Journal of Number Theory 130 (2010) 1537–1559

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

A thermodynamic classification of real numbers

Thomas Garrity

Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267, United States

ARTICLE INFO

Article history:
Received 15 March 2009
Revised 7 January 2010
Communicated by David Goss

Keywords:
Diophantine approximations
Continued fractions
Thermodynamic formalism

ABSTRACT

Text. A new classification scheme for real numbers is given, motivated by ideas from statistical mechanics in general and work of Knauf (1993) [16] and Fiala and Kleban (2005) [8] in particular. Critical for this classification of a real number will be the Diophantine properties of its continued fraction expansion.

Video. For a video summary of this paper, please click here or visit http://www.youtube.com/watch?v=qnPF2QS4cRg.

© 2010 Elsevier Inc. All rights reserved.

1. Introduction

Though this paper is about number theory in general and about a classification scheme for real numbers in particular, it has its roots in thermodynamic formalism, which was developed in the 1960s by Ruelle [32,33], Sinai [35] and others in an attempt to put statistical mechanics on a firm mathematical foundation. Once done, the underlying mathematical scheme can then, in principle, be applied to non-physical situations, using the original real-world interpretations to guide and influence what questions are to be asked and what structure is to be discovered.

This process has been begun in number theory. In [16], Knauf developed a one-dimensional thermodynamic system based on the Farey fractions that exhibited phase transition. In [8], Fiala and Kleban generalized Knauf's work and showed that their generalization has the same free energy as Knauf's. We will put these earlier works into a common linear algebra framework, allowing us to make a seemingly minor, but actually significant, change in the original partition function. We will produce, for each positive real number, a thermodynamic system. Different real numbers will exhibit different free energies, giving us a new classification scheme for positive real numbers. (This classification scheme can easily be extended to also include negative reals.)

In Section 2, we give a brief overview of the parts of the statistical mechanics formalism that we will be using. In particular, we will see the key importance of the partition function. In Section 3, we tie this formalism to number theory, in particular to the Farey matrices. In Section 3.3, we put

E-mail address: tgarrity@williams.edu.

0022-314X/$ – see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.jnt.2010.01.016

Knauf's work into this language and do the same thing in Section 3.4 for Fiala and Kleban's work. In Section 3.5, we show how to alter the earlier partition functions that will put us into the world of Diophantine analysis. This is the section in which we are not just changing the notation from earlier work.

In Section 4.1, we use our Diophantine partition function to give a new classification scheme for real numbers. In particular we develop the idea of a real number having a 1-*free energy limit*. In Section 4.2, we show how this is naturally linked to continued fractions. The rest of Section 4 deals with proving that there are real numbers with 1-free energy limits, that there are reals without a 1-free energy limit, that all algebraic numbers have a k-free energy limits with $k > 1$ and that all quadratic irrationals have 1-free energy limits. We also show that e has a $\sqrt{N}\log N$-free energy limit. We will conclude with open questions in Section 5.

There has been a lot of other work linking statistical mechanics to number theory. There is other work of Knauf [18–20], of Mendès France and Tenenbaum [25,26], of Guerra and Knauf [11], of Contucci and Knauf [5], of Fiala, Kleban and Özlük [9], of Kleban and Özlük [15], of Prellberg, Fiala and Kleban [28], of Feigenbaum, Procaccia and Tel [7] and others.

There is also the transfer operator method, applied primarily to the Gauss map, which allows, in a natural way, tools from functional analysis to be used. We believe this was pioneered by Mayer (see his [24] for a survey), and nontrivially extended by Prellberg [27], by Prellberg and Slawny [29], by Isola [13] and recently by Esposti, Isola and Knauf [6]. An introduction to this work is in chapter nine of Hensley [12]. We will not be following this approach here.

As of August 2008, the web site http://www.secamlocal.ex.ac.uk/people/staff/mrwatkin/zeta/physics.htm offers many other attempts over the years to find links between statistical mechanics and number theory.

Finally, I would like to thank Edward Burger for many interesting conversations about this work, Steven Miller and L. Pedersen for comments on an earlier draft and Michel Mendès France for help on some of the references. Also, I would like to thank Peter Kleban, Ali Özlük and Thomas Prellberg for finding a significant error in an earlier draft. Finally, I would like to thank the referee for help in improving the exposition.

2. The partition function and the free energy

This is a rapid fire overview of basic terms in statistical mechanics. For each $N \in \mathbb{N}$, we have a finite set $\mathcal{S}_N$, called the *state space*. Let

$$E : \mathcal{S}_N \to \mathbb{R}^+$$

be a function that we call *energy*. The *partition function* is defined to be

$$Z_N(\beta) = \sum_{\sigma \in \mathcal{S}_N} e^{-\beta E(\sigma)}.$$

If we were modeling a physical system, the elements in the state space correspond to what can happen. The variable β corresponds to the inverse of the temperature. The underlying physical assumption is that the probability that a system is in a state $\sigma \in \mathcal{S}_N$ will be

$$\text{Probability in state } \sigma = \frac{e^{-\beta E(\sigma)}}{Z_N(\beta)}.$$

While far from a proof, this interpretation makes sense, in that at high temperatures (meaning for β close to zero), all states become increasingly likely, while at low temperatures the most likely state increasingly becomes the state with the lowest energy.

There is a *free energy* if the following limit exists:

$$f(\beta) = \lim_{N\to\infty} \frac{\log(Z_N(\beta))}{N},$$

with the function $f(\beta)$ being called, naturally enough, the *free energy*. It is believed that phase transitions occur at values of β for which $f(\beta)$ fails to be analytic.

For almost all of this paper, our state space will be

$$\mathcal{S}_N = \{\sigma = (\sigma_1, \ldots, \sigma_N):\ \sigma_i = 0 \text{ or } 1\}.$$

Thus each of our $\mathcal{S}_N$ will have order 2^N. We can think of our state space as having N site points, each having value 0 or 1.

The most famous example is the one-dimensional Ising model. For convenience, we let each site have the value of 1 or -1. Thus for the Ising model, we have

$$\mathcal{S}_N = \{\sigma = (\sigma_1, \ldots, \sigma_N):\ \sigma_i = \pm 1\}.$$

The energy function for the Ising model is

$$E(\sigma) = \sum_{i=1}^{N} \sigma_i \sigma_{i+1}.$$

Ising, in his 1925 thesis, showed that for this model there is no phase transition, meaning he showed that the free energy is an analytic function. For the two-dimensional analog, it is one of the great discoveries (originally by Onsager in 1944) that phase transition does occur. Most texts on statistical mechanics, such as [37], describe the Ising model in detail.

Note that in the Ising model, a site will only interact with those other sites that are immediately adjacent to it. This is an example of finite range interaction. Since there is no phase transition for the one-dimensional Ising model, it was long believed that there would be no phase transition for any one-dimensional system. But in the 1960s, it was discovered that phase transition can occur if the interactions are not of finite range but over possibly arbitrarily long distances. A good introduction to this work is in Mayer's *The Ruelle–Araki transfer operator in classical statistical mechanics* [23]. Such interactions are called *long range interactions*. In the following number theoretic models, it is key that the interactions are long range.

3. Number theoretic partition functions

3.1. General set-up

Fix a positive integer k. For each positive integer N, our state space will be

$$\mathcal{S}_N = \{(\sigma_1, \ldots, \sigma_N):\ \sigma_i = 0, 1, \ldots, k-1\}.$$

Thus $\mathcal{S}_N$ contains k^N elements.

We define a new type of product of an N-tuple of $n \times n$ matrices with an M-tuple of such matrices to be the MN-tuple:

$$(A_1, \ldots, A_N)(B_1, \ldots, B_M) = (A_1B_1, A_1B_2, \ldots, A_NB_M).$$

For matrices $A=(a_{ij})$ and $B=(b_{ij})$, we denote the Hilbert–Schmidt product (which is also called the Hadamard product) as

$$A*B=Tr\bigl(AB^T\bigr)=\sum_{1\leqslant i,j\leqslant n}a_{ij}b_{ij}.$$

For example, thinking of a 2×2 matrix as an element of $\mathbb{R}^4$, then $A*B=Tr(AB^T)$ is simply the dot product of the two vectors.

Let $\mathcal{M}_n$ denote the space of $n\times n$ matrices. For a function

$$f:\mathcal{M}_n\to\mathbf{R}$$

and for two $n\times n$ matrices M and A, define

$$f(M)(\beta)|A=\frac{1}{|M*A|^{\beta}}f\bigl(MA^T\bigr),$$

following notation as in [8]. For k $n\times n$ matrices $A_1,A_1,\dots,A_k$, define

$$f(M)(\beta)|(A_1,\dots,A_k)=\sum_{i=1}^{k}f(M)(\beta)|A_i.$$

Consider the map

$$Z:\mathbb{N}\times\mathcal{M}_n\times\mathbb{R}\times\mathcal{M}_n^k\times\Gamma(\mathcal{M}_n,\mathbb{R})\to\mathbb{R}^*,$$

where $\mathbb{N}$ is the natural numbers, $\mathcal{M}_n$ is the space of $n\times n$ matrices, $\Gamma(\mathcal{M}_n,\mathbb{R})$ is the space of functions from $n\times n$ matrices to the real numbers and $\mathbb{R}^*$ is the extended real numbers, defined by setting

$$Z\bigl(N,M,\beta,(A_0,\dots,A_{k-1}),f\bigr)=f(M)(\beta)|(A_0,\dots,A_{k-1})^N.$$

Here the notation $(A_0,\dots,A_{k-1})^N$ is referring to the above newly defined product of tuples of matrices and hence can be viewed as short-hand for all products $A_{i_1}\cdots A_{i_N}$, with $0\leqslant i_l\leqslant k-1$.

We want to link this with partition functions. Fix an $n\times n$ matrix M and also k $n\times n$ matrices $A_0,A_1,\dots,A_{k-1}$. Let the function f be the constant function 1, or in words, let $f(B)=1(B)=1$ for all matrices $B\in\mathcal{M}_n$. Then define the partition function to be

$$Z_N(M)(\beta)=Z\bigl(N,M,\beta,(A_0,\dots,A_{k-1}),1\bigr)=1(M)(\beta)|(A_0,\dots,A_{k-1})^N.$$

The "physical" intuition is as follows. Think of a one-dimensional lattice with n sites. At each site, there are k possible states, each of which can either be indexed by a number σ_i between 0 and $k-1$ or by a matrix A_{σ_i}. Then the states can be viewed as either all possible

$$\sigma=(\sigma_1,\dots,\sigma_n)\in\mathbb{Z}/k\mathbb{Z}$$

or all matrices of the form

$$A_{\sigma_1}\cdots A_{\sigma_n}.$$

In order to get the above partition function, we set the energy of a state to be:

$$E(\sigma) = \log\left|M * (A_{\sigma_1} \cdots A_{\sigma_n})\right|.$$

In our applications, it is more natural not to emphasize the energy function.

Using this language, we have natural recursion relations linking the partition function Z_{n+1} with partition functions for various Z_n, with different choices for the matrix M. More precisely

Lemma 3.1.

$$Z_{N+1}(M)(\beta) = Z_N\big(MA_1^T\big)(\beta) + \cdots + Z_N\big(MA_k^T\big)(\beta).$$

This is a simple calculation. It offers a more general and natural form for the key recursion relation (2) in [8].

3.2. Farey matrices

This section continues the building of needed machinery. (See, also, Section 4.5 in [10]. Another source would be [22].)

We develop the Farey partitioning of the extended real numbers. Start with the set

$$\mathcal{F}_0 = \left\{\frac{1}{0}, \frac{0}{1}\right\}$$

and define the Farey sum of two fractions, in lowest terms, to be

$$\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s}.$$

We now extend a given nth Farey set $\mathcal{F}_n$ to the $(n+1)$st Farey set by adding to $\mathcal{F}_n$ all of the terms obtained by applying the Farey sum. Thus we have,

$$\mathcal{F}_0 = \left\{\frac{1}{0}, \frac{0}{1}\right\},$$

$$\mathcal{F}_1 = \left\{\frac{1}{0}, \frac{1}{1}, \frac{0}{1}\right\},$$

$$\mathcal{F}_2 = \left\{\frac{1}{0}, \frac{2}{1}, \frac{1}{1}, \frac{1}{2}, \frac{0}{1}\right\},$$

$$\mathcal{F}_3 = \left\{\frac{1}{0}, \frac{3}{1}, \frac{2}{1}, \frac{3}{2}, \frac{1}{1}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \frac{0}{1}\right\}.$$

By reversing the order of the terms in $\mathcal{F}_n$, we get a partitioning of $\mathbb{R}^+ \cup \infty$. Here we are thinking of $\frac{1}{0}$ as the point at infinity, which is why we are working with the extended real numbers $\mathbb{R}^*$.

We now describe this partitioning in terms of iterations of matrix multiplication. Let

$$A_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

These two matrices are key to understanding the Farey decomposition of the unit interval and continued fraction expansions. Further, these two matrices will be key to the partition function of

Knauf [16], of Fiala and Kleban [8], and to our eventual use of partition functions for Diophantine approximations.

Note that

$$\begin{pmatrix} p & r \\ q & s \end{pmatrix} A_0 = \begin{pmatrix} p+r & r \\ q+s & s \end{pmatrix}$$

and

$$\begin{pmatrix} p & r \\ q & s \end{pmatrix} A_1 = \begin{pmatrix} p & p+r \\ q & q+s \end{pmatrix}.$$

Then $\mathcal{F}_1$, save for the point at infinity $\frac{1}{0}$, can be obtained in the natural way by examining the right columns of first A_1, then A_0. Likewise, $\mathcal{F}_2$, save for the point at infinity $\frac{1}{0}$, can be obtained in a similar natural way by examining the right columns of first A_1^2, then A_1A_0, then A_0A_1 and finally A_0^2. In general the elements of $\mathcal{F}_n$, save of course for the point $\frac{1}{0}$, are given by the right columns of the 2^n products of matrices of A_1 and A_0.

These allow us to recover the continued fraction of a positive real number α. We know that any real number α can be written

$$\alpha = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}},$$

which is usually denoted by

$$\alpha = [a_0; a_1, a_2, a_3, \ldots],$$

where a_0 is an integer and the remaining a_i are positive integers. The number α is rational if and only if its continued fraction expansion terminates. We say that the rational $\frac{p_m}{q_m}$ is the mth partial fraction for the number α if

$$\frac{p_m}{q_m} = [a_0; a_1, \ldots, a_m].$$

We want to use our Farey matrices to find the nth partial sum $\frac{p_n}{q_n}$ for a given number α. We return to our Farey numbers, but reverse the orders of the numbers:

$$\mathcal{F}_0 = \left\{\frac{0}{1}, \frac{1}{0}\right\},$$

$$\mathcal{F}_1 = \left\{\frac{0}{1}, \frac{1}{1}, \frac{1}{0}\right\},$$

$$\mathcal{F}_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \frac{2}{1}, \frac{1}{0}\right\},$$

$$\mathcal{F}_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \frac{3}{2}, \frac{2}{1}, \frac{3}{1}, \frac{1}{0}\right\}.$$

We can thus view $\mathcal{F}_1$ as providing a splitting of the positive reals into two intervals: $[\frac{0}{1}, \frac{1}{1}]$ and $[\frac{1}{1}, \frac{1}{0}]$. Note that $[\frac{0}{1}, \frac{1}{1}]$ can be thought as flipping the columns of A_0 and $[\frac{1}{1}, \frac{1}{0}]$ can be thought of as the flipping of the columns of A_1.

Likewise, $\mathcal{F}_2$ will split the positive reals into four intervals: $[\frac{0}{1}, \frac{1}{2}]$ (which can be thought of as the flipping of the columns of A_0A_0), $[\frac{1}{2}, \frac{1}{1}]$ (which can be thought of as the flipping of the columns of A_0A_1), $[\frac{1}{1}, \frac{2}{1}]$ (which can be thought of as the flipping of the columns of A_1A_0) and $[\frac{2}{1}, \frac{1}{0}]$ (which can be thought of as the flipping of the columns of A_1A_1).

This pattern continues. Consider the matrix

$$A_1^{a_0}A_0^{a_1}\cdots A_1^{a_{N-1}}A_0^{a_N}$$

with $a_0 \geqslant 0$ and $a_i > 0$. Then the left column of the matrix will correspond to $[a_0; a_1, \ldots, a_N]$ while the right column will correspond to $[a_0; a_1, \ldots, a_{N-1}]$.

For matrices

$$A_1^{a_0}A_0^{a_1}\cdots A_0^{a_{N-1}}A_1^{a_N}$$

with $a_0 \geqslant 0$ and $a_i > 0$, then the left column will correspond to $[a_0; a_1, \ldots, a_{N-1}]$ while now the right column will $[a_0; a_1, \ldots, a_N]$.

Thus to determine the continued fraction expansion for a given positive real number α, we just have to keep track of which interval α is in for a given $\mathcal{F}_N$.

3.3. Earlier work

This section will show how Knauf's number theoretic partition function [16] and Fiala and Kleban's partition function [8] can easily be put into the language of this paper.

Let

$$M^K = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(where the K is not an index but instead stands for Knauf) and let A_0 and A_1 be the above Farey matrices.

Then the Knauf partition function is

$$Z_N^K(\beta) = 1\big(M^K\big)\big|A_0(A_0, A_1)^N.$$

The initial A_0 is just to insure that we are in the unit interval. Also, for $A_0(A_0, A_1)^N$, we are using the new product for matrices defined in Section 3.1 and not traditional matrix multiplication.

As shown in [16], in the limit we get

$$Z^K(\beta) = \lim_{N\to\infty} Z_N^K(\beta) = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^\beta},$$

where $\phi(n)$ is the Euler totient function. In turn, $\sum_{n=1}^{\infty} \phi(n)/n^\beta$ is well known to equal to

$$\frac{\zeta(\beta-1)}{\zeta(\beta)},$$

for $\beta > 2$, where $\zeta(\beta)$ is the Riemann zeta function and is not defined for $\beta \leqslant 2$, showing that there is critical point phenomena for this one-dimensional system. The $\zeta(\beta-1)/\zeta(\beta)$ will show up throughout this paper and is why many of the later theorems are only true for $\beta > 2$.

Fiala and Kleban considered a different number theoretic partition function. In the language of this paper, let

$$M^F = \begin{pmatrix} 0 & 0 \\ x & 1 \end{pmatrix}.$$

(Again, the F is not an index but stands for “Fiala–Kleban”.) Then the new partition function will be

$$Z_N^F(x,\beta) = 1\big(M^F\big)\big|A_0(A_0, A_1)^N.$$

As mentioned earlier, the recursion relation (2) in Section 2 of [8] is simply a special case of Lemma 3.1.

Two partition functions are said to have the same thermodynamics if their free energies are equal. Thus, the Knauf partition function $Z_N^K(\beta)$ and the Fiala–Kleban partition function $Z_N^F(\beta)$ have the same thermodynamics if

$$\lim_{N\to\infty} \frac{\log(Z_N^K(\beta))}{N} = \lim_{N\to\infty} \frac{\log(Z_N^F(\beta))}{N}.$$

This equality is shown in Section 4 of [8], using as an intermediary tool a certain transfer operator and depending on the earlier work of Knauf [16]. In [9], Fiala, Kleban and Özlük showed the thermodynamics of the Knauf partition function is thermodynamically equivalent to a number of other number theoretic partition functions. It is certainly the case, though, that thinking of the various matrices as vectors in $\mathbf{R}^4$ will yield more straightforward proofs of these equivalences.

3.4. A Diophantine approach

We now make a seemingly minor change in our choice for the matrix M that will create a quite different thermodynamics, leading in the next section to a new classification of the real numbers. Set

$$M = \begin{pmatrix} 0 & -1 \\ 0 & \alpha \end{pmatrix}$$

for some real number α. Define the *Diophantine partition function* to be

$$Z_N(\alpha;\beta) = 1(M)|(A_0, A_1)^N.$$

Then we have

$$Z_N(\alpha;\beta) = \sum_{\frac{p}{q}\in\mathcal{F}_N} \frac{1}{|p-\alpha q|^{\beta}}.$$

Note that the terms that dominate the above sum occur when $|p-\alpha q|$ is small. This places us firmly in the realm of Diophantine analysis. We will see that we can classify real numbers α by understanding the existence of free energy limits for the statistical systems associated to the partition function $Z_N(\alpha;\beta)$. Again, while this partition function is cast in the same overall language as Knauf and Fiala and Kleban, its thermodynamic properties will be quite different. This is what separates the present work from [16] and [8].

4. Classifying real numbers via free energy

4.1. The classification

With the notation above, we have for each positive real number α and each positive integer N the partition function $Z_N(\alpha;\beta)$.

Definition 4.1. A real number α has a k-free energy limit if there is a number β_c such that

$$\lim_{N\to\infty} \frac{\log(Z_N(\alpha;\beta))}{\beta N^k}$$

exists for all $\beta > \beta_c$.

For $k = 1$, this is a number-theoretic version of the free energy of the system. By an abuse of notation we will also say that α has a $f(N)$-free energy limit, for an increasing function $f(N)$ if

$$\lim_{N\to\infty} \frac{\log(Z_N(\alpha;\beta))}{\beta f(N)}$$

exists for all $\beta > \beta_c$.

To see that this is a meaningful classification scheme for real numbers, we will establish the following three theorems:

Theorem 4.2. *There exists a number that has a* 1*-free energy limit for* $\beta > 2$.

Theorem 4.3. *There exists a number that does not have a* 1*-free energy limit, for any value of* β.

Theorem 4.4. *Let* α *be a positive real number such that there is a positive constant* C *and constant* $d \geqslant 2$ *with*

$$\frac{C}{q^d} \leqslant |p - \alpha q|$$

for all relatively prime integers p *and* q*. Then* α *has a* k*-free energy limit for* $\beta > 2$*, for any* $k > 1$*, and in fact, the* k*-free energy limit is zero.*

An easy consequence of the above is that all algebraic numbers have k-free energy limits equal to zero, for any $k > 1$.

While we do not know if algebraic numbers have 1-free energy limits, we will show

Theorem 4.5. *All quadratic irrationals have* 1*-free energy limits for* $\beta > 2$.

(There are two reasons that we do not use this theorem to show Theorem 4.2. First, the proof techniques needed for Theorem 4.2 are needed for the proof of Theorem 4.3 and in fact shape the proof for Theorem 4.5. Second, the class of numbers that we construct for Theorem 4.2 is more extensive than just quadratic irrationals.) We will also show, in Section 4.7 that the number e has $\sqrt{N}\log(N)$-free energy limit.

4.2. Links to continued fractions

The goal of this paper is not just to give a new way to classify real numbers but also to show how such a classification scheme follows from the thermodynamical formalism, fitting into a more general framework. But if all we wanted was the classification scheme, then it is possible to reframe our definitions so that there is no need for the language of statistical mechanics. The goal of this section is to state the theorems that would allow us to avoid thermodynamics. They will also be key to proving the five theorems of the previous section.

Let our positive real number α have continued fraction expansion $[a_0; a_1, a_2, \ldots]$. We know that the *best rational approximations* to α are given by the rational numbers $[a_0; a_1, a_2, \ldots, a_m]$. The fractions $[a_0; a_1, a_2, \ldots, a_m, k]$, with $0 < k < a_{m+1}$, are called the *secondary convergents* to α. For a given m, we know that all the vectors corresponding to the $[a_0; a_1, a_2, \ldots, a_{m-1}, k]$, with $0 \leqslant k < a_m$, are on the same side of the line $x = \alpha y$, while $[a_0; a_1, a_2, \ldots, a_m]$ jumps to the other side of the line.

We now set some notation for the rest of the paper. Given the positive real number α, for each positive integer N there is associated a positive integer m and nonnegative integer k such that

$$N = a_0 + a_1 + \cdots + a_m + k,$$

with $0 \leqslant k < a_{m+1}$. We create a subsequence of $\mathbb{N}$, denoted by $N_0, N_1, N_2, \ldots$ by setting

$$N_m = a_0 + a_1 + \cdots + a_m.$$

In this notation, given any $N \in \mathbb{N}$, we have

$$N_m \leqslant N < N_{m+1},$$

or, in other words,

$$a_0 + a_1 + \cdots + a_m \leqslant N = a_0 + a_1 + \cdots + a_m + k < a_0 + a_1 + \cdots + a_m + a_{m+1}.$$

Set

$$\frac{p_N}{q_N} = [a_0; a_1, \ldots, a_m, k],$$

with $0 \leqslant k < a_{m+1}$, with p_N and q_N having no nontrivial common factors. We know that the fractions $\frac{p_{N_m}}{q_{N_m}}$ are the best rational approximations to the initial real α. Denote

$$d_N = \frac{1}{|p_N - \alpha q_N|}.$$

We have the following chain of inequalities that will be key:

$$d_{N_{m-1}} < d_{N_m+1} < d_{N_m+2} < \cdots < d_{N_m+a_{m+1}-1} < d_{N_m},$$

which are well known (for motivation, see the chapter on continued fractions in [36]).

We will show

Theorem 4.6. *For any $\beta > 2$ and for any positive real number α, we have for all positive integers N that*

$$\frac{\log(d_N)}{N} \leqslant \frac{\log(Z_N(\alpha; \beta))}{\beta N} \leqslant \frac{\log(\frac{\zeta(\beta-1)}{\zeta(\beta)} N d_{N_m}^{\beta})}{\beta N}.$$

To show that there are numbers that do not have 1-free energy limits, we will construct an α so that a subsequence of $\frac{\log(d_N)}{N}$ approaches infinity. In turn, to show that there are numbers that have 1-free energy limits, we will construct an α so that $\frac{\log(\frac{\zeta(\beta-1)}{\zeta(\beta)} N d_{N_m}^{\beta})}{\beta N}$ approaches zero, forcing $\frac{\log(Z_N(\alpha;\beta))}{\beta N}$ to approach zero (and thus guaranteeing that the limit exists).

4.3. Preliminaries

The reason why we look at the products of the matrices A_0 and A_1 is that the right columns will correspond to all of the integer lattice points $\binom{p}{q}$ with p and q relatively prime in the the first quadrant of the plane. If $A_{i_1} \cdots A_{i_n}$ has right column $\binom{p}{q}$, then

$$M * (A_{i_1} \cdots A_{i_n}) = (-1, \alpha) \cdot \begin{pmatrix} p \\ q \end{pmatrix} = \alpha q - p.$$

Let

$$v_1 = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}$$

be vectors that satisfy $\det(v_1, v_2) = \pm 1$. Let $C(v_1, v_2)$ denote the cone of integer lattice points defined by:

$$C(v_1, v_2) = \{a v_1 + b v_2\colon a, b \text{ relatively prime nonnegative integers}\}.$$

Let $C_N(v_1, v_2)$ be the subset of $C(v_1, v_2)$ consisting of vectors that are the right columns of all possible $A_{i_1} \cdots A_{i_N}$. Then set

$$Z_N(\alpha; \beta, v_1, v_2) = \sum_{\binom{p}{q} \in C_N(v_1, v_2)} \frac{1}{|\alpha q - p|^{\beta}}.$$

In the same way, we set

$$Z(\alpha; \beta, v_1, v_2) = \sum_{\binom{p}{q} \in C(v_1, v_2)} \frac{1}{|\alpha q - p|^{\beta}}.$$

Suppose we have integer lattice vectors $v_1 = \binom{0}{1}, v_2, \ldots, v_m$ on one side of the line $x = \alpha y$ and integer lattice vectors $w_1 = \binom{1}{0}$, $w_2, \ldots, w_n$ on the other side of the line $x = \alpha y$ such that $\det(v_i, v_{i+1}) = -1$, $\det(w_i, w_{i+1}) = 1$ and $\det(v_m, w_n) = -1$. Then we have

$$Z_N(\alpha; \beta) \leqslant \sum_{i=1}^{m-1} Z_N(\alpha; \beta, v_i, v_{i+1}) + Z_N(\alpha; \beta, v_m, w_n) + \sum_{i=1}^{n-1} Z_N(\alpha; \beta, w_i, w_{i+1}).$$

We need the above to be an inequality since there is "overcounting" on the right-hand side, since, for example, the part of $Z_N(\alpha; \beta, v_{i-1}, v_i)$ coming from the vector v_i also appears as a term in $Z_N(\alpha; \beta, v_i, v_{i+1})$. The key, as we will see, is that the $Z_N(\alpha; \beta, v_m, w_n)$ term contributes the most to the partition function $Z_N(\alpha; \beta)$. For the rest of this section, we want to control the sizes of the various $Z_N(\alpha; \beta, v_i, v_{i+1})$ and $Z_N(\alpha; \beta, w_i, w_{i+1})$.

We first return to the more general case of two vectors $v_1 = \binom{p_1}{q_1}$, $v_2 = \binom{p_2}{q_2}$ with $\det(v_1, v_2) = \pm 1$, under the additional assumption that v_1 and v_2 lie on the same side of the line $x = \alpha y$.

For

$$d_1 = \frac{1}{|p_1 - \alpha q_1|}, \qquad d_2 = \frac{1}{|p_2 - \alpha q_2|},$$

suppose that $d_1 < d_2$, which means that the line through the origin and the point (p_2, q_2) is closer to the line $x = \alpha y$ than the line through the origin and (p_1, q_1).

We want to show that

$$Z(\alpha; \beta, v_1, v_2) < |d_2|^\beta \frac{\zeta(\beta - 1)}{\zeta(\beta)}.$$

We know that

$$\frac{1}{d_1} = \left|(-1, \alpha) \cdot \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}\right|, \qquad \frac{1}{d_2} = \left|(-1, \alpha) \cdot \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}\right|.$$

Let v be some integer lattice point in the cone $C(v_1, v_2)$. Then there are relatively prime positive integers a and b with $v = av_1 + bv_2$.

We have

$$\begin{aligned} (a+b)\big|(-1, \alpha) \cdot v_2\big| &< \big|(-1, \alpha) \cdot (av_1 + bv_2)\big| \\ &= \big|(-1, \alpha) \cdot v\big| \\ &< (a+b)\big|(-1, \alpha) \cdot v_1\big|. \end{aligned}$$

Inverting and raising everything to the power of β, we have

$$\frac{|d_1|^\beta}{(a+b)|^\beta} < \frac{1}{|(-1, \alpha) \cdot v|^\beta} < \frac{|d_2|^\beta}{(a+b)|^\beta}.$$

Summing over every vector in $C(v_1, v_2)$, we get

$$|d_1|^\beta \sum \frac{1}{|a+b|^\beta} < Z(\alpha; \beta, v_1, v_2) < |d_2|^\beta \sum \frac{1}{|a+b|^\beta},$$

where in the first and third summation we are summing over all relatively prime positive integers a and b. It is well known, as mentioned in Section 4.3 of Knauf [16], that,

$$\sum \frac{1}{|a+b|^\beta} = \frac{\zeta(\beta - 1)}{\zeta(\beta)}.$$

It is here that the $\frac{\zeta(\beta-1)}{\zeta(\beta)}$ makes its critical appearance. We have our desired inequality.

4.4. Proof of Theorem 4.6

First, the partition function $Z_N(\alpha;\beta)$ is the sum of many positive terms, including d_N^{β}. Thus we have the lower bound

$$\frac{\log(d_N)}{N} = \frac{\beta\log(d_N)}{\beta N} = \frac{\log(d_N^{\beta})}{\beta N} \leqslant \frac{\log(Z_N(\alpha;\beta))}{\beta N}.$$

Now for the upper bound. We are at the Nth stage. Letting $t \leqslant N$, expressing N as $N = a_0 + \cdots + a_m + k$, with $0 \leqslant k < a_{m+1}$ and using the notation from Section 4.2, we assume that the vectors $\binom{p_{t-1}}{q_{t-1}}$ and $\binom{p_t}{q_t}$ lie on the same side of the line $(x = \alpha y)$. This is equivalent to there being some $s \leqslant m$ with $a_0 + \cdots + a_s \leqslant t-1 < t < a_0 + \cdots + a_{s+1}$. We know that

$$\det\begin{pmatrix} p_{t-1} & p_t \\ q_{t-1} & q_t \end{pmatrix} = \pm 1.$$

We have

$$\begin{aligned} Z_N\left(\alpha;\beta;\begin{pmatrix} p_{t-1} \\ q_{t-1}\end{pmatrix},\begin{pmatrix} p_t \\ q_t\end{pmatrix}\right) &< Z\left(\alpha;\beta;\begin{pmatrix} p_{t-1} \\ q_{t-1}\end{pmatrix},\begin{pmatrix} p_t \\ q_t\end{pmatrix}\right) \\ &< \frac{\zeta(\beta-1)}{\zeta(\beta)} d_t^{\beta} \\ &< \frac{\zeta(\beta-1)}{\zeta(\beta)} d_{N_m}^{\beta}, \end{aligned}$$

using that $d_t \leqslant d_{N_m}$.

Since there will be N such cones, we have

$$\frac{\log(Z_N(\alpha;\beta))}{\beta N} \leqslant \frac{\log(N\frac{\zeta(\beta-1)}{\zeta(\beta)} d_{N_m}^{\beta})}{\beta N},$$

finishing the proof.

4.5. Proof of Theorem 4.2

We know that the best rational approximations to a real number α are the

$$\frac{p_{N_m}}{q_{N_m}} = [a_0; a_1, \ldots, a_m].$$

It is well known that

$$q_{N_{m+1}} = a_{m+1} q_{N_m} + q_{N_{m-1}}.$$

Further (as in Lemma 7.2 of [3])

$$a_{m+1} q_{N_m} \leqslant \frac{1}{|p_{N_m} - \alpha q_{N_m}|} = d_{N_m} \leqslant (a_{m+1} + 2) q_{N_m}.$$

Our goal in this section is to construct a real number α for which

$$\lim_{N\to\infty} \frac{\log(N\frac{\zeta(\beta-1)}{\zeta(\beta)}(d_{N_m})^{\beta})}{\beta N} = 0,$$

for $\beta > 2$, which by Theorem 4.5 will force α to have a 1-free energy limit.

If

$$\lim_{N\to\infty} \frac{\log(N\frac{\zeta(\beta-1)}{\zeta(\beta)}(d_{N_m})^{\beta})}{\beta N}$$

exists, then it equals

$$\lim_{N\to\infty} \frac{\log N}{\beta N} + \lim_{N\to\infty} \frac{\log \frac{\zeta(\beta-1)}{\zeta(\beta)}}{\beta N} + \lim_{N\to\infty} \frac{\log d_{N_m}}{N}.$$

The first two terms in the above certainly go to zero. Thus we must construct a real α so that $\lim_{N\to\infty} \frac{\log d_{N_m}}{N} = 0$. Recalling our notation that $N_m = a_0 + \cdots + a_m \leqslant N = N_m + k$, with $0 \leqslant k < a_{m+1}$, we have for each N,

$$\frac{\log d_{N_m}}{N} \leqslant \frac{\log d_{N_m}}{N_m}.$$

Thus all we have to do is construct an α so that $\lim_{N\to\infty} \frac{\log d_{N_m}}{N_m} = 0$.

For $\alpha = [a_0; a_1, \ldots]$, define the function $f(m)$ by setting

$$a_{m+1} = q_{N_m}^{f(m)}.$$

We have

$$\begin{aligned}
\frac{\log(d_{N_m})}{N} &\leqslant \frac{\log(d_{N_m})}{N_m} \\
&\leqslant \frac{\log[(a_{m+1}+2)q_{N_m}]}{N_m} \\
&\leqslant \frac{\log(2a_{m+1}q_{N_m})}{N_m} \\
&= \frac{\log(2)}{N_m} + \frac{\log(q_{N_m}^{f(m)}q_{N_m})}{N_m} \\
&= \frac{\log(2)}{N_m} + \frac{[f(m)+1]\log(q_{N_m})}{N_m}.
\end{aligned}$$

Since the first term in the last equation goes to zero as $N \to \infty$, we have, if the limits exist, that

$$\lim_{N\to\infty} \frac{\log(d_{N_m})}{N_m} \leqslant \lim_{N\to\infty} \frac{[f(m)+1]\log(q_{N_m})}{N_m}.$$

We now start with a q_0 and a q_1 and a function $f(m)$ and use these to create our number α. Let $q_0 = 1$, $q_1 = 2$, and $f(m) = m$ for $m \geqslant 1$. Then define for $m \geqslant 2$, $a_{m+1} = q_{N_m}^{f(m)}$.

Now

$$\begin{aligned}\log(q_{N_m}) &= \log(a_m q_{N_{m-1}} + q_{N_{m-2}}) \\ &\leqslant \log(2a_m q_{N_{m-1}}) \\ &= \log\bigl(2q_{N_{m-1}}^{f(m-1)+1}\bigr) \\ &= \log(2) + \bigl(f(m-1)+1\bigr)\log(q_{N_{m-1}}).\end{aligned}$$

Then we have

$$\frac{[f(m)+1]\log(q_{N_m})}{N_m} \leqslant \frac{[f(m)+1]\log(2)}{N_m} + \frac{[f(m)+1](f(m-1)+1)\log(q_{N_{m-1}})}{N_m}.$$

We will now use that

$$\begin{aligned}N_m &= a_0 + \cdots + a_m \\ &> a_m \\ &= q_{N_{m-1}}^{f(m-1)}.\end{aligned}$$

Then

$$\frac{[f(m)+1]\log(q_{N_m})}{N_m} < \frac{(m+1)\log(2)}{q_{N_{m-1}}^{m-1}} + \frac{(m+1)m\log(q_{N_{m-1}})}{q_{N_{m-1}}^{m-1}}$$

which has limit zero as $N \to \infty$. Thus with the choice of the function $f(m) = m$ we have constructed a real number that has 1-free energy limit, finishing the proof of Theorem 4.2.

4.6. Proof of Theorem 4.3

We use from Theorem 4.5 that $\frac{\log(d_N)}{N} \leqslant \frac{\log(Z_N(\alpha;\beta))}{\beta N}$. We will construct a real number α so that

$$\lim_{N_m\to\infty} \frac{\log(d_{N_m})}{N_m} = \infty.$$

This will mean that $\lim_{N\to\infty} \frac{\log(Z_N(\beta))}{\beta N}$ will not exist, which is the goal of Theorem 4.3.

Proceeding as in the proof of Theorem 4.2, we define $\alpha = [a_0; a_1, a_2, \ldots]$ inductively on the a'_ms by setting, as before, $a_0 = 1$ and $a_{m+1} = q_{N_m}^{f(m)}$ but now defining $f(m)$ as

$$f(m) = a_0 + \cdots + a_m.$$

We use that $a_{m+1} q_{N_m} \leqslant d_{N_m}$ and $N_m = a_0 + \cdots + a_m$, we have

$$\begin{aligned}\frac{\log d_{N_m}}{N_m} &\geqslant \frac{\log(a_m q_{N_m})}{N_m} \\ &= \frac{\log(q_{N_m}^{f(m)} q_{N_m})}{N_m}\end{aligned}$$

$$= \frac{(f(m)+1)\log q_{N_m}}{N_m}$$

$$= \frac{(a_0+\cdots+a_m+1)\log q_{N_m}}{a_0+\cdots+a_m}.$$

Since the last term goes to infinity as $N_m \to \infty$, we are done.

4.7. Proof of Theorem 4.4

We assume that α is a positive real number such that there is a positive constant C and a constant $d \geqslant 2$ with

$$\frac{C}{q^d} \leqslant |p - \alpha q|$$

for all relatively prime integers p and q.

In particular, we have

$$\frac{C}{q_{N_m}^d} \leqslant \left|\alpha - \frac{p_{N_m}}{q_{N_m}}\right|.$$

In our notation, this is

$$d_{N_m} \leqslant c q_{N_m}^{d-1},$$

where c is a constant depending on α but not on N_m. In a similar argument as in Theorem 4.2, the number α will have a k-free energy limit if we can show that

$$\lim_{N_m\to\infty} \frac{\log(d_{N_m})}{(N_m)^k} = 0.$$

This will certainly happen if we can show

$$\lim_{N_m\to\infty} \frac{\log(c q_{N_m}^{d-1})}{(N_m)^k} = 0$$

and hence if

$$\lim_{N_m\to\infty} \frac{\log(q_{N_m})}{(N_m)^k} = 0.$$

We know that

$$q_{N_m} = a_m q_{N_{m-1}} + q_{N_{m-2}}.$$

Then we get

$$\begin{aligned} q_{N_m} &= a_m q_{N_{m-1}} + q_{N_{m-2}} \\ &\leqslant 2a_m q_{N_{m-1}} \\ &= 2a_m(a_{m-1}q_{N_{m-2}} + q_{N_{m-3}}) \\ &\leqslant 2^2 a_m a_{m-1} q_{N_{m-2}} \\ &\ \vdots \\ &\leqslant 2^{m+1} a_m a_{m-1} \ldots a_0. \end{aligned}$$

Thus we want to examine

$$\lim_{N_m \to \infty} \frac{\log(2^{m+1} a_m a_{m-1} \ldots a_0)}{(N_m)^k}$$

or

$$\lim_{N_m \to \infty} \frac{(m+1)\log(2) + \log(a_0) + \cdots + \log(a_m)}{(a_0 + \cdots + a_m)^k}.$$

Since it is always the case that $a_0 + \cdots + a_m \geqslant m+1$, the above limit must always be zero for any $k > 1$.

Corollary 4.7. *All positive algebraic numbers of degree greater than or equal to two have k-free energy limits equal to zero, for any $k > 1$.*

Liouville's Theorem (see Theorem 1.1 in [4]) states that all irrational algebraic numbers α have the property there is a constant $d \geqslant 2$ with

$$\frac{C}{q^d} \leqslant |p - \alpha q|$$

for all relatively prime integers p and q, allowing us to immediately use the above theorem.

4.8. Quadratic irrationals have 1-free energy limits

The key will be that every quadratic irrational has an eventually periodic continued fraction expansion (see Section 7.6 in [36]). We will show first, for a quadratic irrational, that $\lim_{m\to\infty} \log q_{N_m}/N_m$ exists and then show that the existence of this limit is equivalent to the number having a 1-free energy limit.

Let α be a quadratic irrational. We can write

$$\alpha = [b_0; b_1, \ldots, b_p, c_1, c_2, \ldots, c_l, c_1, c_2, \ldots, c_l, \ldots].$$

The period length is l. For notational convenience, set

$$b = b_0 + \cdots + b_p,$$

$$c = c_1 + \cdots + c_l,$$

$$B = \begin{pmatrix} 0 & 1 \\ 1 & b_p \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & b_0 \end{pmatrix},$$
$$C = \begin{pmatrix} 0 & 1 \\ 1 & c_l \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & c_1 \end{pmatrix}.$$

For each positive $m > p$, there are unique integers $k \geqslant 0$ and n with $0 \leqslant n < l$ such that

$$m = p + kl + n.$$

For this n, set

$$d_n = c_1 + \cdots + c_n,$$
$$D_n = \begin{pmatrix} 0 & 1 \\ 1 & c_n \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & c_1 \end{pmatrix}.$$

Note that there are only a finite number of possibilities for the d_n and the D_n.

From Section 7.6 of Stark, we have

$$\begin{pmatrix} q_{N_{m-1}} & p_{N_{m-1}} \\ q_{N_m} & p_{N_m} \end{pmatrix} = D_n C^k B.$$

Now $N_m = b + kc + d_n$ grows linearly with respect to k. Since q_{N_m} is the lower left term of the matrix $D_n A^k B$, we also have that $\log q_{N_m}$ grows linearly with respect to k, with constant leading coefficient. Thus $\lim_{m\to\infty} \frac{\log q_{N_m}}{N_m}$ must exist.

Now to show why this will imply that α has a 1-free energy limit. Since $d_{N_{m-1}} \leqslant d_N \leqslant d_{N_m}$ and since $N_m \leqslant N \leqslant N_{M+1}$, we have from Theorem 4.6 that

$$\frac{\log(d_{N_{m-1}})}{N_{m+1}} \leqslant \frac{\log(Z_N(\alpha;\beta))}{\beta N} \leqslant \frac{\log(\frac{\zeta(\beta-1)}{\zeta(\beta)} N d_{N_m}^{\beta})}{\beta N_m}.$$

From our earlier work, we know that α will have a 1-free energy limit, for $\beta > 2$, if we can show

$$\lim_{m\to\infty} \frac{\log d_{N_{m-1}}}{N_{m+1}} = \lim_{m\to\infty} \frac{\log d_{N_m}}{N_m}.$$

We will see that these limits will exist if and only if $\lim_{m\to\infty} \frac{\log q_{N_m}}{N_m}$ exists.

Let $\alpha = [a_0; a_1, a_2, \ldots]$. We know that $N_{m+1} = N_{m-1} + a_m + a_{m+1}$. For $m > p$, a_m and a_{m+1} can only be chosen from finite set $\{c_1, \ldots, c_l\}$. Thus

$$\lim_{m\to\infty} \frac{\log d_{N_{m-1}}}{N_{m+1}} = \lim_{m\to\infty} \frac{\log d_{N_{m-1}}}{N_{m-1}}$$

since the denominators, for all m, differ by a fixed amount (and of course since $N_m \to \infty$).

We know that $a_{m+1} q_{N_m} \leqslant d_{N_m} \leqslant (a_{m+1}+2) q_{N_m}$. Then, if the limits exist, we have

$$\lim_{m\to\infty} \frac{\log a_{m+1} q_{N_m}}{N_m} \leqslant \lim_{m\to\infty} \frac{\log d_{N_m}}{N_m} \leqslant \lim_{m\to\infty} \frac{\log(a_{m+1}+2) q_{N_m}}{N_m}.$$

Since the a_{m+1} are bounded and the $N_m \to \infty$, we have

$$\lim_{m\to\infty} \frac{\log a_{m+1}}{N_m} = \lim_{m\to\infty} \frac{\log(a_{m+1}+2)}{N_m} = 0,$$

giving us that

$$\lim_{m\to\infty} \frac{\log d_{N_m}}{N_m} = \lim_{m\to\infty} \frac{\log q_{N_m}}{N_m},$$

concluding the proof.

Corollary 4.8. *The golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ has 1-free energy limit equal to $\log\phi$, for $\beta > 2$.*

For background on ϕ, see Chapter 1.7 in [30]. The key is that continued fraction expansion for ϕ is

$$\phi = [1:1,1,1,1,\ldots].$$

From the above theorem, we know that the 1-free energy limit, for $\beta > 2$ is

$$\lim_{m\to\infty} \frac{\log q_{N_m}}{N_m}.$$

For ϕ, we know that $N_m = m+1$ and that $q_{N_m} = F_m$, where F_m is the mth Fibonacci number. We know that

$$F_m = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^k\right).$$

Then the 1-free energy limit will be

$$\begin{aligned}\lim_{m\to\infty} \frac{\log q_{N_m}}{N_m} &= \lim_{m\to\infty} \frac{\log F_m}{m+1} \\ &= \log\phi,\end{aligned}$$

as desired.

Of course, similar arguments can be used to find the 1-free energy limits for many quadratic irrationals.

4.9. The number e has $\sqrt{N}\log(N)$-free energy limit

We will actually show this for $e-1$. The continued fraction expansion for $e-1$ is

$$[1;1,2,1,1,4,1,1,6,\ldots].$$

Thus for $e-1$, we have

$$a_{3k} = a_{3k+1} = 1 \quad \text{and} \quad a_{3k+2} = 2(k+1).$$

Let q_{N_m} denote the denominators of the fraction $[a_0; a_1, \ldots, a_m]$ associated to $e-1$, where $N_m = a_0 + \cdots + a_m$.

In an analogous fashion to the proof that quadratic irrationals have 1-free energy limits, we will first show for $e-1$ that

$$\lim_{N_m\to\infty}\frac{\log q_{N_m}}{\sqrt{N_m}\log(N_m)}$$

exists and then show that the existence of this limit will give us our result.

We know that

$$a_m q_{N_{m-1}} < q_{N_m} = a_m q_{N_{m-1}} + q_{N_{m-2}}.$$

Then we have

$$a_0 a_1 \dots a_m < q_{N_m} \leqslant 2^{m+1} a_m a_{m-1} \dots a_0.$$

Then, provided the limits exist, we have

$$\lim_{N_m\to\infty}\frac{\log a_0 a_1 \dots a_m}{\sqrt{N_m}\log(N_m)} \leqslant \lim_{N_m\to\infty}\frac{\log q_{N_m}}{\sqrt{N_m}\log(N_m)} \leqslant \lim_{N_m\to\infty}\frac{\log 2^{m+1} a_m a_{m-1}\dots a_0}{\sqrt{N_m}\log(N_m)}.$$

We will show that

$$\lim_{N_m\to\infty}\frac{\log a_0 a_1 \dots a_m}{\sqrt{N_m}\log(N_m)} = \lim_{N_m\to\infty}\frac{\log 2^{m+1} a_m a_{m-1}\dots a_0}{\sqrt{N_m}\log(N_m)},$$

which will force $\lim_{N_m\to\infty}\log q_{N_m}/\sqrt{N_m}\log(N_m)$ to exist.

We now have to look at the explicit values for the a_m. There are three cases, depending on if m is 0, 1 or 2 mod 3. We will let $m=3k+2$ and show that the above limits exist and are equal. The other two cases are similar. The above denominator is

$$1+1+2+1+1+4+1+\cdots+2(k+1) = 2(k+1)+(k+1)(k+2),$$

which has quadratic growth in k. The numerator on the left-hand side of the above limit is

$$\begin{aligned}\log a_0+\cdots+\log a_m &= \log 1+\log 1+\log 2+\log 1+\log 1+\log 4+\cdots+\log 2(k+1)\\ &= k\log 2+\log 1+\log 2+\cdots+\log(k+1),\end{aligned}$$

which has $k\log k$ growth. Similarly for the right-hand side of the above limit, we have

$$\log 2^{m+1} a_m a_{m-1}\dots a_0 = \log(a_{m+1}+2)+(m+1)\log(2)+\log(a_0)+\cdots+\log(a_m),$$

which also has $k\log k$ growth, with the same leading coefficient. The above limits will exist.

Now to show that

$$\lim_{N\to\infty}\frac{\log(Z_N(\alpha;\beta))}{\sqrt{N}\log N}$$

exists.

In an argument similar to the proof of Theorem 4.6, we have

$$\frac{\log(d_{N_{m-1}})}{\sqrt{N_{m+1}}\log N_{m+1}} \leqslant \frac{\log(Z_N(\alpha;\beta))}{\beta\sqrt{N}\log N} \leqslant \frac{\log(\frac{\zeta(\beta-1)}{\zeta(\beta)} N d_{N_m}^{\beta})}{\beta\sqrt{N_m}\log N_m}.$$

Following the proof in the last section, we see that

$$\lim_{m\to\infty}\frac{\log d_{N_{m-1}}}{\sqrt{N_{m+1}}\log N_{m+1}}=\lim_{m\to\infty}\frac{\log d_{N_{m-1}}}{\sqrt{N_{m-1}}\log N_{m-1}},$$

since $N_{m+1}-N_{m-1}$ grows at most linearly with respect to m and since each N_m grows quadratically with respect to m.

Since $a_{m+1}q_{N_m}\leqslant d_{N_m}\leqslant(a_{m+1}+2)q_{N_m}q_{N_m}$, if the limits exist, we have

$$\lim_{m\to\infty}\frac{\log a_{m+1}q_{N_m}}{\sqrt{N_m}\log N_m}\leqslant\lim_{m\to\infty}\frac{\log d_{N_m}}{\sqrt{N_m}\log N_m}\leqslant\lim_{m\to\infty}\frac{\log(a_{m+1}+2)q_{N_m}}{\sqrt{N_m}\log N_m}.$$

Since the a_{m+1} grow at most linearly with respect to m and the N_m grow quadratically, we have

$$\lim_{m\to\infty}\frac{\log a_{m+1}}{\sqrt{N_m}\log N_m}=\lim_{m\to\infty}\frac{\log(a_{m+1}+2)}{\sqrt{N_m}\log N_m}=0,$$

giving us that

$$\lim_{m\to\infty}\frac{\log d_{N_m}}{\sqrt{N_m}\log N_m}=\lim_{m\to\infty}\frac{\log q_{N_m}}{\sqrt{N_m}\log N_m},$$

concluding the proof.

5. Conclusion

There are a number of directions for future work. It would certainly be interesting to know if there are algebraic numbers besides the quadratics that have 1-free energy limits. Also, what type of free energies are there for various other real numbers. We have shown that some real numbers have 1-free energy limits while others do not. This too is just a beginning. What types of limits are possible? Are there numbers α for which the sequence $\frac{\log(Z_N(\alpha;\beta))}{\beta N}$ has any possible limit behavior? For example, certainly we should be able to rig α so that any number can be the limit of the sequence. In fact, we should be able to find such sequences with two accumulation points, three accumulation points, etc. All of these should provide information about the initial real number α.

Once we know that there is a free energy limit, then the most pressing question is to find for which values of β is the free-energy non-analytic. These points will be the analogs to critical point phenomena in physics. For the Knauf approach, there have been a number of good papers (such as in [16–20,2,5,6,9,11,13,14,21]) exploring the nature of these critical points. For our Diophantine partition functions, these types of questions seem to be equally subtle.

The various partition functions $Z_N(\beta)$ depend on the choice of the two-by-two matrix M. Different choices of M give rise to different thermodynamics. Knauf's M^K and Fiala's and Kleban's M^F are of the form

$$\begin{pmatrix}0&0\\x&1\end{pmatrix}$$

while the M introduced in this paper is

$$\begin{pmatrix}0&-1\\0&\alpha\end{pmatrix}.$$

But certainly other M can be chosen and studied.

Also, $Z_N(\beta)$ depends on the choice of the matrices A_0 and A_1. Other choices for these matrices will lead to different thermodynamical systems, each with its own number-theoretic implications.

Of course, why stick to two-by-two matrices. This leads to multi-dimensional continued fractions. There are many different multi-dimensional continued fraction algorithms. See Schweiger [34] for a sampling of some. Also, Major [22] has some preliminary work on this. Each of these will give rise to a thermodynamical system, again with meaning in number theory.

Finally, there is the question of putting these results into the language of transfer operators. (See [31,23,1] for general references.)

Supplementary material

The online version of this article contains additional supplementary material. Please visit doi:10.1016/j.jnt.2010.01.016.

References

[1] V. Baladi, Positive Transfer Operators and Decay of Correlations, Adv. Ser. Nonlinear Dynam., vol. 16, World Scientific, 2000.
[2] F. Boca, Products of matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and the distribution of reduced quadratic irrationals, J. Reine Angew. Math. 606 (2007) 149–165.
[3] E. Burger, Exploring the Number Jungle: A Journey into Diophantine Analysis, Stud. Math. Libr., vol. 8, American Mathematical Society, 2000.
[4] E. Burger, R. Tubbs, Making Transcendence Transparent: An Intuitive Approach to Classical Transcendental Number Theory, Springer, 2004.
[5] P. Contucci, A. Knauf, The phase transition in statistical models defined on Farey fractions, Forum Math. 9 (1997) 547–567.
[6] M. Esposti, S. Isola, A. Knauf, Generalized Farey trees, transfer operators and phase transition, Comm. Math. Phys. 275 (2007) 298–329.
[7] M. Feigenbaum, I. Procaccia, T. Tel, Scaling properties of multifractals as an eigenvalue problem, Phys. Rev. A 39 (1989) 5359–5372.
[8] J. Fiala, P. Kleban, Generalized number theoretic spin-chain conditions to dynamical systems and expectation values, J. Stat. Phys. 121 (2005) 553–577.
[9] J. Fiala, P. Kleban, A. Özlük, The phase transition in statistical models defined on Farey fractions, J. Stat. Phys. 110 (2003) 73–86.
[10] R. Graham, D. Knuth, O. Patashnik, Concrete Mathematics, Addison–Wesley, 1989.
[11] F. Guerra, A. Knauf, Free energy and correlations of the number-theoretical spin chains, J. Math. Phys. 39 (1998) 3188–3202.
[12] D. Hensley, Continued Fractions, World Scientific, 2006.
[13] S. Isola, On the spectrum of Farey and Gauss maps, Nonlinearity 15 (2002) 1521–1539.
[14] J. Kallies, A. Özlük, M. Peter, C. Snyder, On asymptotic properties of a number theoretic function arising out of a spin chain model in statistical mechanics, Comm. Math. Phys. 222 (1) (2001) 9–43.
[15] P. Kleban, A. Ozluk, A Farey fraction spin chain, Comm. Math. Phys. 203 (1999) 635–647.
[16] A. Knauf, On a ferromagnetic chain, Comm. Math. Phys. 153 (1993) 77–115.
[17] A. Knauf, Phases of the number-theoretical spin chain, J. Stat. Phys. 73 (1993) 423–431.
[18] A. Knauf, On a ferromagnetic spin chain, part ii: thermodynamic limit, J. Math. Phys. 35 (1994) 228–236.
[19] A. Knauf, The number-theoretic spin chain and the Riemann zeros, Comm. Math. Phys. 196 (1998) 703–731.
[20] A. Knauf, Number theory, dynamical systems and statistical mechanics, Rev. Math. Phys. 11 (1998) 1027–1060.
[21] P. Manfred, The limit distribution of a number theoretic function arising from a problem in statistical mechanics, J. Number Theory 90 (2) (2001) 265–280.
[22] E. Major, Phase transitions of multidimensional generalizations of the Knauf number-theoretic chain modal, senior thesis, Williams College, 2003.
[23] D. Mayer, The Ruelle–Araki Transfer Operator in Classical Statistical Mechanics, Springer, 1980.
[24] D. Mayer, Continued fractions and related transformations, in: Ergodic Theory, Symbolic Dynamics, and Hyperbolic Spaces, Trieste, 1989, Oxford University Press, 1991, pp. 175–222.
[25] M. Mendès France, G. Tenenbaum, A one-dimensional model with phase transition, Comm. Math. Phys. 154 (1993) 603–611.
[26] M. Mendès France, G. Tenenbaum, Phase transitions and divisors, in: Probability Theory and Mathematical Statistics, Vilnius, 1993, pp. 541–552.
[27] T. Prellberg, Towards a complete determination of the spectrum of the transfer operator associated with intermittency, J. Phys. A 36 (2003) 2455–2461.
[28] T. Prellberg, J. Fiala, P. Kleban, Cluster approximation for the Farey fraction spin chain, J. Stat. Phys. 123 (2006) 455–471.
[29] T. Prellberg, J. Slawny, Maps of intervals with indifferent fixed points: thermodynamic formalism and phase transition, J. Stat. Phys. 66 (1992) 503–514.
[30] A. Rockett, P. Szüsz, Continued Fractions, World Scientific, 1992.
[31] D. Ruelle, Dynamical zeta functions for maps of the interval, Bull. Amer. Math. Soc. (N.S.) 30 (1994) 212–214.

[32] D. Ruelle, Statistical Mechanics: Rigorous Results, World Scientific, 1999.
[33] D. Ruelle, Thermodynamic Formalism: The Mathematical Structure of Equilibrium Statistical Mechanics, second edition, Cambridge University Press, Cambridge, 2004.
[34] F. Schweiger, Multidimensional Continued Fractions, Oxford University Press, 2000.
[35] Y. Sinai, Theory of Phase Transitions, Pergamon Press, 1983.
[36] H. Stark, An Introduction to Number Theory, MIT Press, 1994.
[37] C. Thompson, Mathematical Statistical Mechanics, Princeton University Press, 1972.