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Inequalities between rank moments of overpartitions

Renrong Mao

Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, 21 Nanyang Link, Singapore 637371, Republic of Singapore

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ABSTRACT

F.G. Garvan proved an inequality between crank moments and rank moments of partitions which utilizes an inequality for symmetrized rank and crank moments. In this paper, we study two symmetrized rank moments of overpartitions and prove an inequality between two rank moments of overpartitions.

1. Introduction

Let $N(m,n)$ (resp. $M(m,n)$) be the number of partitions of n whose rank (resp. crank) is m. It was conjectured by F. G. Garvan in [7] that

$$M_{2k}(n) > N_{2k}(n), \tag{1.1}$$

for all $k \geqslant 1$ and $n \geqslant 1$. Here $M_k(n)$ and $N_k(n)$ are the k-th crank and k-th rank moments functions defined by

$$M_k(n) := \sum_{m\in\mathbb{Z}} m^k M(m,n)$$

E-mail address: MAOR0001@e.ntu.edu.sg.

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and

$$N_k(n) := \sum_{m\in\mathbb{Z}} m^k N(m,n).$$

For each fixed k, inequality (1.1) was proved for sufficiently large n in [5]. In [8], by studying symmetrized crank and rank moments, Garvan proved (1.1) for all n and k. Recall that symmetrized rank moments were first discussed by G. E. Andrews in [1]. Andrews defined the k-th symmetrized rank function by

$$\eta_k(n) := \sum_{m=-\infty}^{\infty} \binom{m+\lfloor\frac{k-1}{2}\rfloor}{k} N(m,n).$$

Generalizing the above definition, Garvan defined the k-th symmetrized crank moments by

$$\mu_k(n) := \sum_{m=-\infty}^{\infty} \binom{m+\lfloor\frac{k-1}{2}\rfloor}{k} M(m,n).$$

It was proved in [8] that

$$\mu_{2j}(n) > \eta_{2j}(n), \tag{1.2}$$

for all $n \geqslant j \geqslant 1$. Using (1.2) together with the relations between ranks moments (see [8, Theorem 4.3]), Garvan established inequality (1.1). In this paper, we will establish inequalities between rank moments of overpartitions.

An overpartition [6] is a partition in which the first occurrence of each distinct number may be overlined. There are two distinct ranks of interest: D-rank [9] and M_2-rank [10]. To define these two ranks, we use the notation $l(\cdot)$ to denote the largest part of an object, $n(\cdot)$ to denote the number of parts, and λ_0 for the subpartition of an overpartition consisting of the odd non-overlined parts. Then the D-rank of an overpartition λ is the largest part $l(\lambda)$ minus the number of parts $n(\lambda)$, a straight application of Dyson's partition rank to overpartitions. The M_2-rank of an overpartition λ is given by

$$M_2\text{-rank}(\lambda) := \left\lceil \frac{l(\lambda)}{2} \right\rceil - n(\lambda) + n(\lambda_0) - \chi(\lambda),$$

where $\chi(\lambda) = 1$ if $l(\lambda)$ is odd and non-overlined and $\chi(\lambda) = 0$ otherwise.

Rank moments of overpartitions were studied by Bringmann, Lovejoy and Osburn in [3]. Specifically, let $\overline{N}(m,n)$ (resp. $\overline{N2}(m,n)$) denote the number of overpartitions of n whose D-rank (resp. M_2-rank) is m. Then the rank moments $\overline{N}_k(n)$ and $\overline{N2}_k(n)$, are defined by

$$\overline{N}_k(n) := \sum_{m\in\mathbb{Z}} m^k \overline{N}(m,n)$$

and

$$\overline{N2}_k(n) := \sum_{m\in\mathbb{Z}} m^k \overline{N2}(m,n).$$

It was proved that certain linear combinations of the even rank moments could be written in terms of quasimodular forms (see [3, Theorem 1.1]).

We prove the following inequality between the even rank moments of overpartitions.

Theorem 1.1. *For $n \geqslant 2$ and $k \geqslant 1$, we have*

$$\overline{N}_{2k}(n) > \overline{N2}_{2k}(n).$$

To prove Theorem 1.1, we follow Garvan's approach in [8]. Namely, we study the symmetrized rank moments of overpartitions and establish an inequality between these symmetrized ranks moments in Section 2. By these results together with the relations between the rank moments and the symmetrized rank moments which will be established in Section 3, we prove Theorem 1.1.

2. Symmetrized rank moments of overpartitions

Generalizing the symmetrized rank moments of the ordinary partitions, Bringmann, Lovejoy and Osburn [4] studied the k-th symmetrized rank moment for overpartition pairs. As special cases the two symmetrized rank moments for overpartitions are also discussed. Let $\bar{\eta}_k(n)$ (resp. $\bar{\mu}_k(n)$) denote the symmetrized D-rank (resp. M_2-rank) moments for overpartitions defined by

$$\bar{\eta}_k(n) := \sum_{m=-\infty}^{\infty} \binom{m + \lfloor\frac{k-1}{2}\rfloor}{k} \overline{N}(m,n)$$

and

$$\bar{\mu}_k(n) := \sum_{m=-\infty}^{\infty} \binom{m + \lfloor\frac{k-1}{2}\rfloor}{k} \overline{N2}(m,n).$$

We have the following generating functions

Theorem 2.1.

$$\sum_{n=1}^{\infty} \bar{\eta}_{2k}(n)q^n = \frac{2(-q;q)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}q^{n^2+kn}}{(1-q^n)^{2k}}, \tag{2.1}$$

and

$$\sum_{n=1}^{\infty} \bar{\mu}_{2k}(n)q^n = \frac{2(-q;q)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}q^{n^2+2kn}}{(1-q^{2n})^{2k}}. \tag{2.2}$$

Proof. By the argument in Section 2 and Theorem 2.1 of [4], we see that

$$\sum_{n=1}^{\infty} \bar{\eta}_{2k}(n)q^n = \mathcal{N}_{2k}(0,1;q)$$

and

$$\sum_{n=1}^{\infty} \bar{\mu}_{2k}(n)q^n = \mathcal{N}_{2k}\big(1,1/q;q^2\big),$$

where

$$\mathcal{N}_{2k}(d,e,q) := \frac{(-dq,-eq;q)_\infty}{(q,deq)_\infty} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^{n-1}q^{\frac{n^2+n}{2}+kn}(de)^n(-1/d,-1/e)_n}{(1-q^n)^{2k}(-dq,-eq)_n}. \tag{2.3}$$

Let $d \to 0$ and $e = 1$ in (2.3) and simplifying, we find that

$$\begin{aligned}
\sum_{n=1}^{\infty} \bar{\eta}_{2k}(n)q^n &= \frac{2(-q;q)_\infty}{(q;q)_\infty} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^{n-1}q^{n^2+kn}}{(1+q^n)(1-q^n)^{2k}} \\
&= \frac{2(-q;q)_\infty}{(q;q)_\infty} \left\{ \sum_{n\geqslant 1} \frac{(-1)^{n-1}q^{n^2+kn}}{(1+q^n)(1-q^n)^{2k}} + \sum_{n\leqslant -1} \frac{(-1)^{n-1}q^{n^2+kn}}{(1+q^n)(1-q^n)^{2k}} \right\} \\
&= \frac{2(-q;q)_\infty}{(q;q)_\infty} \left\{ \sum_{n\geqslant 1} \frac{(-1)^{n-1}q^{n^2+kn}}{(1+q^n)(1-q^n)^{2k}} + \sum_{n\geqslant 1} \frac{(-1)^{n-1}q^{n^2-kn}}{(1+q^{-n})(1-q^{-n})^{2k}} \right\} \\
&= \frac{2(-q;q)_\infty}{(q;q)_\infty} \left\{ \sum_{n\geqslant 1} \frac{(-1)^{n-1}q^{n^2+kn}}{(1+q^n)(1-q^n)^{2k}} + \sum_{n\geqslant 1} \frac{(-1)^{n-1}q^{n^2+kn+n}}{(1+q^n)(1-q^n)^{2k}} \right\} \\
&= \frac{2(-q;q)_\infty}{(q;q)_\infty} \sum_{n\geqslant 1} \frac{(-1)^{n-1}q^{n^2+kn}}{(1-q^n)^{2k}}.
\end{aligned}$$

This completes the proof of (2.1). Since the proof of (2.2) is similar, we omit the details. □

As for the symmetrized moments of partitions, an inequality exists between the two symmetrized rank moments of overpartitions.

Theorem 2.2. *For all positive integer k and n, we have*

$$\overline{\eta}_{2k}(n) \geqslant \overline{\mu}_{2k}(n).$$

The inequality is strict for $n \geqslant k+1$.

To prove Theorem 2.2, we need the following lemma.

Lemma 2.3. *For fixed integer r, such that $0 \leqslant r \leqslant k-1$, let*

$$\sum_{j=0}^{\infty} b_{r,k}(j)q^j := \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}q^{n^2+kn}}{(1-q^n)^{2k}} \left\{ \frac{q^{rn}}{(1+q^n)^{2r}} - \frac{q^{(r+1)n}}{(1+q^n)^{2r+2}} \right\}.$$

Then we have $b_{r,k}(j) \geqslant 0$, for all $j \geqslant 0$ and $b_{r,k}(j) > 0$, for all $j \geqslant r+k+1$.

Proof. We have

$$\begin{aligned}
\sum_{j=0}^{\infty} b_{r,k}(j)q^j &= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}q^{n^2+kn}}{(1-q^n)^{2k}} \left\{ \frac{q^{rn}}{(1+q^n)^{2r}} - \frac{q^{(r+1)n}}{(1+q^n)^{2r+2}} \right\} \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}q^{n^2+(r+k)n}}{(1-q^n)^{2k}(1+q^n)^{2r+2}} \left\{ \left(1+q^n\right)^2 - q^n \right\} \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}q^{n^2+(r+k)n}}{(1-q^n)^{2k-2r-2}(1-q^{2n})^{2r+2}} \left\{ 1+q^n+q^{2n} \right\} \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} g_{r,k}\left(q^n\right) \left\{ q^{n^2} + q^{n^2+n} + q^{n^2+2n} \right\},
\end{aligned} \tag{2.4}$$

where

$$g_{r,k}\left(q^n\right) := \frac{q^{(r+k)n}}{(1-q^n)^{2k-2r-2}(1-q^{2n})^{2r+2}}.$$

Since $0 \leqslant r \leqslant k-1$, we have $2k-2r-2 \geqslant 0$ and the following expression

$$g_{r,k}\left(q^n\right) = \sum_{m=0}^{\infty} a_m q^{mn} \tag{2.5}$$

where a_m is non-negative integer depends only on r and k. In particular, we find that $a_{r+k} = 1$ and $a_m = 0$ for $0 \leqslant m < r+k$. Substituting (2.5) into (2.4), we find that

$$\sum_{j=0}^{\infty} b_{r,k}(j)q^j = \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} \times \left(\sum_{m=0}^{\infty} a_m q^{mn} \right) \times \left\{ q^{n^2} + q^{n^2+n} + q^{n^2+2n} \right\}$$

$$= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{m=0}^{\infty} a_m \sum_{n=1}^{\infty} (-1)^{n-1} q^{mn} \{ q^{n^2} + q^{n^2+n} + q^{n^2+2n} \}$$

$$= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{m=0}^{\infty} a_m \sum_{n=1}^{\infty} (-1)^{n-1} \{ q^{n^2+mn} + q^{n^2+(m+1)n} + q^{n^2+(m+2)n} \}$$

$$= \sum_{m=0}^{\infty} a_m \{ h_m(q) + h_{m+1}(q) + h_{m+2}(q) \}, \tag{2.6}$$

where

$$h_m(q) := \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n^2+mn}.$$

Since we have

$$\begin{aligned} h_m(q) &= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n^2+mn} \\ &= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \left(q^{(2n+1)^2+m(2n+1)} - q^{(2n+2)^2+m(2n+2)} \right) \\ &= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} q^{(2n+1)^2+m(2n+1)} \left(1 - q^{4n+m+3} \right) \\ &= \frac{(-q;q)_\infty}{(q^2;q)_\infty} \sum_{n=0}^{\infty} q^{(2n+1)^2+m(2n+1)} \left(\frac{1 - q^{4n+m+3}}{1-q} \right) \\ &= \frac{(-q^2;q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} q^{(2n+1)^2+m(2n+1)} \sum_{j=0}^{4n+m+2} q^{m+j+1} \\ &= q^{m+1} \frac{(-q^2;q)_\infty}{(q;q)_\infty} + \frac{(-q^2;q)_\infty}{(q;q)_\infty} \sum_{j=1}^{m+2} q^j \\ &\quad + \frac{(-q^2;q)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} q^{(2n+1)^2+m(2n+1)} \sum_{j=0}^{4n+m+2} q^j, \end{aligned}$$

it is easy to see that $h_m(q)$ has non-negative coefficients of q^k for $k \geqslant 0$. This together with (2.6) implies $b_{r,k}(j) \geqslant 0$, for all $j \geqslant 0$. Next, since the factor $\frac{q^{m+1}}{1-q}$ appears in the first product of the right side of the above equation, we know that $h_m(q)$ has positive coefficients of q^k when $k \geqslant m+1$. Noting that $a_{r+k} = 1$ and $a_m = 0$ for $0 \leqslant m < r+k$, by (2.6), we see that $b_{r,k}(j) > 0$, for all $j \geqslant r+k+1$. This completes the proof of the lemma. □

Now we prove Theorem 2.2.

Proof of Theorem 2.2. By Theorem 2.1, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\big(\bar{\eta}_{2k}(n)-\bar{\mu}_{2k}(n)\big)q^n \\
&\quad=\frac{2(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n-1}q^{n^2+kn}}{(1-q^n)^{2k}}-\frac{(-1)^{n-1}q^{n^2+2kn}}{(1-q^{2n})^{2k}}\right\} \\
&\quad=\frac{2(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2+kn}}{(1-q^n)^{2k}}\left\{1-\frac{q^{kn}}{(1+q^n)^{2k}}\right\} \\
&\quad=\frac{2(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2+kn}}{(1-q^n)^{2k}}\sum_{r=0}^{k-1}\left\{\frac{q^{rn}}{(1+q^n)^{2r}}-\frac{q^{(r+1)n}}{(1+q^n)^{2r+2}}\right\} \\
&\quad=2\sum_{r=0}^{k-1}\sum_{j=0}^{\infty}b_{r,k}(j)q^j,
\end{aligned}$$

which together with Lemma 2.3 implies $\bar{\eta}_{2k}(n)\geqslant\bar{\mu}_{2k}(n)$ for all positive integers n and k and $\bar{\eta}_{2k}(n)>\bar{\mu}_{2k}(n)$ for all $n\geqslant k+1$. $\square$

3. Rank moments of overpartitions

In this section, we establish certain relations between rank moments and symmetrized rank moments of overpartitions and prove Theorem 1.1. Following the notation in [8], we define the sequence $S^*(n,k)$ $(1\leqslant k\leqslant n)$ recursively by

(1) $S^*(1,1)=1$,

(2) $S^*(n,k)=0$ if $k\leqslant 0$ or $k>n$, and

(3) $S^*(n+1,k)=S^*(n,k-1)+k^2S^*(n,k)$, for $1\leqslant k\leqslant n+1$.

By an argument completely analogous to that in the proof of [8, Theorem 4.3], we can prove the following theorem.

Theorem 3.1. *For $k\geqslant 1$, we have*

$$\overline{N}_{2k}(n)=\sum_{j=1}^{k}(2j)!S^*(k,j)\bar{\eta}_{2j}(n),$$

and

$$\overline{N2}_{2k}(n)=\sum_{j=1}^{k}(2j)!S^*(k,j)\bar{\mu}_{2j}(n).$$

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. By Theorem 3.1, we find that

$$\begin{aligned}
&\overline{N}_{2k}(n) - \overline{N2}_{2k}(n) \\
&\quad = \sum_{j=1}^{k} (2j)! S^*(k,j) \{ \bar{\eta}_{2j}(n) - \bar{\mu}_{2j}(n) \} \\
&\quad = 2S^*(k,1) \{ \bar{\eta}_2(n) - \bar{\mu}_2(n) \} + \sum_{j=2}^{k} (2j)! S^*(k,j) \{ \bar{\eta}_{2j}(n) - \bar{\mu}_{2j}(n) \}. \qquad (3.1)
\end{aligned}$$

By definition, we know that $S^*(k,j)$ is non-negative and $S^*(k,1) \geqslant 1$ for all positive k. This together with Theorem 2.2 implies the first term on the right side of (3.1) is positive for $n \geqslant 2$ and the second term is non-negative for all $n \geqslant 0$. Thus we complete the proof. $\square$

4. Concluding remarks

Since we have

$$\begin{aligned}
M(m,n) &= M(-m,n), \\
N(m,n) &= N(-m,n), \\
\overline{N}(m,n) &= \overline{N}(-m,n)
\end{aligned}$$

and

$$\overline{N2}(m,n) = \overline{N2}(-m,n),$$

all the rank and crank moments discussed in the previous sections are equal to 0 when k is an odd integer. Therefore Andrews, Chan and Kim studied the nontrivial odd crank and rank moments defined by

$$M_k^+(n) = \sum_{m=1}^{\infty} m^j M(m,n),$$

and

$$N_k^+(n) = \sum_{m=1}^{\infty} m^j N(m,n).$$

It was proved in [2] that

$$M_k^+(n) > N_k^+(n), \qquad (4.1)$$

for all positive integers k and n. Noting that $2M_k^+(n) = M_k(n)$ and $2N_k^+(n) = N_k(n)$, we see that (4.1) implies (1.1).

Computer evidence suggests that an analog of (4.1) for overpartitions might hold. Namely, we have the following conjecture.

Conjecture 4.1. *For $n \geqslant 2$ and $k \geqslant 1$, we have*

$$\overline{N}_k^+(n) > \overline{N2}_k^+(n),$$

where

$$\overline{N}_k^+(n) := \sum_{m \geqslant 0} m^k \overline{N}(m,n)$$

and

$$\overline{N2}_k^+(n) := \sum_{m \geqslant 0} m^k \overline{N2}(m,n).$$

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