G})$$

generates a subgroup of $\mathrm{Pic}(Y_G)$.

(b) *If $I_\sigma(X)$ is regular then $\mathscr{D}er_\sigma(\mathscr{O}_X)$ can be extended to an invertible module on all of X, for all $\sigma \in G \setminus \{\mathrm{id}\}$. Hence in this case, γ becomes*

$$G \setminus \{\mathrm{id}\} \to \mathrm{Pic}(X), \quad \sigma \mapsto \mathscr{D}er_\sigma(\mathscr{O}_X)$$

and so generates a subgroup of $\mathrm{Pic}(X)$.

Remark 2.3.

(1) Notice that the only obstruction to $\mathscr{D}er_\sigma(\mathscr{O}_X)$ being locally free on X is the singular points $\mathfrak{p}$ such that $\mathfrak{p} \in I_\sigma$.
(2) If $X \to X/G$ is a G-torsor, then $\mathscr{D}er_\sigma(\mathscr{O}_X)$ is locally free on X.
(3) As the proof below will show, generators of $\mathscr{D}er_\sigma(\mathscr{E}|_{Y_\sigma})$ is generated by elements $(f_i - \sigma(f_i))^{-1}$ over a cover $\{U_i\}$ of X. From this follows that there is an associated Cartier divisor which is the zero locus of $f_i - \sigma(f_i)$ over each U_i.

Proof. Since $\pi : X \twoheadrightarrow X/\langle\sigma\rangle$ is étale on points with trivial inertia, the morphism $X \setminus I_\sigma(X) \to X/\langle\sigma\rangle$ is an étale torsor. The locus where a morphism is étale is open, which means that $I_\sigma(X)$ is closed.

Fix $\sigma \in G\setminus\{\mathrm{id}\}$ and take $\mathfrak{p} \in Y_\sigma := X\setminus I_\sigma(X)$. Notice that G preserves the fibres above S as it acts over S; this means that Y_σ is also G-invariant. Take an affine neighbourhood $U = \mathsf{Spec}(A) \subseteq Y_\sigma$ of $\mathfrak{p}$ such that $\sigma(\mathfrak{p}) \in U$ (the existence of such a neighbourhood is guaranteed by the assumption that X/G exists as an S-scheme). Then there is an $x \in A$ such that $x - \sigma(x) \notin \mathfrak{p}$ (for simplicity we denote the scheme automorphism σ and the induced algebra morphism by the same symbol). Indeed, either (1) we have $\sigma(\mathfrak{p}) \nsubseteq \mathfrak{p}$, or (2) we have $\sigma(\mathfrak{p}) \subseteq \mathfrak{p}$ (i.e., σ is in the decomposition group at $\mathfrak{p}$). In case (1) we can take $x \in \mathfrak{p}$ such that $\sigma(x) \notin \mathfrak{p}$; then $x - \sigma(x) \notin \mathfrak{p}$. For case (2), assume that there is no $t \in A$

such that $(\mathrm{id} - \sigma)(t) \notin \mathfrak{p}$, i.e., for all $t \in A$, $(\mathrm{id} - \sigma)(t) \in \mathfrak{p}$. Then modulo $\mathfrak{p}$, σ reduces to the identity, which is a contradiction since $\mathfrak{p} \in Y_\sigma$, and Y_σ has no points with non-trivial inertia.

We can now apply Proposition 2.5, showing that $\mathscr{D}er_\sigma(\mathscr{E})$ is an invertible sheaf on Y_σ. Hence we have an association $G \to \mathrm{Pic}(Y_\sigma)$ given by $\sigma \mapsto \mathscr{D}er_\sigma(\mathscr{E}|_{Y_\sigma})$. For the last part, since X is integral, [Har77, Prop. II.6.15] states that $\mathrm{CDiv}(Y_\sigma) \simeq \mathrm{Pic}(Y_\sigma)$, and [Har77, Rem. II.6.17.1] shows that $\mathscr{D}er_\sigma(\mathscr{E}|_{Y_\sigma})$ actually gives an effective Cartier divisor since it is locally generated by one element.

For (b), use Proposition 2.7. $\square$

Remark 2.4. The above association gives us, for each $n \in \mathbb{N}$, a map

$$\sigma^n \mapsto \mathscr{D}er_{\sigma^n}(\mathscr{E}) \in \mathrm{Pic}(X).$$

However, note that if $\sigma = \mathrm{id}$, then $\mathscr{D}er_\sigma(\mathscr{E}) = \mathrm{Der}(\mathscr{E}) \notin \mathrm{Pic}(X)$, so the association can certainly not be a group morphism.

Remark 2.5. It would obviously be very interesting to know what kind of subgroup the image of G generates inside $\mathrm{Pic}(X)$. For instance, are there sufficient conditions that $\langle \mathrm{im}(\gamma) \rangle = \mathrm{Pic}(X)$?

Example 2.3. Assume given $\pi : X \to S$ with $X = \underline{\mathfrak{o}_L}$ and $S = \underline{\mathfrak{o}_K}$, where $\mathfrak{o}_L$ and $\mathfrak{o}_K$ are the ring of integers in a Galois extension L/K of number fields. Let $\mathscr{E}$ be a projective $\mathfrak{o}_L$-module (which is automatically torsion-free since $\mathfrak{o}_L$ is a Dedekind domain) and let D be a divisor of S, such that $\pi^{-1}(D)$ includes all the ramified primes in X. In other words, D is a finite set of places in $\mathfrak{o}_K$ including the ramified ones in $\mathfrak{o}_L$. Natural choices for $\mathscr{E}$ are of course $\mathfrak{o}_L$ itself and fractional ideals $\mathscr{J} \in \mathrm{Pic}(\mathfrak{o}_L)$. Then on $X \setminus \pi^{-1}(D)$, $\mathscr{D}er_{\mathrm{Gal}(L/K)}(\mathscr{E})$ is an invertible sheaf. Therefore, we have an association (dependent on D)

$$\mathrm{Gal}(L/K) \to \mathrm{Pic}(\underline{\mathfrak{o}_L} \setminus \pi^{-1}(D)), \quad \sigma \mapsto \mathscr{D}er_\sigma(\mathscr{E}|_{\underline{\mathfrak{o}_L} \setminus \pi^{-1}(D)}),$$

for every $\mathscr{E}$. However, since $\pi^{-1}(D)$ is regular Theorem 2.9(b) applies again, and we can extend to a the whole $\mathfrak{o}_L$,

$$\mathrm{Gal}(L/K) \to \mathrm{Pic}(\mathfrak{o}_L), \quad \sigma \mapsto \mathscr{D}er_\sigma(\mathscr{E}),$$

for every projective $\mathfrak{o}_L$-module $\mathscr{E}$. In fact, in this case we could argue by simply appealing to Theorem 2.6 directly since $\mathfrak{o}_L$, being a Dedekind domain, is automatically regular and hence every localization is a UFD.

We can generalize this example. For this let us briefly recall the definition of a tamely ramified G-covering. We use a slightly more restrictive definition than usual for simplicity.

Definition 2.1. Let $\pi : X \twoheadrightarrow S$ be a *finite* cover with S connected and normal and X normal. We let $D \subset S$ denote a normal crossings divisor such that π is étale over $S \setminus D$ and assume that $\pi^{-1}(D)$ is regular. Then $X \twoheadrightarrow S$ is a (*tamely*) *ramified extension* if for every $s \in D$ of codimension one (in S) and $x \in X$ such that $s = \pi(x)$, $\mathscr{O}_{X,x}/\mathscr{O}_{S,s}$ is a (tamely) ramified extension of discrete valuation rings. If, in addition,

$$X \times_S (S \setminus D) \to S \setminus D$$

is a G-torsor, i.e., Galois covering with $G = \mathrm{Gal}(k(X)/k(S))$, then π is a (*tamely*) *ramified G-covering.*

Example 2.4. Let $\pi : X \twoheadrightarrow S$ be a tamely ramified G-covering, ramified along a divisor D and let $\mathscr{E}$ be a torsion-free sheaf on X. Then D includes the points over which $I_G(X)$ is non-zero. Therefore, the assumptions of Theorem 2.9 are satisfied and so

$$\mathscr{D}er_G\big(\mathscr{E}|_{X\setminus I_G(X)}\big)$$

is a family of invertible sheaves on $X \setminus I_G(X)$. On the other hand, since by assumption $\pi^{-1}(D)$ is regular, by Theorem 2.9(b), we can extend $\mathscr{D}er_G(\mathscr{E})$ to a family of invertible sheaves on the whole of X.

3. Equivariant hom-Lie algebras

3.1. Global equivariant hom-Lie algebras

Let G denote a finite group scheme acting on X over S, and let $\mathscr{A}$ be an $\mathscr{O}_X\{G\}$-sheaf of $\mathscr{O}_X$-algebras. This means that $\mathscr{A}$ is an $\mathscr{O}_X$-algebra together with a G-action, compatible with the G-action on X in the sense that $\sigma(xa) = \sigma(x)\sigma(a)$, $x \in \mathscr{O}_X$, $a \in \mathscr{A}$.

Definition 3.1. Given the above data, a *G-equivariant hom-Lie algebra on X over $\mathscr{A}$* is a $\mathscr{A}\{G\}$-module $\mathscr{L}$ together with, for each open $U \subset X$, an $\mathscr{O}_S$-bilinear product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_U$ on $\mathscr{L}(U)$ such that

(hL1.) $\langle\!\langle a, a \rangle\!\rangle_U = 0$, for all $a \in \mathscr{L}(U)$;
(hL2.) for all $\sigma \in G$ and for each σ a $q_\sigma \in \mathscr{A}(U)$, the identity

$$\circlearrowleft_{a,b,c} \Big\{ \langle\!\langle \sigma(a), \langle\!\langle b, c \rangle\!\rangle_U \rangle\!\rangle_U + q_\sigma \cdot \langle\!\langle a, \langle\!\langle b, c \rangle\!\rangle_U \rangle\!\rangle_U \Big\} = 0,$$

holds.

A morphism of equivariant hom-Lie algebras $(\mathscr{L}, G)$ and $(\mathscr{L}', G')$ is a pair (f, ψ) of a morphism of $\mathscr{O}_X$-modules $f : \mathscr{L} \to \mathscr{L}'$ and $\psi : G \to G'$ such that $f \circ \sigma = \psi(\sigma) \circ f$, and $f(U)\big(\langle\!\langle a, b \rangle\!\rangle_{\mathscr{L};U}\big) = \langle\!\langle f(U)(a), f(U)(b) \rangle\!\rangle_{\mathscr{L}';U}$.

Hence, an equivariant hom-Lie algebra is a family of (possibly isomorphic) products parametrized by G. A product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\sigma$, for fixed $\sigma \in G$, is a *hom-Lie algebra on* $\mathscr{L}$.

Notice that the definition implies that for a morphism

$$(f, \psi) : (\mathscr{L}, G) \to (\mathscr{L}', G')$$

we must have $f(q_\sigma) = q_{\psi(\sigma)}$.

We denote by $\mathsf{EquiHomLie}_{X_{/S}}$ denote the category of all equivariant hom-Lie algebras on X with morphisms given in the definition. The category of hom-Lie algebras over $X_{/S}$ is denoted $\mathsf{HomLie}_{X_{/S}}$.

By the requirements that G is a group, every equivariant hom-Lie algebra includes a Lie algebra, possibly abelian, corresponding to $e \in G$ (see Example 4.1 below). The hom-Lie algebras corresponding to $g \neq e$ in the equivariant hom-Lie algebra can be viewed as "deformations" of the Lie algebra in the equivariant hom-Lie algebra.

Remark 3.1. Clearly, we could use any Grothendieck topology on X for the above definition.

3.2. Base change

Proposition 3.1. *Let $f : X \to Y$ be a morphism in Sch/S and let $\mathscr{A}$ be an $\mathscr{O}_Y$-algebra on Y. Suppose that $\mathscr{L}$ is a hom-Lie algebra over $\mathscr{A}$ on Y. Then,*

$$f^*\mathscr{L} := f^{-1}\mathscr{L} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X, \quad \textit{the } \text{pull-back } \textit{of } \mathscr{L},$$

is a hom-Lie algebra over $f^*\mathscr{A}$ on X.

Proof. This is standard. □

We will now consider what happens when we change the group. For simplicity we consider only the special case. Everything globalizes without problem.

Let L be an A-module, $A \in \mathsf{Com}(k)$ equipped with an equivariant hom-Lie algebra with group G. Suppose we are given a sequence of groups

$$\cdots \to H_{i+1} \xrightarrow{\varphi_{i+1}} H_i \to \cdots \xrightarrow{\varphi_2} H_1 \xrightarrow{\varphi_1} H \xrightarrow{\varphi} G \xrightarrow{\psi} E.$$

Then we have the following proposition.

Proposition 3.2. *The equivariant hom-Lie algebra on L over G descends to a canonical one, φ^*L, over H via φ. In addition, G will act on the invariants $L^{\varphi(H)}$ with induced equivariant hom-Lie algebra over $G/\operatorname{im}(\varphi)$. We also have an induced map $L^G \to L^{\varphi(H)}$, or more generally*

$$\cdots \leftarrow L^{\varphi_{i+1}(H_{i+1})} \leftarrow L^{\varphi_i(H_i)} \leftarrow \cdots \leftarrow L^{\varphi_1(H_1)} \leftarrow L^{\varphi(H)} \leftarrow L^G,$$

with induced hom-Lie structures. Notice that L^G is the trivial (abelian) equivariant hom-Lie algebra. In addition, if $\psi : G \twoheadrightarrow E$ is a surjection, then $L^{\ker\psi} \subseteq L$ is a equivariant hom-Lie algebra over E.

Proof. Obvious. □

Recall that the induced G-module coming from an H-module M, is defined as

$$\mathrm{Ind}_G^H(M) := \{\psi : G \to M \mid \psi(hg) = h\psi(g), \text{ for } h \in H\}.$$

The G-module structure on $\mathrm{Ind}_G^H(M)$ is defined by $(g'.\psi)(g) := \psi(gg')$.

Proposition 3.3. *Suppose that the A-module L is an equivariant hom-Lie algebra over H. Then $\mathrm{Ind}_G^H(L)$ is an equivariant hom-Lie algebra over G with product defined by*

$$\langle\!\langle \psi, \psi' \rangle\!\rangle_{\mathrm{Ind}_G^H(L)}(g) := \langle\!\langle \psi(g), \psi'(g) \rangle\!\rangle_L$$

and A-module structure given by $(a.\psi)(g) := a\psi(g)$.

Proof. Obvious. □

Notice that I allow arbitrary group morphisms when defining induced modules, contrary to the ordinary usage, which restricts to injective morphisms. In general, induced modules are only useful when H is indeed a subgroup of G.

4. Examples

4.1. Basic examples

Example 4.1. Suppose $G = \{\mathrm{id}\}$, the trivial group. Then the above definition amounts to a sheaf of $\mathscr{O}_X$-Lie algebras.

Example 4.2. Let $A \in \mathrm{ob}(\mathsf{Com}(k))$, $L \in \mathrm{ob}(\mathsf{Mod}(A))$ and G a group acting k-linearly on L. Then an equivariant hom-Lie algebra on L over A/k is family of k-bilinear products $\langle\!\langle \cdot, \cdot \rangle\!\rangle_g$, $g \in G$, satisfying

$$\langle\!\langle a, a \rangle\!\rangle_g = 0 \quad \text{and} \quad \circlearrowleft_{a,b,c} \big(\langle\!\langle a^g + a, \langle\!\langle b, c \rangle\!\rangle_g \rangle\!\rangle_g\big) = 0, \quad \text{for all} \quad g \in G.$$

A morphism of equivariant hom-Lie algebras L and L' over A/k is a morphism of k-modules such that $f(a^g) = f(a)^g$ (i.e., G-equivariance) and $f\langle\!\langle a, b \rangle\!\rangle_g^L = \langle\!\langle f(a), f(b) \rangle\!\rangle_g^{L'}$.

If L and L' comes equipped with different group actions, G and G', we demand according to definition, instead of G-equivariance, that $f(a^g) = f(a)^{g'}$, for all $g \neq e \in G$, $g' \neq e' \in G'$.

Example 4.3 *(Twisted derivations).* Let $A \in \mathrm{ob}(\mathsf{Com}(k))$ and assume $M \in \mathrm{ob}(\mathsf{Mod}(A))$ torsion-free. Suppose that $\sigma \in \mathrm{End}(A)$, $\delta_\sigma \in \mathrm{Der}_\sigma(M)$ are such that $\partial = a(\mathrm{id} - \sigma)$, $a \in A$, and

$$\partial \circ \sigma = q \cdot \sigma \circ \partial, \quad \text{with} \quad q \in A.$$

Assume in addition that

$$\sigma \operatorname{Ann}(\partial) \subseteq \operatorname{Ann}(\partial),$$

which is automatic for instance when A is a domain. Then Theorem 2.2 endows $A \cdot \delta_\sigma$ with the structure of hom-Lie algebra. Taking a subgroup $G \subseteq \mathrm{End}(A)$ with a family $\delta_G \subseteq \mathrm{Der}_\sigma(M)$, $\delta_G := \{\delta_\sigma \mid \sigma \in G\}$, such that

$$\partial_\sigma \circ \sigma = q_\sigma \cdot \sigma \circ \partial_\sigma, \quad \text{with} \quad q_\sigma \in A, \quad \text{for each } \sigma \in G,$$

and where $\sigma(am) = \sigma(a)\sigma(m)$. Then Theorem 2.2 gives us an equivariant hom-Lie algebra for G on M. It is easy to see that if $a \in A^\times$, then $q_\sigma := a/\sigma(a)$ satisfies the assumptions on q_σ. Indeed,

$$\sigma \circ \partial_\sigma(b) = \sigma \circ (a(\mathrm{id} - \sigma))(b) = \sigma(a)(\mathrm{id} - \sigma) \circ \sigma(b),$$

so multiplying by $a/\sigma(a)$ gives the desired identity. Fixing $a \in A^\times$, we get an association $G \to A^\times$, $\sigma \mapsto a/\sigma(a)$. In other words, we get an element in $\mathrm{B}^1(G, A^\times)$ (the group of 1-coboundaries in group cohomology). This gives a family $\{(q_\sigma, \partial_\sigma) \mid \sigma \in G, \partial_\sigma = a(\mathrm{id} - \sigma)\}$ satisfying the required conditions of Theorem 2.2.

Notice that we in particular get that if A is a domain, $\mathrm{Der}_G(\mathrm{Fr}(A))$ is an equivariant hom-Lie algebra, where $\mathrm{Fr}(A)$ is the fraction field of A.

We can globalize this in the evident manner. Namely, let X be a scheme, $\mathscr{A}$ a sheaf of $\mathscr{O}_X$-algebras and $\mathscr{E}$ a torsion-free $\mathscr{A}$-module. First, for $U \subseteq X$ an open affine, let ∂ be a section of $\mathscr{D}er_\sigma(\mathscr{A})(U)$ such that $\partial \circ \sigma = q_{\sigma,U} \cdot \sigma \circ \partial$, for some $q_{\sigma,U} \in \mathscr{A}(U)$, and $\sigma \operatorname{Ann}(\partial) \subseteq \operatorname{Ann}(\partial)$. Then to any $\delta \in \mathscr{D}er_\sigma(\mathscr{E})(U)$ such that

$$\delta(am) = \partial(a)m + \sigma(a)\delta(m),$$

is attached a canonical global hom-Lie algebra, $\mathscr{A} \cdot \delta \subseteq \mathscr{D}er_\sigma(\mathscr{A})$, and therefore a global equivariant hom-Lie algebra, $\mathscr{A} \cdot \delta_G$.

4.1.1. Difference equations

In this section we will associate to every difference equation a canonical hom-Lie algebra encoding all the structure of the underlying equation.

There are several, more or less equivalent, ways to represent a difference equation. Let σ be an automorphism on A as usual. Then a difference equation can be given as either

(i) $\sum_{i=0}^{n} a_i\sigma^i$, $a_i \in A$, or as
(ii) $\sum_{i=0}^{n} b_i\Delta_\sigma^i$, with $\Delta_\sigma = c(\mathrm{id} - \sigma)$ a σ-derivation where $c, b_i \in A$, or as
(iii) a finitely generated (most often locally free of finite rank) A-module M together with a σ-linear $\boldsymbol{\sigma} : M \to M$, or as
(iv) a finitely generated (most often locally free of finite rank) A-module M together with a σ-connection

$$\nabla^{(\sigma)} : M \to \sigma^* A \otimes_k M, \quad m \mapsto 1 \otimes a(\mathrm{id} - \boldsymbol{\sigma}),$$

with $\boldsymbol{\sigma} : M \to M$ σ-linear.

It is rather easy to see that these are equivalent for most interesting k-algebras A and difference equations (see for instance [And01,DV02] or [Sau03]).

We will primarily use (iii) here. In this sense a difference equation is given by a matrix Σ with entries in A (once we have chosen a basis for M). A solution to this equation is then a vector $\boldsymbol{f} \in R^m$ such that $\Sigma\boldsymbol{f} = \boldsymbol{f}$. Notice that we might need to enlarge the underlying ring A to a ring R (a so-called *Picard–Vessiot ring* associated to the difference equation) for solutions to exist. In fact, solutions are not guaranteed unless the underlying ring of constants A^σ is an algebraically closed field (see [vdPS97]). Clearly a solution to the difference equation is an element in $\ker(\mathrm{id} - \boldsymbol{\sigma})$.

Form the symmetric A-algebra $S_A(M)$. Clearly $\boldsymbol{\sigma}$ extends to a σ-linear algebra morphism, which we also denote $\boldsymbol{\sigma}$, $S_A(M) \xrightarrow{\boldsymbol{\sigma}} S_A(M)$. The solution space $\ker(\mathrm{id} - \boldsymbol{\sigma})$ generates an ideal in $S_A(M)$ and any generator for this ideal is a solution to the difference equation (possibly after enlarging to a Picard–Vessiot ring).

We now look at the twisted derivation

$$\Delta_{\boldsymbol{\sigma}} := \mathrm{id} - \boldsymbol{\sigma} : \quad S_A(M) \to S_A(M).$$

Then the left $S_A(M)$-module $S_A(M) \cdot \Delta_{\boldsymbol{\sigma}}$ is naturally a hom-Lie algebra by the previous example. Obviously, this algebra encapsulates a lot of information of the difference equation. We will see an example of this in the last section of the paper. For now, let me briefly mention the following:

Example 4.4. Let k be a perfect field of characteristic p, and let $W(k)$ be the ring of Witt vectors of k and K the field of fractions of $W(k)$. The Frobenius automorphism $\sigma(a) = a^p$ of k lifts to an automorphism ϕ of $W(k)$. An *F-crystal* is then a free $W(k)$-module M

together with a ϕ-linear endomorphism $\boldsymbol{\sigma} : M \to M$. An *F-isocrystal* is a free K-module M with a ϕ-linear $\boldsymbol{\sigma}$.

There are versions of this definitions over (formal) schemes and rigid analytic spaces.

The above discussion globalizes in the evident manner.

4.2. Equivariant hom-Lie algebras of morphisms of schemes

4.2.1. Families of equivariant hom-Lie algebras

Let $f : X_{/S} \to Y_{/S}$ be a morphism of S-schemes and let $\mathscr{E}$ be a coherent $\mathscr{O}_X$-module. Assume that G is a group acting on $X_{/S}$ over $Y_{/S}$ (and over S) and equivariantly on $\mathscr{E}$. The fibres $X_y := X \otimes_Y k(y)$ and $\mathscr{E}_y := \mathscr{E} \otimes_{\mathscr{O}_Y} k(y)$, $y \in Y$, are thus invariant under G.

Let $\{U_i\}$ be an affine cover of X and let $\mathscr{D}$ be a rank-one subsheaf of $\mathscr{D}er_\sigma(\mathscr{E})$ with local generators over $\{U_i\}$ given by $\Delta_i := a_i(\mathrm{id} - \sigma)$, with $a_i \in \mathscr{O}_X(U_i)$ satisfying $a_i = q_i\sigma(a_i)$, for some $q_i \in \mathscr{O}_X(U_i)$. Then $\mathscr{D}$ defines a family of hom-Lie algebras, parametrized by Y, by the rule $\mathscr{D}(U_i) = \mathscr{O}_X(U_i) \cdot \Delta_i$ as in Example 4.3. The product is given locally over U_i as

$$\langle\!\langle\, \alpha_i \cdot \Delta_i, \beta_i \cdot \Delta_i \,\rangle\!\rangle = \big(\sigma(\alpha_i)\Delta_i(\beta_i) - \sigma(\beta_i)\Delta_i(\alpha_i)\big) \cdot \Delta_i.$$

Let $y \in Y$. Then

$$\mathscr{D}_y := \mathscr{D} \otimes k(y) = \big(\mathscr{O}_X \cdot \Delta\big) \otimes k(y) = \mathscr{O}_{X_y} \cdot \Delta|_{X_y}$$

is the fibre over $y \in Y$ in this family.

4.2.2. G-covers

Put $X := \mathbf{Spec}_Y(\mathscr{A})$, for $\mathscr{A}$ a finite coherent $\mathscr{O}_Y$-algebra and assume that $f : X_{/S} \to Y_{/S}$ is a (finite) G-cover, with X and Y connected. Notice that this implies that $Y = X/G$ and that $X \to Y$ is étale over the complement of the branch locus. This also implies that $\sigma(\mathscr{A}(U)) \subseteq \mathscr{A}(U)$ for all open $U \subset Y$. Since f is finite, $\mathscr{A}$ is a locally free sheaf of finite rank. Take $\sigma \in G$ and consider $\mathscr{D}er_\sigma(\mathscr{A})$. This is an invertible sheaf over $X \setminus \mathtt{ram}(f)$ which can be extended to an invertible sheaf on the whole X if $\mathtt{ram}(f)$ is regular.

Put, for each $U \subseteq Y$, $\Delta_\sigma(U) := a_U(\mathrm{id} - \sigma)$ and assume that we have $a_U = q_U\sigma(a_U)$, for some $q_U \in \mathscr{A}(U)$. We now look at the submodule $\mathscr{A} \cdot \Delta_\sigma$ inside $\mathscr{D}er_\sigma(\mathscr{A})$. Explicitly,

$$(\mathscr{A} \cdot \Delta_\sigma)(U) = \bigoplus_{i=0}^{n} \mathscr{O}_Y(U)e_i\Delta_\sigma = \bigoplus_{i=0} \mathscr{O}_Y(U)\underline{\varepsilon}_i, \qquad U \subseteq Y,$$

with $\underline{\varepsilon}_i := e_i\Delta_\sigma$, and e_i generating sections of $\mathscr{A}$ over U. We consider the hom-Lie algebra $(\mathscr{A} \cdot \Delta_\sigma, \langle\!\langle\ ,\ \rangle\!\rangle)$.

4.2.3. Witt hom-Lie algebras

Keep the above notation and let $c_{ij}^k \in \mathscr{O}_Y(U)$ be the structure constants for $\mathscr{A}(U)$. Assume in addition that $\sigma \in G$ acts as $\sigma(e_i) = \sum_{k=0}^{n-1} s_{ik} e_k$ with $s_{ik} \in \mathscr{O}_Y(U)$.

Proposition 4.1. *The pair*

$$\mathrm{W}_\sigma^{\mathscr{A}} := (\mathscr{A} \cdot \Delta_\sigma, \langle\!\langle\ ,\ \rangle\!\rangle)$$

defines a hom-Lie algebra on $\mathscr{A}$ over Y and is given by

$$\langle\!\langle\, \underline{\varepsilon}_i, \underline{\varepsilon}_j \,\rangle\!\rangle = a \sum_{\ell=0}^{n-1} \Big(\sum_{k=0}^{n-1} \big(s_{ik} c_{kj}^{\ell} - s_{jk} c_{ki}^{\ell} \big) \Big) \underline{\varepsilon}_\ell .$$

Proof. Simple computation. □

Notice the special case when $\sigma(e_i) = q_i e_i$, with $q_i \in \mathscr{O}_Y^\times(U)$:

$$\langle\!\langle\, \underline{\varepsilon}_i, \underline{\varepsilon}_j \,\rangle\!\rangle = a \sum_{k=0}^{n-1} (q_i - q_j) c_{ij}^k \underline{\varepsilon}_k . \tag{5}$$

We call $\mathrm{W}_\sigma^{\mathscr{A}}$ the (generalized) *Witt hom-Lie algebra* (over $\mathscr{O}_Y$) associated with σ and $\mathscr{A}$.

4.2.4. Kummer–Witt hom-Lie algebras

In this section we study the simplest family of examples of G-covers, namely, cyclic covers. In this case

$$\mathscr{A}(U) \simeq \mathscr{O}_Y(U)[z]/(z^n - B_U) = \bigoplus_{i=0}^{n-1} \mathscr{O}_Y(U) e_i, \quad e_i := z^i,$$

for a section $B_U \in \mathscr{O}_Y(U)$. We assume that $\mathscr{O}_Y$ includes the n-th roots of unity. In fact, $\mathbf{Spec}_Y(\mathscr{A})$ is a cyclic cover of Y with $\sigma(z) := \xi^r z$, $0 \le r \le n-1$, for ξ a primitive n-th root of unity. It is easy to see that B_U represents the branch divisor over U.

Observe that we allow $B_U = 0$ in which case we view $\mathscr{A}$ as an "infinitesimal thickening" of Y.

Put $\underline{\varepsilon}_i := z^i \Delta_\sigma$.

Corollary 4.2. *When $\sigma(z) = \xi^r z$, the hom-Lie algebra structure on $\mathscr{A} \cdot \Delta_\sigma$ is given by*

$$\langle\!\langle\, \underline{\varepsilon}_i, \underline{\varepsilon}_j \,\rangle\!\rangle = \xi^{ri}(1 - \xi^{r(j-i)}) B_U^{\circlearrowleft} \underline{\varepsilon}_{\{i+j \mod n\}}, \quad i \le j,$$

where $B_U^{\circlearrowleft}$ means that B_U is included when $i + j \ge n$.

Proof. Follows immediately from Proposition 4.1. $\square$

We denote the locally free algebra in the proposition by $\mathrm{KW}_{\mathscr{A}}(\xi^r)$ and refer to it as a *Kummer–Witt hom-Lie algebra*.

Here comes a few illustrative examples. Let Y be the projective line:

$$Y = \mathbb{P}^1_k = \mathsf{Spec}(k[s, s^{-1}]) \cup \mathsf{Spec}(k[t, t^{-1}]) = U \cup V,$$

glued (as always) along $s \to t^{-1}$, and let B be the divisor

$$B_U = (s^2 + 1)s^3, \quad B_V = (t^{-2} + 1)t^{-3}.$$

Then $\pi : X \to Y$, with $X := \mathbf{Spec}_Y(\mathscr{A})$, is defined by

$$\pi^{-1}(U) = \mathsf{Spec}\Big(k[s, s^{-1}, z]/\big(z^n - (s^2 + 1)s^3\big)\Big),$$
$$\pi^{-1}(V) = \mathsf{Spec}\Big(k[t, t^{-1}, z]/\big(z^n - (t^{-2} + 1)t^{-3}\big)\Big).$$

Notice that the branch points of π are dependent on whether $\sqrt{-1} \in k$ or not. If $\sqrt{-1} \notin k$ then π is étale over $Y \setminus \{\infty\}$, otherwise it is ramified over $s = \pm\sqrt{-1}$.

Example 4.5. We first look at the example when $n = 3$ and $\sigma(z) = \xi z$, $\xi^3 = 1$. Putting this into the structure-constant-machine in the above corollary gives

$$\langle\!\langle\, \underline{\varepsilon}_0, \underline{\varepsilon}_1 \,\rangle\!\rangle = (1 - \xi)\underline{\varepsilon}_1, \qquad \langle\!\langle\, \underline{\varepsilon}_0, \underline{\varepsilon}_2 \,\rangle\!\rangle = (1 - \xi^2)\underline{\varepsilon}_2, \qquad \langle\!\langle\, \underline{\varepsilon}_1, \underline{\varepsilon}_2 \,\rangle\!\rangle = \xi(1 - \xi)B\underline{\varepsilon}_0.$$

The fibre over $s = 1$ is the spectrum of the algebra

$$A_1 = k(\sqrt[3]{2})f_0 \times k(\sqrt[3]{2})f_1 \times k(\sqrt[3]{2})f_2,$$

since $z^3 - 2 = (z - \sqrt[3]{2})(z - \xi\sqrt[3]{2})(z - \xi^2\sqrt[3]{2})$ over $k(\sqrt[3]{2})$. The induced action of σ on the fibre becomes $f_0 \mapsto f_1 \mapsto f_2 \mapsto f_0$. The hom-Lie algebra over $s = 1$ then has products

$$\langle\!\langle\, \underline{\varepsilon}_0, \underline{\varepsilon}_1 \,\rangle\!\rangle = (1 - \xi)\underline{\varepsilon}_1, \qquad \langle\!\langle\, \underline{\varepsilon}_0, \underline{\varepsilon}_2 \,\rangle\!\rangle = (1 - \xi^2)\underline{\varepsilon}_2, \qquad \langle\!\langle\, \underline{\varepsilon}_1, \underline{\varepsilon}_2 \,\rangle\!\rangle = 2\xi(1 - \xi)\underline{\varepsilon}_0.$$

The fibre over $s = \xi$ is the spectrum of the algebra

$$A_\xi = k(\sqrt[3]{\xi^2 + 1})f_0 \times k(\sqrt[3]{\xi^2 + 1})f_1 \times k(\sqrt[3]{\xi^2 + 1})f_2,$$

and the hom-Lie algebra over $s = \xi$ is

$$\langle\!\langle\, \underline{\varepsilon}_0, \underline{\varepsilon}_1 \,\rangle\!\rangle = (1 - \xi)\underline{\varepsilon}_1, \qquad \langle\!\langle\, \underline{\varepsilon}_0, \underline{\varepsilon}_2 \,\rangle\!\rangle = (1 - \xi^2)\underline{\varepsilon}_2, \qquad \langle\!\langle\, \underline{\varepsilon}_1, \underline{\varepsilon}_2 \,\rangle\!\rangle = (1 - \xi^2)\underline{\varepsilon}_0.$$

The fibre over a branch point is clearly a fat point of order three and there the hom-Lie algebra becomes

$$\langle\!\langle\, \underline{\varepsilon}_0, \underline{\varepsilon}_1 \,\rangle\!\rangle = (1-\xi)\underline{\varepsilon}_1, \qquad \langle\!\langle\, \underline{\varepsilon}_0, \underline{\varepsilon}_2 \,\rangle\!\rangle = (1-\xi^2)\underline{\varepsilon}_2, \qquad \langle\!\langle\, \underline{\varepsilon}_1, \underline{\varepsilon}_2 \,\rangle\!\rangle = 0.$$

Taking instead $\sigma(t) = \xi^2 t$ gives

$$\langle\!\langle\, \underline{\varepsilon}_0, \underline{\varepsilon}_1 \,\rangle\!\rangle = (1-\xi^2)\underline{\varepsilon}_1, \qquad \langle\!\langle\, \underline{\varepsilon}_0, \underline{\varepsilon}_2 \,\rangle\!\rangle = (1-\xi)\underline{\varepsilon}_2, \qquad \langle\!\langle\, \underline{\varepsilon}_1, \underline{\varepsilon}_2 \,\rangle\!\rangle = -\xi(1-\xi)B\underline{\varepsilon}_0,$$

and the products on the fibres over $s = 1$ and $s = \xi$ are

$$\langle\!\langle\, \underline{\varepsilon}_0, \underline{\varepsilon}_1 \,\rangle\!\rangle = (1-\xi^2)\underline{\varepsilon}_1, \qquad \langle\!\langle\, \underline{\varepsilon}_0, \underline{\varepsilon}_2 \,\rangle\!\rangle = (1-\xi)\underline{\varepsilon}_2, \qquad \langle\!\langle\, \underline{\varepsilon}_1, \underline{\varepsilon}_2 \,\rangle\!\rangle = -2\xi(1-\xi)\underline{\varepsilon}_0$$

and

$$\langle\!\langle\, \underline{\varepsilon}_0, \underline{\varepsilon}_1 \,\rangle\!\rangle = (1-\xi^2)\underline{\varepsilon}_1, \qquad \langle\!\langle\, \underline{\varepsilon}_0, \underline{\varepsilon}_2 \,\rangle\!\rangle = (1-\xi)\underline{\varepsilon}_2, \qquad \langle\!\langle\, \underline{\varepsilon}_1, \underline{\varepsilon}_2 \,\rangle\!\rangle = -(1-\xi^2)\underline{\varepsilon}_0.$$

Over a branch point, we get

$$\langle\!\langle\, \underline{\varepsilon}_0, \underline{\varepsilon}_1 \,\rangle\!\rangle = (1-\xi^2)\underline{\varepsilon}_1, \qquad \langle\!\langle\, \underline{\varepsilon}_0, \underline{\varepsilon}_2 \,\rangle\!\rangle = (1-\xi)\underline{\varepsilon}_2, \qquad \langle\!\langle\, \underline{\varepsilon}_1, \underline{\varepsilon}_2 \,\rangle\!\rangle = 0.$$

Obviously, the case when σ is the identity gives the abelian hom-Lie algebra. Notice that the three algebras in the equivariant structure are non-isomorphic over Y.

The reader is invited to study the case $n = 4$, in particular when $\sigma(t) = \xi^2 t = -t$. For all $\sigma \in G$ we have the following subalgebra in the general situation:

Proposition 4.3. *The algebra $J_B(\xi^r)$ given by*

$$\begin{aligned} \langle\!\langle\, \underline{\varepsilon}_0, \underline{\varepsilon}_1 \,\rangle\!\rangle &= (1-\xi^r)\underline{\varepsilon}_1 \\ \langle\!\langle\, \underline{\varepsilon}_0, \underline{\varepsilon}_{n-1} \,\rangle\!\rangle &= (1-\xi^{r(n-1)})\underline{\varepsilon}_{n-1} \\ \langle\!\langle\, \underline{\varepsilon}_1, \underline{\varepsilon}_{n-1} \,\rangle\!\rangle &= \xi(1-\xi^{r(n-2)})B_U\underline{\varepsilon}_0 \end{aligned} \tag{6}$$

is a subalgebra of $\mathrm{KW}_{\mathscr{A}}(\xi^r)$. Furthermore, if $B \neq 0$, it is non-solvable if $n = p > 2$ is a prime. If $n = 2$, $J_B(\xi)$ is clearly solvable. In fact, it is actually a Lie algebra.

Proof. The first statement follows from Corollary 4.2, whereas the second follows from $\langle\!\langle\, J_B(\xi^r), J_B(\xi^r) \,\rangle\!\rangle = J_B(\xi^r)$ and induction. □

Notice the similarity between $J_B(\xi^r)$ and the Jackson-$\mathfrak{sl}_2$ from [LS07]. It is therefore natural to call the algebra $J_B(\xi^r)$ the *Jackson subalgebra* of $\mathrm{KW}_{\mathscr{A}}(\xi^r)$.

Conjecture 1. *If n is composite then there is at least one $\sigma \in G$ such that $\mathrm{KW}_{\mathscr{A}}(\xi^r)$ is solvable.*

In the cases I've investigated this seems to be true and the following proposition gives some support for this claim.

Proposition 4.4. *Let n be composite. Then for some $\sigma \in G$ there are $0 \le i \neq j \le n-1$ such that*

$$\langle\!\langle\, \underline{\varepsilon}_i, \underline{\varepsilon}_j \,\rangle\!\rangle = 0.$$

Proof. Let ξ be a primitive n-th root of unity. Since n is composite there is a $k < n$ such that $k|n$. Consider $\sigma(t) = \xi^k t$. Then there are $i < j < n$ such that $k(j-i) = n$. The claim follows from Corollary 4.2. □

4.3. Hom-Lie algebras associated with t-motives

We need some notation and terminology first. For this, we will primarily follow [Pap08] and [CY07], with some slight modifications.

Let p be a prime and put, $q = p^r$, $A := \mathbb{F}_q[\theta]$ and $k := \mathrm{Fr}(A) = \mathbb{F}_q(\theta)$, where θ is transcendental over $\mathbb{F}_q$. We also use the notation $k_\infty := \mathbb{F}_q((\theta^{-1}))$, with algebraic closure $\overline{k}_\infty$. Finally, we put $\mathbb{C}_\infty := \widehat{\overline{k}}_\infty$, the completion of $\overline{k}_\infty$ under the ∞-norm with $|\theta|_\infty = q$. The Frobenius morphism

$$\sigma : \mathbb{C}_\infty \to \mathbb{C}_\infty, \quad a \mapsto a^{1/q}$$

is extended to $\mathbb{C}_\infty((t))$ by the rule

$$\sigma\Big(\sum_{i\in\mathbb{Z}} f_i t^i\Big) := \sum_{i\in\mathbb{Z}} f_i^{1/q} t^i.$$

Notice that, with this definition,

$$\sigma^r\Big(\sum_{i\in\mathbb{Z}} f_i t^i\Big) := \sum_{i\in\mathbb{Z}} f_i^{1/q^r} t^i.$$

We now have the following two definitions (cf. [Pap08]):

Definition 4.1. An *Anderson t-motive* is a module M together with a σ-linear morphism $\boldsymbol{\sigma} : \mathsf{M} \to \mathsf{M}$, such that M is free of finite rank over both $\overline{k}[t]$ and $\overline{k}[\boldsymbol{\sigma}]$, in addition to $(t-\theta)^n\mathsf{M} \subseteq \boldsymbol{\sigma}(\mathsf{M})$, for some $n \ge 0$.

Definition 4.2. A *pre-t-motive* is a difference equation $\boldsymbol{\sigma}$ over $\overline{k}(t)$. In other words, a $\overline{k}(t)$-vector space of finite dimension together with a σ-linear morphism $\boldsymbol{\sigma} : M \to M$.

These notions define categories with the obvious morphisms.

Fixing a basis for M or M, we can assume that $\boldsymbol{\sigma}$ is given by a matrix Σ with entries in $\overline{k}(t)$ and $\overline{k}[t]$, respectively.

From an Anderson t-motive M we get a pre-t-motive M by base-change along $\overline{k}[t] \to \overline{k}(t)$, with the action of $\boldsymbol{\sigma}$ extended diagonally:

$$M = \mathsf{M} \otimes_{\overline{k}[t]} \overline{k}(t), \quad \boldsymbol{\sigma}(m \otimes f) := \boldsymbol{\sigma}(m) \otimes \sigma(f).$$

We are primarily interested in the following two examples.

Example 4.6. The *Anderson unit t-motive*, $\mathbf{1}$, is defined as the module $\overline{k}[t]\mathbf{e}$, with $\boldsymbol{\sigma}(f\mathbf{e}) := \sigma(f)\mathbf{e}$. The corresponding *unit pre-t-motive*, also denoted $\mathbf{1}$, is naturally $\overline{k}(t)\mathbf{e}$ with $\boldsymbol{\sigma}(f\mathbf{e}) := \sigma(f)\mathbf{e}$. Clearly, $\Sigma = 1$ for the unit motive.

Example 4.7. The other basic example we need is the (*Anderson–*)*Carlitz t-motive*, C, and its associated pre-t-motive C. We define C and C as

$$\mathsf{C} := (\overline{k}[t]\mathbf{e}, \boldsymbol{\sigma}), \quad \boldsymbol{\sigma}(f\mathbf{e}) = \sigma(f)\boldsymbol{\sigma}(e) := \sigma(f)(t-\theta)\mathbf{e},$$

and

$$C := (\overline{k}(t)\mathbf{e}, \boldsymbol{\sigma}), \quad \boldsymbol{\sigma}(f\mathbf{e}) = \sigma(f)\boldsymbol{\sigma}(e) := \sigma(f)(t-\theta)\mathbf{e},$$

respectively. More generally, we can form the *Tate twists*, $n \geq 0$,

$$\mathsf{C}(n) := \mathsf{C}^{\otimes n}, \quad \boldsymbol{\sigma}(c_1 \otimes c_2 \otimes \cdots \otimes c_n) := \boldsymbol{\sigma}(c_1) \otimes \boldsymbol{\sigma}(c_2) \otimes \cdots \otimes \boldsymbol{\sigma}(c_n),$$

with tensor products over $\overline{k}[t]$, and similarly for C. Notice that $\mathsf{C}(0) = \mathbf{1}$, and that

$$\boldsymbol{\sigma}(\mathbf{e}^n) = (t-\theta)^n \mathbf{e}^n, \quad \text{where } \mathbf{e}^n := \mathbf{e} \otimes \mathbf{e} \otimes \cdots \otimes \mathbf{e}.$$

In this case, $\Sigma = (t-\theta)$.

Other examples can be constructed from Drinfel'd modules.

We also need the notion of rigid analytically trivial motives. For this we use the equivalences Propositions 3.3.9(a) and 3.4.7(a) in [Pap08] to simplify our exposition. Let $\mathbb{T}$ be the Tate algebra

$$\mathbb{T} := \mathbb{C}_\infty\langle t\rangle := \{f \in \mathbb{C}_\infty((t)) \mid |f_i|_\infty \to 0,\ i \to \infty\},$$

where $f = \sum f_i t^i$. This is a domain, and we denote the fraction field as $\mathbb{L}$.

An Anderson t-motive M of rank m is called *rigid analytically trivial* if there is a $\Psi \in \mathrm{GL}_m(\mathbb{T})$ such that $\sigma\Psi = \Sigma\Psi$, where σ acts on Ψ element-wise. This notion is stable under change of basis. Observe that Ψ actually gives a fundamental matrix of solutions to the difference equation $\boldsymbol{\sigma}(\boldsymbol{x}) = \Sigma\boldsymbol{x}$, $\boldsymbol{x} \in \mathsf{M}$. Therefore, $\mathbb{T}$ is actually a Picard–Vessiot ring for M. The matrix Ψ is called a *rigid analytic trivialization* for M, and $\Psi(\theta)^{-1}$ is the so-called *period matrix* (for M).

Similarly, a pre-t-motive is rigid analytically trivial if the matrix Ψ is in $\mathrm{Gl}_m(\mathbb{L})$.

Both the unit motive and the Carlitz motive is rigid analytically trivial (see [Pap08, 3.3.3 and 3.3.6]. In fact, a trivialization for C is given by the function

$$\Omega(t) := (-\theta)^{-\frac{q}{q-1}} \prod_{i=1}^{\infty}(1 - t/\theta^{q^i}) \in k_\infty\big((-\theta)^{-\frac{q}{q-1}}\big)[\![t]\!].$$

The period matrix $\tilde{\pi} := -\Omega(\theta)^{-1}$ is called the *Carlitz period* and is fundamental to the theory of t-motives. It is possible to show that the zeros of $\Omega(t)$ are exactly θ^{q^i}, $i \geq 1$.

We noted above that there is a functor (base-change) from the category of Anderson t-motives to the category of pre-t-motives. The images of rigid analytically trivial Anderson t-motives inside the category of pre-t-motives, generate the category of *t-motives*.

4.3.1. The t-motivic hom-Lie algebras

Finally, we come to the connection with hom-Lie algebras. Since an Anderson t-motive (and hence pre-t-motive) can be seen as a difference equation over $\overline{k}[t]$, there is a canonical hom-Lie algebra by the discussion in (4.1.1). We will study this hom-Lie algebra for the Carlitz and unit motive in some detail.

Let M be an Anderson t-motive of rank n. We are interested in the $\overline{k}[t]$-algebra $S_{\overline{k}[t]}(\mathsf{M})$ and the induced operator

$$\Delta_{\boldsymbol{\sigma}} := \mathrm{id} - \boldsymbol{\sigma} : \quad S_{\overline{k}[t]}(\mathsf{M}) \to S_{\overline{k}[t]}(\mathsf{M}),$$

and hom-Lie algebra

$$L(\mathsf{M}) := (S_{\overline{k}[t]}(\mathsf{M}) \cdot \Delta_{\boldsymbol{\sigma}}, \langle\!\langle\ ,\ \rangle\!\rangle\,).$$

See (4.1.1). To be perfectly explicit, we consider the case of the Carlitz motive from now on. For simplicity of notation, put $\mathsf{q} := (t - \theta)$. Notice that

$$S_{\overline{k}[t]}(\mathsf{C}) = \bigoplus_{n=0}^{\infty} \mathsf{C}^{\otimes n} = \bigoplus_{n=0}^{\infty} \mathsf{C}(n) = \bigoplus_{n=0}^{\infty} \overline{k}[t]\mathbf{e}^n = \overline{k}[t][\mathbf{e}],$$

with

$$\boldsymbol{\sigma}(f\mathbf{e}^n) = \sigma(f)\boldsymbol{\sigma}(e^n) = \sigma(f)(t-\theta)^n\mathbf{e}^n.$$

Therefore,

$$\Delta_{\boldsymbol{\sigma}}(f\mathbf{e}^n) = \big(\Delta_\sigma(f) + \sigma(f)(1-\mathsf{q})\{n\}_{\mathsf{q}}\big)\mathbf{e}^n,$$

where we used the "q-number" notation $\{n\}_{\mathsf{q}} := 1 + \mathsf{q} + \mathsf{q}^2 + \cdots + \mathsf{q}^{n-1}$. From this follows, after some computation, that the product on $L(\mathsf{C})$ is given as

$$\langle\!\langle\, f\mathbf{e}^n \cdot \Delta_{\boldsymbol{\sigma}},\, g\mathbf{e}^m \cdot \Delta_{\boldsymbol{\sigma}}\,\rangle\!\rangle_{\mathsf{C}} = \big(\mathsf{q}^n\sigma(f)g - \mathsf{q}^m\sigma(g)f\big)\mathbf{e}^{n+m} \cdot \Delta_{\boldsymbol{\sigma}}.$$

Let us introduce the notation

$$A_{n,m}(f,g)(t) := \big(\mathsf{q}^n \sigma(f) g - \mathsf{q}^m \sigma(g) f\big)(t).$$

Notice the special case $m = -n$:

$$A_{n,-n}(f,g)(t) := \big(\mathsf{q}^n \sigma(f) g - \mathsf{q}^{-n} \sigma(g) f\big)(t).$$

Similarly, we have that $L(\mathbf{1})$ has product

$$\langle\!\langle\, f\mathbf{e}^n \cdot \Delta_{\boldsymbol{\sigma}},\, g\mathbf{e}^m \cdot \Delta_{\boldsymbol{\sigma}} \,\rangle\!\rangle_{\mathbf{1}} = \big(\sigma(f)g - \sigma(g)f\big)\mathbf{e}^{n+m} \cdot \Delta_{\boldsymbol{\sigma}}.$$

Obviously, the same constructions work if we replace $\overline{k}[t]$ with $\overline{k}(t)$, i.e., if we work with pre-t-motives instead of Anderson motives.

As we will see, the structure becomes even richer if we invert the Carlitz motive:

$$\mathsf{C}(-1) := \overline{k}[t]_{(t-\theta)}\mathbf{e}, \quad \boldsymbol{\sigma}(f\mathbf{e}) := \sigma(f)(t-\theta)^{-1}\mathbf{e},$$

similarly with C. It is then clear what $\mathsf{C}(-n)$ (for $n > 1$) should mean. In this way we can form the Laurent polynomial rings

$$S_{\overline{k}[t]}(\mathsf{C})_{(t-\theta)} = \bigoplus_{n\in\mathbb{Z}} \mathsf{C}(n) = \bigoplus_{n\in\mathbb{Z}} \overline{k}[t]_{(t-\theta)}\mathbf{e}^n = \overline{k}[t,(t-\theta)^{-1}][\mathbf{e},\mathbf{e}^{-1}],$$

with the associated hom-Lie algebra structure,

$$L^{\mathrm{loc}}(\mathsf{C}) := (S_{\overline{k}[t]}(\mathsf{C})_{(t-\theta)} \cdot \Delta_{\boldsymbol{\sigma}}, \langle\!\langle\ ,\ \rangle\!\rangle\,).$$

Notice that over $\overline{k}(t)$ we have isomorphisms $\mathsf{C}(-n) \otimes \mathsf{C}(n) \simeq \mathbf{1}$ as pre-t-motives.

We define the function $L_{\alpha,l}(t)$ as

$$L_{\alpha,l}(t) := \alpha + \sum_{i=1}^{\infty} \frac{\alpha^{q^i}}{(t-\theta^q)^l (t-\theta^{q^2})^l \cdots (t-\theta^{q^i})^l},$$

and its evaluation $L_{\alpha,l}(\theta)$ is called the *l-th Carlitz polylogarithm*. From the definition one sees that

$$\sigma(L_{\alpha,l}) = \sigma(\alpha) + \frac{L_{\alpha,l}}{\mathsf{q}^l}.$$

In order for $L_{\alpha,l}$ to converge we need to assume that $|\alpha|_\infty < |\theta|_\infty^{\frac{lq}{q-1}}$.

We are particularly interested in evaluating the products at θ^i:

$$\langle\!\langle\, f\mathbf{e}^n \cdot \Delta_{\boldsymbol{\sigma}}, g\mathbf{e}^m \cdot \Delta_{\boldsymbol{\sigma}} \,\rangle\!\rangle\,|_{t=\theta^i} = A_{n,m}(f,g)(\theta^i)\mathbf{e}^{n+m} \cdot \Delta_{\boldsymbol{\sigma}}.$$

Observe that $\mathsf{q}(\theta) = 0$ so not all products are non-zero.

1. We begin with the "trivial" case $f = g$:

$$A_{n,m}(f,f) = (\mathsf{q}^n - \mathsf{q}^m)\sigma(f)f;$$

in particular, if $f = g = \mathrm{id}$, we get

$$A_{n,m}(\mathrm{id},\mathrm{id}) = (\mathsf{q}^n - \mathsf{q}^m).$$

Compare this with the q-deformed Witt algebra in [HLS06, Example 3.1].

2. $f = \alpha(t-\theta^{q^s})^i$, $g = \mathsf{q}^l\Omega^j$:

$$A_{n,m}(f,g) = \Omega^j\big(\sigma(\alpha)(t-\theta^{q^{s-1}})^i\mathsf{q}^{n+l} - \alpha(t-\theta^{q^s})^i(t-\theta^{1/q})^l\mathsf{q}^{j+m}\big).$$

Notice that if $j \neq 0$, we need to make a base change to $\overline{k}_\infty[\![t]\!]$.

3. By making a base change to $\mathbb{T}$ one can also look at $f = \alpha(t-\theta^{q^s})^i$, $g = \mathsf{q}^j L_{\alpha,l}$:

$$\begin{aligned} A_{n,m}(f,g) = \big(\sigma(\alpha)(t-\theta^{q^{s-1}})^i L_{\alpha,l}\mathsf{q}^{n+j} - \sigma(\alpha)\alpha(t-\theta^{q^s})^i(t-\theta^{1/q})^j\mathsf{q}^m \\ - \alpha(t-\theta^{q^s})^i(t-\theta^{1/q})^j L_{\alpha,l}\mathsf{q}^{m-l}\big). \end{aligned}$$

4. $f = \alpha(t-\theta^{q^s})^i$, $g = \Omega^l L_{\beta,l}$:

$$\begin{aligned} &\langle\!\langle\, \alpha(t-\theta^{q^s})^i\mathbf{e}^n\cdot\Delta_{\boldsymbol{\sigma}}, \Omega^l L_{\beta,l}\mathbf{e}^m\cdot\Delta_{\boldsymbol{\sigma}} \,\rangle\!\rangle \\ &\qquad = \big(\sigma(\alpha)(t-\theta^{q^{s-1}})^i\Omega^l L_{\beta,l}\mathsf{q}^n - \alpha\sigma(\beta)(t-\theta^{q^s})^i\Omega^l\mathsf{q}^{m+l} \\ &\qquad\qquad - \alpha(t-\theta^{q^s})^i\Omega^l L_{\beta,l}\mathsf{q}^m\big)\mathbf{e}^{n+m}\cdot\Delta_{\boldsymbol{\sigma}}. \end{aligned}$$

The reason why this last case is interesting is that when $s = 1$, the elements $\sigma(f_i)$, for $f_i = \alpha_i(t-\theta)^i$, form a difference equation for a t-motive, with $\Omega^l L_{\alpha_i,l}$ a rigid analytic trivialization (i.e., $\beta = \alpha_i$ above). This t-motive is intimately connected to the special values of $L_{\alpha_i,n}$ for $\alpha_i = \theta^i$. See [CY07], for instance.

4.3.2. The "t-motivic $\mathfrak{sl}_2$"

As indicated before the special case $m = -n$ is quite interesting. The reason for this is that the $\overline{k}[t]$-span of $\{\mathbf{e}^{-n}\cdot\Delta_{\boldsymbol{\sigma}}, \mathbf{e}^0\cdot\Delta_{\boldsymbol{\sigma}}, \mathbf{e}^n\cdot\Delta_{\boldsymbol{\sigma}}\}$ generates a subalgebra of $L^{\mathrm{loc}}(\mathsf{C})$.

We will now look at the $\overline{k}[t][f,g,h]$-span of $\{f\mathbf{e}^{-n}\cdot\Delta_{\boldsymbol{\sigma}}, g\mathbf{e}^0\cdot\Delta_{\boldsymbol{\sigma}}, h\mathbf{e}^1\cdot\Delta_{\boldsymbol{\sigma}}\}$.

- We begin in the situation 1 above: $f = h = \Omega^n$. We then get

$$A_{n,-n} = (1-\mathsf{q}^n)\Omega^{2n}, \quad A_{-n,0} = \Omega^n\Delta_{\boldsymbol{\sigma}}(g), \quad A_{n,0} = \Omega^n(\mathsf{q}^{2n}g - \sigma(g)).$$

Evaluating at $t = \theta^a$, for $a \geq 1$, we see that particularly interesting are the cases $t = \theta$ and $t = \theta^{q^i}$:

(i) $t = \theta$:

$$A_{n,-n} = \tilde{\pi}^{-2n}, \quad A_{-n,0} = \tilde{\pi}^{-n}\Delta_{\boldsymbol{\sigma}}(g), \quad A_{n,0} = -\tilde{\pi}^{-2n}\sigma(g);$$

(ii) $t = \theta^{q^i}$: $A_{-n,n} = A_{-n,0} = A_{n,0} = 0$.

- In situation 2, we now replace g with h to follow the current usage of notation. We begin by separating the cases $j = 0$, $j > 0$ and $j < 0$. For simplicity we assume that $s = 1$ throughout. In addition, we assume that $l = 0$. The case $l \neq 0$ is similar but there is a shift in degrees. We invite the reader to analyse this case for her/him-self.
  $j = 0$: Here we have

$$A_{n,m}(f,h) = \sigma(\alpha)(t-\theta)^i \mathsf{q}^n - \alpha(t-\theta^q)^i \mathsf{q}^m,$$

so

$$A_{-n,n} = \sigma(\alpha)(t-\theta)^{i-n} - \alpha(t-\theta^q)^i(t-\theta)^n.$$

The other structure-constants become

$$A_{-n,0}(f,g) = \sigma(\alpha)g(t-\theta)^{i-n} - \sigma(g)(t-\theta^q)^i,$$

and

$$A_{n,0}(h,g) = g(t-\theta)^{i+n} - \sigma(g).$$

Notice that with our assumptions $h = \mathrm{id}$. We now evaluate the different cases for i and n at $t = \theta^a$.
(i) $i = n$:

$$A_{-n,n} = \begin{cases} \sigma(\alpha), & a = 1 \text{ or } q, \text{ else} \\ \sigma(\alpha) - \alpha(\theta^a - \theta^q)^n(\theta^a - \theta)^n, & \end{cases}$$

$$A_{-n,0} = \begin{cases} \sigma(\alpha)g(\theta^q), & a = q, \text{ else} \\ \sigma(\alpha)g(\theta^a) - \sigma(g)(\theta^a)(\theta^a - \theta^q)^n, & \end{cases}$$

$$A_{n,0} = \begin{cases} -\sigma(g)(\theta), & a = 1, \text{ else} \\ g(\theta^a)(\theta^a - \theta)^{2n} - \sigma(g)(\theta^a). & \end{cases}$$

(ii) $i > n$:

$$A_{-n,n} = \begin{cases} 0, & a = 1, \\ \sigma(\alpha)(\theta^q - \theta)^{i-n}, & a = q, \text{ else} \\ \sigma(\alpha)(\theta^a - \theta)^{i-n} - \alpha(\theta^a - \theta^q)^i(\theta^a - \theta)^n, & \end{cases}$$

$$A_{-n,0} = \begin{cases} -\sigma(g)(\theta)(\theta-\theta^q)^i, & a=1, \\ \sigma(\alpha)g(\theta^q)(\theta^q-\theta)^{i-n}, & a=\ q, \text{ else} \\ \sigma(\alpha)g(\theta^a)(\theta^a-\theta)^{i-n} - \sigma(g)(\theta^a)(\theta^a-\theta^q)^i, \end{cases}$$

$$A_{n,0} = \begin{cases} -\sigma(g)(\theta), & a=1, \text{ else} \\ g(\theta^a)(\theta^a-\theta)^{2n} - \sigma(g)(\theta^a). \end{cases}$$

We skip the case $i < n$. In this case we must assume that $g = \beta(t-\theta)^n$.

$j > 0$: Here we have $h = \Omega^j$, so we get

$$\begin{aligned} A_{-n,n} &= \Omega^j\big(\sigma(\alpha)(t-\theta)^{i-n} - \alpha(t-\theta^q)^i(t-\theta)^{j+n}\big), \\ A_{-n,0} &= \sigma(\alpha)g(t-\theta)^{i-n} - \sigma(g)(t-\theta^q)^i, \\ A_{n,0} &= \Omega^j\big(g(t-\theta)^{n+j} - \sigma(g)\big). \end{aligned}$$

When $i = n$ we get

$$A_{-n,n} = \begin{cases} (-\tilde{\pi})^{-j}\sigma(\alpha), & a=1, \\ 0, & a=q, \text{ else} \\ \Omega(\theta^a)^j\big(\sigma(\alpha) - \alpha(\theta^a-\theta^q)^n(\theta^a-\theta)^{j+n}\big), \end{cases}$$

$$A_{-n,0} = \begin{cases} \sigma(\alpha)g(\theta^q), & a=q, \text{ else} \\ \sigma(\alpha)g(\theta^a) - \sigma(g)(\theta^a)(\theta^a-\theta^q)^n, \end{cases}$$

$$A_{n,0} = \begin{cases} -(-\tilde{\pi})^{-j}\sigma(g)(\theta), & a=1, \\ 0, & a=q, \text{ else} \\ \Omega(\theta^a)\big(g(\theta^a)(\theta^a-\theta)^{n+j} - \sigma(g)(\theta^a)\big). \end{cases}$$

We leave it to the reader to write out the other cases $i > n$ and $i < n$.

$j < 0$: This case is actually vacuous since $A_{-n,0}$ is undefined.

- We skip case 3 above and jump to case 4. Remember that g is now h. We assume from the start that $i = n$ and $s = 1$. Hence, $f = \alpha(t-\theta^q)^n$ and $h = \Omega^l L_{\beta,l}$. We find that

$$\begin{aligned} A_{-n,n} &= \Omega^l\big(\sigma(\alpha)L_{\beta,l} - \alpha\sigma(\beta)(t-\theta^q)^n(t-\theta)^{n+l} - \alpha(t-\theta^q)^n L_{\beta,l}(t-\theta)^n, \\ A_{-n,0} &= \sigma(\alpha)g - \alpha\sigma(g)(t-\theta^q)^n, \quad \text{and} \\ A_{n,0} &= \Omega^l\big(\sigma(\beta)(t-\theta)^{n+l}g + L_{\beta,l}g(t-\theta)^n - \sigma(g)L_{\beta,l}\big). \end{aligned}$$

From this we see that

$$A_{-n,n} = \begin{cases} (-\tilde{\pi})^{-l}\sigma(\alpha)L_{\beta,l}(\theta), & a=1, \\ 0, & a=q, \\ \text{and generally by the above formula,} \end{cases}$$

$$A_{-n,0} = \begin{cases} \sigma(\alpha)g(\theta) - \alpha\sigma(g)(\theta)(\theta - \theta^q)^n, & a = 1, \\ \sigma(\alpha)g(\theta^q), & a = q, \\ \text{and generally by the above formula,} \end{cases}$$

$$A_{n,0} = \begin{cases} -(-\tilde{\pi})^{-l}\sigma(g)(\theta)L_{\beta,l}(\theta), & a = 1, \\ 0, & a = q, \\ \text{and generally by the above formula.} \end{cases}$$

Notice the special choice $g = \mathrm{id}$ in the examples above.

The above results indicate that it could possibly be a worthwhile endeavour to continue the study of hom-Lie algebras within the theory of global function fields.

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