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Some divisibility properties of binomial and q-binomial coefficients

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ABSTRACT

We first prove that if a has a prime factor not dividing b then there are infinitely many positive integers n such that $\binom{an+bn}{an}$ is not divisible by $bn+1$. This confirms a recent conjecture of Z.-W. Sun. Moreover, we provide some new divisibility properties of binomial coefficients: for example, we prove that $\binom{12n}{3n}$ and $\binom{12n}{4n}$ are divisible by $6n-1$, and that $\binom{330n}{88n}$ is divisible by $66n-1$, for all positive integers n. As we show, the latter results are in fact consequences of divisibility and positivity results for quotients of q-binomial coefficients by q-integers, generalising the positivity of q-Catalan numbers. We also put forward several related conjectures.

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1. Introduction

The study of arithmetic properties of binomial coefficients has a long history. In 1819, Babbage [6] proved the congruence

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^2}$$

for primes $p \geqslant 3$. In 1862, Wolstenholme [28] showed that the above congruence holds modulo p^3 for any prime $p \geqslant 5$. See [20] for a historical survey on Wolstenholme's theorem. Another famous congruence is

$$\binom{2n}{n} \equiv 0 \pmod{n+1}.$$

The corresponding quotients, the numbers $C_n := \frac{1}{n+1}\binom{2n}{n}$, are called *Catalan numbers*, and they have many interesting combinatorial interpretations; see, for example, [12] and [24, pp. 219–229]. Recently, Ulas and Schinzel [27] studied divisibility problems of Erdős and Straus, and of Erdős and Graham. In [25,26], Sun gave some new divisibility properties of binomial coefficients and their products. For example, Sun proved the following result.

Theorem 1.1. *(See [26, Theorem 1.1].) Let a, b, and n be positive integers. Then*

$$\binom{an+bn}{an} \equiv 0 \mod \frac{bn+1}{\gcd(a, bn+1)}. \tag{1.1}$$

Sun also proposed the following conjecture.

Conjecture 1.2. *(See [26, Conjecture 1.1].) Let a and b be positive integers. If $(bn+1) \mid \binom{an+bn}{an}$ for all sufficiently large positive integers n, then each prime factor of a divides b. In other words, if a has a prime factor not dividing b, then there are infinitely many positive integers n such that $(bn+1) \nmid \binom{an+bn}{an}$.*

Inspired by Conjecture 1.2, Sun [26] introduced a new function $f: \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{N}$. Namely, for positive integers a and b, if $\binom{an+bn}{an}$ is divisible by $bn+1$ for all $n \in \mathbb{Z}^+$, then he defined $f(a,b) = 0$; otherwise, he let $f(a,b)$ be the smallest positive integer n such that $\binom{an+bn}{an}$ is not divisible by $bn+1$. Using *Mathematica*, Sun [26] computed some values of the function f:

$$f(7,36) = 279, \qquad f(10,192) = 362, \qquad f(11,100) = 1187,$$
$$f(22,200) = 6462, \qquad \ldots.$$

The present paper serves several purposes: first of all, we give a proof of Conjecture 1.2 (see Theorem 2.1 below); second, we provide congruences and divisibility results similar to the ones addressed in Theorem 1.1 and Conjecture 1.2 (see Theorems 2.2–2.3 in Section 2); third, we show in Section 3 that among these results there is a significant number which can be "lifted to the q-world"; in other words, there are several such results which follow directly from stronger divisibility results for q-polynomials. In particular, Theorem 1.1 is an easy consequence of Theorem 3.3, and Theorem 2.3 is an easy consequence of Theorem 3.1. On the other hand, Theorem 2.4 hints at the limitations of occurrence of these divisibility phenomena. Sections 4–6 are devoted to the proofs of our results in Sections 2 and 3. We close our paper with Section 7 by posing several open problems.

2. Results, I

Our first result is a more precise version of Conjecture 1.2.

Theorem 2.1. *Conjecture 1.2 is true. Moreover, if p is a prime such that $p \mid a$ but $p \nmid b$, then*

$$f(a,b) \leqslant \frac{p^{\varphi(a+b)}-1}{a+b},$$

where $\varphi(n)$ is Euler's totient function.

For the proof of the above result, we need the following theorem.

Theorem 2.2. *Let a and b be positive integers with $a > b$, and β an integer. Let p be a prime not dividing a. Then there are infinitely many positive integers n such that*

$$\binom{an}{bn+\beta} \equiv \pm 1 \pmod p.$$

Our proofs of Theorems 2.1 and 2.2 are based on Euler's totient theorem and Lucas' classical theorem on the congruence behaviour of binomial coefficients modulo prime numbers, see Section 4.

In [14, Corollary 2.3], the first author proved that

$$\binom{6n}{3n} \equiv 0 \pmod{2n-1}. \tag{2.1}$$

It is easy to see that

$$\binom{2n}{n} = 2\binom{2n-1}{n} = \frac{4n-2}{n}\binom{2n-2}{n-1} \equiv 0 \pmod{2n-1}. \tag{2.2}$$

The next theorem gives congruences similar to (2.1) and (2.2).

Theorem 2.3. *Let n be a positive integer. Then*

$$\binom{12n}{3n} \equiv \binom{12n}{4n} \equiv 0 \pmod{6n-1}, \tag{2.3}$$

$$\binom{30n}{5n} \equiv 0 \pmod{(10n-1)(15n-1)}, \tag{2.4}$$

$$\binom{60n}{6n} \equiv \binom{120n}{40n} \equiv \binom{120n}{45n} \equiv 0 \pmod{30n-1}, \tag{2.5}$$

$$\binom{330n}{88n} \equiv 0 \pmod{66n-1}. \tag{2.6}$$

We shall see that this theorem is the consequence of a stronger result for q-binomial coefficients, cf. Theorem 3.1 in the next section.

It seems that there should exist many more congruences like (2.1)–(2.6). (In this direction, see Conjecture 7.3.) On the other hand, we have the following negative result.

Theorem 2.4. *There are no positive integers a and b such that*

$$\binom{an+bn}{an} \equiv 0 \pmod{3n-1}$$

for all $n \geqslant 1$.

For a possible generalisation of this theorem see Conjecture 7.2 in the last section.

3. Results, II: q-divisibility properties

Recall that the *q-binomial coefficients* (also called *Gaußian polynomials*) are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(1-q^n)(1-q^{n-1})\cdots(1-q)}{(1-q^k)(1-q^{k-1})\cdots(1-q)(1-q^{n-k})(1-q^{n-k-1})\cdots(1-q)}, & \text{if } 0 \leqslant k \leqslant n, \\ 0, & \text{otherwise.} \end{cases}$$

We begin with the announced strengthening of Theorem 2.3.

Theorem 3.1. *Let n be a positive integer. Then all of*

$$\begin{gathered} \frac{1-q}{1-q^{6n-1}} \begin{bmatrix} 12n \\ 3n \end{bmatrix}_q, \qquad \frac{1-q}{1-q^{6n-1}} \begin{bmatrix} 12n \\ 4n \end{bmatrix}_q, \qquad \frac{1-q}{1-q^{30n-1}} \begin{bmatrix} 60n \\ 6n \end{bmatrix}_q, \\ \frac{1-q}{1-q^{30n-1}} \begin{bmatrix} 120n \\ 40n \end{bmatrix}_q, \qquad \frac{1-q}{1-q^{30n-1}} \begin{bmatrix} 120n \\ 45n \end{bmatrix}_q, \qquad \frac{1-q}{1-q^{66n-1}} \begin{bmatrix} 330n \\ 88n \end{bmatrix} \end{gathered} \tag{3.1}$$

are polynomials in q with non-negative integer coefficients. Furthermore,

$$\frac{(1-q)^2}{(1-q^{10n-1})(1-q^{15n-1})}\begin{bmatrix}30n\\5n\end{bmatrix}_q \tag{3.2}$$

is a polynomial in q.

For a conjectural stronger form of the last assertion in the above theorem see Conjecture 7.4 at the end of the paper.

It is obvious that, when $a=b=1$, the numbers $\binom{an+bn}{an}/(bn+1)$ (featured implicitly in Conjecture 1.2 and in Theorem 2.1) reduce to the Catalan numbers C_n. There are various q-analogues of the Catalan numbers. See Fürlinger and Hofbauer [10] for a survey, and see [11,17,16] for the so-called q,t-Catalan numbers.

A natural q-analogue of C_n is

$$C_n(q)=\frac{1-q}{1-q^{n+1}}\begin{bmatrix}2n\\n\end{bmatrix}_q.$$

It is well known that the q-Catalan numbers $C_n(q)$ are polynomials with non-negative integer coefficients (see [1,2,4,10]). Furthermore, Haiman [17, (1.7)] proved (and it follows from Lemma 5.2 below) that the polynomial

$$\frac{1-q}{1-q^{bn+1}}\begin{bmatrix}bn+n\\n\end{bmatrix}_q$$

has non-negative coefficients for all $b,n\geqslant 1$. Another generalisation of $C_n(q)$ was introduced by the first author and Zeng [15]:

$$B_{n,k}(q):=\frac{1-q^k}{1-q^n}\begin{bmatrix}2n\\n-k\end{bmatrix}_q=\begin{bmatrix}2n-1\\n-k\end{bmatrix}_q-\begin{bmatrix}2n-1\\n-k-1\end{bmatrix}_q q^k,\quad 1\leqslant k\leqslant n.$$

They noted that the $B_{n,k}(q)$'s are polynomials in q, but did not address the question whether they are polynomials with non-negative coefficients. As the next theorem shows, this turns out to be the case. The theorem establishes in fact a stronger non-negativity property.

Theorem 3.2. *Let n and k be non-negative integers with $0\leqslant k\leqslant n$. Then*

$$\frac{1-q^{\gcd(k,n)}}{1-q^n}\begin{bmatrix}2n\\n-k\end{bmatrix}_q \tag{3.3}$$

is a polynomial in q with non-negative integer coefficients. Consequently, also $B_{n,k}(q)=\frac{1-q^k}{1-q^n}\left[\begin{smallmatrix}2n\\n-k\end{smallmatrix}\right]_q$ is a polynomial with non-negative coefficients.

Applying the inequality [26, (2.1)], we can also easily deduce that

$$C_{a,b,n}(q) := \frac{1-q^a}{1-q^{bn+1}} \begin{bmatrix} an+bn \\ an \end{bmatrix}_q$$

is a product of certain cyclotomic polynomials, and therefore a polynomial in q. Again, as it turns out, all coefficients in these polynomials are non-negative. Also here, we have actually a stronger result, given in the theorem below. It should be noted that it generalises Theorem 1.1, the latter being obtained upon letting $q \to 1$.

Theorem 3.3. *Let a, b, and n be positive integers. Then*

$$\frac{1-q^{\gcd(an,bn+1)}}{1-q^{bn+1}} \begin{bmatrix} an+bn \\ an \end{bmatrix}_q = \frac{1-q^{\gcd(an,bn+1)}}{1-q^{an+bn+1}} \begin{bmatrix} an+bn+1 \\ an \end{bmatrix}_q$$

is a polynomial in q with non-negative coefficients.

Corollary 3.4. *Let a, b, and n be positive integers. Then*

$$\frac{1-q^a}{1-q^{bn+1}} \begin{bmatrix} an+bn \\ an \end{bmatrix}_q$$

is a polynomial in q with non-negative coefficients.

The proofs of the results in this section are given in Section 5.

4. Proofs of Theorems 2.1 and 2.2

The proof of Theorem 2.2 (from which subsequently Theorem 2.1 is derived) makes essential use of Lucas' classical theorem on binomial coefficient congruences (see, for example, [7,9,13,21]). For the convenience of the reader, we recall the theorem below.

Theorem 4.1 *(Lucas' theorem). Let p be a prime, and let $a_0, b_0, \dots, a_m, b_m \in \{0, 1, \dots, p-1\}$. Then*

$$\binom{a_0 + a_1 p + \cdots + a_m p^m}{b_0 + b_1 p + \cdots + b_m p^m} \equiv \prod_{i=0}^{m} \binom{a_i}{b_i} \pmod{p}.$$

Proof of Theorem 2.2. Note that $\gcd(p, a) = 1$. By Euler's totient theorem (see [23]), we have

$$p^{\varphi(a)} - 1 \equiv 0 \pmod{a}.$$

Since $a > b > 0$, there exists a positive integer N such that $an > bn + \beta > 0$ holds for all $n > N$. Let r be a positive integer such that $p^{r\varphi(a)} - 1 > aN$, and let $n = (p^{r\varphi(a)} - 1)/a$. Then, by Lucas' theorem, we have

$$\binom{an}{bn+\beta} = \binom{p^{r\varphi(a)}-1}{bn+\beta} \equiv \prod_{i=0}^{m} \binom{p-1}{b_i} \equiv (-1)^{b_0+\cdots+b_m} \pmod p,$$

where $bn+\beta = b_0 + b_1 p + \cdots + b_m p^m$ with $0 \leqslant b_0, \ldots, b_m \leqslant p-1$. It is clear that there are infinitely many such r and n. This completes the proof. □

Proof of Theorem 2.1. Suppose that a and b are positive integers and p a prime such that $p \mid a$ but $p \nmid b$. We have the decomposition

$$\frac{1}{bn+1}\binom{an+bn}{an} = \binom{an+bn}{an-1} - \frac{a+b}{a}\binom{an+bn-1}{an-2}. \tag{4.1}$$

It is clear that $p \nmid (a+b)$. By the proof of Theorem 2.2, if we take $n = (p^{r\varphi(a+b)}-1)/(a+b)$ $(r \geqslant 1)$, then

$$\binom{an+bn}{an-1} \equiv \pm 1 \pmod p,$$

and thus

$$(a+b)\binom{an+bn-1}{an-2} = \frac{an-1}{n}\binom{an+bn}{an-1} \equiv \pm(a+b) \not\equiv 0 \pmod p. \tag{4.2}$$

Combining (4.1) and (4.2) gives

$$\frac{1}{bn+1}\binom{an+bn}{an} \notin \mathbb{Z}$$

for all $n = (p^{r\varphi(a+b)}-1)/(a+b)$ $(r = 1, 2, \ldots)$. Namely, Conjecture 1.2 holds and

$$f(a,b) \leqslant \frac{p^{\varphi(a+b)}-1}{a+b},$$

as desired. □

5. Proofs of Theorems 3.1–3.3 and of Corollary 3.4

All the proofs in this section are similar in spirit. They all draw on a lemma from [22, Proposition 10.1.(iii)], which extracts the essentials out of an argument of Andrews [3, Proof of Theorem 2]. (To be precise, Lemma 5.1 below is a slight generalisation of [22, Proposition 10.1.(iii)]. However, the proof from [22] works also for this generalisation. We provide it here for the sake of completeness.) Recall that a polynomial $P(q) = \sum_{i=0}^{d} p_i q^i$ in q of degree d is called *reciprocal* if $p_i = p_{d-i}$ for all i, and that it is called *unimodal* if there is an integer r with $0 \leqslant r \leqslant d$ and $0 \leqslant p_0 \leqslant \cdots \leqslant p_r \geqslant \cdots \geqslant p_d \geqslant 0$.

Lemma 5.1. *Let $P(q)$ be a reciprocal and unimodal polynomial and m and n positive integers with $m \leqslant n$. Furthermore, assume that $A(q) = \frac{1-q^m}{1-q^n} P(q)$ is a polynomial in q. Then $A(q)$ has non-negative coefficients.*

Proof. Since $P(q)$ is unimodal, the coefficient of q^k in $(1-q^m)P(q)$ is non-negative for $0 \leqslant k \leqslant \deg(P)/2$. Consequently, the same must be true for $A(q) = \frac{1-q^m}{1-q^n} P(q)$, considered as a formal power series in q. However, also $A(q)$ is reciprocal, and its degree is at most the degree of $P(q)$. Therefore the remaining coefficients of $A(q)$ must also be non-negative. $\square$

Proof of Theorem 3.1. In view of Lemma 5.1 and the well-known reciprocality and unimodality of q-binomial coefficients (cf. [24, Ex. 7.75.d]), for proving Theorem 3.1 it suffices to show that the expressions in (3.1) and (3.2) are polynomials in q. We are going to accomplish this by a count of the cyclotomic polynomials which divide numerators and denominators of these expressions, respectively.

We begin by showing that $\frac{1-q}{1-q^{6n-1}} \left[{12n \atop 3n} \right]_q$ is a polynomial in q. We recall the well-known fact that

$$q^n - 1 = \prod_{d \mid n} \Phi_d(q),$$

where $\Phi_d(q)$ denotes the d-th cyclotomic polynomial in q. Consequently,

$$\frac{1-q}{1-q^{6n-1}} \begin{bmatrix} 12n \\ 3n \end{bmatrix}_q = \prod_{d=2}^{12n} \Phi_d(q)^{e_d},$$

with

$$e_d = -\chi\big(d \mid (6n-1)\big) + \left\lfloor \frac{12n}{d} \right\rfloor - \left\lfloor \frac{3n}{d} \right\rfloor - \left\lfloor \frac{9n}{d} \right\rfloor, \tag{5.1}$$

where $\chi(\mathcal{S}) = 1$ if $\mathcal{S}$ is true and $\chi(\mathcal{S}) = 0$ otherwise. This is clearly non-negative, unless $d \mid (6n-1)$.

So, let us assume that $d \mid (6n-1)$, which in particular means that $d \geqslant 5$. Let us write $X = \{3n/d\}$, where $\{\alpha\} := \alpha - \lfloor \alpha \rfloor$ denotes the fractional part of α. Using this notation, Eq. (5.1) becomes

$$e_d = -\chi\big(d \mid (2dX-1)\big) + \lfloor 4X \rfloor - \lfloor 3X \rfloor .$$

Since $0 \leqslant X < 1$, we have $-1 \leqslant 2dX - 1 < 2d$. The only integers which are divisible by d in the range $-1, 0, \dots, 2d-1$ are 0 and d. Hence, we must have $X = 1/(2d)$ or $X = (d+1)/(2d)$. The former is impossible since X is a rational number which can be written with denominator d. Thus, the only possibility left is $X = (d+1)/(2d)$. For this choice, it follows that $\lfloor 4X \rfloor - \lfloor 3X \rfloor = 2 - 1 = 1$. (Here we used that $d \geqslant 5$.) This proves that e_d is non-negative also in this case, and completes the proof of polynomiality of $\frac{1-q}{1-q^{6n-1}}\left[\begin{smallmatrix}12n\\3n\end{smallmatrix}\right]_q$.

The proof of polynomiality of $\frac{1-q}{1-q^{6n-1}}\left[\begin{smallmatrix}12n\\4n\end{smallmatrix}\right]_q$ is completely analogous and therefore left to the reader.

We next turn our attention to $\frac{1-q}{1-q^{30n-1}}\left[\begin{smallmatrix}60n\\6n\end{smallmatrix}\right]_q$. Again, we write

$$\frac{1-q}{1-q^{30n-1}}\begin{bmatrix}60n\\6n\end{bmatrix}_q=\prod_{d=2}^{60n}\Phi_d(q)^{e_d},$$

with

$$e_d=-\chi\big(d\mid(30n-1)\big)+\left\lfloor\frac{60n}{d}\right\rfloor-\left\lfloor\frac{6n}{d}\right\rfloor-\left\lfloor\frac{54n}{d}\right\rfloor. \tag{5.2}$$

This is clearly non-negative, unless $d\mid(30n-1)$.

We assume $d\mid(30n-1)$ and note that this implies $d=7$ or $d\geqslant 11$. Here, we write $X=\{6n/d\}$. Using this notation, Eq. (5.2) becomes

$$e_d=-\chi\big(d\mid(5dX-1)\big)+\lfloor 10X\rfloor-\lfloor 9X\rfloor.$$

Since $0\leqslant X<1$, we have $d\mid(5dX-1)$ if and only if X is one of

$$\frac{1}{5d},\quad \frac{1}{5}+\frac{1}{5d},\quad \frac{2}{5}+\frac{1}{5d},\quad \frac{3}{5}+\frac{1}{5d},\quad \frac{4}{5}+\frac{1}{5d}.$$

For the same reason as before, the option $X=1/(5d)$ is impossible. For the other options, the corresponding value of $\lfloor 10X\rfloor-\lfloor 9X\rfloor$ is always 1, except if $X=\frac{1}{5}+\frac{1}{5d}$ and $d=7$. However, in that case, we have $X=\frac{8}{35}$, which cannot be written with denominator $d=7$. Therefore this case can actually not occur. This completes the proof that e_d is non-negative for all $d\geqslant 2$, and, hence, that $\frac{1-q}{1-q^{30n-1}}\left[\begin{smallmatrix}60n\\6n\end{smallmatrix}\right]_q$ is a polynomial in q.

Proceeding in the same style, for the proof of polynomiality of $\frac{1-q}{1-q^{30n-1}}\left[\begin{smallmatrix}120n\\40n\end{smallmatrix}\right]_q$ we have to show that

$$e_d=-\chi\big(d\mid(3dX-1)\big)+\lfloor 12X\rfloor-\lfloor 4X\rfloor-\lfloor 8X\rfloor \tag{5.3}$$

is non-negative for all X of the form $X=x/d$ with $0\leqslant x<d$, x being integral, and $d\geqslant 2$. Clearly, the expression in (5.3) is non-negative, except possibly if $d\mid(3dX-1)$. With the same reasoning as before, we see that the only cases to be examined are $X=(d+1)/(3d)$ and $X=(2d+1)/(3d)$, where $d=7$ or $d\geqslant 11$. If $X=(d+1)/(3d)$, we have

$$\lfloor 12X\rfloor-\lfloor 4X\rfloor-\lfloor 8X\rfloor=4-1-\left\lfloor\frac{8}{3}+\frac{8}{3d}\right\rfloor.$$

So, this will be equal to 1, except if $d = 7$. However, in that case we have $X = \frac{8}{21}$, which cannot be written with denominator $d = 7$, a contradiction. Similarly, if $X = (2d+1)/(3d)$, we have

$$\lfloor 12X \rfloor - \lfloor 4X \rfloor - \lfloor 8X \rfloor = 8 - 2 - 5 = 1.$$

So, again, the exponent e_d in (5.3) is non-negative, which establishes that $\frac{1-q}{1-q^{30n-1}} \left[{120n \atop 40n} \right]_q$ is a polynomial in q.

For the proof of polynomiality of $\frac{1-q}{1-q^{66n-1}} \left[{330n \atop 88n} \right]_q$ we have to show that

$$e_d = -\chi\big(d \mid (3dX - 1)\big) + \lfloor 15X \rfloor - \lfloor 4X \rfloor - \lfloor 11X \rfloor \tag{5.4}$$

is non-negative for all X of the form $X = x/d$ with $0 \leqslant x < d$, x being integral, and $d \geqslant 2$. Clearly, the expression in (5.4) is non-negative, except possibly if $d \mid (3dX - 1)$. With the same reasoning as before, we see that the only cases to be examined are $X = (d+1)/(3d)$ and $X = (2d+1)/(3d)$, where $d = 5$, $d = 7$, or $d \geqslant 13$. If $X = (d+1)/(3d)$, we have

$$\lfloor 15X \rfloor - \lfloor 4X \rfloor - \lfloor 11X \rfloor = 5 + \left\lfloor \frac{5}{d} \right\rfloor - 1 - \left\lfloor \frac{11}{3} + \frac{11}{3d} \right\rfloor.$$

So, this will be equal to 1, except if $d = 7$. However, again, this is an impossible case. Similarly, if $X = (2d+1)/(3d)$, we have

$$\lfloor 15X \rfloor - \lfloor 4X \rfloor - \lfloor 11X \rfloor = 10 + \left\lfloor \frac{5}{d} \right\rfloor - 2 - \left\lfloor \frac{22}{3} + \frac{11}{3d} \right\rfloor = 1.$$

So, again, the exponent e_d in (5.4) is non-negative, which establishes that $\frac{1-q}{1-q^{66n-1}} \left[{330n \atop 88n} \right]_q$ is a polynomial in q.

Turning to (3.2), to prove that $\frac{(1-q)^2}{(1-q^{10n-1})(1-q^{15n-1})} \left[{30n \atop 5n} \right]_q$ is a polynomial in q, we must show that

$$e_d = -\chi\big(d \mid (2dX - 1)\big) - \chi\big(d \mid (3dX - 1)\big) + \lfloor 6X \rfloor - \lfloor 5X \rfloor \tag{5.5}$$

is non-negative for all X of the form $X = x/d$ with $0 \leqslant x < d$, x being integral, and $d \neq 5$.

First of all, we should observe that $\gcd(10n - 1, 15n - 1) = 1$, whence the two truth functions in (5.5) cannot equal 1 simultaneously. Therefore, the expression in (5.5) is non-negative, except possibly if $d \mid (2dX - 1)$ or if $d \mid (3dX - 1)$. With the same reasoning as before, we see that the only cases to be examined are $X = (d+1)/(2d)$, $X = (d+1)/(3d)$, and $X = (2d+1)/(3d)$, where, in the latter two cases, the choice of $d = 3$ is excluded. If $X = (d+1)/(2d)$, then

$$\lfloor 6X \rfloor - \lfloor 5X \rfloor = 3 + \left\lfloor \frac{3}{d} \right\rfloor - \left\lfloor \frac{5}{2} + \frac{5}{2d} \right\rfloor,$$

which always equals 1 for $d \geqslant 2$. If $X = (d+1)/(3d)$, then

$$\lfloor 6X \rfloor - \lfloor 5X \rfloor = 2 + \left\lfloor \frac{2}{d} \right\rfloor - \left\lfloor \frac{5}{3} + \frac{5}{3d} \right\rfloor,$$

which always equals 1 for $d = 2, 6, 7, \ldots$ (sic!). Because of our assumptions, we do not need to consider the cases $d = 3$ and $d = 5$, so the only remaining case is $d = 4$. However, in that case $X = \frac{5}{12}$, which cannot be written with denominator $d = 4$, a contradiction. Finally, if $X = (2d + 1)/(3d)$, then

$$\lfloor 6X \rfloor - \lfloor 5X \rfloor = 4 + \left\lfloor \frac{2}{d} \right\rfloor - \left\lfloor \frac{10}{3} + \frac{5}{3d} \right\rfloor,$$

which always equals 1 for $d \geqslant 2$.

So, again, the exponent e_d in (5.5) is non-negative in all cases, which establishes that $\frac{(1-q)^2}{(1-q^{10n-1})(1-q^{15n-1})} \begin{bmatrix} 30n \\ 15n \end{bmatrix}_q$ is a polynomial in q. □

Proof of Theorem 3.2. By Lemma 5.1, it suffices to establish polynomiality of (3.3). When written in terms of cyclotomic polynomials, expression (3.3) reads

$$\frac{1 - q^{\gcd(k,n)}}{1 - q^n} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q = \prod_{d=2}^{2n} \Phi_d(q)^{e_d},$$

with

$$e_d = \chi\bigl(d \mid \gcd(k,n)\bigr) - \chi(d \mid n) + \left\lfloor \frac{2n}{d} \right\rfloor - \left\lfloor \frac{n-k}{d} \right\rfloor - \left\lfloor \frac{n+k}{d} \right\rfloor. \tag{5.6}$$

Similarly as before, let us write $N = \{n/d\}$ and $K = \{k/d\}$. Using this notation, Eq. (5.6) becomes

$$e_d = \chi\bigl(d \mid \gcd(k,n)\bigr) - \chi(d \mid n) + \lfloor 2N \rfloor - \lfloor N - K \rfloor - \lfloor N + K \rfloor. \tag{5.7}$$

We have to distinguish several cases. If $d \mid n$, then $N = 0$, and (5.7) becomes

$$e_d = \chi(d \mid k) - 1 - \lfloor -K \rfloor.$$

We see that this is zero (and, hence, non-negative) regardless whether $d \mid k$ or not.

On the other hand, if we assume that $d \nmid n$, then (5.7) becomes

$$e_d = \lfloor 2N \rfloor - \lfloor N - K \rfloor - \lfloor N + K \rfloor,$$

and this is always non-negative. We have proven that (3.3) is indeed a polynomial in q.

The statement on $B_{n,k}(q)$ follows immediately from the previous result and the fact that $\gcd(k,n) \mid k$. □

Finally, Theorem 3.3 will follow immediately from the following strengthening of a non-negativity result of Andrews [3, Theorem 2].

Lemma 5.2. *Let a and b be positive integers. Then*

$$\frac{1-q^{\gcd(a,b)}}{1-q^{a+b}}\begin{bmatrix} a+b \\ a \end{bmatrix}_q \tag{5.8}$$

is a polynomial in q with non-negative integer coefficients.

Proof. In view of Lemma 5.1, it suffices to show that the expression in (5.8) is a polynomial in q. Again, we start with the factorisation

$$\frac{1-q^{\gcd(a,b)}}{1-q^{a+b}}\begin{bmatrix} a+b \\ a \end{bmatrix}_q = \prod_{d=2}^{a+b-1} \Phi_d(q)^{e_d},$$

with

$$e_d = \chi\big(d \mid \gcd(a,b)\big) + \left\lfloor \frac{a+b-1}{d} \right\rfloor - \left\lfloor \frac{a}{d} \right\rfloor - \left\lfloor \frac{b}{d} \right\rfloor. \tag{5.9}$$

Next we write $A = \{a/d\}$ and $B = \{b/d\}$. Using this notation, Eq. (5.9) becomes

$$e_d = \chi\big(d \mid \gcd(a,b)\big) + \left\lfloor A + B - \frac{1}{d} \right\rfloor.$$

This is clearly non-negative, unless $A = B = 0$. However, in that case we have $d \mid a$ and $d \mid b$, that is, $d \mid \gcd(a,b)$, so that e_d is non-negative also in this case. □

Proof of Theorem 3.3. Replace a by an and b by $bn+1$ in Lemma 5.2. □

Proof of Corollary 3.4. This follows immediately from Theorem 3.3 and the fact that $a \mid \gcd(a, bn+1) = \gcd(an, bn+1)$. □

6. Proof of Theorem 2.4

The following auxiliary result on the occurrence of prime numbers congruent to 2 modulo 3 in "small" intervals will be crucial.

Lemma 6.1. *If $x \geqslant 530$, there is always at least one prime number congruent to 2 modulo 3 contained in the interval $(x, \frac{20}{19}x)$.*

Proof. Let $\theta(x;3,2)$ denote the classical Chebyshev function, defined by

$$\theta(x;3,2) = \sum_{\substack{p \text{ prime},\ p\leqslant x \\ p\equiv 2 \ (\mathrm{mod}\ 3)}} \log p.$$

McCurley proved the following estimates for this function (see [19, Theorems 5.1 and 5.3]):

$$\theta(y;3,2) < 0.51y, \quad y > 0,$$
$$\theta(y;3,2) > 0.49y, \quad y \geqslant 3761.$$

This implies that, for $x > 3761$, we have

$$\theta\left(\frac{20}{19}x;3,2\right) - \theta(x;3,2) > 0.49 \cdot \frac{20}{19}x - 0.51x > 0.0057x > 1.$$

This means that, if $x > 3761$, there must be a prime number congruent to 2 modulo 3 strictly between x and $\frac{20}{19}x$. (To be completely accurate: the above argument only shows that such a prime number exists in the half-open interval $(x, \frac{20}{19}x]$. However, existence in the open interval $(x, \frac{20}{19}x)$ can be easily established in the same manner, by slightly lowering the value of $\frac{20}{19}$ in the above argument.)

For the remaining range $530 \leqslant x \leqslant 3761$, one can verify the claim directly using a computer. □

Proof of Theorem 2.4. Given a and b, our strategy consists in finding a prime p and a positive integer n such that the p-adic valuation of $\binom{an+bn}{an}/(3n-1)$ is negative, so that $3n-1$ cannot divide $\binom{an+bn}{an}$. We first verified the possibility of finding such p and n for $a, b \leqslant 1850$ using a computer.

To establish the claim for the remaining values of a and b, we have to distinguish several cases, depending on the congruence classes of a and b modulo 3 and the relative sizes of a and b.

First let $(a,b) \in \{(0,0), (0,1), (1,0), (0,2), (1,1), (2,0)\} + (3\mathbb{Z})^2$. By Dirichlet's theorem [8] (see [5]), we know that there are infinitely many primes congruent to 2 modulo 3. Let us take such a prime p with $p > a+b$, and let $3n-1 = p$, that is, $n = (p+1)/3$. Furthermore, let $v_p(\alpha)$ denote the p-adic valuation of α, that is, the maximal exponent e such that p^e divides α. Writing $a = 3a_1 + a_2$ and $b = 3b_1 + b_2$ with $0 \leqslant a_2, b_2 \leqslant 2$, by the well-known formula of Legendre [18, p. 10] for the p-adic valuation of factorials, we then have

$$\begin{aligned} v_p\left(\frac{1}{3n-1}\binom{an+bn}{an}\right) &= -1 + \sum_{\ell \geqslant 1}\left(\left\lfloor \frac{(a+b)n}{p^\ell}\right\rfloor - \left\lfloor \frac{an}{p^\ell}\right\rfloor - \left\lfloor \frac{bn}{p^\ell}\right\rfloor\right) \\ &= -1 + \left\lfloor \frac{a_2+b_2}{3} + \frac{a+b}{3p}\right\rfloor - \left\lfloor \frac{a_2}{3} + \frac{a}{3p}\right\rfloor - \left\lfloor \frac{b_2}{3} + \frac{b}{3p}\right\rfloor. \end{aligned} \tag{6.1}$$

Since, in the current case, we have $0 \leqslant a_2 + b_2 \leqslant 2$ and $\frac{a+b}{3p} < \frac{1}{3}$, all the values of the floor functions on the right-hand side of the above equation are zero. Consequently, the p-adic valuation of $\frac{1}{3n-1}\binom{an+bn}{an}$ equals -1 for our choice of p and n, which in particular means that $\frac{1}{3n-1}\binom{an+bn}{an}$ is not an integer.

For the next case, consider a pair $(a,b) \in \{(2,1),(2,2)\} + (3\mathbb{Z})^2$. Let us first assume that $b \leqslant \frac{9}{10}a$. Since we have already verified the claim for $a,b \leqslant 1850$, we may assume $a \geqslant 1850$. We now choose a prime $p \equiv 2 \pmod 3$ strictly between $\frac{19}{20}a$ and a. Such a prime is guaranteed to exist by Lemma 6.1, because, due to our assumption, we have $\frac{19}{20}a \geqslant 1757.5 > 530$. Furthermore, we choose $n = (p+1)/3$. We have

$$(a+b)n \leqslant \frac{19}{10}a\frac{p+1}{3} < \frac{2}{3}p(p+1) < p^2,$$

and hence, with the same notation as above, Eq. (6.1) holds also in the current case. We have $3 \leqslant a_2 + b_2 \leqslant 4$, $a+b \leqslant \frac{19}{10}a < 2p$, $\frac{a_2}{3} = \frac{2}{3}$, $\frac{1}{3} < \frac{a}{3p} \leqslant \frac{20}{57} < \frac{2}{3}$, hence

$$\left\lfloor \frac{a_2+b_2}{3} + \frac{a+b}{3p} \right\rfloor - \left\lfloor \frac{a_2}{3} + \frac{a}{3p} \right\rfloor - \left\lfloor \frac{b_2}{3} + \frac{b}{3p} \right\rfloor = 1 - 1 - 0 = 0.$$

Consequently, again, the p-adic valuation of $\frac{1}{3n-1}\binom{an+bn}{an}$ equals -1 for our choice of p and n, which in particular means that $\frac{1}{3n-1}\binom{an+bn}{an}$ is not an integer.

Next let again $(a,b) \in \{(2,1),(2,2)\} + (3\mathbb{Z})^2$, but $\frac{9}{10}a < b \leqslant \frac{7}{5}a$. Since we have already verified the claim for $a,b \leqslant 1850$, we may assume $a \geqslant 1300$. (If $a < 1300$, then the above restriction imposes the bound $b \leqslant \frac{7}{5}1300 = 1820 < 1850$.) Here, we choose a prime $p \equiv 2 \pmod 3$ strictly between $\frac{4}{5}a$ and $\frac{9}{10}a$. Such a prime is guaranteed to exist by Lemma 6.1, because, due to our assumption, we have $\frac{4}{5}a = 1040 > 530$. Furthermore, we choose $n = (p+1)/3$. We have still $3 \leqslant a_2 + b_2 \leqslant 4$, moreover $2p < \frac{19}{9}p < \frac{19}{10}a \leqslant a+b \leqslant \frac{12}{5}a < 3p$, $\frac{a_2}{3} = \frac{2}{3}$, $\frac{1}{3} < \frac{a}{3p} \leqslant \frac{5}{12} < \frac{2}{3}$, $\frac{1}{3} < \frac{3a}{10p} \leqslant \frac{b}{3p} \leqslant \frac{7a}{15p} \leqslant \frac{7}{12} < \frac{2}{3}$, hence

$$\left\lfloor \frac{a_2+b_2}{3} + \frac{a+b}{3p} \right\rfloor - \left\lfloor \frac{a_2}{3} + \frac{a}{3p} \right\rfloor - \left\lfloor \frac{b_2}{3} + \frac{b}{3p} \right\rfloor = 1 + \chi(b_2 = 2) - 1 - \chi(b_2 = 2) = 0,$$

implying again that the p-adic valuation of $\frac{1}{3n-1}\binom{an+bn}{an}$ equals -1 for our choice of p and n, as desired.

The next case we discuss is $(a,b) \in \{(2,1)\} + (3\mathbb{Z})^2$ and $\frac{7}{5}a < b \leqslant 3a$. Since we have already verified the claim for $a,b \leqslant 1850$, we may assume $b \geqslant 1850$. Here, we choose a prime $p \equiv 2 \pmod 3$ strictly between $\frac{3}{10}b$ and $\frac{1}{3}b$. Such a prime is guaranteed to exist by Lemma 6.1, because, due to our assumption, we have $\frac{3}{10}b = 555 > 530$. Furthermore, we choose $n = (p+1)/3$. Here we have

$$\begin{aligned} v_p\left(\frac{1}{3n-1}\binom{an+bn}{an}\right) &= -1 + \sum_{\ell \geqslant 1}\left(\left\lfloor \frac{(a+b)n}{p^\ell} \right\rfloor - \left\lfloor \frac{an}{p^\ell} \right\rfloor - \left\lfloor \frac{bn}{p^\ell} \right\rfloor \right) \\ &= -1 + \left\lfloor \frac{(a+b)n}{p^2} \right\rfloor - \left\lfloor \frac{an}{p^2} \right\rfloor - \left\lfloor \frac{bn}{p^2} \right\rfloor \end{aligned}$$

$$+\left\lfloor \frac{a_2+b_2}{3}+\frac{a+b}{3p}\right\rfloor - \left\lfloor \frac{a_2}{3}+\frac{a}{3p}\right\rfloor - \left\lfloor \frac{b_2}{3}+\frac{b}{3p}\right\rfloor .$$

In this case, due to the estimations

$$p^2 < b\frac{p}{3} < (a+b)n = (a+b)\frac{p+1}{3} < \frac{12}{7}b\frac{p+1}{3} < \frac{40}{21}p(p+1) < 2p^2, \quad \text{for } p > 20,$$

and

$$p^2 < b\frac{p}{3} < bn < (a+b)n < 2p^2, \quad \text{for } p > 20,$$

we have

$$\left\lfloor \frac{(a+b)n}{p^2}\right\rfloor - \left\lfloor \frac{an}{p^2}\right\rfloor - \left\lfloor \frac{bn}{p^2}\right\rfloor = 1-0-1=0.$$

Moreover, we have $a_2+b_2=3$, $3p<b<a+b\leqslant \frac{12}{7}b<\frac{120}{21}p<6p$, $\frac{a_2}{3}=\frac{2}{3}$, $\frac{1}{3}<\frac{b}{9p}\leqslant \frac{a}{3p}\leqslant \frac{5b}{21p}<\frac{50}{63}$, $\frac{b_2}{3}=\frac{1}{3}$, $1<\frac{b}{3p}\leqslant \frac{10}{9}$, so that

$$\left\lfloor \frac{a_2+b_2}{3}+\frac{a+b}{3p}\right\rfloor - \left\lfloor \frac{a_2}{3}+\frac{a}{3p}\right\rfloor - \left\lfloor \frac{b_2}{3}+\frac{b}{3p}\right\rfloor = 2-1-1=0,$$

implying again that the p-adic valuation of $\frac{1}{3n-1}\binom{an+bn}{an}$ equals -1 for our choice of p and n, as desired.

The last case to be discussed is $(a,b)\in\{(2,1)\}+(3\mathbb{Z})^2$ and $3a<b$. Again, since we have already verified the claim for $a,b\leqslant 1850$, we may assume $b\geqslant 1850$. Here, we choose a prime $p\equiv 2 \pmod 3$ strictly between $\frac{5}{11}b$ and $\frac{1}{2}b$. Such a prime is guaranteed to exist by Lemma 6.1, because, due to our assumption, we have $\frac{5}{11}b>530$. Furthermore, we choose $n=(p+1)/3$. Since

$$(a+b)n < \frac{4}{3}b\frac{p+1}{3} < \frac{44}{45}p(p+1) < p^2, \quad \text{for } p > 44,$$

for the p-adic valuation of $\frac{1}{3n-1}\binom{an+bn}{an}$ there holds again (6.1). In the current case, we have furthermore $a_2+b_2=3$, $a+b\leqslant\frac{4}{3}b<\frac{44}{15}p<3p$, $\frac{a_2}{3}=\frac{2}{3}$, $\frac{a}{3p}<\frac{b}{9p}<\frac{11}{45}<\frac{1}{3}$, $\frac{b_2}{3}=\frac{1}{3}$, $\frac{2}{3}<\frac{b}{3p}\leqslant\frac{11}{15}$, and hence

$$\left\lfloor \frac{a_2+b_2}{3}+\frac{a+b}{3p}\right\rfloor - \left\lfloor \frac{a_2}{3}+\frac{a}{3p}\right\rfloor - \left\lfloor \frac{b_2}{3}+\frac{b}{3p}\right\rfloor = 1-0-1=0,$$

implying also here that the p-adic valuation of $\frac{1}{3n-1}\binom{an+bn}{an}$ equals -1 for our choice of p and n, as desired.

We have now covered all possible cases (in particular, by symmetry in a and b, we also covered the case $(a,b)\in\{(1,2)\}+(3\mathbb{Z})^2$), and hence this concludes the proof of the theorem. □

7. Concluding remarks and open problems

In the proof of Theorem 2.1, assume that s is the smallest positive integer such that $(a+b) \mid (p^s-1)$. Then we obtain the stronger inequality

$$f(a,b) \leqslant \frac{p^s-1}{a+b}. \tag{7.1}$$

It is easily seen that $s \mid \varphi(a+b)$. However, such an upper bound is still likely much larger than the exact value of $f(a,b)$ given by Sun [26]. For example, the inequality (7.1) gives

$$\begin{aligned}
f(7,36) &\leqslant \frac{7^6-1}{43} = 2736,\\
f(10,192) &\leqslant \frac{5^{25}-1}{202} = 1\,475\,362\,494\,440\,362,\\
f(11,100) &\leqslant \frac{11^6-1}{111} = 15\,960,\\
f(22,200) &\leqslant \frac{11^6-1}{222} = 7980,\\
f(1999,2011) &\leqslant \frac{1999^{400}-1}{4010} \approx 5.272 \times 10^{1316}.
\end{aligned}$$

It seems that Theorem 2.2 can be further generalised in the following way.

Conjecture 7.1. *Let a and b be positive integers with $a > b$, and let α and β be integers. Furthermore, let p be a prime such that $\gcd(p,a) = 1$. Then for each $r = 0,1,\dots,p-1$, there are infinitely many positive integers n such that*

$$\binom{an+\alpha}{bn+\beta} \equiv r \pmod p.$$

In relation to Theorem 2.3, we propose the following two conjectures, the first one generalising Theorem 2.4.

Conjecture 7.2. *For any odd prime p, there are no positive integers $a > b$ such that*

$$\binom{an}{bn} \equiv 0 \pmod{pn-1}$$

for all $n \geqslant 1$.

Conjecture 7.3. *For any positive integer m, there are positive integers a and b such that $am > b$ and*

$$\binom{amn}{bn} \equiv 0 \pmod{an-1}$$

for all $n \geqslant 1$*.*

Note that the congruences (2.1)–(2.6) imply that Conjecture 7.3 is true for $1 \leqslant m \leqslant 5$.

It seems clear that, for each *specific prime* p, a proof of Conjecture 7.2 in the style of the proof of Theorem 2.4 in Section 6 can be given. On the other hand, a proof for *arbitrary* p will likely require a new idea.

We end the paper with the following conjecture, strengthening the last part of Theorem 3.1.

Conjecture 7.4. *For all positive integers* n *and non-negative integers* k *with* $0 \leqslant k \leqslant 125n^2 - 25n + 4$*, the coefficient of* q^k *in the polynomial*

$$\frac{(1-q)^2}{(1-q^{10n-1})(1-q^{15n-1})} \begin{bmatrix} 30n \\ 5n \end{bmatrix}_q$$

is non-negative, except for $k = 1$ *and* $k = 125n^2 - 25n + 3$*, in which case the corresponding coefficient equals* -1*.*

8. Note added in proof

Conjecture 7.2 was proved by Madjid Mirzavaziri and the second author. Conjecture 7.3 was proved by Madjid Mirzavaziri.

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