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General Section

On the x–coordinates of Pell equations which are k–generalized Fibonacci numbers

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A B S T R A C T

For an integer $k \geq 2$, let $\{F_n^{(k)}\}_{n \geq 2-k}$ be the k–generalized Fibonacci sequence which starts with $0, \ldots, 0, 1$ (a total of k terms) and for which each term afterwards is the sum of the k preceding terms. In this paper, for an integer $d \geq 2$ which is square-free, we show that there is at most one value of the positive integer x participating in the Pell equation $x^2 - dy^2 = \pm 1$, which is a k–generalized Fibonacci number, with a couple of parametric exceptions which we completely characterize. This paper extends previous work from [18] for the case $k = 2$ and [17] for the case $k = 3$.

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1. Introduction

Let $d \geq 2$ be a positive integer which is not a perfect square. It is well-known that the Pell equation

$$x^2 - dy^2 = \pm 1 \tag{1}$$

has infinitely many positive integer solutions (x, y). Letting (x_1, y_1) be the smallest positive solution, all solutions are of the form (x_n, y_n) for some positive integer n, where

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n \qquad \text{for all} \quad n \geq 1. \tag{2}$$

Recently, Luca and Togbé [18] considered the Diophantine equation

$$x_n = F_m, \tag{3}$$

where $\{F_m\}_{m\geq 0}$ is the sequence of Fibonacci numbers given by $F_0 = 0$, $F_1 = 1$, and $F_{m+2} = F_{m+1} + F_m$ for all $m \geq 0$. They proved that equation (3) has at most one solution (n, m) in positive integers except for $d = 2$, in which case equation (3) has the three solutions $(n, m) = (1, 1), (1, 2), (2, 4)$.

Luca, Montejano, Szalay, and Togbé [17] considered the Diophantine equation

$$x_n = T_m, \tag{4}$$

where $\{T_m\}_{m\geq 0}$ is the sequence of Tribonacci numbers given by $T_0 = 0$, $T_1 = 1$, $T_2 = 1$, and $T_{m+3} = T_{m+2} + T_{m+1} + T_m$ for all $m \geq 0$. They proved that equation (4) has at most one solution (n, m) in positive integers for all d except for $d = 2$ when equation (4) has the three solutions $(n, m) = (1, 1), (1, 2), (3, 5)$ and when $d = 3$ case in which equation (4) has the two solutions $(n, m) = (1, 3), (2, 5)$.

The purpose of this paper is to generalize the previous results. Let $k \geq 2$ be an integer. We consider a generalization of Fibonacci sequence called the k–generalized Fibonacci sequence $\{F_m^{(k)}\}_{m\geq 2-k}$ defined as

$$F_m^{(k)} = F_{m-1}^{(k)} + F_{m-2}^{(k)} + \cdots + F_{m-k}^{(k)}, \tag{5}$$

with the initial conditions

$$F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0 \quad \text{and} \quad F_1^{(k)} = 1.$$

We call $F_m^{(k)}$ the mth k–generalized Fibonacci number. Note that when $k = 2$, it coincides with the Fibonacci numbers and when $k = 3$ it is the Tribonacci number.

The first $k + 1$ nonzero terms in $F_m^{(k)}$ are powers of 2, namely

$$F_1^{(k)} = 1, \quad F_2^{(k)} = 1, \quad F_3^{(k)} = 2, \quad F_4^{(k)} = 4, \ldots, F_{k+1}^{(k)} = 2^{k-1}.$$

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Furthermore, the next term is $F_{k+2}^{(k)} = 2^k - 1$. Thus, we have that

$$F_m^{(k)} = 2^{m-2} \quad \text{holds for all} \quad 2 \le m \le k+1. \tag{6}$$

We also observe that the recursion (5) implies the three–term recursion

$$F_m^{(k)} = 2F_{m-1}^{(k)} - F_{m-k-1}^{(k)} \quad \text{for all} \quad m \ge 3,$$

which can be used to prove by induction on m that $F_m^{(k)} < 2^{m-2}$ for all $m \ge k+2$ (see also [4], Lemma 2).

2. Main result

In this paper, we show that there is at most one value of the positive integer x participating in (1) which is a k–generalized Fibonacci number, with a couple of parametric exceptions that we completely characterize. This can be interpreted as solving the system of equations

$$x_{n_1} = F_{m_1}^{(k)}, \qquad x_{n_2} = F_{m_2}^{(k)}, \tag{7}$$

with $n_2 > n_1 \ge 1$, $m_2 > m_1 \ge 2$, and $k \ge 2$. The fact that $F_1^{(k)} = F_2^{(k)} = 1$, allows us to assume that $m \ge 2$. That is, if $F_m^{(k)} = 1$ for some positive integer m, then we will assume that $m = 2$. As we already mentioned, the cases $k = 2$ and $k = 3$ have been solved completely by Luca and Togbé [18] and Luca, Montejano, Szalay, and Togbé [17], respectively. So, we focus on the case $k \ge 4$.

We put $\epsilon := x_1^2 - dy_1^2$. Note that $dy_1^2 = x_1^2 - \epsilon$, so the pair (x_1, ϵ) determines d, y_1. Our main result is the following:

Theorem 1. *Let $k \ge 4$ be a fixed integer. Let $d \ge 2$ be a square-free integer. Assume that*

$$x_{n_1} = F_{m_1}^{(k)}, \qquad \text{and} \quad x_{n_2} = F_{m_2}^{(k)} \tag{8}$$

for positive integers $m_2 > m_1 \ge 2$ and $n_2 > n_1 \ge 1$, where x_n is the x–coordinate of the nth solution of the Pell equation (1). Then, either:

(i) $n_1 = 1,\ n_2 = 2,\ m_1 = (k+3)/2,\ m_2 = k+2$, *and* $\epsilon = 1$; *or*
(ii) $n_1 = 1,\ n_2 = 3,\ k = 3 \times 2^{a+1} + 3a - 5,\ m_1 = 3 \times 2^a + a - 1,\ m_2 = 9 \times 2^a + 3a - 5$ *for some positive integer a and* $\epsilon = 1$.

3. Preliminary results

Here, we recall some of the facts and properties of the k–generalized Fibonacci sequence and solutions to Pell equations which will be used later in this paper.

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3.1. Notations and terminology from algebraic number theory

We begin by recalling some basic notions from algebraic number theory.

Let η be an algebraic number of degree d with minimal primitive polynomial over the integers

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d}(x - \eta^{(i)}),$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of η. Then the *logarithmic height* of η is given by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|\eta^{(i)}|, 1\}\right)\right).$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log\max\{|p|, q\}$. The following are some of the properties of the logarithmic height function $h(\cdot)$, which will be used in the next sections of this paper without reference:

$$\begin{aligned} h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2, \\ h(\eta\gamma^{\pm 1}) &\leq h(\eta) + h(\gamma), \\ h(\eta^s) &= |s|h(\eta) \qquad (s \in \mathbb{Z}). \end{aligned} \tag{9}$$

3.2. k-generalized Fibonacci numbers

It is known that the characteristic polynomial of the k–generalized Fibonacci numbers $F^{(k)} := (F_m^{(k)})_{m\geq 2-k}$, namely

$$\Psi_k(x) := x^k - x^{k-1} - \cdots - x - 1,$$

is irreducible over $\mathbb{Q}[x]$ and has just one root outside the unit circle. Let $\alpha := \alpha(k)$ denote that single root, which is located between $2\left(1 - 2^{-k}\right)$ and 2 (see [9]). To simplify notation, in our application we shall omit the dependence on k of α. We shall use $\alpha^{(1)}, \ldots, \alpha^{(k)}$ for all roots of $\Psi_k(x)$ with the convention that $\alpha^{(1)} := \alpha$.

We now consider for an integer $k \geq 2$, the function

$$f_k(z) = \frac{z-1}{2 + (k+1)(z-2)} \qquad \text{for} \qquad z \in \mathbb{C}. \tag{10}$$

With this notation, Dresden and Du presented in [9] the following "Binet–like" formula for the terms of $F^{(k)}$:

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$$F_m^{(k)} = \sum_{i=1}^{k} f_k(\alpha^{(i)}){\alpha^{(i)}}^{m-1}. \tag{11}$$

It was proved in [9] that the contribution of the roots which are inside the unit circle to the formula (11) is very small, namely that the approximation

$$\left|F_m^{(k)} - f_k(\alpha)\alpha^{m-1}\right| < \frac{1}{2} \quad \text{holds for all} \quad m \geqslant 2-k. \tag{12}$$

It was proved by Bravo and Luca in [4] that

$$\alpha^{m-2} \leq F_m^{(k)} \leq \alpha^{m-1} \qquad \text{holds for all} \qquad m \geq 1 \quad \text{and} \quad k \geq 2. \tag{13}$$

The observations from the expressions (11) to (13) lead us to call α the *dominant root* of $F^{(k)}$.

Before we conclude this section, we present some useful lemmas that will be used in the next sections on this paper. The following lemma was proved by Bravo and Luca ([4], pp. 72–73).

Lemma 1 *(Bravo, Luca). Let $k \geq 2$, α be the dominant root of $\{F_m^{(k)}\}_{m\geq 2-k}$, and consider the function $f_k(z)$ defined in (10). Then:*

(i) *The inequalities*

$$\frac{1}{2} < f_k(\alpha) < \frac{3}{4} \qquad \textit{and} \qquad |f_k(\alpha^{(i)})| < 1, \qquad 2 \leq i \leq k$$

hold. In particular, the number $f_k(\alpha)$ is not an algebraic integer.

(ii) *The logarithmic height of $f_k(\alpha)$ satisfies $h(f_k(\alpha)) < 3\log k$.*

Next, we recall the following result due to Cooper and Howard ([8], Theorem 2.5, pp. 234).

Lemma 2 *(Cooper, Howard). For $k \geq 2$ and $m \geq k+2$,*

$$F_m^{(k)} = 2^{m-2} + \sum_{j=1}^{\lfloor \frac{m+k}{k+1} \rfloor - 1} C_{m,j}\, 2^{m-(k+1)j-2},$$

where

$$C_{m,j} = (-1)^j \left[\binom{m-jk}{j} - \binom{m-jk-2}{j-2}\right].$$

In the above, we have denoted by $\lfloor x \rfloor$ the greatest integer less than or equal to x and used the convention that $\binom{a}{b} = 0$ if either $a < b$ or if one of a or b is negative.

Before going further, let us see some particular cases of Lemma 2.

Example 1.

(i) Assume that $m \in [2, k+1]$. Then $1 < \frac{m+k}{k+1} < 2$, so $\lfloor \frac{m+k}{k+1} \rfloor = 1$. In this case,

$$F_m^{(k)} = 2^{m-2},$$

a fact which we already knew.

(ii) Assume that $m \in [k+2, 2k+2]$. Then $2 \le \frac{m+k}{k+1} < 3$, so $\lfloor \frac{m+k}{k+1} \rfloor = 2$. In this case,

$$\begin{aligned} F_m^{(k)} &= 2^{m-2} + C_{m,1} 2^{m-(k+1)-2} \\ &= 2^{m-2} - \left(\binom{m-k}{1} - \binom{m-k-2}{-1} \right) 2^{m-k-3} \\ &= 2^{m-2} - (m-k) 2^{m-k-3}. \end{aligned}$$

(iii) Assume that $m \in [2k+3, 3k+3]$. Then $3 \le \frac{m+k}{k+1} < 4$, so $\lfloor \frac{m+k}{k+1} \rfloor = 3$. In this case,

$$\begin{aligned} F_m^{(k)} &= 2^{m-2} + C_{m,1} 2^{m-(k+1)-2} + C_{m,2} 2^{m-2(k+1)-2} \\ &= 2^{m-2} - (m-k) 2^{m-k-3} + \left(\binom{m-2k}{2} - \binom{m-2k-2}{0} \right) 2^{m-2k-4} \\ &= 2^{m-2} - (m-k) 2^{m-k-3} + \left(\frac{(m-2k)(m-2k-1)}{2} - 1 \right) 2^{m-2k-4} \\ &= 2^{m-2} - (m-k) 2^{m-k-3} + (m-2k+1)(m-2k-2) 2^{m-2k-5}. \end{aligned}$$

Gómez and Luca ([12], Lemma 2, pp. 189) derived from the Cooper and Howard's formula the following asymptotic expansion of $F_m^{(k)}$ valid when $2 \le m < 2^k$.

Lemma 3 *(Gómez, Luca). If $m < 2^k$, then the following estimate holds:*

$$F_m^{(k)} = 2^{m-2} \left(1 + \delta_1(m) \frac{k-m}{2^{k+1}} + \delta_2(m) \frac{f(k,m)}{2^{2k+2}} + \eta(k,m) \right), \tag{14}$$

where $f(k,m) := \frac{1}{2}(z-1)(z+2)$; $z = 2k - m$, $\eta := \eta(k,m)$ is a real number satisfying

$$|\eta| < \frac{4m^3}{2^{3k+3}},$$

and $\delta_i(m)$ is the characteristic function of the set $\{m > i(k+1)\}$ for $i = 1, 2$.

3.3. Linear forms in logarithms and continued fractions

To prove our main result Theorem 1, we need to use several times a Baker–type lower bound for a nonzero linear form in logarithms of algebraic numbers. There are many such in the literature like that of Baker and Wüstholz from [2]. We start by recalling the result of Bugeaud, Mignotte, and Siksek ([6], Theorem 9.4, pp. 989), which is a modified version of the result of Matveev [19]. This result is one of our main tools in this paper.

Theorem 2 *(Matveev according to Bugeaud, Mignotte, Siksek). Let $\gamma_1, \ldots, \gamma_t$ be positive real algebraic numbers in a number field $\mathbb{K}$ of degree D, $b_1, \ldots, b_t$ be nonzero integers, and assume that*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1, \tag{15}$$

is nonzero. Then,

$$\log|\Lambda| > -1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1+\log D)(1+\log B)A_1 \cdots A_t,$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}, \qquad \textit{for all} \qquad i = 1, \ldots, t.$$

When $t = 2$ and γ_1 and γ_2 are positive and multiplicatively independent, we can use a result of Laurent, Mignotte, and Nesterenko [14]. Namely, let in this case B_1 and B_2 be real numbers larger than 1 such that

$$\log B_i \geq \max\left\{h(\gamma_i), \frac{|\log \gamma_i|}{D}, \frac{1}{D}\right\}, \qquad \text{for} \quad i = 1, 2,$$

and put

$$b' := \frac{|b_1|}{D \log B_2} + \frac{|b_2|}{D \log B_1}.$$

Put

$$\Gamma := b_1 \log \gamma_1 + b_2 \log \gamma_2. \tag{16}$$

We note that $\Gamma \neq 0$ because γ_1 and γ_2 are multiplicatively independent. The following result is due to Laurent, Mignotte, and Nesterenko ([14], Corollary 2, pp. 288).

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Theorem 3 *(Laurent, Mignotte, Nesterenko). With the above notations, assuming that η_1 and η_2 are positive and multiplicatively independent, then*

$$\log|\Gamma| > -24.34D^4 \left(\max\left\{\log b' + 0.14, \frac{21}{D}, \frac{1}{2}\right\}\right)^2 \log B_1 \log B_2. \tag{17}$$

Note that with Γ given by (16), we have $e^\Gamma - 1 = \Lambda$, where Λ is given by (15) in case $t = 2$, which explains the connection between Theorem 2 and Theorem 3.

During the course of our calculations, we get some upper bounds on our variables which are too large, thus we need to reduce them. To do so, we use some results from the theory of continued fractions. Specifically, for a nonhomogeneous linear form in two integer variables, we use a slight variation of a result due to Dujella and Pethő ([10], Lemma 5a, pp. 303–304), which itself is a generalization of a result of Baker and Davenport [1].

For a real number X, we write $||X|| := \min\{|X - n| : n \in \mathbb{Z}\}$ for the distance from X to the nearest integer.

Lemma 4 *(Dujella, Pethő). Let M be a positive integer, p/q be a convergent of the continued fraction of the irrational number τ such that $q > 6M$, and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Furthermore, let $\varepsilon := ||\mu q|| - M||\tau q||$. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w},$$

in positive integers u, v, and w with

$$u \le M \quad \text{and} \quad w \ge \frac{\log(Aq/\varepsilon)}{\log B}.$$

The above lemma cannot be applied when $\mu = 0$ (since then $\varepsilon < 0$). In this case, we use the following classical result in the theory of Diophantine approximation, which is the well-known Legendre criterion.

Lemma 5 *(Legendre). Let τ be real number and x, y integers such that*

$$\left|\tau - \frac{x}{y}\right| < \frac{1}{2y^2}. \tag{18}$$

Then $x/y = p_k/q_k$ is a convergent of τ. Furthermore,

$$\left|\tau - \frac{x}{y}\right| \ge \frac{1}{(a_{k+1} + 2)y^2}. \tag{19}$$

Finally, the following lemma is also useful. It is a result due to Gúzman and Luca ([13], Lemma 7, pp. 173).

Lemma 6 *(Gúzman, Luca). If $m \geqslant 1$, $T > (4m^2)^m$, and $T > x/(\log x)^m$, then*

$$x < 2^m T(\log T)^m.$$

3.4. Pell equations and Dickson polynomials

Let $d \geq 2$ be square-free. We put $\delta := x_1 + \sqrt{x_1^2 - \epsilon}$ for the minimal positive integer x_1 such that

$$x_1^2 - dy_1^2 = \epsilon, \qquad \epsilon \in \{\pm 1\}$$

for some positive integer y_1. Then,

$$x_n + \sqrt{d}y_n = \delta^n \qquad \text{and} \qquad x_n - \sqrt{d}y_n = \eta^n, \qquad \text{where} \qquad \eta := \epsilon\delta^{-1}.$$

From the above, we get

$$2x_n = \delta^n + (\epsilon\delta^{-1})^n \qquad \text{for all} \qquad n \geq 1. \tag{20}$$

There is a formula expressing $2x_n$ in terms of $2x_1$ by means of the Dickson polynomial $D_n(2x_1, \epsilon)$, where

$$D_n(x, \nu) = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i} \binom{n-i}{i} (-\nu)^i x^{n-2i}.$$

These polynomials appear naturally in many number theory problems and results, most notably in a result of Bilu and Tichy [3] concerning polynomials $f(X), g(X) \in \mathbb{Z}[X]$ such that the Diophantine equation $f(x) = g(y)$ has infinitely many integer solutions (x, y).

Example 2.

(i) $n = 2$. We have

$$2x_2 = \sum_{i=0}^{1} \frac{2}{2-i} \binom{2-i}{i} (-\epsilon)^i (2x_1)^{2-2i} = 4x_1^2 - 2\epsilon, \qquad \text{so} \qquad x_2 = 2x_1^2 - \epsilon.$$

(ii) $n = 3$. We have

$$2x_3 = \sum_{i=0}^{1} \frac{3}{3-i} \binom{3-i}{i} (-\epsilon)^i (2x_1)^{3-2i} = (2x_1)^3 - 3\epsilon(2x_1), \qquad \text{so} \qquad x_3 = 4x_1^3 - 3\epsilon x_1.$$

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(iii) $n \geq 4$. We have

$$2x_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i}\binom{n-i}{i}(-\epsilon)^i(2x_1)^{n-2i}$$

$$= (2x_1)^n - n\epsilon(2x_1)^{n-2} + \frac{n(n-3)}{2}(2x_1)^{n-4} + \sum_{i\geq 3}^{\lfloor n/2 \rfloor} \frac{n(-\epsilon)^i}{n-i}\binom{n-i}{i}(2x_1)^{n-2i}.$$

The following variation of a result of Luca [16] is useful. Let $P(m)$ denote the largest prime factor of the positive integer m.

Lemma 7. *If $P(x_n) \leq 5$, then either $n = 1$, or $n = 2$ and $x_2 \in \{3, 9, 243\}$.*

Proof. In [16], it was shown that if $\varepsilon = 1$ and $P(x_n) \leq 5$, then $n = 1$. We give here a proof for both cases $\epsilon \in \{\pm 1\}$. Since $x_n = y_{2n}/y_n$, where $y_m = (\delta^m - \eta^m)/(\delta - \eta)$, it follows, by Carmichael's Primitive Divisor Theorem [7], that if $n \geq 7$, then x_n has a prime factor which is primitive for y_{2n}. In particular, this prime is $\geq 2n - 1 > 5$. Thus, $n \leq 6$. Next, assume that $n > 1$. If $n \in \{3, 6\}$, then x_n is of the form $x(4x^2 \pm 3)$, where $x = x_\ell$ with $\ell = n/3 \in \{1, 2\}$. The factor $4x^2 \pm 3$ is larger than 1 (since $x_n > x_\ell$) odd (hence, coprime to 2), not a multiple of 9, and coprime to 5 since $\left(\frac{\pm 3}{5}\right) = -1$. Thus, the only possibility is $4x^2 \pm 3 = 3$, equation which does not have a positive integer solution x. If $n \in \{2, 4\}$, then $x_n = 2x^2 \pm 1$, where $x = x_\ell$ and $\ell = n/2 \in \{1, 2\}$. Further, if $\ell = 2$ only the case with the -1 on the right is possible. The expression $2x^2 - 1$ is odd, and coprime to both 3 and 5 since $\left(\frac{2}{3}\right) = \left(\frac{2}{5}\right) = -1$, so the case $x_n = 2x_\ell^2 - 1$ is not possible. Finally, if $x_n = 2x_\ell^2 + 1$, then $n = 2,\ \ell = 1$. Further, $2x^2 + 1$ is coprime to 2 and 5 so we must have $2x^2 + 1 = 3^b$ for some exponent b. Thus, $x^2 = (3^b - 1)/(3 - 1)$, and the only solutions are $b \in \{1, 2, 5\}$ by a result of Ljunggren [15]. $\square$

Since none of $3,\ 9,\ 243$ are of the form $F_m^{(k)}$ for any $m \geq 1,\ k \geq 4$, for our practical purpose we will use the implication that if $x_n = F_m^{(k)}$ and $P(x_n) \leq 5$, then $n = 1$.

4. A small linear form in logarithms

We assume that (x_1, y_1) is the fundamental solution of the Pell equation (1). As in Subsection 3.4, we set

$$x_1^2 - dy_1^2 =: \epsilon, \qquad \epsilon \in \{\pm 1\},$$

and put

$$\delta := x_1 + \sqrt{d}y_1 \qquad \text{and} \qquad \eta := x_1 - \sqrt{d}y_1 = \epsilon\delta^{-1}.$$

From (2) (or (20)), we get

$$x_n = \frac{1}{2}\left(\delta^n + \eta^n\right). \tag{21}$$

Since $\delta \geq 1 + \sqrt{2} > 2 > \alpha$, it follows that the estimate

$$\frac{\delta^n}{\alpha^2} \leq x_n < \delta^n \quad \text{holds for all} \quad n \geq 1. \tag{22}$$

We now assume, as in the hypothesis of Theorem 1, that (n_1, m_1) and (n_2, m_2) are pairs of positive integers with $n_1 < n_2$, $2 \leq m_1 < m_2$ and

$$x_{n_1} = F_{m_1}^{(k)} \qquad \text{and} \qquad x_{n_2} = F_{m_2}^{(k)}.$$

By setting $(n, m) = (n_j, m_j)$ for $j \in \{1, 2\}$ and using the inequalities (13) and (22), we get that

$$\alpha^{m-2} \leq F_m^{(k)} = x_n < \delta^n \qquad \text{and} \qquad \frac{\delta^n}{\alpha^2} \leq x_n = F_m^{(k)} \leq \alpha^{m-1}. \tag{23}$$

Hence,

$$nc_1 \log\delta \leq m + 1 \leq nc_1 \log\delta + 3, \qquad c_1 := 1/\log\alpha. \tag{24}$$

Next, by using (11) and (21), we get

$$\frac{1}{2}\left(\delta^n + \eta^n\right) = f_k(\alpha)\alpha^{m-1} + (F_m^{(k)} - f_k(\alpha)\alpha^{m-1}),$$

so

$$\delta^n (2f_k(\alpha))^{-1}\alpha^{-(m-1)} - 1 = \frac{-\eta^n}{2f_k(\alpha)\alpha^{m-1}} + \frac{(F_m^{(k)} - f_k(\alpha)\alpha^{m-1})}{f_k(\alpha)\alpha^{m-1}}.$$

Hence, by using (12) and Lemma 1(i), we have

$$|\delta^n (2f_k(\alpha))^{-1}\alpha^{-(m-1)} - 1| \leq \frac{1}{\alpha^{m-1}\delta^n} + \frac{1}{\alpha^{m-1}} < \frac{1.5}{\alpha^{m-1}}. \tag{25}$$

In the above, we have used the facts that $1/f_k(\alpha) < 2$, $|F_m^{(k)} - f_k(\alpha)\alpha^{m-1}| < 1/2$, $|\eta| = \delta^{-1}$, as well as the fact that $\delta > 2$. We let Λ be the expression inside the absolute value of the left–hand side above. We put

$$\Gamma := n\log\delta - \log(2f_k(\alpha)) - (m-1)\log\alpha. \tag{26}$$

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Note that $e^{\Gamma} - 1 = \Lambda$. Inequality (25) implies that

$$|\Gamma| < \frac{3}{\alpha^{m-1}}. \tag{27}$$

Indeed, for $m \geq 3$, we have that $\frac{1.5}{\alpha^{m-1}} < \frac{1}{2}$, and then inequality (27) follows from (25) via the fact that

$$|e^{\Gamma} - 1| < x \qquad \text{implies} \qquad |\Gamma| < 2x, \qquad \text{whenever} \qquad x \in (0, 1/2), \tag{28}$$

with $x := \frac{1.5}{\alpha^{m-1}}$. When $m = 2$, we have $x_n = F_m^{(k)} = 1$, so $n = 1$, $\epsilon = 1$, $\delta = 1 + \sqrt{2}$, and then

$$|\Gamma| = |\log(1+\sqrt{2}) - \log(2f_k(\alpha)\alpha)| < \max\{\log(1+\sqrt{2}), \log(2f_k(\alpha)\alpha)\} < \log 3 < \frac{3}{\alpha},$$

where we used the fact that $1 < 2f_k(\alpha)\alpha < 3$ (see Lemma 1, (i)). Hence, inequality (26) holds for all pairs (n, m) with $x_n = F_m^{(k)}$ with $m \geq 2$.

Let us recall what we have proved, since this will be important later-on.

Lemma 8. *If (n, m) are positive integers with $m \geq 2$ such that $x_n = F_m^{(k)}$, then with $\delta = x_1 + \sqrt{x_1^2 - \epsilon}$, we have*

$$|n \log \delta - \log(2f_k(\alpha)) - (m-1)\log \alpha| < \frac{3}{\alpha^{m-1}}. \tag{29}$$

5. Bounding n in terms of m and k

We next apply Theorem 2 on the left-hand side of (25). First we need to check that

$$\Lambda = \delta^n (2f_k(\alpha))^{-1} \alpha^{-(m-1)} - 1$$

is nonzero. Well, if it were, then $\delta^n = 2f_k(\alpha)\alpha^{m-1}$. So, $2f_k(\alpha) = \delta^n \alpha^{-(m-1)}$ is a unit. To see that this is not so, we perform a norm calculation of the element $2f_k(\alpha)$ in $\mathbb{L} := \mathbb{Q}(\alpha)$. For $i \in \{2, \ldots, k\}$, we have that $|\alpha^{(i)}| < 1$, so that, by the absolute value inequality, we have

$$|2f_k(\alpha^{(i)})| = \frac{2|\alpha^{(i)} - 1|}{|2 + (k+1)(\alpha^{(i)} - 2)|} \leq \frac{4}{(k+1)(2 - |\alpha^{(i)}|) - 2} < \frac{4}{k-1} \leq \frac{4}{5} \quad \text{for} \quad k \geq 6.$$

Thus, for $k \geq 6$, using also Lemma 1 (i), we get

$$|\mathcal{N}_{\mathbb{L}/\mathbb{Q}}(2f_k(\alpha))| < |2f_k(\alpha)| \prod_{i=2}^{k} |2f_k(\alpha^{(i)})| < \frac{3}{2}\left(\frac{4}{5}\right)^{k-1} \leq \frac{3}{2}\left(\frac{4}{5}\right)^{5} < 1.$$

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This is for $k \geq 6$. For $k = 4,\ 5$ one checks that $|\mathcal{N}_{\mathbb{L}/\mathbb{Q}}(2f_k(\alpha))| < 1$ as well. In fact, the norm of $2f_k(\alpha)$ has been computed (for all $k \geq 2$) in [11], and the formula is

$$|\mathcal{N}_{\mathbb{L}/\mathbb{Q}}(2f_k(\alpha))| = \frac{2^k(k-1)^2}{2^{k+1}k^k - (k+1)^{k+1}}.$$

One can check directly that the above number is always smaller than 1 for all $k \geq 2$ (in particular, for $k = 4, 5$). Thus, $\Lambda \neq 0$, and we can apply Theorem 2. We take

$$t = 3, \quad \gamma_1 = \delta, \quad \gamma_2 = 2f_k(\alpha), \quad \gamma_3 = \alpha, \quad b_1 = n, \quad b_2 = -1, \quad b_3 = -(m-1).$$

We take $\mathbb{K} = \mathbb{Q}(\sqrt{d}, \alpha)$ which has degree $D \leq 2k$. Since $\delta \geq 1 + \sqrt{2} > \alpha$, the second inequality in (23) tells us right-away that $n \leq m$, so we can take $B := m$. We have $h(\gamma_1) = (1/2)\log\delta$ and $h(\gamma_3) = (1/k)\log\alpha$. Further,

$$h(\gamma_2) = h(2f_k(\alpha)) \leq h(2) + h(f_k(\alpha)) < 3\log k + \log 2 < 4\log k \tag{30}$$

by Lemma 1 (ii). So, we can take $A_1 := k\log\delta$, $A_2 := 8k\log k$ and $A_3 := 2\log 2$. Now Theorem 2 tells us that

$$\begin{aligned}\log|\Lambda| &> -1.4 \times 30^6 \times 3^{4.5} \times (2k)^2(1+\log 2k)(1+\log m)(k\log\delta)(8k\log k)(2\log 2),\\ &> -1.6 \times 10^{13}k^4(\log k)^2\log(\delta)(1+\log m).\end{aligned}$$

In the above, we used the fact that $k \geq 4$, therefore $2k \leq k^{3/2}$, so

$$1 + \log(2k) \leq 1 + 1.5\log k < 2.5\log k.$$

By comparing the above inequality with inequality (25), we get

$$(m-1)\log\alpha - \log 3 < 1.6 \times 10^{13}k^4(\log k)^2(\log\delta)(1+\log m).$$

Thus,

$$(m+1)\log\alpha < 1.7 \times 10^{13}k^4(\log k)^2(\log\delta)(1+\log m).$$

Since $\alpha^{m+1} \geq \delta^n$ by the second inequality in (23), we get that

$$n < 1.7 \times 10^{13}k^4(\log k)^2(1+\log m). \tag{31}$$

Furthermore, since $\alpha > 1.927$, we get

$$m < 2.6 \times 10^{13}k^4(\log k)^2(\log\delta)(1+\log m). \tag{32}$$

We now record what we have proved so far, which are estimates (31) and (32).

Lemma 9. *If* $x_n = F_m^{(k)}$ *and* $m \geq 2$, *then*

$$n < 1.7 \times 10^{13} k^4 (\log k)^2 (1 + \log m) \quad \text{and} \quad m < 2.6 \times 10^{13} k^4 (\log k)^2 (\log \delta)(1 + \log m).$$

Note that in the above bound, n is bounded only in terms of m and k (but not δ).

6. Bounding m_1, n_1, m_2, n_2 in terms of k

Next, we write down inequalities (29) for both pairs $(n, m) = (n_j, m_j)$ with $j = 1, 2$, multiply the one for $j = 1$ with n_2 and the one with $j = 2$ with n_1, subtract them and apply the triangle inequality to the result to get that

$$\begin{aligned} & |(n_2 - n_1) \log(2f_k(\alpha)) - (n_1 m_2 - n_2 m_1 + n_2 - n_1) \log \alpha| \\ \leq\ & n_2 |n_1 \log \delta - \log(2f_k(\alpha)) - (m_1 - 1) \log \alpha| \\ +\ & n_1 |n_2 \log \delta - \log(2f_k(\alpha)) - (m_2 - 1) \log \alpha| \\ \leq\ & \frac{3n_2}{\alpha^{m_1-1}} + \frac{3n_1}{\alpha^{m_2-1}} < \frac{6n_2}{\alpha^{m_1-1}}. \end{aligned}$$

Therefore, we have

$$|(n_2 - n_1) \log(2f_k(\alpha)) - (n_1 m_2 - n_2 m_1 + n_2 - n_1) \log \alpha| < \frac{6n_2}{\alpha^{m_1-1}}. \tag{33}$$

We are now set to apply Theorem 3 with

$$\gamma_1 = 2f_k(\alpha), \quad \gamma_2 = \alpha, \quad b_1 = n_2 - n_1, \quad b_2 = -(n_1 m_2 - n_2 m_1 + n_2 - n_1).$$

The fact that γ_1 and γ_2 are multiplicatively independent follows because α is a unit and $2f_k(\alpha)$ isn't by a previous argument. Next, we observe that $n_2 - n_1 < n_2$, while by the absolute value of the inequality in (33), we have

$$|n_1 m_2 - n_2 m_1 + n_2 - n_1| \leq (n_2 - n_1) \frac{\log(2f_k(\alpha))}{\log \alpha} + \frac{6n_2}{\alpha^{m_1-1} \log \alpha} < 6n_2.$$

In the above, we used that

$$\frac{\log(2f_k(\alpha))}{\log \alpha} < \frac{\log(1.5)}{\log \alpha} < 1 \qquad \text{and} \qquad \frac{6}{\alpha^{m_1-1} \log \alpha} < 5,$$

because $\alpha \geq \alpha_4 > 1.92$ and $m_1 \geq 2$. We take $\mathbb{K} := \mathbb{Q}(\alpha)$ which has degree $D = k$. So, we can take

$$\log B_1 = 4 \log k > \max \left\{ h(\gamma_1), \frac{|\log \gamma_1|}{k}, \frac{1}{k} \right\}$$

(see inequality (30)), and

$$\log B_2 = \frac{1}{k} = \max\left\{h(\gamma_2), \frac{|\log \gamma_2|}{k}, \frac{1}{k}\right\}.$$

Thus,

$$b' = \frac{(n_2 - n_1)}{k \times (1/k)} + \frac{|n_1 m_2 - n_2 m_1 + n_2 - n_1|}{4k \log k} < n_2 + \frac{6n_2}{4k \log k} < 1.3 n_2.$$

Now Theorem 3 tells us that with

$$\Gamma := (n_2 - n_1)\log(2f_k(\alpha)) - (n_1 m_2 - n_2 m_1 + n_2 - n_1)\log \alpha,$$

we have

$$\log|\Gamma| > -24.34 \times k^4 \left(\max\left\{\log(1.3n_2) + 0.14, \frac{21}{k}, \frac{1}{2}\right\}\right)^2 (4 \log k)\left(\frac{1}{k}\right).$$

Thus,

$$\log|\Gamma| > -97.4 k^3 \log k \left(\max\left\{\log(1.5n_2), \frac{21}{k}, \frac{1}{2}\right\}\right)^2,$$

where we used the fact that $\log(1.3n_2) + 0.14 = \log(1.3 \times e^{0.14} n_2) < \log(1.5n_2)$. By combining the above inequality with (33), we get

$$(m_1 - 1)\log \alpha - \log(6n_2) < 97.4 k^3 \log k \left(\max\left\{\log(1.5n_2), \frac{21}{k}, \frac{1}{2}\right\}\right)^2. \tag{34}$$

Since $\log(1.5n_2) \geq \log 3 > 1.098$, the maximum in the right–hand side above cannot be $1/2$. If it is not $\log(1.5n_2)$, we then get

$$1.098 < \log(1.5n_2) \leq \frac{21}{k} \leq 5.25, \qquad \text{so} \quad k \leq 19 \quad \text{and} \quad n_2 \leq 127. \tag{35}$$

Then, the above inequality (34) gives

$$\begin{aligned}(m_1 + 1)\log \alpha &< 97.4 \times 21^2 k \log k + \log(6 \times 127) + 2\log \alpha \\ &< 4.3 \times 10^5 k \log k. \end{aligned} \tag{36}$$

Since $\alpha \geq 1.927$, we get that

$$m_1 + 1 < 6.6 \times 10^5 k \log k. \tag{37}$$

Further, we have

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$$\begin{aligned}(\alpha^{(m_1+1)})^{n_2} &> (3F_{m_1}^{(k)})^{n_2} \ge (2F_{m_1}^{(k)}+1)^{n_2} = (2x_{n_1}+1)^{n_2}\\ &= (\delta^{n_1}+(1+\eta^{n_1}))^{n_2} > \delta^{n_1 n_2} = (\delta^{n_2})^{n_1}\\ &= (2x_{n_2}-\eta^{n_2})^{n_1} > 2x_{n_2}-1 > x_{n_2} = F_{m_2}^{(k)} > \alpha^{m_2-2},\end{aligned}$$

so

$$m_2 \le 1+n_2(m_1+1) < 8.4\times 10^7 k\log k. \tag{38}$$

Since $n_1<n_2$, inequalities (35), (37) and (38) bound m_1,n_1,m_2,n_2 in terms of k when the maximum in the right–hand side of (34) is $21/k$.

Assume next that the maximum in the right–hand side of (34) is $\log(1.5n_2)$. Then

$$\begin{aligned}(m_1+1)\log\alpha &< 97.4k^3\log k(\log(1.5n_2))^2+2\log\alpha+\log(6n_2)\\ &< 97.4k^3(\log k)(\log 1.5+\log n_2)^2+\log(24n_2)\\ &< 97.5\times 2.56k^3(\log k)(\log n_2)^2+6\log n_2\\ &< 249.6k^3(\log k)(\log n_2)^2+6\log n_2\\ &< 249.6k^3(\log k)(\log n_2)^2\left(1+\frac{6}{249.6k^3(\log k)(\log n_2)}\right)\\ &< 250k^3(\log k)(\log n_2)^2.\end{aligned} \tag{39}$$

For the above inequality, we used that $2\log\alpha+\log(6n_2)<\log(24n_2)\le 6\log n_2$ (since $n_2\ge 2$ and $\alpha<2$), the fact that $\log(1.5n_2)<1.6\log n_2$ holds for $n_2\ge 2$ and the fact that

$$1+\frac{6}{249.6k^3(\log k)(\log n_2)}<1.0004 \qquad \text{holds for}\quad k\ge 4\quad\text{and}\quad n_2\ge 2.$$

In turn, since $\alpha\ge\alpha_4\ge 1.927$, (39) yields

$$m_1<4\times 10^2k^3(\log k)(\log n_2)^2. \tag{40}$$

Since $\alpha^{m_1+1}>\delta^{n_1}\ge\delta$ (see the second relation in (25)), we get

$$\log\delta\le n_1\log\delta<(m_1+1)\log\alpha<250k^3(\log k)(\log n_2)^2. \tag{41}$$

By combining the above inequality with Lemma 9 for $(n,m):=(n_2,m_2)$ together with the fact that $n_2<m_2$, we get

$$\begin{aligned}m_2 &< 2.6\times 10^{13}k^4(\log k)^2(\log\delta)(1+\log m_2)\\ &< 2.6\times 10^{13}k^4(\log k)^2(2.5\times 10^2k^3(\log k))(\log m_2)^2(1.92\log m_2)\\ &< 1.25\times 10^{16}k^7(\log k)^3(\log m_2)^3.\end{aligned} \tag{42}$$

In the above, we used that $1+\log m_2 \le 1.92\log m_2$ holds for all $m_2 \ge 3$. We now apply Lemma 6 with $m := 3$ and $T := 1.25\times 10^{16}k^7(\log k)^3$ (which satisfies the hypothesis $T > (4\cdot m^2)^m$), to get

$$\begin{aligned} m_2 &< 8\times 1.25\times 10^{16}k^7(\log k)^3(\log T)^3 \\ &< 10^{17}k^7(\log k)^3(7\log k+3\log\log k+\log(1.25\times 10^{16}))^3 \\ &< 10^{17}\times(4.1\times 10^5)k^7(\log k)^6 \\ &< 4.1\times 10^{22}k^7(\log k)^6. \end{aligned} \tag{43}$$

In the above calculation, we used that

$$\left(\frac{7\log k+3\log\log k+\log(10^{16})}{\log k}\right)^3 < 4.1\times 10^5 \qquad \text{for all} \qquad k\ge 4.$$

By substituting the upper bound (43) for m_2 in the first inequality of Lemma 9, we get

$$\begin{aligned} n_2 &< 1.7\times 10^{13}k^4(\log k)^2(1+\log m_2) \\ &< 1.7\times 10^{13}k^4(\log k)^2(1+\log(4.1\times 10^{22})+7\log k+6\log\log k) \\ &< 1.7\times 10^{13}\times 48k^4(\log k)^3 \\ &< 8.2\times 10^{14}k^4(\log k)^3, \end{aligned} \tag{44}$$

where we used the fact that

$$\frac{7\log k+6\log\log k+\log(4.1\times 10^{22})+1}{\log k} < 48 \quad \text{for all} \quad k\ge 4.$$

Finally, if we substitute the upper bound (44) for n_2 into the inequality (39), we get

$$\begin{aligned} (m_1+1)\log\alpha &< 2.5\times 10^2k^3(\log k)(\log n_2)^2 \\ &< 2.5\times 10^2k^3(\log k)(1+\log(4\times 10^{16})+4\log k+3\log\log k)^2 \\ &< 2.5\times 10^2(9.2\times 10^2)k^3(\log k)^3 \\ &< 2.3\times 10^5k^3(\log k)^3. \end{aligned} \tag{45}$$

In the above, we used that

$$\left(\frac{4\log k+3\log\log k+\log(3.4\times 10^{16})+1}{\log k}\right)^2 < 9.2\times 10^2 \qquad \text{for all} \qquad k\ge 4.$$

Thus, using $\alpha > 1.927$, we get

$$m_1 < 3.6\times 10^5k^3(\log k)^3. \tag{46}$$

Thus, inequalities (43), (44), (46) give upper bounds for m_2, n_2 and m_1, respectively, in the case in which the maximum in the right–hand side of inequality (34) is $\log(1.5n_2)$. Comparing inequalities (43) with (38), (44) with (35), and (45) with (37), respectively, we conclude that (43), (44) and (46) always hold. Let us summarise what we have proved again, which are the bounds (43), (44) and (46).

Lemma 10. *If $x_{n_j} = F^{(k)}_{m_j}$ for $j \in \{1,2\}$ with $2 \le m_1 < m_2$, and $n_1 < n_2$, then*

$$m_1 < 3.6 \times 10^5 k^3 (\log k)^3, \quad m_2 < 4.1 \times 10^{22} k^7 (\log k)^6, \quad n_2 < 8.2 \times 10^{14} k^4 (\log k)^3.$$

Since $n_1 \le m_1$, the above lemma gives bounds for all of m_1, n_1, m_2, n_2 in terms of k only.

7. The case $k > 500$

Lemma 11. *If $k > 500$, then*

$$8m_2^3 < 2^k. \tag{47}$$

Proof. In light of the upper bound given by Lemma 10 on m_2, this is implied by

$$4.1 \times 10^{22} k^7 (\log k)^6 < 2^{k/3-1},$$

which indeed holds for all $k \ge 462$ as confirmed by *Mathematica*. □

From now on, we assume that $k > 500$. Thus, (47) holds. The main result of this section is the following.

Lemma 12. *If $k > 500$, then $m_1 \le k+1$. In particular, $x_{n_1} = F^{(k)}_{m_1} = 2^{m_1-2}$, and $n_1 = 1$.*

For the proof, we go to Lemma 3 and write for $m := m_j$ with $j = 1, 2$ the following approximations

$$F^{(k)}_m = 2^{m-2}(1+\zeta_m) = 2^{m-2}\left(1 + \delta_m\left(\frac{k-m}{2^{k+1}}\right) + \gamma_m\right), \tag{48}$$

where $\delta_m \in \{0, 1\}$ and

$$|\zeta_m| \le \frac{m}{2^{k+1}} + \frac{m^2}{2^{2k+2}} + \frac{4m^3}{2^{3k+3}} < \frac{1}{2^{2k/3}}\left(\frac{1}{2} + \frac{1}{2^{2+2k/3}} + \frac{1}{2^{4+4k/3}}\right) < \frac{1}{2^{2k/3}}, \tag{49}$$

$$|\gamma_m| \le \frac{m^2}{2^{2k+2}} + \frac{4m^3}{2^{3k+3}} < \frac{1}{2^{4k/3}}\left(\frac{1}{2^2} + \frac{1}{2^{2k/3+4}}\right) < \frac{1}{2^{4k/3}},$$

where we used that $m < 2^{k/3-1}$ (see (47)) and $k \ge 4$. We then write

$$|F_m^{(k)} - x_n| = 0,$$

from where we deduce

$$|2^{m-1}(1+\zeta_m) - \delta^n| = \frac{1}{\delta^n}. \tag{50}$$

Thus,

$$|2^{m-1} - \delta^n| = \frac{1}{\delta^n} + |\zeta_m| 2^{m-1},$$

so

$$|1 - \delta^n 2^{-(m-1)}| = \frac{1}{2^{m-1}\delta^n} + |\zeta_m| < \frac{1}{2^m} + \frac{1}{2^{2k/3}} \le \frac{1}{2^{\min\{2k/3-1, m-1\}}}. \tag{51}$$

In the above, we used that $\delta^n \ge \delta \ge 1 + \sqrt{2} > 2$. The right–hand side above is $< 1/2$, so we may pass to logarithmic form as in (28) to get that

$$|n \log \delta - (m-1)\log 2| < \frac{1}{2^{\min\{2k/3-2, m-2\}}}. \tag{52}$$

We write the above inequality for (n_1, m_1) and (n_2, m_2) cross-multiply the one for (n_1, m_1) by n_2 and the one for (n_2, m_2) by n_1 and subtract them to get

$$|(n_1(m_2-1) - n_2(m_1-1))\log 2| < \frac{n_2}{2^{\min\{2k/3-2, m_1-2\}}} + \frac{n_1}{2^{\min\{2k/3-2, m_2-2\}}}.$$

Assume $n_1(m_2-1) \neq n_2(m_1-1)$. Then the left–hand side above is $\ge \log 2 > 1/2$. In particular, either

$$2^{\min\{2k/3-2, m_1-2\}} < 4n_2 \quad \text{or} \quad 2^{\min\{2k/3-2, m_2-2\}} < 4n_1.$$

The first one is weaker than the second one and is implied by the second one, so the first one must hold. If the minimum is $2k/3-2$, we then get

$$2^{2k/3-2} \le 4n_2 < 2^{k/3+1},$$

because $n_2 \le m_2 < 2^{k/3-1}$, so $2k/3 - 2 < k/3 + 1$, or $k < 9$, a contradiction. Thus,

$$2^{m_1-2} < 4n_2 < 2^{k/3+1},$$

getting

$$m_1 < k/3 + 3 < k + 2.$$

Thus, by Example 1 (i), we get that $x_{n_1} = F_{m_1}^{(k)} = 2^{m_1-2}$, which by Lemma 7, implies that $n_1 = 1$.

So, we got the following partial result.

Lemma 13. *For $k > 500$, either $n_1 = 1$ and $m_1 < k/3+3$, or $n_1/n_2 = (m_1-1)/(m_2-1)$.*

To finish the proof of Lemma 12, assume for a contradiction that $m_1 \geq k+2$. Lemma 13 shows that $n_1/n_2 = (m_1-1)/(m_2-1)$. Further, in (48), we have $\delta_{m_1} = \delta_{m_2} = 1$. Thus, we can rewrite equation (50) using γ_m for both $m \in \{m_1, m_2\}$. We get

$$\left| 2^{m-1}\left(1+\frac{k-m}{2^{k+1}}+\gamma_m\right) - \delta^n \right| = \frac{1}{\delta^n},$$

so

$$\left| 2^{m-1}\left(1+\frac{k-m}{2^{k+1}}\right) - \delta^n \right| \leq \frac{1}{\delta^n} + 2^{m-1}|\gamma_m|,$$

therefore

$$\left| \left(1+\frac{k-m}{2^{k+1}}\right) - \delta^n 2^{-(m-1)} \right| \leq \frac{1}{2^{m-1}\delta^n} + |\gamma_m|.$$

Now $\delta^n \geq \alpha^{m-2}$ by the first inequality in (23). Thus,

$$2^{m-1}\delta^n \geq 2^{m-1}\alpha^{m-2} \geq 2^{m-1}2^{0.9(m-2)} > 2^{1.9m-3} > 2^{1.9k} > 2^{4k/3},$$

where we used the fact that $m \geq k+2$ and that $\alpha \geq \alpha_4 = 1.9275\ldots > 2^{0.9}$. Since also $|\gamma_m| \leq \frac{1}{2^{4k/3}}$, we get that

$$\left| \left(1+\frac{k-m}{2^{k+1}}\right) - \delta^n 2^{-(m-1)} \right| < \frac{2}{2^{4k/3}}.$$

The expression $1+(k-m)/2^{k+1}$ is in $[1/2, 2]$. Thus,

$$\left| 1 - \delta^n 2^{-(m-1)}(1+(k-m)/2^{k+1})^{-1} \right| < \frac{4}{2^{4k/3}}.$$

The right–hand side is $< 1/2$ for all $k \geq 4$. We pass to logarithms via implication (28) getting that

$$\left| n\log\delta - (m-1)\log 2 - \log\left(1+\frac{k-m}{2^{k+1}}\right) \right| < \frac{8}{2^{4k/3}}.$$

We evaluate the above in $(n,m) := (n_j, m_j)$ for $j = 1, 2$. We multiply the expression for $j = 1$ with n_2, the one with $j = 2$ with n_1, subtract them and use $n_2(m_1-1) = n_1(m_2-1)$, to get

$$\left| n_1 \log\left(1+\frac{k-m_2}{2^{k+1}}\right) - n_2\left(1+\frac{k-m_1}{2^{k+1}}\right) \right| < \frac{16 n_2}{2^{4k/3}}. \tag{53}$$

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One checks that in our range we have

$$16n_2 < 2^{k/4}. \tag{54}$$

By Lemma 10, this is fulfilled if

$$16 \times 8.2 \times 10^{14} k^4 (\log k)^3 < 2^{k/4},$$

and *Mathematica* checks that this is so for all $k \geq 346$. Thus, inequality (53) implies

$$\left| n_1 \log\left(1 + \frac{k - m_2}{2^{k+1}}\right) - n_2 \left(1 + \frac{k - m_1}{2^{k+1}}\right)\right| < \frac{2^{k/4}}{2^{4k/3}} < \frac{1}{2^{13k/12}}.$$

Using the fact that the inequality

$$|\log(1 + x) - x| < 2x^2 \quad \text{holds for} \quad |x| < 1/2,$$

with $x_j := (k - m_j)/2^{k+1}$ for $j = 1, 2$, and noting that $2x_j^2 < 2m_2^2/2^{2k+2}$ holds for both $j = 1, 2$, we get

$$\left| \frac{n_1(k - m_2)}{2^{k+1}} - \frac{n_2(k - m_1)}{2^{k+1}} \right| < \frac{4n_2 m_2^2}{2^{2k+2}} + \frac{1}{2^{13k/12}}.$$

In the right–hand side, we have

$$\frac{4n_2 m_2^2}{2^{2k+2}} < \frac{2^{2+(k/4-4)+2(k/3-1)}}{2^{2k+2}} = \frac{1}{2^{13k/12+5}}.$$

Hence,

$$\left| \frac{n_1(k - m_2)}{2^{k+1}} - \frac{n_2(k - m_1)}{2^{k+1}} \right| < \frac{2}{2^{13k/12}},$$

which implies

$$|n_1(k - m_2) - n_2(k - m_1)| < \frac{4}{2^{k/12}}.$$

Since $k > 500$, the right–hand side is smaller than 1. Since the left–hand side is an integer, it must be the zero integer. Thus,

$$n_1/n_2 = (k - m_1)/(k - m_2).$$

Since also $n_1/n_2 = (m_1 - 1)/(m_2 - 1)$, we get that $(m_1 - 1)/(m_2 - 1) = (m_1 - k)/(m_2 - k)$, or $(m_1 - 1)/(m_1 - k) = (m_2 - 1)/(m_2 - k)$. This gives $1 + (k - 1)/(m_1 - k) = 1 + (k - 1)/(m_2 - k)$, so $m_1 = m_2$, a contradiction.

Thus, $m_1 \leq k + 1$. By Example 1 (i), we get that $x_{n_1} = 2^{m_1 - 2}$, which by Lemma 7 implies that $n_1 = 1$. This finishes the proof of Lemma 12.

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8. The case $m_1 > 376$

Since $k > 500$, we know, by Lemma 12, that $m_1 \leq k+1$ and $n_1 = 1$. In this section, we prove that if $m_1 > 376$, then the only solutions are the ones shown at (i) and (ii) of the Theorem 1. This finishes the proof of Theorem 1 in the case $k > 500$ and $m_1 > 376$. The remaining cases are handled computationally in the next section.

8.1. A lower bound for m_1 in terms of m_2

The main goal of this subsection is to prove the following result.

Lemma 14. *Assume that $m_1 > 376$. Then $2^{m_1-6} > \max\{k^4, n_2^2\}$.*

Proof. Assume $m_1 > 376$. We evaluate (51) in $(n, m) := (n_2, m_2)$. Further, by Lemma 7, x_{n_2} is not a power of 2, so $m_2 \geq k+2$, therefore $\min\{2k/3-2, m_2-2\} = 2k/3-2$, getting

$$|n_2 \log \delta - (m_2-1)\log 2| < \frac{1}{2^{2k/3-2}}. \tag{55}$$

We write a lower bound for the left–hand side using Theorem 3. Let

$$\Lambda := n_2 \log \delta - (m_2-1)\log 2. \tag{56}$$

We have

$$\gamma_1 = \delta, \quad \gamma_2 = 2, \quad b_1 = n_2, \quad b_2 = -(m_2-1).$$

We have $\mathbb{K} := \mathbb{Q}(\delta)$ has $D = 2$. Further, $h(\gamma_1) = (\log \delta)/2$ and $h(\gamma_2) = \log 2$. Thus, we can take $\log B_1 = (\log \delta)/2$, $\log B_2 = \log 2$,

$$b' = \frac{n_2}{2\log 2} + \frac{m_2-1}{\log \delta} < m_2\left(\frac{1}{2\log 2} + \frac{1}{\log(1+\sqrt{2})}\right) < 2m_2.$$

Furthermore, Theorem 3 is applicable since γ_1 and γ_2 are real positive and multiplicatively independent (this last condition follows because δ is a unit and 2 isn't). Theorem 3 shows that

$$\log|\Lambda| > -24.34 \cdot 2^4 E^2 (\log \delta/2)\log 2 > -195\log 2(\log \delta)E^2, \quad E := \max\{\log(3m_2), 10.5\}^2,$$

where we used $\log(3m_2) > 0.14 + \log(2m_2) > 0.14 + \log b'$. Thus,

$$|\Lambda| > 2^{-195(\log \delta)E^2}. \tag{57}$$

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Comparing (55) and (57), we get

$$195(\log\delta)E^2 > 2k/3 - 2. \tag{58}$$

Since

$$2^{m_1-1} = 2x_1 = \delta + \frac{\varepsilon}{\delta} > \frac{\delta}{2},$$

we get $\delta < 2^{m_1}$, so $\log\delta < m_1\log 2$. Thus,

$$(m_1\log 2)(195E^2) > 2k/3 - 2.$$

Now let us assume that in fact the inequality $2^{m_1-6} < \max\{k^4, n_2^2\}$ holds. Assume first that the above maximum is n_2^2. Then $m_1\log 2 < \log(2^6 n_2^2)$. We thus get that

$$2k/3 - 2 < 195\log(64n_2^2)E^2.$$

Since by Lemma 10, $64n_2^2 < 64 \times 8.2^2 \times 10^{28}k^8(\log k)^6$, and $3m_2 < 12.3 \times 10^{22}k^7(\log k)^5$, we get that

$$2k/3 - 2 < 195\log(64 \times 8.2^2 \times 10^{28}k^8(\log k)^6)\max\{10.5, \log(12.3 \times 10^{22}k^7(\log k)^5)\}^2,$$

which gives $k < 4 \times 10^9$. Thus,

$$n_2 < 8.2 \times 10^{14}k^4(\log k)^3 < 5 \times 10^{55},$$

and since

$$2^{m_1-6} \le n_2^2 < (5 \times 10^{55})^2,$$

we get $m_1 < 6 + 2(\log 5 \times 10^{55})/(\log 2) < 377$, contradicting the fact that $m_1 > 376$. This was in the case $n_2 \ge k^2$. But if $n_2 < k^2$, then $\max\{n_2^2, k^4\} = k^4$ and the same argument gives us an even smaller bound on k; hence, on m_1. This contradiction finishes the proof of this lemma. $\square$

8.2. We have $m_2 - 1 = n_2(m_1 - 1)$

The aim of this subsection is to prove the following result.

Lemma 15. *If $k > 500$ and $m_1 > 376$, then $n_2(m_1 - 1) = m_2 - 1$.*

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For the proof, we write

$$2x_1 = \delta + \frac{\epsilon}{\delta} = 2F_{m_1}^{(k)} = 2^{m_1-1};$$
$$2x_{n_2} = \delta^{n_2} + \left(\frac{\epsilon}{\delta}\right)^{n_2} = 2F_{m_2}^{(k)}.$$

Thus,

$$\begin{aligned} 2F_{m_2}^{(k)} &= \sum_{i=0}^{\lfloor n_2/2 \rfloor} \frac{n_2}{n_2-i}\binom{n_2-i}{i}(-\epsilon)^i 2^{(m_1-1)(n_2-2i)} \\ &= 2^{(m_1-1)n_2}\left(1+\sum_{i=1}^{\lfloor n_2/2\rfloor}\frac{n_2}{n_2-i}\binom{n_2-i}{i}\left(-\frac{\epsilon}{2^{2(m_1-1)}}\right)^i\right). \end{aligned}$$

Note that

$$\frac{n_2}{n_2-i}\binom{n_2-i}{i} < n_2^i.$$

Thus,

$$\left|\frac{n_2}{n_2-i}\binom{n_2-i}{i}\left(-\frac{\epsilon}{2^{2(m_1-1)}}\right)^i\right| < \left(\frac{n_2}{2^{2(m_1-1)}}\right)^i. \tag{59}$$

Since $m_1 > 376$, we have $2^{m_1-6} > n_2^2$ by Lemma 14. In this case, (59) tells us that

$$\left|\frac{n_2}{n_2-i}\binom{n_2-i}{i}\left(-\frac{\epsilon}{2^{2(m_1-1)}}\right)^i\right| < \frac{1}{2^{1.5m_1 i}}\left(\frac{n_2}{2^{0.5m_1-2}}\right)^i < \frac{1}{2^{1.5m_1 i}}\left(\frac{1}{2^i}\right). \tag{60}$$

Combining (60) with (49),

$$\begin{aligned} 2x_{n_2} &= 2^{(m_1-1)n_2}\left(1+\sum_{i=1}^{\lfloor n_2/2\rfloor}\frac{n_2}{n_2-i}\binom{n_2-i}{i}\left(-\frac{\epsilon}{2^{2(m_1-1)}}\right)^i\right) \\ &:= 2^{(m_1-1)n_2}(1+\zeta'_{n_2}) \\ 2F_{m_2}^{(k)} &= 2^{m_2-1}\left(1+\zeta_{m_2}\right), \end{aligned}$$

where

$$\zeta'_{n_2} := \sum_{i=1}^{\lfloor n_2/2\rfloor}\frac{n_2}{n_2-i}\binom{n_2-i}{i}\left(-\frac{\epsilon}{2^{2(m_1-1)}}\right)^i.$$

Since $2x_{n_2} = 2F_{m_2}^{(k)}$, we then have

$$|2^{(m_1-1)n_2} - 2^{m_2-1}| \le 2^{(m_1-1)n_2}|\zeta'_{n_2}| + 2^{m_2-1}|\zeta_{m_2}|.$$

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If $(m_1-1)n_2 \neq m_2-1$, then putting $R := \max\{2^{(m_1-1)n_2}, 2^{m_2-1}\}$, the left–hand side above is $\geq R/2$, while the right-side above is $< R/2$, since

$$|\zeta_{m_2}| < \frac{1}{2^{2k/3}} < \frac{1}{4} \quad \text{and} \quad |\zeta'_{n_2}| < \sum_{i\geq 1} \frac{1}{2^{1.5m_1 i}} \left(\frac{1}{2^i}\right) < \frac{1}{2^{1.5m_1}} \sum_{i\geq 1} \frac{1}{2^i} < \frac{1}{2^{1.5m_1}} < \frac{1}{4}.$$

This contradiction shows that $m_2-1 = n_2(m_1-1)$, which finishes the proof of Lemma 15.

8.3. The case $n_2 = 2$

By Lemma 15, we get $m_2 = 2m_1 - 1$. Since $m_1 \leq k+1$, we get that $m_2 \leq 2k+1$. Also, $m_2 \geq k+2$. By Example 1 (ii), we have

$$F_{m_2}^{(k)} = 2^{m_2-2} - (m_2-k)2^{m_2-k-3} = x_2 = 2x_1^2 - \epsilon = 2(2^{m_1-2})^2 - \epsilon.$$

We thus get

$$2^{2m_1-3} - (2m_1-k-1)2^{2m_1-k-4} = 2^{2m_1-3} - \epsilon.$$

We get that the $\epsilon = 1$, and further $(2m_1-k-1)2^{2m_1-k-4} = 1$, so $m_1 = (k+3)/2$. This gives the parametric family (i) from Theorem 1.

8.4. The case $n_2 = 3$

By Lemma 15, we get $m_2 = 3(m_1-1)+1 = 3m_1-2$. Since $m_1 \leq k+1$, we get that $m_2 = 3m_1 - 2 \leq 3k+1$. Further, $m_2 \geq k+2$. If $m_2 \in [k+2, 2k+2]$, then, by Example 1 (ii), we have

$$F_{m_2}^{(k)} = 2^{m_2-2} - (m_2-k)2^{m_2-k-3} = x_3 = 4x_1^3 - 3\epsilon x_1 = 4(2^{m_1-2})^3 - 3\epsilon 2^{m_1-2},$$

so $\epsilon = 1$, and $(3m_1-k-2)2^{3m_1-k-5} = 3 \times 2^{m_1-2}$. This gives

$$(3m_1-k-2)2^{2m_1-k-3} = 3.$$

By unique factorisation, we get

$$3m_1 - k - 2 = 3 \times 2^a \qquad \text{and} \qquad 2m_1 - k - 3 = -a$$

for some integer $a \geq 0$. Solving, we get

$$m_1 = 3 \times 2^a + a - 1,$$
$$k = 3 \times 2^{a+1} + 3a - 5,$$

and then $m_2 = 3m_1 - 2 = 9 \times 2^a + 3a - 5$. The case $a = 0$ gives $k = 1$, which is not convenient so $a \geq 1$. This is the parametric family (ii).

It can also be the case that $m_2 \in [2k+3, 3k+1]$. By Example 1 (iii), we get

$$4(2^{m_1-2})^3 - 3\epsilon 2^{m_1-2} = 2^{m_2-2} - (m_2-k)2^{m_2-k-3} - (m_2-2k+1)(m_2-2k-2)2^{m_2-2k-5}.$$

This leads to

$$3\epsilon 2^{m_1-2} = (3m_1 - k - 2)2^{3m_1-k-5} - (3m_1 - 2k - 1)(3m_1 - 2k - 4)2^{3m_1-2k-7}.$$

Simplifying 2^{3m_1-2k-7} from both sides of the above equation we get

$$3\epsilon 2^{2k+5-2m_1} = (3m_1 - k - 2)2^{k+2} - (3m_1 - 2k - 1)(3m_1 - 2k - 4).$$

Since $m_2 = 3m_1 - 2 \geq 2k+3$, it follows that $m_2 \geq (2k+5)/3$, so $2k+5-2m_1 \leq (2k+5)/3$. It thus follows, by the absolute value inequality, that

$$\begin{aligned} 2^{k+2} < (3m_1 - k - 2)2^{k+2} &\leq 3 \cdot 2^{2k+5-2m_1} + (3m_1 - 2k - 1)(3m_1 - 2k - 4) \\ &\leq 3 \cdot 2^{(2k+5)/3} + (k+2)(k-1), \end{aligned}$$

an inequality which fails for $k \geq 5$. Thus, there are no other solutions in this range for $n_2 = 3$ except for the ones indicated in (ii) of Theorem 1.

8.5. The case $n_2 = 4$

In this case, we have $m_2 = 4(m_1 - 1) + 1 = 4m_1 - 3$. Since $m_1 \leq k+1$, we have $m_2 \leq 4k+1$. Note that

$$x_4 = 2x_2^2 - 1 = 2(2x_1^2 - \epsilon)^2 - 1 = 8x_1^4 - 8\epsilon x_1^2 + 1 = 8(2^{m_1-2})^4 - 8\epsilon(2^{m_1-2})^2 + 1 \quad (61)$$

is odd. Assume first that $m_2 \in [k+2, 2k+2]$. We then have, by Example 1,

$$F_{m_2}^{(k)} = 2^{m_2-2} - (m_2 - k)2^{m_2-k-3} = 2^{4m_1-5} - (4m_1 - k - 3)2^{4m_1-k-6}. \quad (62)$$

Comparing (62) with (61), we get

$$(4m_1 - k - 3)2^{4m_1-k-6} = \epsilon 2^{2m_1-1} - 1.$$

First, $\epsilon = 1$. Second, the right–hand side above is odd. This implies that the left–hand side is also odd. Thus, the left–hand side is in $\{1, 3\}$. This is impossible since the right–hand side is at least 2^{753}. Thus, this instance does not give us any solution.

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Assume next that $m_2 \in [2k+3, 3k+3]$. Then

$$\begin{aligned} F_{m_2}^{(k)} &= 2^{m_2-2} - (m_2-k)2^{m_2-k-3} + (m_2-2k+1)(m_2-2k-2)2^{m_2-2k-5} \\ &= 8(2^{m_1-2})^4 - 8\epsilon(2^{m_1-2})^2 + 1. \end{aligned}$$

Identifying, we get

$$(4m_1-k-3)2^{4m_1-k-6} - (4m_1-2k-2)(4m_1-2k-5)2^{4m_1-2k-8} = \epsilon 2^{2m_1-1} - 1.$$

Note that $4m_1-2k-8$ is even. If $4m_1-2k-8 \geq 0$, then the left–hand side is even and the right–hand side is odd, a contradiction. Thus, we must have $4m_1-2k-8 = -2$. This gives $4m_1 = 2k+6$, so $m_1 = (k+3)/2$. We thus get

$$(k+3)2^k - 1 = \epsilon 2^{k+2} - 1.$$

This implies that $\epsilon = 1$ and $(k+3)2^k = 2^{k+2}$, which leads to $k+3 = 4$, so $k = 1$, which is impossible. Thus, this instance does not give us a solution either.

Assume finally that $m_2 \in [3k+4, 4k+1]$. Applying the Cooper-Howard formula from Lemma 2, we get

$$F_{m_2}^{(k)} = 2^{m_2-2} + \sum_{j=1}^{3} C_{m_2,j} 2^{m_2-(k+1)j-2}.$$

Eliminating the main term in the equality $F_{m_2}^{(k)} = x_4$ and changing signs in the remaining equation, we get

$$\sum_{j=1}^{3} -C_{m_2,j} 2^{m_2-(k+1)j-2} = \epsilon 2^{2m_1-1} - 1. \tag{63}$$

At $j = 3$, the exponent of 2 is $m_2 - 3j - 5$. If this is positive, the left hand side is even and the right–hand side is odd, a contradiction. Thus, $m_2 \in \{3k+4, 3k+5\}$. In this case,

$$-C_{m_2,3} 2^{m_2-3k-5} = \left(\binom{m_2-3k}{3} - \binom{m_2-3k-2}{1} \right) 2^{m_2-3k-5} \in \{1, 7\}.$$

For $j \in \{1, 2\}$, $m_2 - j(k+1) - 2 \geq m_2 - 2k - 4 \geq k > 500$. Thus, the left–hand side in (63) is congruent to $1, 7 \pmod{2^{500}}$, while the right–hand side of (63) is congruent to $-1 \pmod{2^{500}}$ because $m_1 > 500$. We thus get $1, 7 \equiv -1 \pmod{2^{500}}$, a contradiction. Hence, there are no solutions with $n_2 = 4$.

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8.6. The case $n_2 \geq 5$

The goal here is to prove the following result.

Lemma 16. *If $k > 500$ and $m_1 > 376$, then there is no solution with $n_2 \geq 5$.*

We write again the two series for $2x_{n_2} = 2F_{m_2}^{(k)}$:

$$2F_{m_2}^{(k)} = 2^{m_2-1}\left(1+\frac{k-m_2}{2^{k+1}}+\gamma_{m_2}\right) = 2^{n_2(m_1-1)}\left(1+\frac{-\epsilon n_2}{2^{2(m_1-1)}}+\gamma'_{n_2}\right),$$

where

$$|\gamma_{m_2}| < \frac{1}{2^{4k/3}} \qquad \text{and} \qquad |\gamma'_{n_2}| \leq \sum_{i\geq 2}\frac{1}{2^{1.5m_1 i}}\left(\frac{1}{2^i}\right) < \frac{1}{2^{3m_1}}.$$

By Lemma 15, we have $m_2 - 1 = n_2(m_1 - 1)$ so the leading powers of 2 above cancel, and we get

$$\frac{k-m_2}{2^{k+1}}+\gamma_{m_2} = \frac{-\epsilon n_2}{2^{2(m_1-1)}}+\gamma'_{n_2}.$$

We would like to derive that this implies that

$$\frac{k-m_2}{2^{k+1}} = \frac{-\epsilon n_2}{2^{2(m_1-1)}}. \tag{64}$$

Well, we distinguish two cases.

Case 1. *Suppose that $2(m_1 - 1) \geq k + 1$.*

We then write

$$\left|\frac{k-m_2}{2^{k+1}}+\frac{n_2\epsilon}{2^{2(m_1-1)}}\right| \leq |\gamma_{m_2}|+|\gamma'_{n_2}| \leq \frac{1}{2^{4k/3}}+\frac{1}{2^{3m_1}}. \tag{65}$$

Since $2m_1 \geq k + 3$, we get $3m_1 > 3k/2 > 4k/3$. Thus,

$$\left|\frac{k-m_2}{2^{k+1}}+\frac{n_2\epsilon}{2^{2(m_1-1)}}\right| \leq \frac{2}{2^{4k/3}}. \tag{66}$$

Suppose further that $m_1 \leq 2k/3$. Multiplying inequality (66) across by $2^{2(m_1-1)}$, we get

$$|2^{2(m_1-1)-(k+1)}(k-m_2)+\epsilon n_2| \leq \frac{2^{2m_1-1}}{2^{4k/3}} \leq \frac{1}{2},$$

and since the left–hand side above is an integer, it must be the zero integer. This proves (64) in the current case assuming that $m_1 \leq 2k/3$. If $m_1 > 2k/3$, we deduce from (66) that

$$\frac{m_2-k}{2^{k+1}} < \frac{2}{2^{4k/3}} + \frac{n_2}{2^{2(m_1-1)}} < \frac{2+4n_2}{2^{4k/3}} < \frac{5n_2}{2^{4k/3}} < \frac{1}{2^{13k/12}},$$

where in the right–above we used the fact that $8n_2 < 2^{k/4}$ (see (54)). We thus get

$$2 \le m_2 - k < \frac{2^{k+1}}{2^{13k/12}} < \frac{2}{2^{k/12}} < 1,$$

where the right–most inequality holds since $k > 500$. This is a contradiction, so the $m_1 > 2k/3$ cannot occur in this case. This completes the proof of (64) in Case 1.

Case 2. *Assume that* $2(m_1-1) < k+1$.

We then write

$$\frac{n_2}{2^{2(m_1-1)}} \le \frac{m_2-k}{2^{k+1}} + |\gamma_{m_2}| + |\gamma'_{n_2}|.$$

Since $|\gamma_{m_2}| < 1/2^{4k/3} < 1/2^{k+1}$ and $|\gamma'_{n_2}| \le 1/2^{3m_1} < 1/2^{2(m_1-1)}$, we get that

$$\frac{1}{2^{2(m_1-1)}} < \left|\frac{n_2-1}{2^{2(m_1-1)}}\right| \le \frac{n_2}{2^{2(m_1-1)}} - |\gamma'_{n_2}| \le \frac{m_2-k}{2^{k+1}} + |\gamma_{m_2}| < \frac{m_2}{2^{k+1}},$$

where we also used that $n_2 > 1$ and $k \ge 2$. Thus,

$$2^{k+1-2(m_1-1)} < m_2.$$

We now go back to (65) and write that

$$\left|\frac{k-m_2}{2^{k+1}} + \frac{n_2\epsilon}{2^{2(m_1-1)}}\right| < \frac{2}{2^{\min\{4k/3,3m_1\}}}.$$

We multiply across by 2^{k+1} getting

$$|(k-m_2) + 2^{k+1-2(m_1-1)}\epsilon n_2| < \frac{2^{k+2}}{2^{\min\{4k/3,3m_1\}}}.$$

If the minimum on the right above is $4k/3$, then the right–hand side above is smaller than $4/2^{k/3} < 1/2$ since k is large, so the number on the left is zero. If the minimum is $3m_1$, on the right above then

$$|(k-m_2) + 2^{k+1-2(m_1-1)}\epsilon n_2| < \frac{1}{2}\left(\frac{2^{k+1-2(m_1-1)}}{2^{m_1}}\right).$$

Since

$$2^{k+1-2(m_1-1)} < m_2 = n_2(m_1-1) < kn_2 \le \max\{k^2, n_2^2\} < 2^{m_1-6} < 2^{m_1}$$

(here, we used Lemma 14 for the inequality in the right–hand side above), it follows that

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$$|(k-m_2)+2^{k+1-2(m_1-1)}\epsilon n_2|<\frac{1}{2},$$

so again the left–hand side is 0. Since $m_2>k$, this implies that $\epsilon=1$. We record what we just proved.

Lemma 17. *If $k>500$, $m_1>376$ and $n_2\geq 5$, then $m_1\leq k+1$, $n_1=1$, $\epsilon=1$, $m_2-1=n_2(m_1-1)$ and*

$$\frac{m_2-k}{2^{k+1}}=\frac{n_2}{2^{2(m_1-1)}}.$$

We now get an extra relation. First, from Lemma 17, we get that

$$n_2=\begin{cases}2^{2(m_1-1)-(k+1)}(m_2-k) & \text{if} \quad 2(m_1-1)\geq k+1;\\ \frac{m_2-k}{2^{k+1-2(m_1-1)}} & \text{if} \quad 2(m_1-1)<k+1.\end{cases} \tag{67}$$

Since $n_2\geq 5$, we can write more terms.

$$2F_{m_2}^{(k)}=2^{m_2-1}\left(1+\frac{k-m_2}{2^{k+1}}+\delta_{m_2}\frac{(m_2-2k+1)(m_2-2k-2)}{2^{2k+2}}+\eta_{m_2}\right)$$
$$2x_{n_2}=2^{n_2(m_1-1)}\left(1+\frac{-\epsilon n_2}{2^{2(m_1-1)}}+\frac{n_2(n_2-3)}{2^{4(m_1-1)+1}}+\eta'_{n_2}\right)$$

In the formula for $F_{m_2}^{(k)}$, we have $\delta_{m_2}=\zeta_{m_2}=0$ if $m_2\leq 2k+2$. But $m_2\leq 2k+2$ is not possible since then the only terms in the first expansion of $2F_{m_2}^{(k)}$ are the first two which already coincide with the first two terms of the expansion of $2x_{n_2}$, but in the second expansion we have additional terms since $n_2\geq 5$ while in the first we do not, which is a contradiction. Thus, $m_2\geq 2k+3$.

Assume that $2(m_1-1)\geq k+1$. In this case, from (67), we deduce that

$$n_2=2^{2(m_1-1)-(k+1)}(m_2-k)=\frac{m_2-1}{m_1-1}.$$

So, $m_2-k\mid m_2-1$. Thus, $m_2-k\mid (m_2-1)-(m_2-k)=k-1$. This shows that $m_2-k\leq k-1$, so $m_2\leq 2k-1$, a contradiction. Thus, $k+1>2(m_1-1)$.

Simplifying again the power of 2 from the two representations of $2x_{n_2}=2F_{m_2}^{(k)}$ and eliminating the first two terms we get

$$\frac{(m_2-2k+1)(m_2-2k-2)}{2^{2k+3}}+\eta_{m_2}=\frac{n_2(n_2-3)}{2^{4(m_1-1)+1}}+\eta'_{n_2}.$$

Here,

$$|\eta_{m_2}|<\frac{4m_2^3}{2^{3k+3}}<\frac{1}{2^{2k+4}} \quad \text{and} \quad |\eta'_{n_2}|\leq\sum_{i\geq 3}\frac{1}{2^{1.5mi}}\left(\frac{1}{2^i}\right)<\frac{1}{2^{4.5m_1+1}},$$

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by (47) and (60). Thus,

$$\left|\frac{(m_2-2k+1)(m_2-2k-2)}{2^{2k+3}}-\frac{n_2(n_2-3)}{2^{4(m_1-1)+1}}\right| \le |\eta_{m_2}|+|\eta'_{n_2}| < \frac{2}{\min\{2^{2k+4},2^{4.5m_1+1}\}}. \tag{68}$$

Recall that $2(m_1-1)<k+1$. Then, by (67), we have $n_2 \mid m_2-k$. Since also $n_2 \mid m_2-1$, it follows that $n_2 \mid (m_2-1)-(m_2-k)=k-1$. Thus, $n_2<k$, and since $2^{(k+1)-2(m_1-1)}$ is a divisor of n_2, we conclude that $2^{(k+1)-2(m_1-1)}<k$. We multiply (68) across by $2^{2(k+1)}$. We get

$$\left|\frac{(m_2-2k+1)(m_2-2k-2)}{2}-2^{2(k+1)-4(m_1-1)}\frac{n_2(n_2-3)}{2}\right| \le \frac{2^{2k+3}}{\min\{2^{2k+4},2^{4.5m_1+1}\}}.$$

If the minimum above is 2^{2k+4}, then the right–hand side is $<\frac{1}{2}<1$. The left–hand side is an integer, so it equals 0. If the minimum is $2^{4.5m_1+1}$, then we can rewrite it as

$$\frac{2^{2k+3}}{2^{4.5m_1+1}}=\frac{2^{2(k+1)-4(m_1-1)}}{2^{0.5m_1+4}}<\frac{k^2}{2^{0.5m_1+5}}<1.$$

The right–most inequality holds because $2^{m_1-6}>k^4$ by Lemma 14. Hence, the left–hand side above is again 0. We get that

$$(m_2-2k+1)(m_2-2k-2)=2^{2(k+1)-4(m_1-1)}n_2(n_2-3). \tag{69}$$

So, let us record the equations we have:

$$\begin{cases} m_2-1 &= n_2(m_1-1);\\ b &= (k+1)-2(m_1-1);\\ n_2 &= \frac{m_2-k}{2^b};\\ (m_2-2k+1)(m_2-2k-2) &= 2^{2b}n_2(n_2-3), \end{cases} \tag{70}$$

with $b>0$. To finish, we need to prove the following lemma.

Lemma 18. *There are no integer solutions (b,k,m_1,m_2,n_1,n_2) to system (70) with $n_2\ge 5$ in the range $k>500$ and $m_1>376$.*

Now that we are seeing the light at the end of the tunnel, let's prove Lemma 18. As we saw, $n_2 \mid (k-1)$. The last equation in system (70) is

$$\left(\frac{m_2-1}{n_2}-\frac{2(k-1)}{n_2}\right)(m_2-2k-2)=2^{2b}(n_2-3),$$

or, using the first equation in system (70),

$$\left(m_1-1-\frac{2(k-1)}{n_2}\right)(m_2-2k-2)=n_2-3.$$

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Now $n_2 < k$ and $m_1 \le k+1$, so from the first equation $m_2 < k^2$. Since $2^b \mid m_2 - k$, we get that $2^b < k^2$, so $b < 2(\log k)/(\log 2) < 3\log k$. Since $b = (k+1) - 2(m_1 - 1)$, we get that

$$m_1 = \frac{k+3-b}{2} \in \left(\frac{k+3-3\log k}{2}, \frac{k+3}{2} \right).$$

In the last equation in the left, at most one of $m_1 - 1 - 2(k-1)/n_2$ (divisor of $m_2 - 2k + 1$) and $m_2 - 2k - 2$ is even. If the first one is even, then $m_2 - 2k - 2$ is a divisor of $n_2 - 3$. Thus,

$$n_2 - 3 \ge m_2 - 2k - 2 = n_2(m_1 - 1) - 2k - 1 \ge n_2 \left(\frac{k+1-3\log k}{2} \right) - 2k - 1,$$

giving

$$2k - 2 \ge n_2 \left(\frac{k+1-3\log k}{2} - 1 \right) = n_2 \left(\frac{k-1-3\log k}{2} \right).$$

Since $n_2 \ge 5$, we get

$$4k - 4 \ge 5(k - 1 - 3\log k), \qquad \text{or} \qquad k \le 15\log k + 1,$$

giving $k \le 63$, a contradiction. Thus, $2^{2b} \mid m_2 - 2k - 2$. Hence,

$$\left(m_1 - 1 - \frac{2(k-1)}{n_2} \right) \left(\frac{m_2 - 2k - 2}{2^{2b}} \right) = n_2 - 3,$$

and all fractions above are in fact integers. The left–most integer is

$$m_1 - 1 - \frac{2(k-1)}{n_2} \ge \frac{k+1-3\log k}{2} - \frac{2(k-1)}{5} > \frac{k-1}{12} - 3$$

since $k > 500$. Since this number is a divisor of (so, at most as large as) the number $n_2 - 3 = (k-1)/D - 3$ for some integer D, we get that $D \in \{1, 2, \ldots, 11\}$. Thus, $(k-1)/D \in \{1, \ldots, 11\}$, so

$$m_1 - 1 - \frac{2(k-1)}{n_2} \ge \frac{k+1-3\log k}{2} - 22 = \frac{k-43-3\log k}{2}.$$

Now let us look at the integer $(m_2 - 2k - 2)/2^{2b}$. Assume that it is at least 3. We then get

$$3\left(\frac{k-43-3\log k}{2} \right) \le n_2 - 3 \le k - 4, \qquad \text{or} \qquad k \le 121 + 9\log k,$$

and this is false for $k \ge 500$. Thus, $(m_2 - 2k - 2)/2^{2b} \in \{1, 2\}$.

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Assume that $(m_2 - 2k - 2)/2^{2b} = 1$. Then

$$m_1 - 1 - \frac{2(k-1)}{n_2} = n_2 - 3.$$

The number in the left hand side is

$$m_1 - 1 - \frac{2(k-1)}{n_2} \geq \frac{k+1-3\log k}{2} - 22 = \frac{k-43-3\log k}{2} > \frac{k-1}{3} - 3$$

(since $k > 500$) and also

$$m_1 - 1 - \frac{2(k-1)}{n_2} \leq m_1 - 3 \leq \frac{k-3}{2} < k - 4.$$

Thus, writing again $n_2 = (k-1)/D$, we get that

$$n_1 - 3 = \frac{k-1}{D} - 3 \in \left(\frac{k-1}{3} - 3, \frac{k-1}{1} - 3\right),$$

showing that $1 < D < 3$, so $D = 2$. Thus, $n_2 = (k-1)/2$, and we get that

$$\frac{k-7}{2} = \frac{k-1}{2} - 3 = n_2 - 3 = m_1 - 1 - \frac{2(k-1)}{n_2} = m_1 - 1 - 4 = m_1 - 5,$$

so

$$m_1 = \frac{k+3}{2}, \qquad \text{so} \qquad b = 0,$$

which is impossible.

Assume next that $(m_2 - 2k - 2)/2^b = 2$. In this case, we get

$$n_2 - 3 = 2\left(m_1 - 1 - \frac{2(k-1)}{n_2}\right).$$

Proceeding as before, we have

$$\begin{aligned}\frac{k-1}{D} - 3 = n_2 - 3 = 2\left(m_1 - 1 - \frac{2(k-1)}{n_2}\right) &\geq 2\left(\frac{k+1-3\log k}{2} - 22\right) \\ = k - 43 - 3\log k &> \frac{k-1}{2} - 3,\end{aligned}$$

showing that $D < 2$. Thus, $D = 1$ and so $n_2 = k - 1$. Hence,

$$k - 4 = n_2 - 3 = 2\left(m_1 - 1 - \frac{2(k-1)}{n_2}\right) = 2(m_1 - 1 - 2) = 2(m_1 - 3),$$

so

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$$m_1 = \frac{k+2}{2}, \qquad \text{therefore} \qquad b = 1.$$

Thus, $m_2 - 2k - 2 = 2^{2b+1} = 8$. Consequently,

$$8 = (m_2 - 1) - 2k - 1 = n_2(m_1 - 1) - 2k - 1 = \frac{(k-1)k}{2} - 2k - 1 = \frac{k^2 - 5k - 2}{2},$$

giving $k^2 - 5k - 18 = 0$, which is impossible.

So, indeed there are no solutions with $k > 500$ and $m_1 > 376$ other than the ones from (i) and (ii) of Theorem 1. $\square$

9. The computational part $k \leq 500$ or $m_1 \leq 376$

Throughout this section, we make the following definition.

Definition 1. Assume that $k \geq 4$, $x_1 \geq 1$, $\epsilon \in \{\pm 1\}$ are given such that there exist $n_1 \geq 1$ and $m_1 \geq 2$ such that $x_{n_1} = F_{m_1}^{(k)}$. We say that n_1 is minimal if there are no positive integers $n_0 < n_1$ and $m_0 < m_1$ such that the equality $x_{n_0} = F_{m_0}^{(k)}$ also holds.

The aim of this section is to first show that in the range $k \leq 500$ or $m_1 \leq 376$, all solutions of $x_{n_1} = F_{m_1}^{(k)}$ with n_1 minimal have $n_1 = 1$. Then we finish the calculations.

9.1. The case $k \leq 500$

Here, we exploit inequality (33), which we consider convenient to remind:

$$|(n_2 - n_1)\log(2f_k(\alpha)) - (n_1 m_2 - n_2 m_1 + n_2 - n_1)\log\alpha| < \frac{6n_2}{\alpha^{m_1-1}}. \tag{71}$$

Thus,

$$\left|\chi_k - \frac{N}{n_2 - n_1}\right| < \frac{6n_2}{(n_2 - n_1)\alpha^{m_1-1}\log\alpha}, \qquad \chi_k := \frac{\log(2f_k(\alpha))}{\log\alpha}, \tag{72}$$

with $N := n_1 m_2 - n_2 m_1 + n_2 - n_1$. Lemma 10 shows that

$$n_2 - n_1 < n_2 < 8.2 \times 10^{14} k^4 (\log k)^3 < 10^{29}.$$

The right–hand side of (72) can be rewritten as

$$\frac{1}{2(n_2 - n_1)^2}\left(\frac{\alpha^{m_1-1}\log\alpha}{12 n_2 (n_2 - n_1)}\right)^{-1}. \tag{73}$$

Assume that

$$\frac{\alpha^{m_1-1}}{\log\alpha} > 12(8.2 \times 10^{14} k^4 (\log k)^3)^2. \tag{74}$$

Using $\alpha > 1.927$, inequality (74) holds with $k \le 500$ for all $m_1 \ge 203$. In this case, inequalities (73), (72) and Lemma 5 (i) show that $N/(n_1 - n_1) = p_j^{(k)}/q_j^{(k)}$ for some $j \ge 0$, where $p_j^{(k)}/q_j^{(k)}$ is the jth convergent of χ_k. Note that $\chi_k \in (0,1)$ because by Lemma 1 (i), we have $1 < 2f_k(\alpha) < 1.5 < \alpha$.

We distinguish two cases.

Case 1. $N \neq 0$.

In this case, $j \ge 1$. Since

$$n_2 - n_1 \le 10^{29} < F_{150} \le q_{150}^{(k)},$$

where F_{150} is the 150th member of the Fibonacci sequence, it follows that if we take

$$Q := \max\{a_i^{(k)} : 2 \le i \le 150; 4 \le k \le 500\},$$

then Lemma 5 (ii) implies that

$$\frac{1}{(Q+2)(q_j^{(k)})^2} < \left|\chi_k - \frac{N}{n_2 - n_1}\right| < \frac{6n_2}{(n_2 - n_1)\alpha^{m_1-1}\log\alpha}.$$

A computer calculation shows that $Q = 433576$, so $Q + 2 < 10^6$. Hence,

$$\begin{aligned}\alpha^{m_1-1}\log\alpha &< 6n_2(Q+2)(q_j^{(k)})^2(n_2 - n_1) < 6 \times 10^6 n_2^2 \\ &< 6 \times 10^6(8.2 \times 10^{14}500^4(\log 500)^3)^2,\end{aligned}$$

and using $\alpha \ge 1.927$, we get $m_1 \le 221$.

Case 2. $N = 0$.

In this case, inequality (72) gives

$$\alpha^{m_1-1}\log\alpha < 6n_2\chi_k^{-1} < 6 \times (8.2 \times 10^{14}500^4(\log 500)^3)\chi_k^{-1}.$$

A computation with *Mathematica* reveals that $\chi_k^{-1} < 10^{148}$ for $k \le 500$. Feeding this into the above inequality, we get $m_1 \le 720$. Note that since $N = 0$, we also have $n_1(m_2 - 1) = n_2(m_1 - 1)$. In particular, $n_1 = 1$ is not possible in this case.

Let us record what we just proved.

Lemma 19. *If $k \le 500$, then the following hold:*

(i) $m_1 \le 221$;
(ii) $m_1 \in [222, 720]$, *but* $n_1 > 1$.

For reasons that will become clear later, we allow $m \le 1049$ (instead of just $m \le 720$). To continue, assume first that $x_1 \in \{1, 2, 3, \ldots, 20\}$. We then generate all values of $\delta = x_1 + \sqrt{x_1^2 - \epsilon}$ for $\epsilon \in \{\pm 1\}$. We generate $x_{n_1} = (\delta^{n_1} + \eta^{n_1})/2$, where η is the Galois

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conjugate of δ in the quadratic field $\mathbb{Q}(\delta)$, for all $1 \le n \le m \le 1049$ and we test for the equation

$$x_n = F_m^{(k)} \qquad 4 \le k \le 500, \quad 2 \le m \le 1049.$$

The only solutions we find computationally have:

(i) $n = 1$ and $x_1 \in \{1, 2, 4, 8, 15, 16\}$;
(ii) $n = 2$ and $x_2 \in \{31, 127, 511\}$. These are not minimal because $x_2 = 31 = F_7^{(5)}$ has $\epsilon = 1$ and for it $x_1 = 4 = F_7^{(4)}$, $x_2 = 127 = F_9^{(7)}$ has $\epsilon = 1$ and for it $x_1 = 8 = F_9^{(5)}$, while $x_2 = 511 = F_{11}^{(9)}$ has $\epsilon = 1$ and for it $x_1 = 16 = F_{11}^{(6)}$, as stated in (i) of Theorem 1 with $k = 7$, 9, and 11, respectively.
(iii) $n = 3$ and $x_3 = 16336 = F_{19}^{(13)}$. This is not minimal since $x_1 = 16 = F_6^{(13)}$, as stated in (ii) of Theorem 1 with $a = 1$.

Assume now that $x_1 \ge 21$. Then $\delta \ge 21 + \sqrt{440}$. Inequality (24) together with the fact that $m_1 \le 1050$ gives

$$n_1 \le \frac{(m_1 + 1)\log\alpha}{\log\delta} \le \frac{1051 \log 2}{\log(21 + \sqrt{440})},$$

so $n_1 \le 194$. Our next goal is to show that in our range $k \le 500$ and $m \le 1049$, we must have $n \in \{1, 2, 3\}$. For this, assume that $n > 3$. Every positive integer > 3 is either divisible by 4, 6, 9 or a prime $p \ge 5$. Thus, we generate the set

$$\mathcal{B} = \{4, 6, 9, p_k : 3 \le k \le 44\},$$

a set with 45 elements, where p_k is the kth prime. We use the fact that if $a \mid b$, then x_b is the ath solution of the Pell equation whose first (smallest) x-coordinate is $x_{b/a}$ (that is, δ gets replaced by $\delta^{b/a}$). In particular, x_{n_1} is x_b for some $b \in B$ and some value of x_1. Further, say $y = F_m^{(k)}$ for some $m \in [2, 1049]$ and $k \in [3, 500]$. We then need to solve $x_b = y$. Note that if $z \ge 1$ and $n \ge 2$, then

$$(z^n + 1)^{1/n} - z = z\left(\left(1 + \frac{1}{z^n}\right)^{1/n} - 1\right) < \frac{1}{nz^{n-1}} \le \frac{1}{2}. \tag{75}$$

Thus,

$$x_b = \left(x_1 + \sqrt{x_1^2 - \epsilon}\right)^b + \left(x_1 - \sqrt{x_1^2 - \epsilon}\right)^b = 2y$$

implies

$$x_1 + \sqrt{x_1^2 - 1} \in ((2y - 1/2)^{1/b}, (2y + 1/2)^{1/b}).$$

Further, this leads to

$$2x_1 \in ((2y - 1/2)^{1/n} - 1/2, (2y + 1/2)^{1/n} + 1/2).$$

The length of the interval on the right above is, by (75), at most 2, so it contains at most one even integer $2x_1$ and if it contains one, it must be such that

$$x_1 = \left\lfloor \frac{1}{2}\left(\left(2y + \frac{1}{2}\right)^{1/b} + \frac{1}{2}\right)\right\rfloor. \tag{76}$$

So what we did was for each $y = F_m^{(k)}$ and each $b \in B$, we calculated the last 10-digits of the integer shown at (76) (that is, we only calculated it modulo 10^{10}). Then we picked $\epsilon \in \{\pm 1\}$ and generated $\{x_n\}_{n\geq 0}$ as the sequence given by $x_0 := 1$, x_1 given by (76) modulo 10^{10} and $x_{n+1} = (2x_1)x_n - \epsilon x_{n-1} \pmod{10^{10}}$ for all $n \geq 1$. In this way, we never kept more than last 10 digits of x_n. And we checked whether indeed $x_b \equiv y \pmod{10^{10}}$. Unsurprisingly, no solution was found. We used the same program for $n_1 = 2, 3$. For these we got that all solutions of (i) in our range were candidates for $n_1 = 2$ and all solutions (ii) in our range were candidates for $n_1 = 3$. By candidates we meant that we only checked out these equalities modulo 10^{10}. They turn out to be actual solutions for $\epsilon = 1$ (and they are not solutions with $\epsilon = -1$ just because a number of the form $2^{2j+1} - 1$ with $j \geq 2$ cannot be also of the form $2z^2 + 1$ for some integer z, while a number of the form $4x^3 - 3x$ for some integer $x > 1$ then it cannot be also of the form $4z^3 + 3z$ for some integer z). Finally, one word about "recognising" y as number of the form $F_m^{(k)}$. It follows from a result of Bravo and Luca [5] that the equation $F_m^{(k)} = F_n^{(\ell)}$ with $m \geq k+2$, $n \geq \ell+2$ and $k > \ell \geq 4$ has no solutions (m, k, n, ℓ). Thus, if we already know a representation of a representation of y as $F_m^{(k)}$ for some m and $k \geq 4$, then it is unique. In particular, for $j \geq 2$, $F_{2j+3}^{(2j+1)}$ is the only representation of $2^{2j+1} - 1$ as a $F_m^{(k)}$ for some positive integers m and $k \geq 4$.

9.2. The case $m_1 \leq 376$

We may assume that $k > 500$, otherwise we are in the preceding case. Thus, $k > m_1$, so $n_1 = 1$. Thus, $\delta = 2^{m_1-2} + \sqrt{2^{2m_1-2} - \epsilon}$ for all $m_1 \geq 2$ and $\epsilon \in \{\pm 1\}$ (except for $m_1 = 2$, case in which only $\epsilon = 1$ is possible). We now go back to the proof of Lemma 14 to get that the inequality (55), recalled below

$$|n_2 \log \delta - (m_2 - 1)\log 2| < \frac{1}{2^{2k/3-2}} \tag{77}$$

implies (58), namely

$$2k/3 - 2 < 195(\log \delta) \max\{10.5, \log(3m_2)\}^2.$$

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For us, $\log\delta \le m_1 \log 2 \le 376\log 2$. Using also the upper bound from Lemma 10 on m_2, we get

$$2k/3 - 2 < 195 \times 376(\log 2)\max\left\{10.5, \log(3\times 4.1\times 10^{22}k^7(\log k)^6)\right\}^2,$$

leading to $k < 4\times 10^9$. Thus, by Lemma 10 again,

$$n_2 < 8.2\times 10^{14}k^4(\log k)^3 < 8.2\times 10^{14}(4\times 10^9)^4(\log(4\times 10^9))^3 < 10^{58}.$$

Now (77) gives

$$\left|\frac{\log\delta}{\log 2} - \frac{m_2-1}{n_2}\right| < \frac{1}{(\log 2)2^{2k/3-1}n_2}. \tag{78}$$

In our range, the right–hand side above is smaller than $1/(2n_2^2)$. Indeed, this is equivalent to $n_2 < 2^{2k/3-3}(\log 2)$, which holds provided that

$$8.2\times 10^{14}k^4(\log k)^3 < 2^{2k/3-3}(\log 2),$$

which indeed holds for all $k > 500$. Thus, $(m_2-1)/n_2 = p_j/q_j$ is some convergent of $\log\delta/\log 2$. Since its denominator q_j divides n_2 and

$$q_j \le n_2 < 10^{58} < F_{299},$$

where F_{299} is the 299th term of the Fibonacci sequence, it follows that $j \le 298$. We generated the continued fractions of all $\log\delta/\log 2$ for all possibilities for $m_1 \le 376$, $\epsilon\in\{\pm 1\}$ and $j \le 299$ and collected together the obtained values of a_j. The maximum value obtained was 1033566. Hence,

$$\frac{1}{1.1\times 10^7 n_2^2} < \frac{1}{(a_{j+1}+2)n_2^2} < \left|\frac{\log\delta}{\log 2} - \frac{m_2-1}{n_2}\right| < \frac{1}{(\log 2)2^{2k/3-1}n_2},$$

giving

$$2^{2k/3-2}\log 2 < 1.1\times 10^7 n_2 < 1.1\times 10^7\times(8.2\times 10^{14}k^4(\log k)^3),$$

giving $k \le 166$, a contradiction.

Thus, this case leads to no solution, and we must have $k \le 500$, $n_1 = 1$ and $m_1 \le 221$ by Lemma 19.

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9.3. The final computations

Now we go to inequality (29) for $(n,m)=(n_2,m_2)$:

$$|n_2\log\delta-\log(2f_k(\alpha))-(m_2-1)\log\alpha|<\frac{3}{\alpha^{m_2-1}}. \tag{79}$$

We divide both sides by $\log\alpha$ and get

$$|n_2\tau-(m_2-1)-\mu|<\frac{A}{B^{m_2-1}},\quad (\tau,\mu,A,B):=\left(\frac{\log\delta}{\log\alpha},\frac{\log(2f_k(\alpha))}{\log\alpha},\frac{3}{\log(1.92)},1.92\right).$$

We have

$$n_2\leq 8.2\times10^{14}k^4(\log k)^3\leq 8.2\times10^{14}(500)^4(\log 500)^3<1.3\times10^{28}:=M.$$

Since $6M<10^{30}<F_{150}$, we try q_λ for some $\lambda\geq150$. A computer code ran through the range $k\in[4,500]$, $m_1\in[2,221]$ and $\epsilon\in\{\pm1\}$, generated $\delta=2^{m_1-2}+\sqrt{2^{2(m_1-2)}-\epsilon}$ (except for $m_1=2$, when only $\epsilon=1$ is possible), and confirmed the following:

(i) For $4\leq k\leq500$ and $\lambda=200$, we have $\varepsilon>0$ in all cases.
(ii) The maximal value of $1+\lfloor\log(Aq_\lambda/\varepsilon)/\log B\rfloor$ in (i) above is 1049.

Applying Lemma 4, we got that in all cases $m_2\leq1049$ by using q_{200}. By the calculations from Subsection 9.1 where in fact we treated the case $m\leq1049$, we get that (n_2,m_2) is one of the solutions listed in (i) or (ii) of Theorem 1. This finishes the proof of the theorem.

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