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A Diophantine approximation problem with two primes and one k-th power of a prime

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Abstract

We refine a result of the last two Authors of [8] on a Diophantine approximation problem with two primes and a k-th power of a prime which was only proved to hold for $1 < k < 4/3$. We improve the k-range to $1 < k \leq 3$ by combining Harman's technique on the minor arc with a suitable estimate for the L^4-norm of the relevant exponential sum over primes.

Keywords: Diophantine inequalities, Goldbach-type problems, Hardy-Littlewood method
2000 MSC: primary 11D75,
2000 MSC: secondary 11J25, 11P32, 11P55

1. Introduction

This paper deals with an improvement of the result contained in [8], which is due to the last two Authors: we refer to its introduction for a more thorough description of the general Diophantine problem with prime variables. Here we just recall that the goal is to prove that the inequality

$$|\lambda_1 p_1^{k_1} + \cdots + \lambda_r p_r^{k_r} - \omega| \leq \eta,$$

where $k_1, \ldots, k_r$ are fixed positive numbers, $\lambda_1, \ldots, \lambda_r$ are fixed non-zero real numbers and $\eta > 0$ is arbitrary, has infinitely many solutions in prime variables $p_1, \ldots, p_r$ for any given real number ω, under as mild Diophantine assumptions on $\lambda_1, \ldots, \lambda_r$ as possible. In some cases, it is even possible to prove that the above inequality holds when η is a small negative power of the largest prime occurring in it, usually when $1/k_1 + \cdots + 1/k_r$ is large enough.

The problem tackled in [8] had $r = 3$, $k_1 = k_2 = 1$, $k_3 = k \in (1, 4/3)$. Assuming that λ_1/λ_2 is irrational and that the coefficients λ_j are not all of the same sign, the last two Authors proved

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that one can take $\eta = \left(\max\{p_1, p_2, p_3^k\}\right)^{-\phi(k)+\varepsilon}$ for any fixed $\varepsilon > 0$, where $\phi(k) = (4-3k)/(10k)$. Our purpose in this paper is to improve on this result both in the admissible range for k and in the exponent, replacing $\phi(k)$ by a larger value in the common range. More precisely, we prove the following Theorem.

Theorem 1. *Assume that $1 < k \le 3$, λ_1, λ_2 and λ_3 are non-zero real numbers, not all of the same sign, that λ_1/λ_2 is irrational and let ω be a real number. The inequality*

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3^k - \omega| \le \left(\max\{p_1, p_2, p_3^k\}\right)^{-\psi(k)+\varepsilon} \tag{1}$$

has infinitely many solutions in prime variables p_1, p_2, p_3 for any $\varepsilon > 0$, where

$$\psi(k) = \begin{cases} (3-2k)/(6k) & \text{if } 1 < k \le \frac{6}{5}, \\ 1/12 & \text{if } \frac{6}{5} < k \le 2, \\ (3-k)/(6k) & \text{if } 2 < k < 3, \\ 1/24 & \text{if } k = 3. \end{cases} \tag{2}$$

We point out that in the common range $1 < k < 4/3$ we have $\psi(k) > \phi(k)$. We also remark that the strong bounds for the exponential sum S_k, defined in (3) below, that recently became available for integral k (see Bourgain [1] and Bourgain, Demeter & Guth [2]) are not useful in our problem.

The technique used to tackle this problem is the variant of the circle method introduced in the 1940's by Davenport & Heilbronn [4], where the integration on a circle, or equivalently on the interval $[0, 1]$, is replaced by integration on the whole real line. Our improvement is due to the use of the Harman technique on the minor arc and to the fourth-power average for the exponential sum S_k for $k \ge 1$.

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2. Outline of the proof

Throughout this paper p_i denotes a prime number, $k \ge 1$ is a real number, ε is an arbitrarily small positive number whose value may vary depending on the occurrences and ω is a fixed real number. In order to prove that (1) has infinitely many solutions, it is sufficient to construct an increasing sequence X_n that tends to infinity such that (1) has at least one solution with $\max\{p_1, p_2, p_3^k\} \in [\delta X_n, X_n]$, with a fixed $\delta > 0$ which depends only on the choice of λ_1, λ_2 and λ_3. Let q be a denominator of a convergent to λ_1/λ_2 and let $X_n = X$ (dropping the suffix n) run through the sequence $X = q^3$. The main quantities we will use are:

$$S_k(\alpha) = \sum_{\delta X \le p^k \le X} \log p \, e(p^k \alpha), \quad U_k(\alpha) = \sum_{\delta X \le n^k \le X} e(n^k \alpha) \quad \text{and} \quad T_k(\alpha) = \int_{(\delta X)^{1/k}}^{X^{1/k}} e(t^k \alpha) \, \mathrm{d}t, \tag{3}$$

where $e(\alpha) = e^{2\pi i\alpha}$. We will approximate S_k with T_k and U_k. By the Prime Number Theorem and first derivative estimates for trigonometric integrals we have

$$S_k(\alpha) \ll_{k,\delta} X^{1/k}, \qquad T_k(\alpha) \ll_{k,\delta} X^{1/k-1} \min\{X, |\alpha|^{-1}\}. \tag{4}$$

Moreover the Euler summation formula implies that

$$T_k(\alpha) - U_k(\alpha) \ll_{k,\delta} 1 + |\alpha|X. \tag{5}$$

We also need a continuous function we will use to detect the solutions of (1), so we introduce

$$\widehat{K}_\eta(\alpha) := \max\{0, \eta - |\alpha|\}, \quad \text{where } \eta > 0,$$

which is the Fourier transform of the function K_η defined by

$$K_\eta(\alpha) = \left(\frac{\sin(\pi\alpha\eta)}{\pi\alpha}\right)^2$$

for $\alpha \neq 0$ and, by continuity, $K_\eta(0) = \eta^2$. A well-known estimate is

$$K_\eta(\alpha) \ll \min\{\eta^2, |\alpha|^{-2}\}. \tag{6}$$

Let now

$$\mathcal{P}(X) = \{(p_1, p_2, p_3) : \delta X < p_1, p_2 \leq X,\ \delta X < p_3^k \leq X\}$$

and

$$\mathcal{I}(\eta, \omega, \mathfrak{X}) = \int_{\mathfrak{X}} S_1(\lambda_1\alpha) S_1(\lambda_2\alpha) S_k(\lambda_3\alpha) K_\eta(\alpha) e(-\omega\alpha)\, \mathrm{d}\alpha,$$

where $\mathfrak{X}$ is a measurable subset of $\mathbb{R}$. From (3) and using the Fourier transform of $K_\eta(\alpha)$, we get

$$\begin{aligned} \mathcal{I}(\eta, \omega, \mathbb{R}) &= \sum_{(p_1,p_2,p_3)\in\mathcal{P}(X)} \log p_1 \log p_2 \log p_3\ \max\left\{0, \eta - |\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3^k - \omega|\right\} \\ &\leq \eta(\log X)^3 \mathcal{N}(X), \end{aligned}$$

where $\mathcal{N}(X)$ actually denotes the number of solutions of the inequality (1) with $(p_1, p_2, p_3) \in \mathcal{P}(X)$. In other words $\mathcal{I}(\eta, \omega, \mathbb{R})$ provides a lower bound for the quantity we are interested in; therefore it is sufficient to prove that $\mathcal{I}(\eta, \omega, \mathbb{R}) > 0$.

We now decompose $\mathbb{R}$ into subsets such that $\mathbb{R} = \mathfrak{M} \cup \mathfrak{M}^* \cup \mathfrak{m} \cup \mathfrak{t}$ where $\mathfrak{M}$ is the major arc, $\mathfrak{M}^*$ is the intermediate arc (which is non-empty only for some values of k, see section 6), $\mathfrak{m}$ is the minor arc and $\mathfrak{t}$ is the trivial arc. The decomposition is the following: if $1 < k < 5/2$ we consider

$$\begin{aligned} \mathfrak{M} &= [-P/X, P/X], & \mathfrak{M}^* &= \emptyset, \\ \mathfrak{m} &= [P/X, R] \cup [-R, -P/X], & \mathfrak{t} &= \mathbb{R} \setminus (\mathfrak{M} \cup \mathfrak{M}^* \cup \mathfrak{m}), \end{aligned} \tag{7}$$

while, for $5/2 \le k \le 3$, we set

$$\mathfrak{M} = [-P/X, P/X], \qquad \mathfrak{M}^* = [P/X, X^{-3/5}] \cup [-X^{-3/5}, -P/X],$$
$$\mathfrak{m} = [X^{-3/5}, R] \cup [-R, -X^{-3/5}], \qquad \mathfrak{t} = \mathbb{R} \setminus (\mathfrak{M} \cup \mathfrak{M}^* \cup \mathfrak{m}), \tag{8}$$

where the parameters $P = P(X) > 1$ and $R = R(X) > 1/\eta$ are chosen later (see (15) and (16)) as well as $\eta = \eta(X)$, that, as we explained before, we would like to be a small negative power of $\max\{p_1, p_2, p_3^k\}$ (and so of X). We have to distinguish two cases in the previous decomposition of the real line in order to avoid a gap between the end of the major arc and the beginning of the minor arc, where we can prove Lemma 12 in the form that we need: see the comments at the beginning of section 6 and just before the statement of Lemma 12. As we will see later in section 6, we need to introduce intermediate arc only for $k \ge 5/2$.

The constraints on η are in (18), (20) and (21), according to the value of k. In any case, we have $\mathcal{I}(\eta, \omega, \mathbb{R}) = \mathcal{I}(\eta, \omega, \mathfrak{M}) + \mathcal{I}(\eta, \omega, \mathfrak{M}^*) + \mathcal{I}(\eta, \omega, \mathfrak{m}) + \mathcal{I}(\eta, \omega, \mathfrak{t})$. We expect that $\mathfrak{M}$ provides the main term with the right order of magnitude without any special hypothesis on the coefficients λ_j. It is necessary to prove that $\mathcal{I}(\eta, \omega, \mathfrak{M}^*)$, $\mathcal{I}(\eta, \omega, \mathfrak{m})$ and $\mathcal{I}(\eta, \omega, t)$ are $o\big(\mathcal{I}(\eta, \omega, \mathfrak{M})\big)$ as $X \to +\infty$ on the particular sequence chosen: we show that the contribution from trivial arc is "tiny" with respect to the main term. The main difficulty is to estimate the minor arc contribution; in this case we will need the full force of the hypothesis on the coefficients λ_j and the theory of continued fractions.

Remark: from now on, anytime we use the symbol $\ll$ or $\gg$ we drop the dependence of the approximation from the constants λ_j, δ and k. We use the notation $f = \infty(g)$ for $g = o(f)$.

3. Lemmas

In their original paper [4] Davenport and Heilbronn approximate directly the difference $|S_k(\alpha) - T_k(\alpha)|$ estimating it with $\mathcal{O}(1)$. The L^2-norm estimation approach (see Brüdern, Cook & Perelli [3] and [8]) improves on this taking the L^2-norm of $|S_k(\alpha) - T_k(\alpha)|$: this leads to the possibility of having a wider major arc compared to the original approach. We introduce the generalized version of the Selberg integral

$$\mathcal{J}_k(X, h) = \int_X^{2X} \big(\theta((x+h)^{1/k}) - \theta(x^{1/k}) - ((x+h)^{1/k} - x^{1/k})\big)^2 \, \mathrm{d}x,$$

where $\theta(x) = \sum_{p \le x} \log p$ is the usual Chebyshev function. We have the following lemmas.

Lemma 1 ([7], Theorem 3.1). *Let $k \ge 1$ be a real number. For $0 < Y \le 1/2$ we have*

$$\int_{-Y}^{Y} |S_k(\alpha) - U_k(\alpha)|^2 \, \mathrm{d}\alpha \ll \frac{X^{2/k-2} \log^2 X}{Y} + Y^2 X + Y^2 \mathcal{J}_k\Big(X, \frac{1}{2Y}\Big).$$

Lemma 2 ([7], Theorem 3.2). *Let $k \geq 1$ be a real number and ε be an arbitrarily small positive constant. There exists a positive constant $c_1(\varepsilon)$, which does not depend on k, such that*

$$\mathcal{J}_k(X, h) \ll h^2 X^{2/k-1} \exp\Big(-c_1\Big(\frac{\log X}{\log\log X}\Big)^{1/3}\Big)$$

uniformly for $X^{1-5/(6k)+\varepsilon} \leq h \leq X$.

In order to prove our crucial Lemma 4 on the L^4-norm of $S_k(\alpha)$, we need the following technical result.

Lemma 3. *Let $\varepsilon > 0$, $k > 1$ and $\gamma > 0$. Let further $\mathcal{B}(X^{1/k}; k; \gamma)$ denote the number of solutions of the inequalities*

$$|n_1^k + n_2^k - n_3^k - n_4^k| < \gamma, \qquad X^{1/k} < n_1, n_2, n_3, n_4 \leq 2X^{1/k}.$$

Then

$$\mathcal{B}(X^{1/k}; k; \gamma) \ll \big(X^{2/k} + \gamma X^{4/k-1}\big) X^{\varepsilon}.$$

Proof. This is an immediate consequence of Theorem 2 of Robert & Sargos [9]; we just need to choose $M = X^{1/k}$, $\alpha = k$ and $\gamma = \delta M^k$ there. □

Lemma 4. *Let $\varepsilon > 0$, $\delta > 0$, $k > 1$, $n \in \mathbb{N}$ and $\tau > 0$. Then we have*

$$\int_{-\tau}^{\tau} |S_k(\alpha)|^4 \,\mathrm{d}\alpha \ll \big(\tau X^{2/k} + X^{4/k-1}\big) X^{\varepsilon} \quad \textit{and} \quad \int_{n}^{n+1} |S_k(\alpha)|^4 \,\mathrm{d}\alpha \ll \big(X^{2/k} + X^{4/k-1}\big) X^{\varepsilon}.$$

Proof. A direct computation gives

$$\begin{aligned}
\int_{-\tau}^{\tau} |S_k(\alpha)|^4 \,\mathrm{d}\alpha &= \sum_{\delta X < p_1^k, p_2^k, p_3^k, p_4^k \leq X} (\log p_1) \cdots (\log p_4) \int_{-\tau}^{\tau} e((p_1^k + p_2^k - p_3^k - p_4^k)\alpha) \,\mathrm{d}\alpha \\
&\ll (\log X)^4 \sum_{\delta X < p_1^k, p_2^k, p_3^k, p_4^k \leq X} \min\Big\{\tau, \frac{1}{|p_1^k + p_2^k - p_3^k - p_4^k|}\Big\} \\
&\ll (\log X)^4 \sum_{\delta X < n_1^k, n_2^k, n_3^k, n_4^k \leq X} \min\Big\{\tau, \frac{1}{|n_1^k + n_2^k - n_3^k - n_4^k|}\Big\} \\
&\ll U\tau(\log X)^4 + V(\log X)^4, \qquad (9)
\end{aligned}$$

where

$$U := \sum_{\substack{\delta X < n_1^k, n_2^k, n_3^k, n_4^k \leq X \\ |n_1^k + n_2^k - n_3^k - n_4^k| \leq 1/\tau}} 1, \quad \text{and} \quad V := \sum_{\substack{\delta X < n_1^k, n_2^k, n_3^k, n_4^k \leq X \\ |n_1^k + n_2^k - n_3^k - n_4^k| > 1/\tau}} \frac{1}{|n_1^k + n_2^k - n_3^k - n_4^k|},$$

say. Using Lemma 3 on U we get

$$U \ll \mathscr{B}(X^{1/k}; k; 1/\tau) \ll \Big(X^{2/k} + \frac{1}{\tau} X^{4/k-1}\Big) X^{\varepsilon}. \tag{10}$$

Concerning V, by a dyadic argument we get

$$\begin{aligned} V &\ll \log X \Big(\max_{1/\tau < W \ll X} \sum_{\substack{\delta X < n_1^k, n_2^k, n_3^k, n_4^k \le X \\ W < |n_1^k + n_2^k - n_3^k - n_4^k| \le 2W}} \frac{1}{|n_1^k + n_2^k - n_3^k - n_4^k|} \Big) \\ &\ll \log X \max_{1/\tau < W \ll X} \Big(\frac{1}{W} \mathscr{B}(X^{1/k}; k; 2W) \Big) \ll \max_{1/\tau < W \ll X} \Big(X^{4/k-1} + \frac{X^{2/k}}{W} \Big) X^{\varepsilon} \\ &\ll (\tau X^{2/k} + X^{4/k-1}) X^{\varepsilon}. \end{aligned} \tag{11}$$

Combining (9)-(11), the first part of the lemma follows. The second part can be proved in a similar way. □

We need the following result in the proof of Lemma 9 and also when dealing with $\mathfrak{M}^*$; see section 6.

Lemma 5. *Let $\delta > 0$, $k > 1$, $n \in \mathbb{N}$ and $\tau > 0$. Then*

$$\int_{-\tau}^{\tau} |S_k(\alpha)|^2 \, \mathrm{d}\alpha \ll \big(\tau X^{1/k} + X^{2/k-1}\big)(\log X)^3 \quad \text{and} \quad \int_{n}^{n+1} |S_k(\alpha)|^2 \, \mathrm{d}\alpha \ll X^{1/k} (\log X)^3.$$

Proof. It follows directly from the proof of Lemma 7 of Tolev [10] by letting $c = k$ and using $X^{1/k}$ instead of X there. We explicitly remark that the condition $c \in (1, 15/14)$ in Tolev's original version of this lemma depends on other parts of his paper; in fact the proof of Lemma 7 of [10] holds for every $c > 1$. □

We now state some other lemmas which will be mainly useful on the minor and trivial arcs.

Lemma 6 (Vaughan [11], Theorem 3.1). *Let α be a real number and a, q be positive integers satisfying $(a, q) = 1$ and $|\alpha - a/q| < 1/q^2$. Then*

$$S_1(\alpha) \ll \Big(\frac{X}{\sqrt{q}} + \sqrt{Xq} + X^{4/5} \Big) (\log X)^4.$$

Lemma 7. *Let $X^{-1} \ll |\alpha| \ll X^{-3/5}$. Then $S_1(\alpha) \ll X^{1/2} |\alpha|^{-1/2} (\log X)^4$.*

Proof. It follows immediately from Lemma 6 by choosing $q = \lfloor 1/\alpha \rfloor$ and $a = 1$. □

Lemma 8. *Let* $\lambda \in \mathbb{R} \setminus \{0\}$, $X \geq Z \geq X^{4/5}(\log X)^5$ *and* $|S_1(\lambda\alpha)| > Z$. *Then there are coprime integers* $(a, q) = 1$ *satisfying*

$$1 \leq q \ll \left(\frac{X(\log X)^4}{Z}\right)^2, \qquad |q\lambda\alpha - a| \ll \frac{X(\log X)^{10}}{Z^2}.$$

Proof. Let Q be a parameter that we will choose later. By Dirichlet's theorem there exist coprime integers $(a, q) = 1$ such that $1 \leq q \leq Q$ and $|q\lambda\alpha - a| \ll Q^{-1} \leq q^{-1}$. The choice

$$Q = \frac{Z^2}{X(\log X)^{10}}$$

allows us to prove the second part of the statement and to neglect some terms in the estimations of $|S_1(\lambda\alpha)|$. Using Lemma 6, knowing that $Z \geq X^{4/5}(\log X)^5$ and $|S_1(\lambda\alpha)| > Z$, we can rewrite the bound for $|S_1(\lambda\alpha)|$ neglecting the term $X^{4/5}$:

$$Z < |S_1(\lambda\alpha)| \ll (Xq^{-1/2} + X^{1/2}q^{1/2})(\log X)^4.$$

The condition $q \leq Q$ allows us to neglect the term $X^{1/2}q^{1/2}$ and deal with small values of q; in fact, if $q > X^{1/2}$ then we would have a contradiction

$$Z < |S_1(\lambda\alpha)| \ll X^{1/2}q^{1/2}(\log X)^4 \leq X^{1/2}\frac{Z}{X^{1/2}(\log X)^5}(\log X)^4 = o(Z).$$

Then $q \leq \min\{X^{1/2}, Q\} = X^{1/2}$, since $Z = X^{4/5}(\log X)^5 > X^{3/4}(\log X)^5$. Moreover, we can rewrite the inequality on $|S_1(\lambda\alpha)|$ as

$$Z < |S_1(\lambda\alpha)| \ll Xq^{-1/2}(\log X)^4$$

and finally we get $q^{1/2}Z \ll X(\log X)^4$, which completes the proof. □

The optimizations in section 7 depend either on L^2 or on L^4 averages of S_k, according to the value of k; these are provided by the following Lemmas. For brevity, we skip the proof of the first one, remarking that it requires Lemma 5.

Lemma 9 (Lemma 5 of [8]). *Let* $\lambda \in \mathbb{R} \setminus \{0\}$, $k > 1$, $0 < \eta < 1$, $R > 1/\eta$ *and* $1 < P < X$. *We have*

$$\int_{P/X}^{R} |S_1(\lambda\alpha)|^2 K_\eta(\alpha)\,\mathrm{d}\alpha \ll \eta X \log X \quad \text{and} \quad \int_{P/X}^{R} |S_k(\lambda\alpha)|^2 K_\eta(\alpha)\,\mathrm{d}\alpha \ll \eta X^{1/k}(\log X)^3.$$

Lemma 10. *Let* $\lambda \in \mathbb{R} \setminus \{0\}$, $\varepsilon > 0$, $k > 1$, $0 < \eta < 1$, $R > 1/\eta$ *and* $1 < P < X$. *Then*

$$\int_{P/X}^{R} |S_k(\lambda\alpha)|^4 K_\eta(\alpha)\,\mathrm{d}\alpha \ll \eta \max\{X^{2/k}, X^{4/k-1}\}X^\varepsilon.$$

Proof. Using (6) we obtain

$$\int_{P/X}^{R} |S_k(\lambda\alpha)|^4 K_\eta(\alpha)\,\mathrm{d}\alpha \ll \eta^2 \int_{P/X}^{1/\eta} |S_k(\lambda\alpha)|^4\,\mathrm{d}\alpha + \int_{1/\eta}^{R} |S_k(\lambda\alpha)|^4 \frac{\mathrm{d}\alpha}{\alpha^2} = A + B, \tag{12}$$

say. By Lemma 4, we immediately get

$$A \ll \eta^2 \int_{-|\lambda|/\eta}^{|\lambda|/\eta} |S_k(\alpha)|^4\,\mathrm{d}\alpha \ll \eta \max\{X^{2/k}, \eta X^{4/k-1}\} X^{\varepsilon}. \tag{13}$$

Moreover, again by Lemma 4, we have that

$$\begin{aligned} B &\ll \int_{|\lambda|/\eta}^{+\infty} |S_k(\alpha)|^4 \frac{\mathrm{d}\alpha}{\alpha^2} \ll \sum_{n\geq|\lambda|/\eta} \frac{1}{(n-1)^2} \int_{n-1}^{n} |S_k(\alpha)|^4\,\mathrm{d}\alpha \\ &\ll \eta \max\{X^{2/k}, X^{4/k-1}\} X^{\varepsilon}. \end{aligned} \tag{14}$$

Combining (12)-(14) and using $0 < \eta < 1$, the lemma follows. □

As we remarked in the introduction, stronger bounds are now available for larger integral k, but they are not useful for our purpose. The next Lemma provides the additional information that enables us to give a non-trivial result also when $k = 3$.

Lemma 11. *Let $\lambda \in \mathbb{R} \setminus \{0\}$, $\varepsilon > 0$, $0 < \eta < 1$, $R > 1/\eta$ and $1 < P < X$. Then*

$$\int_{P/X}^{R} |S_3(\lambda\alpha)|^8 K_\eta(\alpha)\,\mathrm{d}\alpha \ll \eta X^{5/3+\varepsilon}.$$

Proof. Inserting Hua's estimate in [6], i.e. $\int_0^1 |S_3(\alpha)|^8\,\mathrm{d}\alpha \ll X^{5/3+\varepsilon}$, in the body of the proof of Lemma 10 and exploiting the periodicity of $S_3(\alpha)$, the result follows immediately. □

Another lemma on the minor arc is inserted in the body of section 7.

4. The major arc

We recall the definitions in (7) and (8). The major arc computation is the same as in [8]:

$$\begin{aligned} \mathcal{J}(\eta,\omega,\mathfrak{M}) &= \int_{\mathfrak{M}} S_1(\lambda_1\alpha)S_1(\lambda_2\alpha)S_k(\lambda_3\alpha)K_\eta(\alpha)e(-\omega\alpha)\,\mathrm{d}\alpha \\ &= \int_{\mathfrak{M}} T_1(\lambda_1\alpha)T_1(\lambda_2\alpha)T_k(\lambda_3\alpha)K_\eta(\alpha)e(-\omega\alpha)\,\mathrm{d}\alpha \\ &\quad + \int_{\mathfrak{M}} (S_1(\lambda_1\alpha) - T_1(\lambda_1\alpha))T_1(\lambda_2\alpha)T_k(\lambda_3\alpha)K_\eta(\alpha)e(-\omega\alpha)\,\mathrm{d}\alpha \end{aligned}$$

$$
\begin{aligned}
&+ \int_{\mathfrak{M}} S_1(\lambda_1\alpha)(S_1(\lambda_2\alpha) - T_1(\lambda_2\alpha))T_k(\lambda_3\alpha)K_\eta(\alpha)e(-\omega\alpha)\,\mathrm{d}\alpha \\
&+ \int_{\mathfrak{M}} S_1(\lambda_1\alpha)S_1(\lambda_2\alpha)(S_k(\lambda_3\alpha) - T_k(\lambda_3\alpha))K_\eta(\alpha)e(-\omega\alpha)\,\mathrm{d}\alpha \\
= {}& J_1 + J_2 + J_3 + J_4,
\end{aligned}
$$

say.

4.1. Main term: lower bound for J_1

As the reader might expect the main term is given by the summand J_1.

Let $H(\alpha) = T_1(\lambda_1\alpha)T_1(\lambda_2\alpha)T_k(\lambda_3\alpha)K_\eta(\alpha)e(-\omega\alpha)$ so that

$$
J_1 = \int_{\mathbb{R}} H(\alpha)\,\mathrm{d}\alpha + \mathcal{O}\Big(\int_{P/X}^{+\infty} |H(\alpha)|\,\mathrm{d}\alpha\Big).
$$

Using (6) and (4), we get

$$
\int_{P/X}^{+\infty} |H(\alpha)|\,\mathrm{d}\alpha \ll \eta^2 X^{1/k-1} \int_{P/X}^{+\infty} \frac{\mathrm{d}\alpha}{\alpha^3} \ll \eta^2 \frac{X^{1+1/k}}{P^2} = o\big(\eta^2 X^{1+1/k}\big),
$$

provided that $P \longrightarrow +\infty$. Let now $D = [\delta X, X]^2 \times [(\delta X)^{1/k}, X^{1/k}]$. We obtain

$$
\begin{aligned}
\int_{\mathbb{R}} H(\alpha)\,\mathrm{d}\alpha &= \iiint_D \int_{\mathbb{R}} e((\lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_3^k - \omega)\alpha)K_\eta(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3 \\
&= \iiint_D \max\{0, \eta - |\lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_3^k - \omega)|\}\,\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3.
\end{aligned}
$$

Apart from trivial changes of sign, there are essentially two cases:

1. $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 < 0$
2. $\lambda_1 > 0, \lambda_2 < 0, \lambda_3 < 0$.

We deal with the first one. We warn the reader that here it may be necessary to adjust the value of δ in order to guarantee the necessary set inclusions. After a suitable change of variables, letting $D' = [\delta X, (1-\delta)X]^3$, we find that

$$
\begin{aligned}
\int_{\mathbb{R}} H(\alpha)\,\mathrm{d}\alpha &\gg \iiint_{D'} \max\{0, \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3)|\}\, u_3^{1/k-1}\,\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3 \\
&\gg X^{1/k-1} \iiint_{D'} \max\{0, \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3)|\}\,\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3.
\end{aligned}
$$

Apart from sign, the computation is essentially symmetrical with respect to the coefficients λ_j: we assume, as we may, that $|\lambda_3| \ge \max\{\lambda_1, \lambda_2\}$, the other cases being similar. Now, for $j = 1, 2$ let $a_j = \dfrac{2\delta|\lambda_3|}{|\lambda_j|}$, $b_j = \dfrac{3}{2}a_j$ and $\mathscr{D}_j = [a_j X, b_j X]$; if $u_j \in \mathscr{D}_j$ for $j = 1, 2$ then

$$\lambda_1 u_1 + \lambda_2 u_2 \in [4|\lambda_3|\delta X, 6|\lambda_3|\delta X]$$

so that, for every choice of (u_1, u_2) the interval $[a, b]$ with endpoints $\pm\eta/|\lambda_3| + (\lambda_1 u_1 + \lambda_2 u_2)/|\lambda_3|$ is contained in $[\delta X, (1-\delta)X]$. In other words, for $u_3 \in [a, b]$ the values of $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$ cover the whole interval $[-\eta, \eta]$. Hence for any $(u_1, u_2) \in \mathscr{D}_1 \times \mathscr{D}_2$ we have

$$\int_{\delta X}^{(1-\delta)X} \max\{0, \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3|\}\, \mathrm{d}u_3 = |\lambda_3|^{-1} \int_{-\eta}^{\eta} \max\{0, \eta - |u|\}\, \mathrm{d}u \gg \eta^2.$$

Summing up, we get

$$J_1 \gg \eta^2 X^{1/k-1} \iint_{\mathscr{D}_1 \times \mathscr{D}_2} \mathrm{d}u_1 \mathrm{d}u_2 \gg \eta^2 X^{1/k-1} X^2 = \eta^2 X^{1+1/k},$$

which is the expected lower bound.

4.2. Bound for J_2, J_3 and J_4

The computations for J_2 and J_3 are similar to and simpler than the corresponding one for J_4; moreover the most restrictive condition on P arises from J_4; hence we will skip the computation for both J_2 and J_3. Using the triangle inequality and (6),

$$\begin{aligned}
J_4 &\ll \eta^2 \int_{\mathfrak{M}} |S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)||S_k(\lambda_3\alpha) - T_k(\lambda_3\alpha)|\, \mathrm{d}\alpha \\
&\le \eta^2 \int_{\mathfrak{M}} |S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)||S_k(\lambda_3\alpha) - U_k(\lambda_3\alpha)|\, \mathrm{d}\alpha \\
&\qquad + \eta^2 \int_{\mathfrak{M}} |S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)||U_k(\lambda_3\alpha) - T_k(\lambda_3\alpha)|\, \mathrm{d}\alpha \\
&= \eta^2(A_4 + B_4),
\end{aligned}$$

say, where $U_k(\lambda_3\alpha)$ and $T_k(\lambda_3\alpha)$ are defined in (3). Using the Cauchy-Schwarz inequality, Lemmas 1-2 and trivial bounds yields, for any fixed $A > 0$,

$$\begin{aligned}
A_4 &\ll X\Big(\int_{\mathfrak{M}} |S_1(\lambda_1\alpha)|^2\, \mathrm{d}\alpha\Big)^{1/2} \Big(\int_{\mathfrak{M}} |S_k(\lambda_3\alpha) - U_k(\lambda_3\alpha)|^2\, \mathrm{d}\alpha\Big)^{1/2} \\
&\ll X^{1+1/k}(\log X)^{(1-A)/2} = o\big(X^{1+1/k}\big)
\end{aligned}$$

as long as $A > 1$, provided that $P \leq X^{5/(6k)-\varepsilon}$. Using again the Cauchy-Schwarz inequality, (5) and trivial bounds, we see that

$$\begin{aligned} B_4 &\ll \int_0^{1/X} |S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)| \, d\alpha + X \int_{1/X}^{P/X} \alpha |S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)| \, d\alpha \\ &\ll X + P\Big(\int_{1/X}^{P/X} |S_1(\lambda_1\alpha)|^2 \, d\alpha \int_{1/X}^{P/X} |S_1(\lambda_2\alpha)|^2 \, d\alpha \Big)^{1/2} \ll PX \log X. \end{aligned}$$

Taking $P = o\big(X^{1/k}(\log X)^{-1}\big)$ we get $\eta^2 B_4 = o\big(\eta^2 X^{1+1/k}\big)$. We may therefore choose

$$P = X^{5/(6k)-\varepsilon}. \tag{15}$$

5. The trivial arc

We recall that the trivial arc is defined in (7) and (8). Using the Cauchy-Schwarz inequality and (4), we see that

$$\begin{aligned} |\mathcal{I}(\eta,\omega,\mathfrak{t})| &\ll \int_R^{+\infty} |S_1(\lambda_1\alpha)S_1(\lambda_2\alpha)S_k(\lambda_3\alpha)| K_\eta(\alpha) \, d\alpha \\ &\ll X^{1/k} \Big(\int_R^{+\infty} |S_1(\lambda_1\alpha)|^2 K_\eta(\alpha) \, d\alpha \Big)^{1/2} \Big(\int_R^{+\infty} |S_1(\lambda_2\alpha)|^2 K_\eta(\alpha) \, d\alpha \Big)^{1/2} \\ &\ll X^{1/k} C_1^{1/2} C_2^{1/2}, \end{aligned}$$

say. Using the PNT and the periodicity of $S_1(\alpha)$, for every $j = 1, 2$ we have that

$$C_j = \int_R^{+\infty} |S_1(\lambda_j\alpha)|^2 \frac{d\alpha}{\alpha^2} \ll \int_{|\lambda_j|R}^{+\infty} |S_1(\alpha)|^2 \frac{d\alpha}{\alpha^2} \ll \sum_{n \geq |\lambda_j|R} \frac{1}{(n-1)^2} \int_{n-1}^{n} |S_1(\alpha)|^2 \, d\alpha \ll \frac{X \log X}{|\lambda_j| R}.$$

Hence, recalling that $|\mathcal{I}(\eta,\omega,\mathfrak{t})|$ has to be $o(\eta^2 X^{1+1/k})$, the choice

$$R = \eta^{-2}(\log X)^{3/2} \tag{16}$$

is admissible.

6. The intermediate arc: $5/2 \leq k \leq 3$

In section 7 we apply Harman's technique to the minor arc, using Lemma 8 as the starting point. We remark that in the course of the proof of Lemma 12 it is crucial that both the integers a_1 and a_2 appearing in (22) below do not vanish; in fact, if $a_1 = 0$, say, then α is very small ($\alpha \ll X^{-2/3}$) and, according to our definitions above, it belongs to $\mathfrak{M} \cup \mathfrak{M}^*$.

For small k we do not need an intermediate arc, because the major arc is wide enough to rule out the possibility that $a_1a_2 = 0$ for $\alpha \in \mathfrak{m}$. For larger values of k, the constraint (15) implies that there is a gap between the major arc and the minor arc which we need to fill: see the definition in (8). Using the intermediate arc $\mathfrak{M}^*$, we are able to cover more than needed.

Let $5/2 \le k \le 3$: we now show that the contribution of $\mathfrak{M}^*$ is negligible. Using (6), Lemma 7, the Cauchy-Schwarz inequality and (15) we get

$$\begin{aligned}
\mathcal{J}(\eta,\omega,\mathfrak{M}^*) &\ll \eta^2 \int_{P/X}^{X^{-3/5}} |S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)||S_k(\lambda_3\alpha)| \,\mathrm{d}\alpha \\
&\ll \eta^2 X(\log X)^8 \int_{P/X}^{X^{-3/5}} |S_k(\lambda_3\alpha)| \frac{\mathrm{d}\alpha}{\alpha} \\
&\ll \eta^2 X(\log X)^8 \Bigl(\int_{-X^{-3/5}}^{X^{-3/5}} |S_k(\lambda_3\alpha)|^2 \,\mathrm{d}\alpha\Bigr)^{1/2} \Bigl(\int_{P/X}^{X^{-3/5}} \frac{\mathrm{d}\alpha}{\alpha^2}\Bigr)^{1/2} \\
&\ll \eta^2 X (X^{1/k-3/5})^{1/2} (X^{1-5/(6k)})^{1/2} X^{\varepsilon} \ll \eta^2 X^{6/5+1/(12k)+\varepsilon},
\end{aligned}$$

where we also used Lemma 5 with $\tau = X^{-3/5}$ and the fact that $k \ge 5/2$. The last estimate is $o\bigl(\eta^2 X^{1+1/k}\bigr)$ for every $5/2 \le k < 55/12$.

7. The minor arc

Here we use Harman's technique as described in [5]. The minor arc $\mathfrak{m}$ is defined in (7) and (8), according to the value of k. In view of using Lemma 8, we now split $\mathfrak{m}$ into subsets $\mathfrak{m}_1$, $\mathfrak{m}_2$ and $\mathfrak{m}^* = \mathfrak{m} \setminus (\mathfrak{m}_1 \cup \mathfrak{m}_2)$, where

$$\mathfrak{m}_i = \{\alpha \in \mathfrak{m} \colon |S_1(\lambda_i\alpha)| \le X^{5/6}(\log X)^5\} \quad \text{for } i = 1, 2.$$

In order to obtain the optimization, we chose to split the range for k into two intervals in which to take advantage of the L^2-norm of $S_k(\alpha)$ in one case (Lemma 9) and the L^4-norm of $S_k(\alpha)$ in the other one (Lemma 10). The same choice will be made later in the discussion of the arc $\mathfrak{m}^*$. We will see that it is not possible to split the minor arc in another way in order to get a better result, in the present state of knowledge on exponential sums.

7.1. *Bounds on* $\mathfrak{m}_1 \cup \mathfrak{m}_2$

Using Hölder's inequality and Lemma 9, for $1 < k \le 6/5$ we obtain

$$\begin{aligned}
|\mathcal{J}(\eta,\omega,\mathfrak{m}_i)| &\ll \int_{\mathfrak{m}_i} |S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)||S_k(\lambda_3\alpha)| K_\eta(\alpha)\,\mathrm{d}\alpha \\
&\ll \Bigl(\max_{\alpha\in\mathfrak{m}_i} |S_1(\lambda_1\alpha)|\Bigr) \Bigl(\int_{\mathfrak{m}_i} |S_1(\lambda_2\alpha)|^2 K_\eta(\alpha)\,\mathrm{d}\alpha\Bigr)^{1/2}
\end{aligned}$$

$$\times \Big(\int_{\mathfrak{m}_i} |S_k(\lambda_3\alpha)|^2 K_\eta(\alpha)\,\mathrm{d}\alpha \Big)^{1/2}$$
$$\ll X^{5/6}(\log X)^5(\eta X \log X)^{1/2}(\eta X^{1/k}(\log X)^3)^{1/2}$$
$$\ll \eta X^{4/3+1/(2k)+\varepsilon}. \tag{17}$$

The estimate in (17) should be $o(\eta^2 X^{1+1/k})$; hence this leads to the constraint

$$\eta = \infty(X^{1/3-1/(2k)+\varepsilon}), \tag{18}$$

where $f = \infty(g)$ means $g = o(f)$.

Using Hölder's inequality and Lemmas 9 and 10, for $6/5 < k < 3$ we obtain

$$\begin{aligned} |\mathcal{I}(\eta,\omega,\mathfrak{m}_i)| &\ll \int_{\mathfrak{m}_i} |S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)||S_k(\lambda_3\alpha)|K_\eta(\alpha)\,\mathrm{d}\alpha \\ &\ll \Big(\max_{\alpha\in\mathfrak{m}_i} |S_1(\lambda_1\alpha)|^{1/2}\Big)\Big(\int_{\mathfrak{m}_i} |S_1(\lambda_1\alpha)|^2 K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{1/4} \\ &\quad\times \Big(\int_{\mathfrak{m}_i} |S_k(\lambda_3\alpha)|^4 K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{1/4}\Big(\int_{\mathfrak{m}_i} |S_1(\lambda_2\alpha)|^2 K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{1/2} \\ &\ll X^{5/12}(\log X)^{5/2}(\eta X \log X)^{1/4}(\eta \max\{X^{2/k}, X^{4/k-1}\})^{1/4}(\eta X \log X)^{1/2} \\ &\ll \eta \max\big\{X^{7/6+1/(2k)}, X^{11/12+1/k}\big\} X^{\varepsilon}. \end{aligned} \tag{19}$$

The estimate in (19) should be $o(\eta^2 X^{1+1/k})$; hence this leads to

$$\eta = \infty\big(\max\{X^{1/6-1/(2k)+\varepsilon}, X^{-1/12+\varepsilon}\}\big). \tag{20}$$

If $k = 3$ we use Lemmas 9 and 11 thus getting

$$\begin{aligned} |\mathcal{I}(\eta,\omega,\mathfrak{m}_i)| &\ll \int_{\mathfrak{m}_i} |S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)||S_3(\lambda_3\alpha)|K_\eta(\alpha)\,\mathrm{d}\alpha \\ &\ll \Big(\max_{\alpha\in\mathfrak{m}_i} |S_1(\lambda_1\alpha)|^{1/4}\Big)\Big(\int_{\mathfrak{m}_i} |S_1(\lambda_1\alpha)|^2 K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{3/8} \\ &\quad\times \Big(\int_{\mathfrak{m}_i} |S_3(\lambda_3\alpha)|^8 K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{1/8}\Big(\int_{\mathfrak{m}_i} |S_1(\lambda_2\alpha)|^2 K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{1/2} \\ &\ll \eta X^{31/24+\varepsilon}. \end{aligned}$$

This bound leads to the constraint

$$\eta = \infty\big(X^{-1/24+\varepsilon}\big), \tag{21}$$

which justifies the last line of (2).

7.2. Bound on $\mathfrak{m}^*$

We recall our definitions in (7) and (8). It remains to discuss the set $\mathfrak{m}^*$ where the following bounds hold simultaneously

$$|S_1(\lambda_1\alpha)| > X^{5/6}(\log X)^5, \quad |S_1(\lambda_2\alpha)| > X^{5/6}(\log X)^5, \quad T \le |\alpha| \le \eta^{-2}(\log X)^{3/2} = R,$$

where $T = P/X = X^{5/(6k)-1-\varepsilon}$ by our choice in (15) if $k < 5/2$, and $T = X^{-3/5}$ otherwise. Using a dyadic dissection, we split $\mathfrak{m}^*$ into disjoint sets $E(Z_1, Z_2, y)$ in which, for $\alpha \in E(Z_1, Z_2, y)$, we have

$$Z_i < |S_1(\lambda_i\alpha)| \le 2Z_i, \qquad y < |\alpha| \le 2y,$$

where $Z_i = 2^{k_i}X^{5/6}(\log X)^5$ and $y = 2^{k_3}X^{5/(6k)-1-\varepsilon}$ for some non-negative integers k_1, k_2, k_3.

It follows that the number of disjoint sets is, at most, $\ll (\log X)^3$. Let us write $\mathscr{A}$ as a shorthand for the set $E(Z_1, Z_2, y)$. We need an upper bound for the Lebesgue measure of $\mathscr{A}$. In the following Lemma, it is crucial that both the integers a_1 and a_2 appearing in (22) below do not vanish; in fact, if $a_1 = 0$, say, then $q_1 = 1$ and α is so small that it can not belong to $\mathfrak{m}$. If k is large, we treat the range $[P/X, X^{-3/5}]$ and its symmetrical by means of the argument in section 6: this is needed because, in this case, the inequalities (22) below do not rule out the possibility that $a_1a_2 = 0$, unless $|\alpha|$ is large enough.

Lemma 12. *Let $\varepsilon > 0$. We have that $\mu(\mathscr{A}) \ll yX^{8/3+\varepsilon}Z_1^{-2}Z_2^{-2}$, where $\mu(\cdot)$ denotes the Lebesgue measure.*

Proof. If $\alpha \in \mathscr{A}$, by Lemma 8 there are coprime integers (a_1, q_1) and (a_2, q_2) such that

$$1 \le q_i \ll \Big(\frac{X(\log X)^4}{Z_i}\Big)^2, \qquad |q_i\lambda_i\alpha - a_i| \ll \frac{X(\log X)^{10}}{Z_i^2}. \tag{22}$$

We remark that $a_1a_2 \ne 0$ otherwise we would have $\alpha \in \mathfrak{M} \cup \mathfrak{M}^*$. In fact, if $a_1a_2 = 0$, recalling the definitions of Z_i and (22), $\alpha \ll q_i^{-1}X(\log X)^{10}Z_i^{-2} \ll X^{-2/3}$.

Now, we can further split $\mathfrak{m}^*$ into sets $I = I(Z_1, Z_2, y, Q_1, Q_2)$ where, on each set, $Q_j \le q_j \le 2Q_j$. Note that a_i and q_i are uniquely determined by α; in the opposite direction, for a given quadruple a_1, q_1, a_2, q_2, the inequalities (22) define an interval of α of length

$$\ll \min\left\{\frac{X(\log X)^{10}}{Q_1Z_1^2}, \frac{X(\log X)^{10}}{Q_2Z_2^2}\right\} \ll \frac{X(\log X)^{10}}{Q_1^{1/2}Q_2^{1/2}Z_1Z_2},$$

by taking the geometric mean.

Now we need a lower bound for Q_1Q_2: by (22) we obtain

$$\left|a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2\right| = \left|\frac{a_2}{\lambda_2\alpha}(q_1\lambda_1\alpha - a_1) - \frac{a_1}{\lambda_2\alpha}(q_2\lambda_2\alpha - a_2)\right|$$

$$\ll q_2|q_1\lambda_1\alpha - a_1| + q_1|q_2\lambda_2\alpha - a_2|$$
$$\ll Q_2\frac{X(\log X)^{10}}{Z_1^2} + Q_1\frac{X(\log X)^{10}}{Z_2^2}.$$

Recalling that $Q_i \ll (X(\log X)^4/Z_i)^2$ and that $Z_i \gg X^{5/6}(\log X)^5$, we have

$$\left|a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2\right| \ll \left(\frac{X(\log X)^4}{X^{5/6}(\log X)^5}\right)^2\left(\frac{X^{1/2}(\log X)^5}{X^{5/6}(\log X)^5}\right)^2 \ll X^{-1/3}(\log X)^{-2} < \frac{1}{4q}. \tag{23}$$

We recall that $q = X^{1/3}$ is a denominator of a convergent of λ_1/λ_2. Hence by (23), Legendre's law of best approximation for continued fractions implies that $|a_2q_1| \geq q$ and by the same token, for any pair α, α' having distinct associated products a_2q_1,

$$|a_2(\alpha)q_1(\alpha) - a_2(\alpha')q_1(\alpha')| \geq q;$$

thus, by the pigeon-hole principle, there is at most one value of a_2q_1 in the interval $[rq, (r+1)q)$ for any positive integer r. Furthermore a_2q_1 determines a_2 and q_1 to within $X^{\varepsilon/2}$ possibilities (from the bound for the divisor function) and consequently also a_2q_1 determines a_1 and q_2 to within $X^{\varepsilon/2}$ possibilities from (23).

Hence we got a lower bound for q_1q_2, since, using $Q_j \leq q_j \leq 2Q_j$, we get

$$q_1q_2 = a_2q_1\frac{q_2}{a_2} \gg \frac{rq}{|\alpha|} \gg rqy^{-1}.$$

for the quadruple under consideration.

As a consequence we obtain that the total length of the part of $I(Z_1, Z_2, y, Q_1, Q_2)$ with $a_2q_1 \in [rq, (r+1)q)$ is

$$\ll X^{1+\varepsilon/2}(\log X)^{10}Z_1^{-1}Z_2^{-1}r^{-1/2}q^{-1/2}y^{1/2}.$$

Now we need a bound for r: since $a_2q_1 \in [rq, (r+1), q)$, we have

$$rq \leq |a_2q_1| \ll q_1q_2|\alpha| \ll y\left(\frac{X(\log X)^4}{Z_1}\right)^2\left(\frac{X(\log X)^4}{Z_2}\right)^2 \ll \frac{yX^4(\log X)^{16}}{Z_1^2Z_2^2}$$

and hence we get

$$r \ll q^{-1}yX^4(\log X)^{16}Z_1^{-2}Z_2^{-2}.$$

Next, we sum on every interval to get an upper bound for the measure of $\mathscr{A}$: we get

$$\mu(\mathscr{A}) \ll \frac{X^{1+\varepsilon/2}y^{1/2}(\log X)^{10}}{Z_1Z_2q^{1/2}}\sum_{1\leq r\ll q^{-1}yX^4(\log X)^{16}Z_1^{-2}Z_2^{-2}} r^{-1/2}.$$

Standard estimates imply that the sum on the right is $\ll (q^{-1}yX^4(\log X)^{16}Z_1^{-2}Z_2^{-2})^{1/2}$, and recalling that $q = X^{1/3}$ we can finally write

$$\mu(\mathscr{A}) \ll yX^{3+\varepsilon/2}(\log X)^{18}Z_1^{-2}Z_2^{-2}q^{-1} \ll yX^{8/3+\varepsilon}Z_1^{-2}Z_2^{-2}.$$

This proves the lemma. □

8. Conclusion

Here we finally justify the choice of the function ψ in the statement of the main Theorem. Using Lemmas 9-10-12 we are now able to estimate $\mathcal{J}(\eta,\omega,\mathcal{A})$ for $1 < k \le 3$. For $k \ge \frac{5}{2}$, we also need the result in section 6.

If $1 < k \le 6/5$ we proceed as follows:

$$\begin{aligned}
|\mathcal{J}(\eta,\omega,\mathcal{A})| &\ll \int_{\mathcal{A}} |S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)||S_k(\lambda_3\alpha)|K_\eta(\alpha)\,\mathrm{d}\alpha \\
&\ll \Big(\int_{\mathcal{A}} |S_1(\lambda_1\alpha)S_1(\lambda_2\alpha)|^2 K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{1/2}\Big(\int_{\mathcal{A}} |S_k(\lambda_3\alpha)|^2 K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{1/2} \\
&\ll \big(\min\{\eta^2, y^{-2}\}\big)^{1/2}\big((Z_1Z_2)^2\mu(\mathcal{A})\big)^{1/2}\big(\eta X^{1/k+\varepsilon}\big)^{1/2} \\
&\ll \big(\min\{\eta^2, y^{-2}\}\big)^{1/2} Z_1Z_2\big(yX^{8/3+\varepsilon}Z_1^{-2}Z_2^{-2}\big)^{1/2}\eta^{1/2}X^{1/(2k)+\varepsilon/2} \\
&\ll \eta X^{4/3+1/(2k)+\varepsilon}.
\end{aligned}$$

Hence we need $\eta = \infty\big(X^{1/3-1/(2k)+\varepsilon}\big)$, which is the same condition we got in (18).

If $6/5 < k < 3$,

$$\begin{aligned}
|\mathcal{J}(\eta,\omega,\mathcal{A})| &\ll \int_{\mathcal{A}} |S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)||S_k(\lambda_3\alpha)|K_\eta(\alpha)\,\mathrm{d}\alpha \\
&\ll \Big(\int_{\mathcal{A}} |S_1(\lambda_1\alpha)S_1(\lambda_2\alpha)|^{4/3} K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{3/4}\Big(\int_{\mathcal{A}} |S_k(\lambda_3\alpha)|^4 K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{1/4} \\
&\ll \big(\min\{\eta^2, y^{-2}\}\big)^{3/4}\big((Z_1Z_2)^{4/3}\mu(\mathcal{A})\big)^{3/4}\big(\eta\max\{X^{2/k}, X^{4/k-1}\}X^{\varepsilon}\big)^{1/4} \\
&\ll \big(\min\{\eta^2, y^{-2}\}\big)^{3/4} Z_1Z_2\big(yX^{8/3+\varepsilon}Z_1^{-2}Z_2^{-2}\big)^{3/4}\eta^{1/4}\max\{X^{1/(2k)}, X^{1/k-1/4}\}X^{\varepsilon/4} \\
&\ll \eta Z_1^{-1/2}Z_2^{-1/2}X^{2+\varepsilon}\max\{X^{1/(2k)}, X^{1/k-1/4}\} \\
&\ll \eta\max\{X^{7/6+1/(2k)}, X^{11/12+1/k}\}X^{\varepsilon}.
\end{aligned}$$

Hence we need $\eta = \infty\big(\max\{X^{1/6-1/(2k)+\varepsilon}, X^{-1/12+\varepsilon}\}\big)$, which is the same condition we got in (20). If $k = 3$, using Lemmas 11 and 12 we obtain

$$\begin{aligned}
|\mathcal{J}(\eta,\omega,\mathcal{A})| &\ll \int_{\mathcal{A}} |S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)||S_3(\lambda_3\alpha)|K_\eta(\alpha)\,\mathrm{d}\alpha \\
&\ll \Big(\int_{\mathcal{A}} |S_1(\lambda_1\alpha)S_1(\lambda_2\alpha)|^{8/7} K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{7/8}\Big(\int_{\mathcal{A}} |S_3(\lambda_3\alpha)|^8 K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{1/8} \\
&\ll \eta Z_1^{-3/4}Z_2^{-3/4}X^{7/3+5/24+\varepsilon} \ll \eta X^{31/24+\varepsilon}.
\end{aligned}$$

This leads to the same constraint for η that we had in (21).

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