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A description of continued fraction expansions of quadratic surds represented by polynomials

R.A. Mollin[a,*] and B. Goddard[b]

[a] Department of Mathematics and Statistics, University of Calgary, 2500 University Drive N.W., MS 588, Calgary, Alberta, Canada T2N 1N4
[b] Mathematics Department, Concordia University, Austin, TX 78705, USA

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Abstract

The principal thrust of this investigation is to provide families of quadratic polynomials $\{D_k(X) = f_k^2X^2 + 2e_kX + C\}_{k\in\mathbb{N}}$, where $e_k^2 - f_k^2C = n$ (for *any* given nonzero integer n) satisfying the property that for any $X \in \mathbb{N}$, the period length $\ell_k = \ell(\sqrt{D_k(X)})$ of the simple continued fraction expansion of $\sqrt{D_k(X)}$ is constant for fixed k and $\lim_{k\to\infty}\ell_k = \infty$. This generalizes, and completes, numerous results in the literature, where the primary focus was upon $|n| = 1$, including the work of this author, and coauthors, in Mollin (Far East J. Math. Sci. Special Vol. 1998, Part III, 257–293; Serdica Math. J. 27 (2001) 317–342) Mollin and Cheng (Math. Rep. Acad. Sci. Canada 24 (2002) 102–108; Internat Math J 2 (2002) 951–956) and Mollin et al. (JP J. Algebra Number Theory Appl. 2 (2002) 47–60).

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1. Introduction

One may find, in a perennial favorite reference Dickson [1, Chapter 12], that there is an informative description of early investigations into infinite parametric families

*Corresponding author. Fax: +1-403-282-5150.
E-mail addresses: ramollin@math.ucalgary.ca (R.A. Mollin), goddardb@concordia.edu (B. Goddard).
URL: http://www.math.ucalgary.ca/~ramollin.

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of nonsquare integers D_k such that a unit (often the fundamental unit) of the order $\mathbb{Z}[\sqrt{D_k}]$ can be explicitly (and easily) determined. This problem has a distinguished and long history. One of the most informative and lucrative vehicles for such investigations is the continued fraction approach. In particular, in the early 1960s, Schinzel [9,10], studied simple continued fraction expansions of $\sqrt{D(X)}$ where $D(X)$ is an integral polynomial. Since our focus in this paper is upon the quadratic case, we highlight what Schinzel found in that arena. In particular, for quadratic integral polynomials, he showed that $\overline{\lim}_{X\to\infty}\ell(\sqrt{D(X)}) = \infty$ when the leading coefficient of $D(X)$ is not a perfect square. He also examined some criteria for $\overline{\lim}_{X\to\infty}\ell(\sqrt{D(X)}) < \infty$. Some examples of the latter are those $D(X) = A^2X^2 + E$ where $E \in \{\pm A, \pm 2A, \pm 4A\}$, called *Extended Richaud–Degert* types (see [2, Chapter 3] for a description of these well-studied objects). In earlier work [8], we looked at $D(X) = A^2X^2 + 2BX + C$, where

$$B^2 - A^2C = 1 \tag{1}$$

for nonsquare $C \in \mathbb{N}$. Therein, we showed that,

$$\text{for any } N \in \mathbb{N}, \text{there exist } A, B, C \in \mathbb{N} \text{ such that } \ell(\sqrt{D(X)}) > N. \tag{2}$$

Moreover,

$$\text{for fixed } A, B, C \in \mathbb{N}, \quad \ell(\sqrt{D(X)}) \text{ is constant for any } X \in \mathbb{N}. \tag{3}$$

For each family, we explicitly found the fundamental unit of the quadratic order $\mathbb{Z}[\sqrt{D(X)}]$, which turns out to be relatively small, and this implies large class number $h_{D(X)}$ for the order (as a result of Siegel's class number result—see [2, p. 173]). In [5–7], we investigated more general types such as those of the form

$$D(X) = (B \pm 1)^2 A^2 X^2 + 2(B \pm 1)^2 X + C,$$

where Eq. (1) holds. Again, we were able to explicitly determine the fundamental unit and proved (2) and (3) for this class of families as well.

In this paper, the goal is to generalize all of the above by substituting the value of 1 in Eq. (1) by *any* nonzero value $n \in \mathbb{Z}$. This yields a significant number of results, in the literature on this topic, as immediate consequences.

2. Preliminaries

We will be dealing with quadratic irrationals $\alpha = (P_0 + \sqrt{D})/Q_0$ where D is a nonsquare positive integer. In this case, the *complete quotients* are given by $(P_j + \sqrt{D})/Q_j$ where the P_j and Q_j are given by the recursive formulae as follows for any $j \geqslant 0$:

$$q_j^{(\alpha)} = \left\lfloor \frac{P_j + \sqrt{D}}{Q_j} \right\rfloor, \tag{4}$$

$$P_{j+1}^{(\alpha)} = q_j^{(\alpha)} Q_j^{(\alpha)} - P_j^{(\alpha)} \tag{5}$$

and

$$D = P_{j+1}^{(\alpha)^2} + Q_j^{(\alpha)} Q_{j+1}^{(\alpha)}. \tag{6}$$

The superscripts (α) are used to distinguish various values of $P_j^{(\alpha)}, Q_j^{(\alpha)}$, etc. for different quadratic irrationals α in a given argument. We will suppress the superscripts when the context is clear. Thus, we may write

$$\alpha = \langle q_0; q_1, \ldots, q_k, (P_{k+1} + \sqrt{D})/Q_{k+1} \rangle. \tag{7}$$

We will also use the following terminology. By a *positive solution* of $x^2 - Dy^2 = n$, $D \in \mathbb{N}$, we mean that $x, y \in \mathbb{N}$. Moreover, if (x_1, y_1) and (x_2, y_2) are both positive solutions of this equation, then the following are equivalent (1) $x_1 < x_2$, (2) $y_1 < y_2$, (3) $x_1 + y_1\sqrt{D} < x_2 + y_2\sqrt{D}$. Hence, there exists a *least* positive solution, which we call the *fundamental solution* of the equation.

Let $C \in \mathbb{N}$, not a perfect square, $n' \in \mathbb{Z}$ nonzero, and let

$$e_0' + f_0'\sqrt{C}$$

be the fundamental solution of

$$x^2 - y^2 C = n'$$

(so $\gcd(e_0', f_0') = 1$). Also, let

$$T_0 + U_0\sqrt{C}$$

be the fundamental solution of the Pell equation

$$x^2 - y^2 C = 1$$

and set

$$T_k + U_k\sqrt{C} = (T_0 + U_0\sqrt{C})^k$$

for any integer $k \geqslant 0$. Also, for $g \in \mathbb{N}$, set

$$e_k + f_k\sqrt{C} = g(e_0' + f_0'\sqrt{C})(T_0 + U_0\sqrt{C})^k,$$

$$e_k/f_k = \langle q_0; q_1, \ldots, q_\ell \rangle = \langle q_0; \vec{E_k} \rangle, \tag{8}$$

with

$$\vec{E}_k = q_1, q_2, \dots, q_\ell$$

and where ℓ is chosen to be odd when $n > 0$, and ℓ is chosen to be even when $n < 0$. (Recall that $\langle q_0; q_1, \dots, q_\ell \rangle = \langle q_0; q_1, \dots, q_{\ell-1}, q_\ell - 1, 1 \rangle$ when $q_\ell \neq 1$.) We also will need the symbol

$$\overleftarrow{E}_k = q_\ell, q_{\ell-1}, \dots, q_1.$$

Thus,

$$e_k^2 - f_k^2 C = g^2 n' = n.$$

Also, set

$$D_k(X) = f_k^2 X^2 + 2e_k X + C.$$

We will need the following.

Proposition 1. *If $\alpha_0 = x_0 + y_0\sqrt{D}$ is a primitive solution of $x^2 - Dy^2 = n$ for a radicand D, then there exists a unique primitive element $\alpha = x + y\sqrt{D}$ such that $\alpha\alpha_0 = P_0 + \sqrt{D}$ with $-|n|/2 < P_0 \leqslant |n|/2$. Thus, if $\beta = (P_0 + \sqrt{D})/|n|$ We say that α_0* belongs to $P_0 = P_0^{(\beta)}$.

Proof. See [4, Proposition 1. p. 269]. □

Lastly, the symbol ε_D will denote the fundamental unit of the real quadratic order $\mathbb{Z}[\sqrt{D}]$.

3. Results

Theorem 1. *With the setup in Section 2, we assume that $|n| \mid 2(f_k^2 X + e_k)$*[1] *with $|n| \neq (f_k^2 X + e_k)$. Then*

(a) If $g^2 = |n|$, (in other words $|n'| = 1$), then

$$\sqrt{D_k(X)} = \left\langle f_k X + q_0; \overline{\vec{E}_k, 2(f_k X + q_0)} \right\rangle \tag{9}$$

and

$$\ell(\sqrt{D_k(X)}) = \ell + 1.$$

[1] Note that this implies $g^2 | 2e_k$.

(b) If $g^2 \neq n$, then

$$\sqrt{D_k(X)} = \left\langle f_k X + q_0; \overline{\overrightarrow{E}_k, \frac{2P_{\ell+1}^{(\sqrt{D_k(X)})}}{|n'|}, \overleftarrow{E}_k, 2(f_k X + q_0)} \right\rangle \tag{10}$$

and

$$\ell(\sqrt{D_k(X)}) = 2\ell + 2.$$

Here,

$$P_{\ell+1}^{(\sqrt{D_k(X)})} = \left\lfloor \frac{P_0^{(\beta)} + \sqrt{D_k(X)}}{|n'|} \right\rfloor |n'| - P_0^{(\beta)}$$

with $P_0^{(\beta)}$ being the unique solution mandated by Proposition 1, which is determined via

$$P_0^{(\beta)} \equiv -((f_k/g)^2 X + e_k/g)(f_k/g)^{-1} (\text{mod}\, |n'|).$$

Furthermore, in either case, the fundamental solution of $x^2 - D_k(X)y^2 = n'$ *is given by*

$$(X_\ell, Y_\ell) = \left(\frac{f_k^2 X + e_k}{g}, \frac{f_k}{g} \right). \tag{11}$$

Proof. For part (a), we assume that $|n| = g^2$. We prove only the result for the case $n' = -1$ since the case $n' = 1$ is similar. By [2, Theorem 3.2.1, p. 78],

$$\varepsilon_{4(f_k/g)^2 D_k(X)} = \frac{f_k^2 X + e_k}{g} + \frac{f_k}{g} \sqrt{D_k(X)}.$$

Moreover, if X_j/Y_j denotes the jth convergent of $\sqrt{D_k(X)}$ and $\bar{\ell} = \ell(\sqrt{D_k(X)})$, then by [3, Corollary 5.3.3, p. 249], there is an $i \in \mathbb{N}$ such that $X_{i\bar{\ell}-1} = (f_k^2 X + e_k)/g$ and $Y_{i\bar{\ell}-1} = f_k/g$. However, by [3, Exercise 5.1.9, p. 227],

$$\langle f_k X + q_0; q_1, \ldots, q_\ell \rangle = \langle f_k X + e_k/f_k \rangle = \frac{f_k^2 X + e_k}{f_k}$$

and there is no smaller value that yields this, so $\ell = i\bar{\ell} - 1$ and $i = 1$ is forced, so $\bar{\ell} = \ell + 1$. This establishes (9). It also shows that

$$\varepsilon_{4D_k(X)} = \varepsilon_{4(f_k/g)^2 D_k(X)},$$

so (11) follows. We have secured part (a).

Now we establish part (b). Since $(f_k/g)^2 D_k(X) = ((f_k^2 X + e_k)/g)^2 - n'$, then $f_k^2 D_k(X)$ is an ERD type. Since $|n'| \neq 1$, then by [2, Theorem 3.2.1, p. 78], the fundamental unit of $\mathbb{Z}[\sqrt{f_k^2 D_k(X)}]$ is

$$\frac{(f_k^2 X + e_k + f_k\sqrt{D_k(X)})^2}{|n|} = \frac{2(f_k^2 X + e_k)^2 - n + 2f_k(f_k^2 X + e_k)\sqrt{D_k(X)}}{|n|}.$$

Thus, by [3, Corollary 5.3.3, p. 249], if we can show that

$$\langle f_k + q_0; \vec{E}_k, 2P_{\ell+1}^{(\sqrt{D_k(X)})}/|n|, \overleftarrow{E}_k \rangle = \frac{2(f_k^2 X + e_k)^2 - n}{2f_k(f_k^2 X + e_k)} = \frac{X_{2\ell-1}}{Y_{2\ell-1}}, \tag{12}$$

we will have shown that the above is also the fundamental unit of $\mathbb{Z}[\sqrt{D_k(X)}]$ and that (10) and (11) are valid. To see this, let $\bar{\ell} = \ell(\sqrt{D_k(X)})$. Then if (12) holds, by [3, Corollary 5.3.3, p. 249], there exists a $j \in \mathbb{N}$ such that $X_{j\bar{\ell}-1} = X_{2\ell-1}$ and $Y_{j\bar{\ell}-1} = Y_{2\ell-1}$, where X_i/Y_i is the ith convergent in the simple continued fraction expansion of $\sqrt{D_k(X)}$. Thus, $j\bar{\ell} - 1 = 2\ell - 1$, so $j = 1$ and $\bar{\ell} = 2\ell + 2$ since no smaller value than $2\ell + 1$ yields (12), which we now seek to prove.

First, if $|n| > \sqrt{D_k(X)}$, then $D_k(X)/|n| < \sqrt{D_k(X)}$, so by [2, Theorem 3.2.1, p. 78], $|n| = f_k^2 X + e_k$, contradicting the hypothesis, so $|n| < \sqrt{D_k(X)}$, which forces $|n| = Q_j^{(\sqrt{D_k(X)})}$ for some $j \geqslant 0$ in the simple continued fraction expansion of $\sqrt{D_k(X)}$ (see [3, Remark 5.5.1, p. 267]). However, $|n|\,|2(f_k^2 X + e_k)$, so $|n|\,|2D_k(X)$, and as above

$$\frac{X_\ell}{Y_\ell} = \langle f_k X + q_0; q_1, \ldots, q_\ell \rangle = \langle f_k X + e_k/f_k \rangle = \frac{f_k^2 X + e_k}{f_k},$$

so $|n| = Q_{\ell+1}^{(\sqrt{D_k(X)})}$. For the balance of he proof, we will let $Q_{\ell+1} = Q_{\ell+1}^{(\sqrt{D_k(X)})}$ and $P_{\ell+1} = P_{\ell+1}^{(\sqrt{D_k(X)})}$.

Now, by [3, Exercise 5.1.9, p. 227] we have

$$\begin{aligned} &\langle f_k X + q_0; q_1, \ldots, q_\ell, 2P_{\ell+1}/|n|, q_\ell, \ldots, q_1 \rangle \\ &\quad = \langle f_k X + q_0; q_1, \ldots, q_\ell, 2P_{\ell+1}/|n| + Y_{\ell-1}/Y_\ell \rangle. \end{aligned} \tag{13}$$

However, $2P_{\ell+1}/|n| + Y_{\ell-1}/Y_\ell = (2P_{\ell+1}Y_\ell + Q_{\ell+1}Y_{\ell-1})/(Q_{\ell+1}Y_\ell)$, and by [3, Exercise 5.3.10, p. 251], $P_{\ell+1}Y_\ell + Q_{\ell+1}Y_{\ell-1} = X_\ell$. Thus, if we set

$$M = (P_{\ell+1}Y_\ell + X_\ell)/(Q_{\ell+1}Y_\ell),$$

then by [3, Theorem 5.1.2, p. 224] (13) equals

$$\begin{aligned}\frac{MX_\ell + X_{\ell-1}}{MY_\ell + Y_{\ell-1}} &= \frac{(P_{\ell+1}Y_\ell + X_\ell)X_\ell + X_{\ell-1}Q_{\ell+1}Y_\ell}{(P_{\ell+1}Y_\ell + X_\ell)Y_\ell + Y_{\ell-1}Q_{\ell+1}Y_\ell} \\ &= \frac{P_{\ell+1}X_\ell Y_\ell + X_\ell^2 + X_{\ell-1}Q_{\ell+1}Y_\ell}{P_\ell Y_\ell^2 + X_\ell Y_\ell + Q_{\ell+1}Y_{\ell-1}Y_\ell}. \end{aligned} \tag{14}$$

However, by [2, Exercise 2.1.13(g)(iii), p. 55], $P_{\ell+1}X_\ell + X_{\ell-1}Q_{\ell+1} = D_k(X)Y_\ell$. Therefore, by [3, Theorem 5.3.4, p. 246], the numerator of (14) equals

$$X_\ell^2 + D_k(X)Y_\ell^2 = 2X_\ell^2 - (X_\ell^2 - Y_\ell^2 D_k(X)) = 2X_\ell^2 - n$$

and by [3, Exercise 5.3.10, p. 251], the denominator of (14) equals

$$Y_\ell(P_{\ell+1}Y_\ell + X_\ell + Y_{\ell-1}Q_{\ell+1}) = Y_\ell(2X_\ell).$$

Hence, (14) equals

$$\frac{2X_\ell^2 - n}{2X_\ell Y_\ell} = \frac{2(f_k^2 X + e_k)^2 - n}{2f_k(f_k^2 X + e_k)},$$

which verifies (12) and confirms that $q_j = q_j^{(\sqrt{D_k(X)})}$ for $j = 1, \ldots, \ell$.

It remains to show that $\lfloor (P_0^{(\beta)} + \sqrt{D_k(X)})/|n'| \rfloor |n'| - P_0^{(\beta)} = P_{\ell+1}$, but this is a consequence of [4, Lemma 1 and Corollary 5, pp. 285–286]. $\square$

Note that in Theorem 1, there need not be a $j \geqslant 0$ such that $|n| = Q_j$ in the simple continued fraction expansion of $\sqrt{C}$ as there did for us in earlier work such as [5–8]. Indeed in those works, $|n| = 1$. Thus, the above result substantially generalizes earlier work and allows for any $n \in \mathbb{Z}$ to be considered (subject to the divisibility condition cited above). For instance, we have the following illustration where $\sqrt{C}$ has *only* $Q_j = 1$ for any $j \geqslant 0$, yet any integer $n \in \mathbb{Z}$ may be considered.

Example 1. If $C = 5$, $n = 4$, then $T_0 + U_0\sqrt{C} = 9 + 4\sqrt{5}$ and $e_0 + f_0\sqrt{C} = 3 + \sqrt{5}$. Thus, $e_1 + f_1\sqrt{C} = 47 + 21\sqrt{5}$. Since $e_1/f_1 = \langle 2; 4, 4, 1 \rangle$ and $4 \mid 2(f_1^2 X + 21)$ precisely when X is odd, then we consider $k = X = 1$ first,

$$D_1(X) = 21^2 X^2 + 2 \cdot 47X + 5 = 441X^2 + 94X + 5,$$

so

$$\sqrt{D_1(1)} = \sqrt{540} = \langle 23; \overline{4, 4, 1, 10, 1, 4, 4, 46} \rangle.$$

Here, $P_0^{(\beta)} = 0$ via $-(f_1^2 + e_1)f_1^{-1} \equiv 0 \pmod 4$. Also, $(X_\ell, Y_\ell) = (488, 21)$ and

$$(488 + 21\sqrt{540})(-2835 + 122\sqrt{540}) = \sqrt{540}.$$

Example 1 deals with the case where $n>0$ so ℓ is necessarily odd. We now illustrate the case where $n<0$, so ℓ is even. To allow for economy of notation, we will denote $q_1, q_2, \dots, q_\ell$ in (8) by E_k.

Example 2. Let $C = 43$, $n = -19$, for which $e_0 + f_0\sqrt{C} = 72 + 11\sqrt{43}$, $T_0 + U_0\sqrt{C} = 3482 + 531\sqrt{43}$, $e_1 + f_1\sqrt{C} = 501867 + 76534\sqrt{43}$, and

$$e_1/f_1 = \langle 6; 1, 1, 3, 1, 5, 1, 3, 1, 1, 12, 1, 1, 4, 1 \rangle = \langle 6; E_1 \rangle,$$

where $\ell = 14$. Hence,

$$\begin{aligned} D_1(X) &= 76534^2 X^2 + 2 \times 501867X + 43 \\ &= 5857453156X^2 + 1003734X + 43, \end{aligned}$$

where $|n| = 19 | (76534^2 X + 501867)$ if and only if $X \equiv 14 \pmod{19}$. Thus, for $k = 1$ and $X = 14$, we have,

$$\sqrt{D_1(14)} = \sqrt{1148074870895} = \langle 1071482; \overline{E_1, 112786, E_1, 2(1071482)} \rangle.$$

Here, $P_0^{(\beta)} = 0$ and $P_1^{(\beta)} = 1071467$. Thus,

$$q_{15} = q_{\ell+1} = 2P_{\ell+1}/|n| = 2P_1^{(\beta)}/|n| = 112786.$$

Notice as well that $Q_j^{(\sqrt{43})} \neq 19$ for any $j \geqslant 0$ since $19 > 2\sqrt{43}$. Also, the fundamental solution of $x^2 - D_1(14)y^2 = -19$ is

$$14f_1^2 + e_1 + f_1\sqrt{D_1(14)} = 82004846051 + 76534\sqrt{1148074870895}.$$

Examples 1 and 2 deal only with the case where $g = 1$. We now produce a result that deals with the more general case and thereby simplifies a result we obtained elsewhere.

Corollary 1 (Mollin [5, Theorem 4.1, p. 325]). *Let U_k, T_k, C, be as above, and set*

$$D_k(X) = U_k^2 (T_k - 1)^2 X^2 + 2(T_k - 1)^2 X + C$$

and

$$\frac{T_k - 1}{U_k} = \langle q_0; q_1, \dots, q_\ell \rangle = \langle q_0; \vec{E_k} \rangle.$$

Then the fundamental solution of $x^2 - D_k(X)y^2 = 1$ *is*

$$(x,y) = ((T_k - 1)(U_k^2 X + 1)^2 + 1, U_k^3 X + U_k).$$

If $g = \gcd(T_k - 1, U_k)$, *and* $g^2 = 2(T_k - 1)$,

$$\sqrt{D_k(X)} = \left\langle (T_k - 1)U_k X + q_0; \overline{\vec{E}_k, 2((T_k - 1)U_k X + q_0)} \right\rangle$$

and if $g^2 \neq 2(T_k - 1)$ *and* $2(T_k - 1) = g^2 n'$, *then*

$$\sqrt{D_k(X)} = \left\langle U_k(T_k - 1)X + q_0; \overline{\vec{E}_k, \frac{2P_{\ell+1}^{(\sqrt{D_k(X)})}}{|n'|}, \overleftarrow{E}_k, 2(U_k(T_k - 1)X + q_0)} \right\rangle,$$

where

$$P_{\ell+1}^{(\sqrt{D_k(X)})} = \left\lfloor \frac{P_0^{(\beta)} + \sqrt{D_k(X)}}{|n'|} \right\rfloor |n'| - P_0^{(\beta)}$$

with $P_0^{(\beta)}$ *being the unique solution given by Proposition* 1, *which is determined via*

$$P_0^{(\beta)} \equiv -((U_k/g)^2(T_k - 1)X + (T_k - 1)/g)(U_k/g)^{-1} (\text{mod } |n'|).$$

Proof. Let

$$D_k(Y) = U_k^2 Y^2 + 2(T_k - 1)Y + C,$$

where $Y = (T_k - 1)X$. Then, we may apply Theorem 1 with $f_k = U_k$ and $e_k = T_k - 1$, whence

$$e_k^2 - f_k^2 C = -2(T_k - 1) = -2e_k = n.$$

Hence, $e_k/f_k = \langle q_0; q_1, \ldots, q_\ell \rangle$ where ℓ is even. By hypothesis we have that $g = \gcd(e_k, f_k) = \sqrt{2e_k}$. Thus, $n = -g^2$, and $n' = -1$. By Theorem 1

$$\sqrt{D_k(X)} = \left\langle (T_k - 1)U_k X + q_0; \overline{\vec{E}_k, 2(T_k - 1)U_k X + q_0)} \right\rangle$$

and the fundamental solution of $x^2 - D_k(X)y^2 = -1$ is

$$(X_\ell, Y_\ell) = \left(\frac{U_k^2 X(T_k - 1) + T_k - 1}{\sqrt{2e_k}}, \frac{U_k}{\sqrt{2e_k}} \right).$$

Hence, the fundamental solution of $x^2 - D_k(X)y^2 = 1$ is given by

$$(x,y) = ((T_k - 1)(U_k^2 X + 1)^2 + 1, U_k^3 X + U_k).$$

When $g^2 \neq n'$, the result follows as above via Theorem 1. $\square$

In the above proof, we actually get more than the corollary mandates, namely we get the fundamental unit of $\mathbb{Z}[\sqrt{D_k(X)}]$, which we illustrate as follows.

Example 3. Let $C = 85$, $e_1 = T_1 - 1 = 285768$, $f_1 = U_1 = 30996$, $n = -2e_1 = -571536$. Then $g = 756$ and $g^2 = |n|$ with $n' = -1$, and

$$e_1/f_1 = \langle 9; 4, 1, 1, 4 \rangle = \langle q_0; \vec{E_1} \rangle,$$

where $\ell = 4$, and for $Y = e_1 X$,

$$D_1(Y) = 30996^2 Y^2 + 2 \times 285768\, Y + 85,$$

so for $Y = e_1$, $X = 1$,

$$\begin{aligned}\sqrt{D_1(285768)} &= \sqrt{78458228140047944917} = \langle e_1 f_1 X + q_0; \overline{\vec{E_1}, 2(e_1 f_1 X + q_0)} \rangle \\ &= \langle 8857664937; \overline{4, 1, 1, 4, 2(8857664937)} \rangle.\end{aligned}$$

Moreover, the fundamental unit of $\mathbb{Z}[\sqrt{D_1(X)}]$ is

$$(X_4, Y_4) = \left(\frac{f_1^2 e_1 + e_1}{g}, \frac{f_1}{g} \right) = (363164262426, 41),$$

with $X_4^2 - Y_4^2 D_1(e_1) = -1$, whereas the value $(X_4 + Y_4 \sqrt{D_1(e_1)})^2$ yields the value for the Pell equation in the Corollary.

In [5, Theorem 4.1, p. 325], numerous cases were involved that are simplified by Corollary 1. We now illustrate with a couple of examples of those cases.

Example 4. Let $C = 21$, $T_1 = 55$, $U_1 = 12$. Then $e_1/f_1 = (T_1 - 1)/U_1 = \langle 4; 1, 1 \rangle = \langle q_0; \vec{E_1} \rangle$, so $\ell = 2$. Also $g = 6$, $2(T_1 - 1) \neq g^2$, and $n' = 3$. Thus, for

$$D_1(Y) = 12^2 Y^2 + 2 \times 54\, Y + 21,$$

$$\begin{aligned}\sqrt{D_1(e_1)} &= \sqrt{425757} = \langle 652; \overline{1, 1, 434, 1, 1, 1304} \rangle \\ &= \langle U_1(T_1 - 1)X + q_0; \overline{\vec{E_1}, 2P_3/3, \overleftarrow{E_1}, 2(U_1(T_1 - 1)X + q_0)} \rangle.\end{aligned}$$

Here $P_{\ell+1} = 651 = \lfloor \sqrt{D_1(e_1)}/3 \rfloor 3$, with $P_0^{(\beta)} = 0$, so $q_{\ell+1} = q_3 = 434 = 2P_{\ell+1}/|n|'$. Moreover, the fundamental solution of $x^2 - D_1(e_1)y^2 = 1$ is given by

$$(x, y) = (((T_1 - 1)U_1^2 + 1)^2 + 1, U_1^3 + U_1) = (1135351, 1740).$$

Example 5. Let $C = 3$, $T_4 = 97$, and $U_4 = 56$. Then

$$e_4/f_4 = (T_4 - 1)/U_4 = \langle 1; 1, 2, 1, 1 \rangle = \langle q_0; \vec{E_4} \rangle.$$

Hence, $\ell = 4$, $g = 3$, and $n' = 3$. Thus, for

$$D_4(Y) = 56^2 Y^2 + 2 \times 96 Y + 3 = 3136 Y^2 + 192 Y + 3,$$

$$\sqrt{D_4(e_4)} = \sqrt{28919811} = \langle 5377; \overline{1, 2, 1, 1, 3584, 1, 1, 2, 1, 10754} \rangle$$

$$= \langle U_4(T_4 - 1)X + q_0; \overline{\vec{E_4}, 2P_5/3, \overleftarrow{E_4}, 2(U_4(T_4 - 1)X + q_0)} \rangle.$$

Here $P_0^{(\beta)} = 0$ and $P_{\ell+1} = 5376 = \lfloor \sqrt{D_4(e_4)}/3 \rfloor 3$, so $q_{\ell+1} = q_5 = 3584 = 2P_{\ell+1}/3$.

Other results in the literature are easily obtained from Theorem 1 as well.

Corollary 2 (Mollin et al. [8, Theorem 3.1]). *With T_k, U_k, C as above, set*

$$D_k(X) = U_k^2 X^2 + 2T_k X + C$$

and

$$T_k/U_k = \langle q_0; q_1, \ldots, q_\ell \rangle = \left\langle q_0; \vec{E_k} \right\rangle.$$

The fundamental unit of $x^2 - D_k(X)y^2 = 1$ is

$$(x, y) = (U_k^2 X + T_k, U_k) \tag{15}$$

and

$$\sqrt{D_k(X)} = \langle U_k X + q_0; \overline{\vec{E_k}, 2(U_k X + q_0)} \rangle. \tag{16}$$

Proof. Set $e_k = T_k, f_k = U_k$ in Theorem 1. Thus, $n = 1$, so (15)–(16) both hold. $\square$

The following is proved in a similar fashion to that of Corollary 1.

Corollary 3 (Mollin [5, Theorem 4.2, p. 337]). *Suppose that T_k, U_k, C, are given as above, $(T_k + 1)/U_k = \langle q_0; q_1, \ldots, q_\ell \rangle = \langle q_0; \vec{E_k} \rangle$, $D_k(X) = (T_k + 1)^2 U_k^2 X^2 + 2(T_k + 1)^2 X + C$, and $a_0 = (T_k + 1)U_k X + q_0$. Then if $g = \gcd(T_k + 1, U_k)$ and*

$g^2 = 2(T_k + 1)$,

$$\sqrt{D_k(X)} = \langle a_0; \overline{\vec{E}_k, 2a_0} \rangle$$

and the fundamental solution of $x^2 - D_k(X)y^2 = 1$ *is given by*

$$(x, y) = \left(\frac{(T_k + 1)(U_k^2 X + 1)}{g}, \frac{U_k}{g} \right).$$

If $g^2 \neq 2(T_k + 1)$, $2(T_k + 1) = g^2 n'$,

$$\sqrt{D_k(X)} = \langle a_0; \overline{\vec{E}_k, 2P_{\ell+1}^{(\sqrt{D_k(X)})}/|n'|, \overleftarrow{E}_k, 2a_0} \rangle,$$

where

$$P_{\ell+1}^{(\sqrt{D_k(X)})} = \left\lfloor \frac{P_0^{(\beta)} + \sqrt{D_k(X)}}{|n'|} \right\rfloor |n'| - P_0^{(\beta)},$$

with $P_0^{(\beta)}$ *being the unique solution given by Proposition* 1, *which is determined via*

$$P_0^{(\beta)} \equiv -((U_k/g)^2(T_k + 1)X + (T_k + 1)/g)(U_k/g)^{-1} (\mathrm{mod}\ |n'|).$$

Lastly, the fundamental solution of $x^2 - D_k(X)y^2 = 1$ *is given by*

$$(x, y) = ((T_k + 1)(U_k^2 X + 1)^2 - 1, U_k^3 X + U_k).$$

Example 6. From Example 5, $C = 3$, $e_4 = T_4 + 1 = 98$, $f_4 = U_4 = 56$, $2(T_4 + 1) = g^2 = 14^2$, $e_4/f_4 = \langle 1; 1, 2, 1 \rangle = \langle q_0; \vec{E}_4 \rangle$, so

$$D_4(X) = 30118144X^2 + 19208X + 3$$

and

$$\sqrt{D_4(1)} = \sqrt{30137355} = \langle 5489; \overline{1, 2, 1, 10978} \rangle = \langle (T_4 + 1)U_4 + q_0; \overline{\vec{E}_4, 2a_0} \rangle.$$

Also, the fundamental solution of $x^2 - D_4(1)y^2 = 1$ is given by

$$(x, y) = \left(\frac{(T_4 + 1)(U_4^2 + 1)}{g}, \frac{U_4}{g} \right) = (21959, 4).$$

Example 7. From Example 4, $C=21$, $e_1=T_1+1=56$, $f_1=U_1=12$, $g=4$, $n'=7$, and $e_1/f_1=\langle 4;1,1,1\rangle$. Thus,

$$D_1(X)=451584X^2+6272X+21$$

and

$$\sqrt{D_1(1)}=\sqrt{457877}=\langle 672;\overline{1,1,1,192,1,1,1,1352}\rangle=\left\langle a_0;\overline{\vec{E}_1,\frac{2P_{\ell+1}^{(\sqrt{D_1(1)})}}{n'},\overleftarrow{E}_1,2a_0}\right\rangle,$$

where $a_0=(T_1+1)U_1+q_0$, and $P_{\ell+1}^{(\sqrt{D_1(1)})}=P_4^{(\sqrt{D_1(1)})}=672=\lfloor\sqrt{457877}/7\rfloor\cdot 7$, with $P_0^{(\beta)}=0$, so $q_{\ell+1}=q_4=2P_{\ell+1}^{(\sqrt{D_1(1)})}/n'=192$. Also, the fundamental solution of $x^2-D_1(1)y^2=1$ is given by

$$(x,y)=((T_1+1)(U_1^2X+1)^2-1,\,U_1^3X+U_1)=(4677175,3468).$$

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