Journal of Number Theory 167 (2016) 430–480

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Number of prime ideals in short intervals

Emre Alkan *, Tevekkül Mehreliyev

Department of Mathematics, Koç University, Rumelifeneri Yolu, 34450, Sarıyer, Istanbul, Turkey

A R T I C L E   I N F O

Article history:
Received 8 September 2015
Received in revised form 1 March 2016
Accepted 1 March 2016
Available online 28 April 2016
Communicated by David Goss

MSC:
11R44
11N13
11M26
11J86

Keywords:
Cyclotomic extension
Prime ideal
Primes in a progression
Dedekind zeta function
Dirichlet L-function
Branch of complex logarithm
Linear forms in logarithms
Siegel zero

A B S T R A C T

Assuming a weaker form of the Riemann hypothesis for Dedekind zeta functions by allowing Siegel zeros, we extend a classical result of Cramér on the number of primes in short intervals to prime ideals of the ring of integers in cyclotomic extensions with norms belonging to such intervals. The extension is uniform with respect to the degree of the cyclotomic extension. Our approach is based on the arithmetic of cyclotomic fields and analytic properties of their Dedekind zeta functions together with a lower bound for the number of primes over progressions in short intervals subject to similar assumptions. Uniformity with respect to the modulus of the progression is obtained and the lower bound turns out to be best possible, apart from constants, as shown by the Brun–Titchmarsh theorem.

* Corresponding author.
E-mail addresses: ealkan@ku.edu.tr (E. Alkan), tmehreliyev@ku.edu.tr (T. Mehreliyev).

http://dx.doi.org/10.1016/j.jnt.2016.03.003

1. Introduction

Assuming the Riemann hypothesis, the prime number theorem refines to

$$\pi(x) = li(x) + O(\sqrt{x}\log x) \tag{1.1}$$

for $x \geq 2$, where $\pi(x)$ is the number of primes that are $\leq x$ and

$$li(x) = \int_2^x \frac{dt}{\log t}.$$

It is not possible to obtain directly from (1.1) that there are primes in short intervals of the form

$$(x, x + c\sqrt{x}\log x]$$

for an absolute positive constant c. Despite this drawback, Cramér [5] was still able to show that

$$p_{n+1} - p_n = O(p_n^{1/2}\log p_n)$$

under the Riemann hypothesis, where $\{p_n\}$ is the sequence of primes. For interesting connections between correlation of zeros and gaps in primes, the reader is referred to the work of Languasco, Perelli and Zaccagnini [11]. Cramér actually had the stronger result

$$\pi(x + c\sqrt{x}\log x) - \pi(x) > \sqrt{x} \tag{1.2}$$

for some absolute constant $c > 0$ as a consequence of the Riemann hypothesis. By providing a new proof of (1.2), Dudek [7] showed that for any $\epsilon > 0$, one may take $c = 3+\epsilon$ for all sufficiently large x. However, the optimal value of c for all sufficiently large x is still unknown. In the same paper, Cramér [5] suggested a convenient probabilistic model which predicted that the distribution of primes in short intervals is Poissonian. This was nicely confirmed by Gallagher [9] who assumed a uniform version of the k-tuple conjecture on primes. Recently, Tsai and Zaharescu [17] went along this direction and discovered that the Poissonian behavior of prime elements over small regions persists in number fields by assuming a general form of the k-tuple conjecture adapted to finite extensions of $\mathbb{Q}$. With this motivation, we are curious to see how far (1.2) could be extended to the number field case. It turns out that one can provide a satisfactory answer (see Theorem 1 below) by establishing a uniform version of (1.2) for all cyclotomic extensions of $\mathbb{Q}$. To be precise, let K be a number field over $\mathbb{Q}$ of degree n with $\mathcal{O}_k$ denoting the ring of integers in K. For any integral ideal $\mathfrak{a}$ of $\mathcal{O}_k$, let $N(\mathfrak{a}) = |\mathcal{O}_k : \mathfrak{a}| = |\mathcal{O}_k/\mathfrak{a}|$ be the norm of $\mathfrak{a}$. Then the Dedekind zeta function of K is given by

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} = \prod_{\mathfrak{p}} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1}$$

for $\Re(s) > 1$, where the summation and the Euler product are over the integral and prime ideals of $\mathcal{O}_k$, respectively. $\zeta_K(s)$ has an analytic continuation to the whole complex plane with the exception of a simple pole at $s = 1$ and has a functional equation relating $\zeta_K(s)$ to $\zeta_K(1-s)$ (see p. 467 of [14]). By a critical zero of $\zeta_K(s)$, a zero s with $0 < \Re(s) < 1$ is understood. Then the Riemann hypothesis for $\zeta_K(s)$ claims that any critical zero of $\zeta_K(s)$ has real part 1/2. The statement of the main result is now as follows.

Theorem 1. *Assume that all of the nonreal critical zeros of Dedekind zeta functions corresponding to cyclotomic extensions of $\mathbb{Q}$ have real part 1/2. Let $K_q = \mathbb{Q}(\zeta_q)$ be a cyclotomic extension with ζ_q being a primitive qth root of unity. Further let $\pi_{K_q}(x)$ be the number of prime ideals of the ring of integers in K_q whose norm is $\leq x$. Given any number $A > 0$, there exists an effective absolute constant $c > 0$ such that for all sufficiently large x with $q \leq (\log x)^A$, we have the inequality*

$$\pi_{K_q}(x + c\,\varphi(q)\sqrt{x}\log x) - \pi_{K_q}(x) > \varphi(q)\sqrt{x},$$

where $\varphi(q)$ is Euler's function. The inequality also holds with an ineffective constant c_A in place of c for all x with $q \leq (\log x)^A$. Finally, assuming the Riemann hypothesis for all Dedekind zeta functions of cyclotomic extensions, there exist effective positive constants c and λ such that the same inequality holds for all x with $q \leq x^\lambda$.

An interesting feature of Theorem 1 is its uniformity over the order of primitive roots, hence over the degree of cyclotomic extensions. Concerning this uniformity, Ledoan, Roy and Zaharescu [12] gave asymptotic formulas for the number of nonreal zeros (in the upper half plane with imaginary part $\leq T$) of partial sums converging to $\zeta_K(s)$ when $K = \mathbb{Q}(\zeta_q)$ with a fixed $q \geq 2$. Their asymptotic formulas turn out to be sharper than the classical case of the Riemann zeta function. It remains to be seen whether analogs of their results with more uniformity in q, such as varying q with T, can be studied. One could also speculate that Theorem 1 holds for any number field K over $\mathbb{Q}$ in a way that the degree of the extension, namely $[K:\mathbb{Q}]$, plays the role of $\varphi(q)$ and $[K:\mathbb{Q}] \leq f(x)$ for a suitable increasing function $f(x)$ tending to infinity. Clearly the only additive input for counting prime numbers is the number of positive integers that are $\leq x$ and this is $x + O(1)$. The situation significantly changes in the case of a number field K over $\mathbb{Q}$ with $[K:\mathbb{Q}] \geq 2$ since the count of prime ideals depends ultimately on the number of integral ideals with norm $\leq x$ which is known to be

$$c_K x + O_K\left(x^{1-\frac{1}{[K:\mathbb{Q}]}}\right), \tag{1.3}$$

where $c_K > 0$ is the residue of $\zeta_K(s)$ at $s = 1$, also called the ideal density of K. Although the error term in (1.3) gets progressively worse with increasing $[K:\mathbb{Q}]$, we still have, subject to the Riemann hypothesis for $\zeta_K(s)$, that

$$\pi_K(x) = li(x) + O_K(\sqrt{x}\log x).$$

Unconditionally, Landau's prime ideal theorem (see [10]) gives that

$$\pi_K(x) = li(x) + O_K\left(xe^{-c\sqrt{\log x}}\right)$$

for some $c > 0$ depending only on K. Moreover, it is possible to further relax the assumptions in Theorem 1. Indeed assuming a quasi-Riemann hypothesis for Dedekind zeta functions of cyclotomic extensions by requiring that $\Re(s) \leq \frac{1}{2} + \delta$ for a small $\delta > 0$ and for any critical zero s (such an assumption is not unreasonable as by a celebrated result of Bohr and Landau, see section 9.6 of [8], all but an infinitesimal proportion of the critical zeros of the Riemann zeta function lie within δ of the critical line for any given $\delta > 0$), one can show by modifying our approach that

$$\pi_{K_q}(x + c\,\varphi(q)x^{\frac{1}{2}+\delta}\log x) - \pi_{K_q}(x) > \varphi(q)x^{\frac{1}{2}+\delta}$$

with uniformity in q. We have simply selected to present our results in terms of the shortest possible intervals. Besides finding numerical values of c for all sufficiently large x (as in [7]) in Theorem 1 is an interesting problem. Referring to the general theory, it is known that (see p. 466 of [14]) the completed zeta function

$$Z_K(s) = |d_K|^{s/2}\pi^{-ns/2}2^{(1-s)r_2}\Gamma(s/2)^{r_1}\Gamma(s)^{r_2}\zeta_K(s)$$

satisfies the functional equation

$$Z_K(s) = Z_K(1-s),$$

where d_K is the discriminant of K, r_1 and, respectively, $2r_2$ denote the number of real and complex embeddings of K with $r_1 + 2r_2 = [K : \mathbb{Q}]$ and $\Gamma(s)$ is the gamma function. Consequently $\zeta_K(s)$ belongs to the Selberg class and by a result of Selberg [16]

$$N_K(T) \sim \frac{[K:\mathbb{Q}]}{2\pi}T\log T$$

follows, $N_K(T)$ being the number of critical zeros with imaginary part in $(0, T)$. It is the specific structure and arithmetic of cyclotomic fields that we exploit in the course of proving Theorem 1. However, we also make use of the following result on the number of primes over progressions in short intervals assuming a slightly weaker form of the Riemann hypothesis for Dirichlet L-functions by allowing Siegel zeros.

Theorem 2. *Assume that all of the nonreal critical zeros of Dirichlet L-functions have real part $1/2$. Let $\pi(x, q, a)$ be the number of primes $p \leq x$ satisfying $p \equiv a \pmod{q}$, where $1 \leq a \leq q$ and $(a, q) = 1$. Given any positive number A, there exists an effective absolute constant $c > 0$ such that for all sufficiently large x and for all a, q with $1 \leq a \leq q \leq (\log x)^A$ and $(a, q) = 1$, we have*

$$\pi(x + c\,\varphi(q)\sqrt{x}\log x, q, a) - \pi(x, q, a) > \sqrt{x}.$$

The inequality also holds with an ineffective constant c_A *in place of* c *for all* x *with* $q \leq (\log x)^A$*. Finally, assuming the Riemann hypothesis for all Dirichlet* L*-functions, there exist effective positive constants* c *and* λ *such that the same inequality holds for all* x *with* $q \leq x^{\lambda}$*.*

As will be clear from the proof of Theorem 2, $\lambda < 1/4$. Besides being flexible in the modulus aspect, the conclusion of Theorem 2 is partially influenced by the presence of exceptional real zeros of Dirichlet L-functions. Another instance of this influence is clearly seen in the classical asymptotic formula (see p. 123 of [6])

$$\pi(x,q,a) = \frac{li(x)}{\varphi(q)} - \frac{\chi_1(a)li(x^{\beta_1})}{\varphi(q)} + O\left(xe^{-c_1\sqrt{\log x}}\right)$$

which holds in the range $q \leq e^{c\sqrt{\log x}}$, where χ_1 is the exceptional real character modulo q, β_1 is the Siegel zero of the corresponding Dirichlet L-function

$$L(s,\chi_1) = \sum_{n=1}^{\infty} \frac{\chi_1(n)}{n^s}$$

for $\Re(s) > 1$ and $c_1 > 0$ depends only on the arbitrary constant $c > 0$. As a final remark on Theorem 2, note that $q < c\,\varphi(q)\sqrt{x}\log x$ holds for all $q \leq x$, $x \geq 2$ and for a suitable constant $c > 0$. Then the Brun–Titchmarsh theorem gives that

$$\pi(x + c\,\varphi(q)\sqrt{x}\log x, q, a) - \pi(x,q,a) = O\left(\frac{\sqrt{x}\log x}{\log\left(\frac{c\,\varphi(q)\sqrt{x}\log x}{q}\right)}\right) = O(\sqrt{x}).$$

Therefore, the lower bound in Theorem 2 is best possible apart from constants. In light of this observation, it is natural to expect more uniformity over q in Theorem 2 (and in Theorem 1) such as $q \leq x$. Although we are far from confirming uniformity for all $q \leq x$, some heuristics are worth mentioning in an almost all q sense. A number $\theta > 0$ is called admissible if there exists a positive constant $C(\theta)$ such that

$$\pi(x,q,a) \geq \frac{C(\theta)x}{\varphi(q)\log x} \tag{1.4}$$

holds for almost all $q \leq x^{\theta} = Q$, allowing $\frac{Q}{(\log x)^K}$ exceptions for some $K > 0$. Inequality (1.4) can be viewed as a lower bound version of the Brun–Titchmarsh theorem adapted to the case of almost all moduli. Bombieri–Vinogradov theorem shows that any $\theta < 1/2$ is admissible. Rousselet [15] was the first to go beyond $1/2$ and obtained that any $\theta \leq \frac{1}{2} + \frac{1}{10^{100}}$ is admissible. Let us assume that

$$\frac{\sqrt{x}}{\log\log x} \leq q \leq x^{\theta} \tag{1.5}$$

for some $1/2 \le \theta < 1$. By the Brun–Titchmarsh theorem, we easily get from (1.5) that

$$\pi(x,q,a) \ll \frac{x}{\varphi(q)\log\left(\frac{x}{q}\right)} \ll \frac{\sqrt{x}(\log\log x)^2}{\log x} = o(\sqrt{x}). \tag{1.6}$$

Heuristically, as $x = o(\varphi(q)\sqrt{x}\log x)$, (1.4) suggests that

$$\pi(x + c\,\varphi(q)\sqrt{x}\log x, q, a) \ge \frac{C(\theta)(x + c\,\varphi(q)\sqrt{x}\log x)}{\varphi(q)\log(x + c\,\varphi(q)\sqrt{x}\log x)} > 2\sqrt{x} \tag{1.7}$$

for many values of q if c is large enough. Thus we would expect from (1.6) and (1.7) that

$$\pi(x + c\,\varphi(q)\sqrt{x}\log x, q, a) - \pi(x,q,a) > \sqrt{x}$$

still holds for many values of q subject to (1.5) (actually we require also the condition $q \ll (\varphi(q)\sqrt{x}\log x)^{\theta}$). This ends our digression on the uniformity of q in an almost all sense. For research on other types of denser sequences such as the B-free numbers and application to the size of gaps between consecutive nonzero Fourier coefficients of modular forms via sieve methods, see [1,2,13].

2. Proof of Theorem 1 assuming Theorem 2

First of all the ring of integers in $K_q = \mathbb{Q}(\zeta_q)$ is (see p. 60 of [14])

$$\mathbb{Z}[\zeta_q] = \mathbb{Z} + \mathbb{Z}\zeta_q + \cdots + \mathbb{Z}\zeta_q^{\varphi(q)-1}.$$

We will use an elegant decomposition law of primes into prime ideals of the Dedekind domain $\mathbb{Z}[\zeta_q]$ (see p. 61 of [14]). To state this decomposition law, let

$$q = \prod_p p^{\nu_p}$$

be the prime factorization of q, where the product is over all primes with the convention that $\nu_p = 0$ if p does not divide q. For every prime p, let f_p be the smallest positive integer satisfying

$$p^{f_p} \equiv 1 \pmod{q/p^{\nu_p}}. \tag{2.1}$$

Then the factorization

$$(p) = p\mathbb{Z}[\zeta_q] = (\mathfrak{p}_1 \dots \mathfrak{p}_{r_p})^{\varphi(p^{\nu_p})} \tag{2.2}$$

holds for the principal ideal generated by p, where $\mathfrak{p}_1, \dots, \mathfrak{p}_{r_p}$ are distinct prime ideals of $\mathbb{Z}[\zeta_q]$ lying above p with $N(\mathfrak{p}_j) = p^{f_p}$ for all j. Since K_q is a Galois extension, we know that for each prime p,

$$efr = \varphi(p^{\nu_p}) f_p r_p = \varphi(q) \tag{2.3}$$

holds. Now consider a prime p with $p \equiv 1 \pmod q$. Since p does not divide q, $\nu_p = 0$ and (2.1)–(2.3) give that $f_p = 1$ so that there are exactly $\varphi(q)$ distinct prime ideals lying above p each having norm p. As K_q is a cyclotomic extension, the factorization of the Dedekind zeta function of K_q (see p. 468 of [14]) in the form

$$\zeta_{K_q}(s) = \prod_{\mathfrak{p}|q} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1} \prod_{\chi} L(s,\chi) \tag{2.4}$$

is valid over the complex plane, where the first product in (2.4) is over the prime ideals lying above the prime divisors of q and the second product in (2.4) is over all Dirichlet characters modulo q. Since we are assuming that all of the nonreal critical zeros of $\zeta_{K_q}(s)$ have real part $1/2$, it follows from (2.4) that all of the nonreal critical zeros of $L(s,\chi)$ have real part $1/2$. Thus Theorem 2 applies and gives the existence of an effective absolute constant $c > 0$ satisfying

$$\pi(x + c\,\varphi(q)\sqrt{x}\log x, q, 1) - \pi(x, q, 1) > \sqrt{x} \tag{2.5}$$

for all sufficiently large x with $q \le (\log x)^A$. As we have seen above, for each prime $p \equiv 1 \pmod q$ belonging to the interval $(x, x + c\,\varphi(q)\sqrt{x}\log x]$, there are $\varphi(q)$ distinct prime ideals lying above p with norm p. Consequently the norms of these prime ideals also belong to the same interval. Note that for different primes, the set of prime ideals lying above them are disjoint as their norms are different. Using (2.5), this contributes more than $\varphi(q)\sqrt{x}$ prime ideals of $\mathbb{Z}[\zeta_q]$ with norms in $(x, x + c\,\varphi(q)\sqrt{x}\log x]$ and

$$\pi_{K_q}(x + c\,\varphi(q)\sqrt{x}\log x) - \pi_{K_q}(x) > \varphi(q)\sqrt{x}$$

follows for all sufficiently large x with $q \le (\log x)^A$ and some absolute constant $c > 0$. The remaining claims in Theorem 1 can be shown in the same way using Theorem 2. This completes the proof of Theorem 1 assuming Theorem 2. The rest of the paper will be devoted to the proof of Theorem 2.

As the proof of Theorem 2 is rather long, it would be worthwhile to give an outline of the argument. Our approach is mainly based on obtaining explicit formulas representing certain exponential sums over the critical zeros of the Riemann zeta function in the case of the principal character (see (3.86) below) and over the critical zeros of L-functions in the case of nonprincipal characters (see (3.129) below). For the principal character, our explicit formula differs from Cramér's explicit formula [5] in the respect that additional sums over the divisors of the modulus appear and these create further technical complications. The verification of such formulas proceeds by complex integration. To overcome difficulties arising from accumulated zeros of finite Euler products, we resort to strong lower bounds for the linear forms in logarithms. At the same time, one has to keep track of the argument change resulting from logarithms of L-functions. Moreover, some detailed analysis based on zero-free regions is needed to justify the interchange of the summation with the integral in (3.82). For the fusion of (3.86) and (3.129), the influence of a possible Siegel zero has to be taken into account and this is the most delicate part of the proof. Here finding the exact value of the argument (see (3.167) below) is indispensable. Using a consequence of the fusion of explicit formulas (see (3.172) below), we then pass to the arithmetic setting (see (3.183) below) with the help of asymptotic formulas for the number of critical zeros with ordinates in $(-T, T)$ as T tends to infinity, where $L(s, \chi)$ and $L(s, \overline{\chi})$ are grouped for the count of their critical zeros. The proof is then completed by deducing lower bounds for suitable weighted sums over prime powers in a progression, where the main contribution comes from the primes, by employing the Brun–Titchmarsh theorem.

3. Proof of Theorem 2

Let χ_0 be the principal character modulo q. Consider

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right) \tag{3.1}$$

which holds for all $s \in \mathbb{C}$, $\zeta(s)$ denoting the Riemann zeta function. Two obvious consequences of (3.1) are the facts that $L(s, \chi_0)$ has a simple pole at $s = 1$ with residue $\varphi(q)/q$ and all of the critical zeros of $L(s, \chi_0)$ coincide with the critical zeros of $\zeta(s)$ as the zeros of the finite product in (3.1) are purely imaginary. For any complex number $z = x + iy$ with $x, y > 0$, define

$$V(z) := \sum_{\gamma > 0} e^{\rho z}, \tag{3.2}$$

where the sum in (3.2) is over the critical zeros $\rho = \sigma + i\gamma$ of $L(s, \chi_0)$ (hence of the Riemann zeta function) with positive imaginary part. The convergence issue in (3.2) can be settled easily as by partial integration, we see that

$$\left| \sum_{0<\gamma<T} e^{\rho z} \right| \leq e^x \sum_{0<\gamma<T} e^{-\gamma y} = e^x \int_0^T e^{-ty}\, dN(t) = \frac{e^x N(T)}{e^{Ty}} + e^x y \int_0^T \frac{N(t)}{e^{ty}}\, dt \tag{3.3}$$

with $N(T)$ denoting the number of critical zeros of the Riemann zeta function with imaginary part in $(0, T)$. Since $N(t) = O(t \log t)$ for $t \geq 2$ (see p. 98 of [6]) and $y > 0$, we have

$$\int_0^T \frac{N(t)}{e^{ty}}\, dt = O\left(\int_2^T \frac{t \log t}{e^{ty}}\, dt \right) = O(1). \tag{3.4}$$

Thus (3.3) and (3.4) give that $V(z)$ in (3.2) is a well-defined function. Let us now work towards relating $V(z)$ to certain explicit formulas involving primes. To this end, let $\mathcal{S}$ be the set of all purely imaginary zeros of $L(s,\chi_0)$ with nonnegative imaginary part. Precisely, we may write

$$\mathcal{S} = \left\{ \frac{2\pi i k}{\log p} \,:\, p \mid q,\ k \in \mathbb{Z},\ k \geq 0 \right\}. \tag{3.5}$$

Clearly, (3.5) shows that $\mathcal{S}$ is a countable set and may be represented as the sequence $\{z_n\}_{n\geq 1}$ with $0 = \Im(z_1) < \Im(z_2) < \ldots$. Let $\delta_{\mathcal{S}} = \inf |z_{n+1} - z_n|$. If q is a prime power, then $\mathcal{S}$ is a discrete set of equally spaced points. On the other hand, if $\omega(q) \geq 2$, $\omega(q)$ denoting the number of distinct prime divisors of q, then let us see that $\mathcal{S}$ is not discrete. Assume that r and p are distinct primes dividing q. Then $\log r/\log p$ is irrational and by Dirichlet's theorem, there are infinitely many $l/k \in \mathbb{Q}$ satisfying

$$\left|\frac{\log r}{\log p} - \frac{l}{k}\right| < \frac{1}{k^2} \quad \text{and} \quad \left|\frac{2\pi k}{\log p} - \frac{2\pi l}{\log r}\right| < \frac{2\pi}{k \log r}. \tag{3.6}$$

As k can be arbitrarily large, (3.6) gives that $\delta_{\mathcal{S}} = 0$ and $\mathcal{S}$ is therefore not discrete in this case. Next for a positive parameter T, let C_T be an almost rectangular path with positive orientation having vertices at 0, 1, $1 + iT$ and iT, where in addition there are half circular indentations of radius $\epsilon_n > 0$ directed to the right of the imaginary axis with center at z_n for $n \geq 2$. Moreover, C_T has quarter circle indentations of radius $\epsilon > 0$ both at 0 and 1, where at 0, the direction of the indentation is to the right of the imaginary axis and at 1, it is to the left of the axis $\Re(s) = 1$. ϵ and ϵ_n are chosen small enough so that the indentations do not intersect with each other and all the critical zeros with imaginary part in $(0,T)$ lie in the region enclosed by C_T. The choice of T is delicate, especially when $\mathcal{S}$ is not discrete, and demands further consideration. In particular, as a first step, one may take T in such a way that (see p. 108 of [6]) the interval

$$\left(T - \frac{c_0}{\log T}, T + \frac{c_0}{\log T}\right) \tag{3.7}$$

does not contain the imaginary parts of critical zeros for some constant $c_0 > 0$ and all large T. With this choice of T, the estimates (see p. 108 and p. 116 of [6])

$$\frac{\zeta'(\sigma + iT)}{\zeta(\sigma + iT)} = O(\log^2 T), \quad \frac{L'(\sigma \pm iT, \chi)}{L(\sigma \pm iT, \chi)} = O(\log^2 qT) \tag{3.8}$$

hold uniformly for $-1 \leq \sigma \leq 2$ and any primitive character χ modulo q. We will need (3.8) repeatedly in the sequel. Observe that there is a unique integer $n \geq 1$ such that $\Im(z_n) \leq T < \Im(z_{n+1})$. If $\mathcal{S}$ is discrete, then

$$\Im(z_{n+1}) - \Im(z_n) = \frac{2\pi}{\log p}$$

is fixed, where p is the unique prime divisor of q. Otherwise, we may assume without loss of generality that

$$\Im(z_n) = \frac{2\pi k}{\log v} \quad \text{and} \quad \Im(z_{n+1}) = \frac{2\pi l}{\log r},$$

where $v \neq r$ are prime divisors of $q \geq 6$ and l, k are positive integers. Let us find a positive lower bound for

$$\Im(z_{n+1}) - \Im(z_n) = \frac{2\pi}{\log r \log v}(l \log v - k \log r). \tag{3.9}$$

For this purpose we shall use an estimate of Waldschmidt that complements the seminal work of Baker [3,4] on the transcendentality of linear forms in logarithms (see [18]). Given an algebraic number α with minimal polynomial

$$a_0 \prod_{j=1}^{d} (x - \alpha_j)$$

over $\mathbb{Z}$, define the Mahler measure of α by

$$M(\alpha) = a_0 \prod_{j=1}^{d} \max(1, |\alpha_j|),$$

and the absolute logarithmic height by

$$h(\alpha) = \frac{1}{d} \log M(\alpha).$$

Then the precise formulation of the result we need is as follows.

Lemma 1. *Let $m \geq 1$ and let K be a number field of degree d over $\mathbb{Q}$. Let $\alpha_1, \ldots, \alpha_m$ be nonzero elements of K and $\beta_0, \beta_1, \ldots, \beta_m \in K$. Consider*

$$\kappa = \beta_0 + \beta_1 \log \alpha_1 + \ldots + \beta_m \log \alpha_m.$$

Assume that the numbers $V_1, \ldots, V_m, W$ are subject to the conditions

$$\frac{1}{d} \leq V_1 \leq \ldots \leq V_m, \quad V_{m-1} \geq 1,$$
$$V_j \geq \max(h(\alpha_j), |\log \alpha_j|/d) \quad \text{for } 1 \leq j \leq m,$$
$$W \geq \max_{1 \leq j \leq m} h(\beta_j).$$

If $\kappa \neq 0$, then

$$|\kappa| \geq \exp\left(-2^{8m+51} m^{2m} d^{m+2} V_1 \ldots V_m (W + \log(edV_m)) \log(edV_{m-1})\right).$$

Noting that $l \log v - k \log r > 0$, we may apply Lemma 1 to $\kappa = l \log v - k \log r$. In this case, $m = 2$ and $d = 1$. Taking $V_1 = 2 \log v$, $V_2 = 2 \log r$ and $W = \max(\log l, \log k)$, one obtains that

$$l \log v - k \log r \geq \exp\left(-2^{73} \log v \log r (\max(\log l, \log k) + \log(2e \log r)) \log(2e \log v)\right).$$

Consequently, as $\log v, \log r \leq \log q$, the lower bound

$$l \log v - k \log r \geq \frac{e^{-C_1 (\log q)^2 (\log \log q)^2}}{(\max(l,k))^{C_2 (\log q)^2 \log \log q}} \tag{3.10}$$

follows for some positive constants C_1, C_2 independent of q. Since only arbitrarily large values of T are under consideration, one may also suppose that $\Im(z_{n+1}) \leq 2\Im(z_n)$. Hence $T \log q \gg \max(l, k)$ and (3.10) becomes

$$l \log v - k \log r \geq C(q) T^{-C(\log q)^2 \log \log q} \tag{3.11}$$

for some positive constant $C(q)$ depending on q and C. Assembling (3.9) and (3.11), one deduces that

$$\Im(z_{n+1}) - \Im(z_n) \gg_q T^{-C(\log q)^2 \log \log q}. \tag{3.12}$$

As a result of (3.12), we may vary T by an amount which is a constant multiple of $T^{-C(\log q)^2 \log \log q}$. When doing this, it is possible to stay in the interval of (3.7) as

$$T^{-C(\log q)^2 \log \log q} = o((\log T)^{-1}).$$

In this way one may ensure that for any prime $p \mid q$, if $T \log p = 2\pi j \pm \tau_p$ holds with $2\pi j$ being the closest integer multiple of 2π to $T \log p$ and $0 \leq \tau_p \leq \pi$, then

$$\tau_p \gg_q T^{-C(\log q)^2 \log \log q}. \tag{3.13}$$

As a consequence of (3.13), we now have

$$|p^{iT} - 1| = |e^{iT \log p} - 1| = |\sin(\tau_p/2)| \gg_q T^{-C(\log q)^2 \log \log q}. \tag{3.14}$$

Once T is chosen with the above restrictions, ϵ_n and ϵ_{n+1} are taken to be small enough so that the indentations at z_n and z_{n+1} do not intersect the horizontal line passing through iT. This completes our digression on the choice of T and the construction of C_T. As $\zeta(s)$ has no zeros in $[0, 1]$ and on the lines $\Re(s) = 0$, $\Re(s) = 1$, by the residue theorem applied to C_T, one obtains

$$\int_{C_T} e^{sz} \frac{L'(s, \chi_0)}{L(s, \chi_0)} \, ds = 2\pi i \sum_{0 < \gamma < T} e^{\rho z} \tag{3.15}$$

for all $z = x + iy$ with $x, y > 0$. Next let us show that

$$\int_{1+iT}^{iT} e^{sz} \frac{L'(s,\chi_0)}{L(s,\chi_0)}\, ds \to 0 \tag{3.16}$$

as T tends to infinity subject to the conditions above. First we have

$$\left| \int_{1+iT}^{iT} e^{sz} \frac{L'(s,\chi_0)}{L(s,\chi_0)}\, ds \right| \le e^{x-Ty} \int_0^1 \left(\left| \frac{\zeta'(\sigma+iT)}{\zeta(\sigma+iT)} \right| + \sum_{p|q} \frac{\log p}{|p^{\sigma+iT}-1|} \right) d\sigma. \tag{3.17}$$

Using (3.8), (3.14) and the elementary inequality $|p^{\sigma+iT}-1| \ge |p^{iT}-1|$ for all primes p dividing q, one gets

$$\left| \int_{1+iT}^{iT} e^{sz} \frac{L'(s,\chi_0)}{L(s,\chi_0)}\, ds \right| = O_q\left(e^{x-Ty} (\log^2 T + T^{C(\log q)^2 \log\log q}) \right). \tag{3.18}$$

Thus (3.16) follows from (3.18). Let $C_T^{\times}$ be the remaining part of C_T with the same orientation after deleting the open segment from $1 + iT$ to iT. Integrating by parts, we see that

$$\begin{aligned} \int_{C_T^{\times}} e^{sz} \frac{L'(s,\chi_0)}{L(s,\chi_0)}\, ds &= e^{(1+iT)z} \log L(1+iT,\chi_0) - e^{iTz} \log L(iT,\chi_0) \\ &\quad - z \int_{C_T^{\times}} e^{sz} \log L(s,\chi_0)\, ds. \end{aligned} \tag{3.19}$$

Gathering (3.15) and (3.19),

$$\begin{aligned} 2\pi i \sum_{0<\gamma<T} e^{\rho z} &= \int_{1+iT}^{iT} e^{sz} \frac{L'(s,\chi_0)}{L(s,\chi_0)}\, ds - z \int_{C_T^{\times}} e^{sz} \log L(s,\chi_0)\, ds \\ &\quad + e^{(1+iT)z} \log L(1+iT,\chi_0) - e^{iTz} \log L(iT,\chi_0) \end{aligned} \tag{3.20}$$

follows. Let us remark that in (3.19) and (3.20), $\log L(s,\chi_0)$ is the unique branch of the complex logarithm defined on the simply connected region

$$\mathbb{C} - (-\infty, 1] - \left[\frac{1}{2} + i\gamma_0, \frac{1}{2} + i\infty\right) - \left[\frac{1}{2} - i\gamma_0, \frac{1}{2} - i\infty\right) - \bigcup_{p|q, k\neq 0} \left(-\infty + \frac{2\pi i k}{\log p}, \frac{2\pi i k}{\log p}\right]$$

(assuming the Riemann hypothesis with $\frac{1}{2} + i\gamma_0$ denoting the critical zero of $\zeta(s)$ with least positive ordinate) and satisfies

$$\log L(\sigma, \chi_0) = \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi_0(n)}{n^\sigma \log n} \to 0$$

as $\sigma \to \infty$, where $\Lambda(n)$ is the von Mangoldt function. Note that this branch is analytic on $C_T^{\times}$ and differs from the classical branch defined on the region

$$\mathbb{C} - (-\infty, 1] - \bigcup_{\gamma} \left(-\infty + i\gamma, \frac{1}{2} + i\gamma\right] - \bigcup_{p|q, k\neq 0} \left(-\infty + \frac{2\pi ik}{\log p}, \frac{2\pi ik}{\log p}\right]$$

(with $\frac{1}{2} + i\gamma$ denoting the critical zeros of $\zeta(s)$) by an amount in the argument which is easily seen to be $O(t \log t)$ at points of the form $\sigma \pm it$ with $0 \leq \sigma \leq 1/2$ and $t \geq \gamma_0$. For other points of the form $\sigma + it$ with $\sigma \geq 0$, the two branches coincide. Therefore, one can still carry out calculations with respect to the classical branch but then add the difference in the argument which is $O(t \log t)$ whenever necessary. We have to show that both of the terms $e^{(1+iT)z} \log L(1 + iT, \chi_0)$ and $e^{iTz} \log L(iT, \chi_0)$ in (3.20) tend to zero as T tends to infinity. To begin with, note that

$$\log L(1 + iT, \chi_0) = \log |L(1 + iT, \chi_0)| + i \arg L(1 + iT, \chi_0). \tag{3.21}$$

Here $\arg L(\sigma + it, \chi_0) = \Im \log L(1 + it, \chi_0)$ is defined by continuous variation from $\infty + it$ to $\sigma + it$, namely that

$$\arg L(\sigma + it, \chi_0) = \int_{\infty}^{\sigma} \Im \frac{\zeta'}{\zeta}(\alpha + it)\, d\alpha.$$

If $\sigma + it$ is a zero, then we set

$$\arg L(\sigma + it, \chi_0) = \frac{1}{2} \left(\arg L(\sigma + it^+, \chi_0) + \arg L(\sigma + it^-, \chi_0)\right)$$

in the case of a horizontal branch cut at $\sigma + it$ and

$$\arg L(\sigma + it, \chi_0) = \frac{1}{2} \left(\arg L(\sigma^+ + it, \chi_0) + \arg L(\sigma^- + it, \chi_0)\right)$$

in the case of a vertical branch cut at $\sigma + it$. Let us write from (3.1) that

$$\log |L(1 + iT, \chi_0)| = \log |\zeta(1 + iT)| + \sum_{p|q} \log \left|1 - \frac{1}{p^{1+iT}}\right|. \tag{3.22}$$

Using $\log |\zeta(2 + iT)| = O(1)$ and (3.8), we have

$$\log |\zeta(1 + iT)| = \log |\zeta(2 + iT)| + \Re \int_{2+iT}^{1+iT} \frac{\zeta'(s)}{\zeta(s)}\, ds \ll \log^2 T. \tag{3.23}$$

Observing that

$$\log\left|1-\frac{1}{p^{1+iT}}\right| \le \max\left(\log\left(1+\frac{1}{p}\right), -\log\left(1-\frac{1}{p}\right)\right) = -\log\left(1-\frac{1}{p}\right),$$

one obtains

$$\sum_{p|q}\log\left|1-\frac{1}{p^{1+iT}}\right| \le \log\frac{q}{\varphi(q)} \ll \log\log\log q. \tag{3.24}$$

Thus from (3.22)–(3.24),

$$\log|L(1+iT,\chi_0)| = O(\log^2 T + \log\log\log q) \tag{3.25}$$

follows. Moreover, we may write

$$\arg L(1+iT,\chi_0) = \arg L(2+iT,\chi_0) + \Im\int_{2+iT}^{1+iT}\frac{L'(s,\chi_0)}{L(s,\chi_0)}\,ds. \tag{3.26}$$

First of all since $\arg L(2,\chi_0)=0$,

$$\arg L(2+iT,\chi_0) = \Im\int_{2}^{2+iT}\frac{L'(s,\chi_0)}{L(s,\chi_0)}\,ds$$

holds. By uniform convergence, one gets

$$\int_{2}^{2+iT}\frac{L'(s,\chi_0)}{L(s,\chi_0)}\,ds = -i\int_0^T\sum_{n=1}^{\infty}\frac{\Lambda(n)\chi_0(n)}{n^{2+it}}\,dt = \sum_{n=1}^{\infty}\frac{\Lambda(n)\chi_0(n)}{n^2\log n}(e^{-iT\log n}-1).$$

This implies that

$$\arg L(2+iT,\chi_0) = -\sum_{n=1}^{\infty}\frac{\Lambda(n)\chi_0(n)}{n^2\log n}\sin(T\log n) = O(1). \tag{3.27}$$

It remains to consider

$$\Im\int_{2+iT}^{1+iT}\frac{L'(s,\chi_0)}{L(s,\chi_0)}\,ds = \Im\int_{2+iT}^{1+iT}\frac{\zeta'(s)}{\zeta(s)}\,ds + \Im\sum_{p|q}\log p\int_{2+iT}^{1+iT}\frac{1}{p^s-1}\,ds. \tag{3.28}$$

Noticing

$$\Im \int_{2+iT}^{1+iT} \frac{\zeta'(s)}{\zeta(s)}\, ds = \arg \zeta(1+iT) - \arg \zeta(2+iT) \ll \log T,$$

$$\Im \sum_{p|q} \log p \int_{2+iT}^{1+iT} \frac{1}{p^s - 1}\, ds \ll \omega(q)$$

and gathering (3.26)–(3.28), we see that

$$\arg L(1+iT, \chi_0) = O(\log T + \omega(q)). \tag{3.29}$$

Therefore, from (3.21), (3.25) and (3.29), one has

$$\log L(1+iT, \chi_0) = O(\log^2 T + \log\log\log q + \omega(q)) \tag{3.30}$$

and (3.30) gives that

$$|e^{(1+iT)z} \log L(1+iT, \chi_0)| = e^{x-Ty} |\log L(1+iT, \chi_0)| \to 0$$

as $T \to \infty$. Next consider

$$\log L(iT, \chi_0) = \log |L(iT, \chi_0)| + i \arg L(iT, \chi_0). \tag{3.31}$$

Similarly as above, we have

$$\log |L(iT, \chi_0)| = \log |\zeta(iT)| + \sum_{p|q} \log \left| 1 - \frac{1}{p^{iT}} \right|. \tag{3.32}$$

Using (3.8) and (3.14), one obtains that

$$\log |\zeta(iT)| = \log |\zeta(2+iT)| + \Re \int_{2+iT}^{iT} \frac{\zeta'(s)}{\zeta(s)}\, ds \ll \log^2 T, \tag{3.33}$$

$$\sum_{p|q} \log \left| 1 - \frac{1}{p^{iT}} \right| \ll_q \log T. \tag{3.34}$$

Hence by (3.32)–(3.34), we may write

$$\log |L(iT, \chi_0)| = O_q(\log^2 T). \tag{3.35}$$

Finally observing that

$$\arg L(iT,\chi_0) = O(1) + \Im \int\limits_{2+iT}^{iT} \frac{\zeta'(s)}{\zeta(s)}\,ds + \Im \sum_{p|q} \log p \int\limits_{2+iT}^{iT} \frac{1}{p^s-1}\,ds, \tag{3.36}$$

using

$$\Im \int\limits_{2+iT}^{iT} \frac{\zeta'(s)}{\zeta(s)}\,ds \ll \log T, \quad \Im \sum_{p|q} \log p \int\limits_{2+iT}^{iT} \frac{1}{p^s-1}\,ds \ll_q T^{C(\log q)^2 \log\log q},$$

and the argument difference between the branches which is $\ll T\log T$, we deduce from (3.36) that

$$\arg L(iT,\chi_0) = O_q(T^{C(\log q)^2 \log\log q}). \tag{3.37}$$

Combining (3.31), (3.35) and (3.37), one easily infers that

$$|e^{iTz}\log L(iT,\chi_0)| = e^{x-Ty}|\log L(iT,\chi_0)| \to 0$$

as $T\to\infty$. Next for $n \geq 2$, let

$$I_n = \int\limits_{C_{\epsilon_n}} e^{sz}\log L(s,\chi_0)\,ds, \tag{3.38}$$

where C_{ϵ_n} is the half circular indentation of radius ϵ_n at z_n. Let us decompose the right hand side of (3.38) as

$$\int\limits_{C_{\epsilon_n}} e^{sz}\log|L(s,\chi_0)|\,ds + i\int\limits_{C_{\epsilon_n}} e^{sz}\arg L(s,\chi_0)\,ds. \tag{3.39}$$

First note that

$$\int\limits_{C_{\epsilon_n}} e^{sz}\log|L(s,\chi_0)|\,ds = \int\limits_{C_{\epsilon_n}} e^{sz}\log|\zeta(s)|\,ds + \sum_{p|q}\int\limits_{C_{\epsilon_n}} e^{sz}\log\left|1-\frac{1}{p^s}\right|\,ds. \tag{3.40}$$

As $\log|\zeta(s)|$ is continuous and uniformly bounded on C_{ϵ_n} when $\epsilon_n\to 0$, we see that

$$\lim_{\epsilon_n\to 0}\int\limits_{C_{\epsilon_n}} e^{sz}\log|\zeta(s)|\,ds = 0. \tag{3.41}$$

Assuming $z_n = \frac{2\pi ik}{\log p}$ for some prime $p \mid q$ and $k \geq 1$, we also have

$$\lim_{\epsilon_n \to 0} \sum_{\substack{r|q \\ r \neq p}} \int\limits_{C_{\epsilon_n}} e^{sz} \log \left| 1 - \frac{1}{r^s} \right| = 0. \tag{3.42}$$

Moreover, when ϵ_n is small enough, one may write

$$1 - \frac{1}{p^s} = (s - z_n) F(s).$$

Here $F(s)$ is a bounded analytic function as $\epsilon_n \to 0$, satisfying

$$\lim_{s \to z_n} F(s) = \log p.$$

Therefore,

$$\log \left| 1 - \frac{1}{p^s} \right| = \log |s - z_n| + \log |F(s)| = \log |s - z_n| + O(1)$$

holds when $s \to z_n$ and

$$\lim_{\epsilon_n \to 0} \int\limits_{C_{\epsilon_n}} e^{sz} \log \left| 1 - \frac{1}{p^s} \right| ds = \lim_{\epsilon_n \to 0} O(\epsilon_n |\log \epsilon_n|) = 0 \tag{3.43}$$

follows. Combining (3.40)–(3.43), one gets

$$\lim_{\epsilon_n \to 0} \int\limits_{C_{\epsilon_n}} e^{sz} \log |L(s, \chi_0)| \, ds = 0. \tag{3.44}$$

For all $s = \sigma + it \in C_{\epsilon_n}$, we know that

$$\arg L(s, \chi_0) = O(1) + \Im \int\limits_{2+it}^{\sigma+it} \frac{\zeta'(s)}{\zeta(s)} \, ds + \sum_{p|q} \Im \int\limits_{2+it}^{\sigma+it} \frac{\log p}{p^s - 1} \, ds. \tag{3.45}$$

Next we have

$$\Im \int\limits_{2+it}^{\sigma+it} \frac{\zeta'(s)}{\zeta(s)} \, ds = O(\log(|t| + 4)) = O(\log(|z_n| + 4)). \tag{3.46}$$

On the other hand, over the strip region defined by $\Im(s) \in (|z_n| - 2\epsilon_n, |z_n| + 2\epsilon_n)$ and $\Re(s) \geq 0$, one may write for all small ϵ_n that

$$\frac{\log p}{p^s - 1} = \frac{1}{s - z_n} + G(s),$$

where $G(s)$ is analytic and bounded as $s \to z_n$ in this strip. Consequently we obtain that

$$\Im \int_{2+it}^{\sigma+it} \frac{\log r}{r^s - 1}\, ds = O(1) \tag{3.47}$$

for all primes $r \mid q$ with $r \neq p$ and

$$\Im \int_{2+it}^{\sigma+it} \frac{\log p}{p^s - 1}\, ds = \Im \int_{2+it}^{\sigma+it} \left(\frac{1}{s - z_n} + G(s) \right) ds = O(1). \tag{3.48}$$

Assembling (3.45)–(3.48) and taking into account the argument difference between the branches, one infers that

$$\arg L(s, \chi_0) = O_q((|z_n| + 4) \log(|z_n| + 4))$$

is uniformly bounded on C_{ϵ_n} for all small ϵ_n and

$$\lim_{\epsilon_n \to 0} \int_{C_{\epsilon_n}} e^{sz} \arg L(s, \chi_0)\, ds = 0. \tag{3.49}$$

Thus from (3.38), (3.39), (3.44) and (3.49),

$$\lim_{\epsilon_n \to 0} I_n = 0 \tag{3.50}$$

is justified for $n \geq 2$. Similarly, we can show that

$$\lim_{\epsilon \to 0} \int_{C_{0,\epsilon}} e^{sz} \log L(s, \chi_0)\, ds = 0, \tag{3.51}$$

where $C_{0,\epsilon}$ is the quarter circle centered at $z_1 = 0$ with radius $\epsilon > 0$, by taking into account the singularities of $\log(1 - p^{-s})$ at $s = 0$ for all $p \mid q$. On the other hand, if $C_{1,\epsilon}$ is the quarter circle centered at 1 with radius $\epsilon > 0$, then by writing

$$L(s, \chi_0) = \frac{f(s)}{s - 1},$$

where $f(s)$ is an analytic function bounded in a neighborhood of 1 with

$$\lim_{s \to 1} f(s) = \frac{\varphi(q)}{q},$$

one can easily obtain

$$\lim_{\epsilon \to 0} \int\limits_{C_{1,\epsilon}} e^{sz} \log |L(s, \chi_0)| \, ds = 0. \tag{3.52}$$

Moreover, on $C_{1,\epsilon}$, we have $\arg L(s, \chi_0) = O_q(1)$ uniformly for all small $\epsilon > 0$ and this implies that

$$\lim_{\epsilon \to 0} \int\limits_{C_{1,\epsilon}} e^{sz} \arg L(s, \chi_0) \, ds = 0. \tag{3.53}$$

From (3.52) and (3.53), it follows that

$$\lim_{\epsilon \to 0} \int\limits_{C_{1,\epsilon}} e^{sz} \log L(s, \chi_0) \, ds = 0. \tag{3.54}$$

Now letting $T \to \infty$ in (3.20), we get

$$\begin{aligned} 2\pi i \sum_{\gamma > 0} e^{\rho z} = z \int\limits_{CBA} e^{sz} \log L(s, \chi_0) \, ds - z \int\limits_{CD} e^{sz} \log L(s, \chi_0) \, ds \\ - z \int\limits_{DEF} e^{sz} \log L(s, \chi_0) \, ds, \end{aligned} \tag{3.55}$$

where

$$\begin{aligned} &A = +i\infty, \quad B = +i\epsilon, \quad C = \epsilon \\ &D = 1 - \epsilon, \quad E = 1 + i\epsilon, \quad F = 1 + i\infty. \end{aligned}$$

Using (3.1) and the functional equation for $\zeta(s)$, one has

$$L(s, \chi_0) = 2^s \pi^{s-1} \sin\left(\frac{s\pi}{2}\right) \Gamma(1-s) L(1-s, \chi_0) \prod_{p|q} \left(1 - \frac{1}{p^s}\right) \left(1 - \frac{1}{p^{1-s}}\right)^{-1}. \tag{3.56}$$

Our goal is now to exploit some symmetry on the path of integration by relating $\log L(s, \chi_0)$ to $\log L(1-s, \chi_0)$ through (3.56). Thus it will be necessary to keep track of the initial difference between the arguments of the relevant logarithmic terms. Precisely, let us show that as s traverses from C to B, then $\arg L(s, \chi_0) = -\pi$ at $s = C$, whereas $1 - s$ traverses from D to $1 - i\epsilon = \overline{E}$ with $\arg L(1-s, \chi_0) = \pi$ at $1 - s = D$. This results in a 2π initial difference in the arguments and by continuous variation of the argument, the same holds on the path CBA. To verify our claim, for $0 < s < 1$, let $C = C_1 \cup C_2$ be the circle of radius $1 - s$ centered at 1, positively oriented, where C_1 is the upper half and C_2 is the lower half. As $L(s,\chi_0)$ has no zeros in C and a simple pole at $s=1$, it follows that

$$\int_{C_1} \frac{L'}{L}(s,\chi_0)\,ds + \int_{C_2} \frac{L'}{L}(s,\chi_0)\,ds = -2\pi i. \tag{3.57}$$

Using

$$\frac{L'}{L}(\bar{s},\chi_0) = \overline{\frac{L'}{L}(s,\chi_0)},$$

one sees that

$$\int_{C_1} \frac{L'}{L}(s,\chi_0)\,ds = -\overline{\int_{C_2} \frac{L'}{L}(s,\chi_0)\,ds}.$$

Now as $\arg L(2-s,\chi_0)=0$, by continuous variation on C_1 and using (3.57), we have

$$\arg L(s,\chi_0) = \arg L(s,\chi_0) - \arg L(2-s,\chi_0) = \Im \int_{C_1} \frac{L'}{L}(s,\chi_0)\,ds = -\pi. \tag{3.58}$$

Similarly, by continuous variation on C_2, we get

$$-\arg L(s,\chi_0) = \arg L(2-s,\chi_0) - \arg L(s,\chi_0) = \Im \int_{C_2} \frac{L'}{L}(s,\chi_0)\,ds = -\pi, \tag{3.59}$$

which implies that $\arg L(s,\chi_0)=\pi$. From (3.58) and (3.59), one concludes that the argument of $\log L(s,\chi_0)$ is sensitive to the direction of the path. It is $-\pi$ if we traverse the path from right to left indenting around 1 in the positive direction and it is π if we traverse from left to right indenting around 1 in the positive direction. In addition the arguments of the logarithms of all the other terms appearing on the right hand side of (3.56) are 0 for $0<s<1$. Consequently, the following equation is justified via (3.56) by formally taking logarithms of both sides.

$$\begin{aligned} &z \int_{CBA} e^{sz} \log L(s,\chi_0)\,ds \\ &\quad = z \int_{CBA} e^{sz} \Big(s\log 2 + (s-1)\log \pi + \log \sin\left(\frac{s\pi}{2}\right) + \log \Gamma(1-s) + \log L(1-s,\chi_0) \\ &\qquad + \sum_{p|q} \log\left(1-\frac{1}{p^s}\right) - \sum_{p|q} \log\left(1-\frac{1}{p^{1-s}}\right) - 2\pi i \Big)\,ds. \end{aligned} \tag{3.60}$$

Letting $\epsilon \to 0$ and $\epsilon_n \to 0$ for all $n \geq 1$, (3.50), (3.51) and (3.54) give that the path of integration in (3.55) and (3.60) can be assumed to be unions of straight lines. Hence

CBA becomes the straight line from 0 to $i\infty$, DEF becomes the straight line from 1 to $1+i\infty$ and CD becomes the straight line from 0 to 1. To analyze the right hand side of (3.60) further, let us consider

$$\int_{0}^{i\infty} e^{sz}\log\left(1-\frac{1}{p^s}\right)ds. \tag{3.61}$$

Note that for any $k \geq 0$,

$$\int_{\frac{2\pi ik}{\log p}}^{\frac{2\pi i(k+1)}{\log p}} \left|\log\left(1-\frac{1}{p^s}\right)\right| ds = \int_{0}^{\frac{2\pi}{\log p}} \left|\log\left(2\sin\frac{t\log p}{2}\right)\right| dt + O\left(\frac{1}{\log p}\right) = O(1).$$

Using this, (3.61) becomes

$$\int_{0}^{i\infty} e^{sz}\log\left(1-\frac{1}{p^s}\right)ds = \int_{0}^{iT} e^{sz}\log\left(1-\frac{1}{p^s}\right)ds + O(e^{-yT}) \tag{3.62}$$

(here the parameter T has nothing to do with the earlier uses of it in the proof). But

$$\log\left(1-\frac{1}{p^s}\right) = -\sum_{m=1}^{\infty}\frac{1}{mp^{ms}} = -\sum_{m=1}^{\infty}\frac{e^{-ms\log p}}{m}, \tag{3.63}$$

for all $s = it$, $t \neq \frac{2\pi k}{\log p}$, where both

$$\Re\log\left(1-\frac{1}{p^s}\right) = -\sum_{m\geq 1}\frac{\cos(mt\log p)}{m}$$

and

$$\Im\log\left(1-\frac{1}{p^s}\right) = \sum_{m\geq 1}\frac{\sin(mt\log p)}{m}$$

are boundedly convergent except at finitely many points (so almost everywhere on $[0,T]$). Thus interchange is permissible in (3.62) and by (3.63), we obtain

$$\begin{aligned}\int_{0}^{iT} e^{sz}\log\left(1-\frac{1}{p^s}\right)ds &= -\sum_{m\geq 1}\frac{1}{m}\int_{0}^{iT} e^{s(z-m\log p)}\,ds \\ &= \sum_{m\geq 1}\left(\frac{1}{m(z-m\log p)} - \frac{e^{iT(z-m\log p)}}{m(z-m\log p)}\right).\end{aligned}$$

As $e^{iT(z-m\log p)} = O(e^{-yT})$, letting $T \to \infty$ termwise in the above equation,

$$z \int_0^{i\infty} e^{sz} \log\left(1 - \frac{1}{p^s}\right) ds = \sum_{m=1}^{\infty} \frac{z}{m(z - \log p^m)} \tag{3.64}$$

follows. Similarly, one shows that

$$z \int_0^{i\infty} e^{sz} \log\left(1 - \frac{1}{p^{1-s}}\right) ds = \sum_{m=1}^{\infty} \frac{z}{mp^m(z + \log p^m)}. \tag{3.65}$$

Gathering (3.55), (3.60), (3.64) and (3.65), the following formula is verified.

$$\begin{aligned}
2\pi i V(z) = &-z \int_1^{1+i\infty} e^{sz} \log L(s,\chi_0)\, ds + z \int_0^{i\infty} e^{sz} \log L(1-s,\chi_0)\, ds \\
&- z(\log\pi + 2\pi i) \int_0^{i\infty} e^{sz}\, ds + z\log 2\pi \int_0^{i\infty} s e^{sz}\, ds - z\int_0^1 e^{sz} \log L(s,\chi_0)\, ds \\
&+ z \int_0^{i\infty} e^{sz} \log\sin\left(\frac{s\pi}{2}\right) ds + z \int_0^{i\infty} e^{sz} \log\Gamma(1-s)\, ds \\
&+ \sum_{p|q} \sum_{m=1}^{\infty} \frac{z}{m(z-\log p^m)} - \sum_{p|q} \sum_{m=1}^{\infty} \frac{z}{mp^m(z+\log p^m)}.
\end{aligned} \tag{3.66}$$

Next some terms on the right hand side of (3.66) are to be estimated. To begin with, note that

$$\begin{aligned}
z \int_0^{i\infty} e^{sz} \log\sin\left(\frac{s\pi}{2}\right) ds &= \log 2 - \frac{\pi i}{2} + \frac{\pi i}{2z} + iz \int_0^{\infty} e^{itz} \log t\, dt + iz \int_0^{\infty} e^{itz} \log \frac{e^{\pi t} - 1}{t}\, dt \\
&= U + \log 2 - \pi i + \frac{\pi i}{2z} + \log z - \log\pi - \int_0^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t} + 1\right) e^{-\frac{zt}{\pi i}}\, dt \\
&= U + \log 2 + \frac{\pi i}{2}\left(\frac{1}{z} - 1\right) + \Psi\left(\frac{z}{\pi i}\right),
\end{aligned} \tag{3.67}$$

where $U = -\int_0^{\infty} e^{-t} \log t\, dt$ is Euler's constant and

$$\Psi(s) = \frac{\Gamma'(s)}{\Gamma(s)} = -U + \int_0^{\infty} \frac{e^{-t} - e^{-ts}}{1 - e^{-t}}\, dt \tag{3.68}$$

is the digamma function. By Stirling's asymptotic formula for the gamma function, we have

$$|e^{sz}\log\Gamma(1-s)| = e^{\sigma x - ty}|\log\Gamma(1-s)| = O(e^{-k|s|}|s|\log|s|) \tag{3.69}$$

for some positive constant k when s is in the second quadrant. Therefore, residue theorem, (3.68) and (3.69) give

$$\begin{aligned} z\int_0^{i\infty} e^{sz}\log\Gamma(1-s)\,ds &= z\int_0^{-\infty} e^{sz}\log\Gamma(1-s)\,ds \\ &= -ze^{z}\int_1^{\infty} e^{-sz}\log\Gamma(s)\,ds = -e^{z}\int_1^{\infty}\Psi(s)e^{-sz}\,ds. \end{aligned} \tag{3.70}$$

From (3.68) and (3.70),

$$\begin{aligned} z\int_0^{i\infty} e^{sz}\log\Gamma(1-s)\,ds &= \frac{U}{z} - e^{z}\int_0^{\infty}\frac{dt}{1-e^{-t}}\int_1^{\infty}\left(e^{-t-sz} - e^{-(t+z)s}\right)ds \\ &= \frac{U}{z} - \frac{1}{z}\int_0^{\infty}\frac{t}{e^{t}-1}\cdot\frac{dt}{t+z} \end{aligned} \tag{3.71}$$

follows. For any character χ modulo q, consider the Dirichlet series

$$\sum_{n=2}^{\infty}\frac{\Lambda(n)\chi(n)}{n^{s}\log n} \tag{3.72}$$

which represents $\log L(s,\chi)$ for $\Re(s) > 1$. Assume that $s = 1 + it$ with $0 < \epsilon_0 \le t \le M$, where ϵ_0 and M are fixed but arbitrary. Let us see that (3.72) converges uniformly to $\log L(s,\chi)$ for all such s. If x is half an odd integer (which we may assume without loss of generality), then by Perron's formula, we have

$$\sum_{2\le n\le x}\frac{\Lambda(n)\chi(n)}{n^{s}\log n} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{x^{w}}{w}\log L(s+w,\chi)\,dw + O\left(\frac{\log x}{T}\right), \tag{3.73}$$

where in (3.73), $c = 1/\log x$ and T is a parameter that can be taken to be arbitrarily large and will be chosen as an increasing function of x later (here the parameter T has nothing to do with the earlier uses of it in the proof). Note that $|\Im(s+w)| \le T + M$ and $\Re(s+w) = 1 + \Re(w)$. Let us recall that $L(s+w,\chi)$ has no zeros when

$$1 + \Re(w) \ge 1 - \frac{c_1}{\log(q(|\Im(s+w)|+4))}$$

for some absolute constant $c_1 > 0$ with the possible exception of a single simple real zero satisfying

$$1 + \Re(w) < 1 - \frac{c_2}{\sqrt{q}\log^2 q}$$

for some absolute constant $c_2 > 0$. We may choose $\delta > 0$ small enough so that $L(s+w, \chi)$ has no zeros for $\Re(w) \geq -\delta$ and $|\Im(s+w)| \leq T + M$. Indeed it suffices to take

$$\delta = \frac{A}{\log T}, \tag{3.74}$$

where $A > 0$ is a constant depending only on q and M. Hence we see that if χ is a nonprincipal character modulo q, then $L(s+w, \chi)$ has no poles or zeros in the rectangular region with vertices at $c - iT$, $c + iT$, $-\delta + iT$, $-\delta - iT$. Then by the residue theorem, one has

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{x^w}{w}\log L(s+w,\chi)\,dw \\ = \log L(s,\chi) + \frac{1}{2\pi i}\left(\int_{-\delta+iT}^{c+iT} + \int_{-\delta-iT}^{-\delta+iT} + \int_{c-iT}^{-\delta-iT}\right).\end{aligned} \tag{3.75}$$

In the above zero-free region, we know that

$$|\log L(s+w,\chi)| \ll_{M,q} \log\log T$$

for all large T. Thus

$$\begin{aligned}\int_{-\delta-iT}^{-\delta+iT} \frac{x^w}{w}\log L(s+w,\chi)\,dw &= O_{M,q}\left(x^{-\delta}\log\log T\int_{-T}^{T}\frac{dt}{\sqrt{\delta^2+t^2}}\right) \\ &= O_{M,q}\left(x^{-\delta}\log T\log\log T\right).\end{aligned} \tag{3.76}$$

Moreover, we have

$$\begin{aligned}\int_{-\delta+iT}^{c+iT} \frac{x^w}{w}\log L(s+w,\chi)\,dw &= O_{M,q}\left(\frac{\log\log T}{T}\int_{-\delta}^{c} x^u\,du\right) \\ &= O_{M,q}\left(\frac{x^c\log\log T}{T}\right) = O_{M,q}\left(\frac{\log\log T}{T}\right)\end{aligned} \tag{3.77}$$

and similarly

$$\int_{c-iT}^{-\delta-iT} \frac{x^w}{w} \log L(s+w,\chi)\, dw = O_{M,q}\left(\frac{\log\log T}{T}\right) \tag{3.78}$$

follows. Assembling (3.73), (3.75)–(3.78), one infers that

$$\sum_{2\le n\le x} \frac{\Lambda(n)\chi(n)}{n^s \log n}$$
$$= \log L(s,\chi) + O\left(\frac{\log x}{T}\right) + O_{M,q}\left(x^{-\delta}\log T \log\log T\right) + O_{M,q}\left(\frac{\log\log T}{T}\right). \tag{3.79}$$

Taking for example $T = \log^2 x$ in (3.79), we deduce that the convergence is uniform for all $s = 1 + it$ with $0 < \epsilon_0 \le t \le M$. If $\chi = \chi_0$ is the principal character modulo q, then a minor modification of the argument is required as $L(s+w,\chi_0)$ has a simple pole at $w = -it$. Precisely, by the residue theorem

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{x^w}{w} \log L(s+w,\chi_0)\, dw$$
$$= \log L(s,\chi_0) + \frac{1}{2\pi i}\left(\int_{-\delta+iT}^{c+iT} + \int_{-\delta-iT}^{-\delta+iT} + \int_{c-iT}^{-\delta-iT}\right) + \frac{1}{2\pi i}\int_C, \tag{3.80}$$

where C in (3.80) is a loop starting and finishing at $-\delta - it$ encircling $-it$ in the positive direction. In particular, C consists of two oppositely oriented straight lines joining $-\delta - it$ to $-\frac{c}{2} - it$ and a circle of radius $c/2$ positively oriented with center $-it$. We also have

$$\int_C \frac{x^w}{w} \log L(s+w,\chi_0)\, dw = O_{\epsilon_0,q}\left(x^c \log(1/c)(c+\delta)\right)$$
$$= O_{\epsilon_0,q}\left(\log\log x\left(\frac{1}{\log T} + \frac{1}{\log x}\right)\right). \tag{3.81}$$

As a result of (3.76)–(3.78), taking $T = (\log x)^{\log\log x}$ in (3.79)–(3.81), the uniform convergence of

$$\sum_{n=2}^{\infty} \frac{\Lambda(n)\chi_0(n)}{n^s \log n}$$

to $\log L(s,\chi_0)$ is obtained for all $s = 1 + it$ with $0 < \epsilon_0 \le t \le M$. Observe that

$$\begin{aligned} -z \int_{1}^{1+i\infty} e^{sz} \log L(s,\chi_0)\, ds &= -z \lim_{\substack{\epsilon_0 \to 0 \\ M\to\infty}} \int_{1+i\epsilon_0}^{1+iM} e^{sz} \log L(s,\chi_0)\, ds \\ &= -z \lim_{\substack{\epsilon_0 \to 0 \\ M\to\infty}} \int_{1+i\epsilon_0}^{1+iM} e^{sz} \left(\sum_{m,p} \frac{\chi_0(p^m)}{mp^{ms}} \right) ds \\ &= -z \lim_{\substack{\epsilon_0 \to 0 \\ M\to\infty}} \sum_{m,p} \frac{\chi_0(p^m)}{m} \int_{1+i\epsilon_0}^{1+iM} e^{s(z-m\log p)}\, ds, \end{aligned} \tag{3.82}$$

where the change of order of sum and integral in (3.82) is possible by the uniform convergence of the series on the range of integration. (3.82) further gives

$$\begin{aligned} &-z \int_{1}^{1+i\infty} e^{sz} \log L(s,\chi_0)\, ds \\ &= -ze^{z} \lim_{\substack{\epsilon_0 \to 0 \\ M\to\infty}} \sum_{m,p} \frac{\chi_0(p^m)}{mp^m(z-m\log p)} \left(e^{iM(z-m\log p)} - e^{i\epsilon_0(z-m\log p)} \right). \end{aligned} \tag{3.83}$$

As

$$\left| e^{iM(z-m\log p)} - e^{i\epsilon_0(z-m\log p)} \right| = O(1)$$

and the series

$$\sum_{m,p} \frac{\chi_0(p^m)}{mp^m(z-m\log p)}$$

is convergent by comparison with $\sum_p \frac{1}{p\log p}$, one may let $\epsilon_0 \to 0$ and $M \to \infty$ termwise on the right hand side of (3.83) to get

$$-z \int_{1}^{1+i\infty} e^{sz} \log L(s,\chi_0)\, ds = ze^{z} \sum_{m,p} \frac{\chi_0(p^m)}{mp^m(z-\log p^m)}. \tag{3.84}$$

Similarly, we have

$$z \int_{0}^{i\infty} e^{sz} \log L(1-s,\chi_0)\, ds = -z \sum_{m,p} \frac{\chi_0(p^m)}{mp^m(z+\log p^m)}. \tag{3.85}$$

Feeding (3.71), (3.84) and (3.85) into (3.66), one derives the following explicit formula over primes:

$$
\begin{aligned}
2\pi i V(z) = {} & z e^{z} \sum_{m,p} \frac{\chi_0(p^m)}{mp^m(z-\log p^m)} - z \sum_{m,p} \frac{\chi_0(p^m)}{mp^m(z+\log p^m)} \\
& + \sum_{p|q} \sum_{m=1}^{\infty} \frac{z}{m(z-\log p^m)} - \sum_{p|q} \sum_{m=1}^{\infty} \frac{z}{mp^m(z+\log p^m)} \\
& - z \int_0^1 e^{sz} \log L(s,\chi_0)\, ds + z \int_0^{i\infty} e^{sz} \log \sin\left(\frac{s\pi}{2}\right) ds \\
& + \log \pi + \frac{1}{z}(U + \log 2\pi) + 2\pi i - \frac{1}{z} \int_0^{\infty} \frac{t}{e^t-1} \cdot \frac{dt}{t+z}. \qquad (3.86)
\end{aligned}
$$

The case of the principal character settled, we may now proceed to the case of a nonprincipal character χ modulo q and arrive at an analog of (3.86). Assume that χ is induced by the primitive character χ_1 modulo q_1 (so that q_1 divides q). First we have the identity

$$
L(s,\chi) = L(s,\chi_1) \prod_{p|q} \left(1 - \frac{\chi_1(p)}{p^s}\right) \qquad (3.87)
$$

for all complex s. Let us define for any $z = x + iy$ with $x, y > 0$

$$
V(z,\chi) := \sum_{\gamma>0} e^{\rho_\chi z}, \qquad (3.88)
$$

where the sum in (3.88) is over the critical zeros of $L(s,\chi)$ with positive imaginary part. The convergence in (3.88) can be checked easily as before since the number of critical zeros of $L(s,\chi)$ with imaginary part in $(0,T)$ coincide with that of $L(s,\chi_1)$ by (3.87) and this number is $O(T\log qT)$. Let $\mathcal{S}_\chi$ be the set of purely imaginary zeros of $L(s,\chi)$ with nonnegative imaginary part. Clearly each $\chi_1(p)$ is a qth root of unity and we write $\chi_1(p) = e^{2\pi i r_p}$ with $r_p = a_p/q$, $0 \le a_p < q$. Then $\mathcal{S}_\chi$ can be described precisely as

$$
\mathcal{S}_\chi = \left\{ \frac{2\pi i}{\log p}(k + r_p) \,:\, p \mid q,\ k \ge 0 \right\}. \qquad (3.89)
$$

In general $\mathcal{S}_\chi$ is not a discrete set but can be represented as a sequence $\{z_n\}_{n\ge1}$ with $0 \le \Im(z_1) < \Im(z_2) < \dots$. For a positive parameter T, let $C_{T,\chi}$ be an almost rectangular path with positive orientation having vertices at 0, 1, $1+iT$, iT and half circular indentations of radius $\epsilon_n > 0$ directed to the right of the imaginary axis with center at z_n for $n \ge 2$ (for $n \ge 1$ if $0 < \Im(z_1)$). In addition, $C_{T,\chi}$ has a quarter circle indentation of radius $\epsilon > 0$ at 0, where the direction of the indentation is to the right of the imaginary axis (the indentation at 0 might be necessary when 0 is a member of $\mathcal{S}_\chi$ or when χ is even so that $L(0,\chi) = 0$). In the case of a pair of Siegel zeros in $(0,1)$, say at β and $1-\beta$, we make half circular indentations of radius ϵ at β and $1-\beta$, where the indentations are above the real axis. Here ϵ and ϵ_n are chosen small enough so that the indentations do not interfere with each other and all the critical zeros with imaginary part in $(0, T)$ lie in the region enclosed by $C_{T,\chi}$. The choice of T is again delicate and is subject to (3.7) and (3.8). We may assume that $\Im(z_n) \leq T < \Im(z_{n+1})$,

$$\Im(z_n) = \frac{2\pi}{\log v}(k + r_v) \quad \text{and} \quad \Im(z_{n+1}) = \frac{2\pi}{\log u}(l + r_u),$$

where $v \neq u$ are prime divisors of q. As $\log v$ and $\log u$ are linearly independent over $\mathbb{Q}$, it follows that $(l + r_u)\log v - (k + r_v)\log u > 0$. It remains to find a positive lower bound for this combination in terms of T and q. This can be done by applying Lemma 1. Only the choice W changes and becomes

$$W = \max(\log(ql + a_u), \log(qk + a_v)).$$

Thus similarly as in (3.10) and (3.11), one obtains

$$(l + r_u)\log v - (k + r_v)\log u \geq \frac{e^{-C_1(\log q)^2(\log\log q)^2}}{(\max(ql + a_u, qk + a_v))^{C_2(\log q)^2 \log\log q}}. \tag{3.90}$$

As $0 \leq a_u, a_v < q$, it follows from (3.90) that

$$\Im(z_{n+1}) - \Im(z_n) \gg_q T^{-C(\log q)^2 \log\log q}. \tag{3.91}$$

Compared to (3.12), only the constant depending on q worsens in (3.91). Nevertheless, varying T subject to (3.7), we may guarantee that for any prime $p \mid q$, if $T\log p - 2\pi r_p = 2\pi j \pm \tau_p$ holds with $2\pi j$ being the closest integer multiple of 2π to $T\log p - 2\pi r_p$ and $0 \leq \tau_p \leq \pi$, then

$$\tau_p \gg_q T^{-C(\log q)^2 \log\log q}. \tag{3.92}$$

Consequently from (3.92),

$$|p^{iT} - \chi_1(p)| = |e^{iT\log p} - e^{2\pi i r_p}| \gg_q T^{-C(\log q)^2 \log\log q} \tag{3.93}$$

follows for all $p \mid q$. After choosing T with these conditions, we take ϵ_n and ϵ_{n+1} small enough so that the indentations at z_n and z_{n+1} do not intersect the horizontal line passing through iT. This completes the construction of $C_{T,\chi}$ and gives

$$\int\limits_{C_{T,\chi}} e^{sz}\frac{L'(s,\chi)}{L(s,\chi)}\,ds = 2\pi i \sum_{0<\gamma<T} e^{\rho_\chi z} \tag{3.94}$$

since $L(s, \chi_1)$ has no zeros on the lines $\Re(s) = 0$, $\Re(s) = 1$ except possibly at $s = 0$. From (3.87), one gets

$$\frac{L'(s,\chi)}{L(s,\chi)} = \frac{L'(s,\chi_1)}{L(s,\chi_1)} + \sum_{p|q} \frac{\chi_1(p)\log p}{p^s - \chi_1(p)}. \tag{3.95}$$

Using (3.8), (3.93), (3.95) and $|p^{\sigma+iT} - \chi_1(p)| \geq |p^{iT} - \chi_1(p)|$ for $\sigma \geq 0$, we easily see that

$$\left| \int_{1+iT}^{iT} e^{sz} \frac{L'(s,\chi)}{L(s,\chi)}\, ds \right| = O_q\left(e^{x-Ty} (\log^2 T + T^{C(\log q)^2 \log\log q}) \right). \tag{3.96}$$

Let $C_{T,\chi}^{\times}$ be the remaining part of $C_{T,\chi}$ with the same orientation after deleting the open segment from $1+iT$ to iT. Integrating by parts, we see that

$$\begin{aligned} &\int_{C_{T,\chi}^{\times}} e^{sz} \frac{L'(s,\chi)}{L(s,\chi)}\, ds \\ &\quad = e^{(1+iT)z} \log L(1+iT,\chi) - e^{iTz} \log L(iT,\chi) - z \int_{C_{T,\chi}^{\times}} e^{sz} \log L(s,\chi)\, ds. \end{aligned} \tag{3.97}$$

Thus by (3.94) and (3.97),

$$\begin{aligned} 2\pi i \sum_{0<\gamma<T} e^{\rho_\chi z} &= \int_{1+iT}^{iT} e^{sz} \frac{L'(s,\chi)}{L(s,\chi)}\, ds - z \int_{C_{T,\chi}^{\times}} e^{sz} \log L(s,\chi)\, ds \\ &\quad + e^{(1+iT)z} \log L(1+iT,\chi) - e^{iTz} \log L(iT,\chi) \end{aligned} \tag{3.98}$$

follows. Here $\log L(s,\chi)$ is the unique branch of the complex logarithm defined on the simply connected region

$$\begin{aligned} &\mathbb{C} - (-\infty,\beta] - \left[\frac{1}{2} + i\gamma_{0,\chi}, \frac{1}{2} + i\infty\right) - \left[\frac{1}{2} - i\gamma_{1,\chi}, \frac{1}{2} - i\infty\right) \\ &\quad - \bigcup_{p|q, k\in\mathbb{Z}} \left(-\infty + \frac{2\pi i}{\log p}(k + r_p), \frac{2\pi i}{\log p}(k + r_p)\right] \end{aligned}$$

(where β is the largest real zero of $L(s,\chi)$, possibly a Siegel zero and $\frac{1}{2} + i\gamma_{0,\chi}$, $\frac{1}{2} - i\gamma_{1,\chi}$ are the first critical zeros above and below the real axis, respectively) and satisfies

$$\log L(\sigma,\chi) = \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)}{n^\sigma \log n} \to 0$$

as $\sigma \to \infty$. If $\sigma + it$ is a zero, then the limiting conventions for $\arg L(\sigma+it,\chi)$ in the case of a vertical or a horizontal branch cut are defined similarly as in $\arg L(\sigma+it,\chi_0)$. Again the classical branch requires horizontal cuts at each critical zero. But for all points of the form $\sigma + it$ with $0 \le \sigma \le 1/2$ and $t \ge \gamma_{0,\chi}$, we have to take into account a change in the argument between the two branches which is $O((t+4)\log q(t+4))$. This being said, we can still use the classical branch of $\log L(s,\chi)$ in the rest of the argument. Let us write

$$\log L(iT,\chi) = \log|L(iT,\chi)| + i\arg L(iT,\chi). \tag{3.99}$$

We also have

$$\log|L(iT,\chi)| = \log|L(iT,\chi_1)| + \sum_{p|q}\log\left|1 - \frac{\chi_1(p)}{p^{iT}}\right|. \tag{3.100}$$

Since χ_1 is primitive, using (3.8) and (3.93), one obtains that

$$\log|L(iT,\chi_1)| = \log|L(2+iT,\chi_1)| + \Re\int_{2+iT}^{iT}\frac{L'(s,\chi_1)}{L(s,\chi_1)}\,ds \ll \log^2 qT, \tag{3.101}$$

$$\sum_{p|q}\log\left|1 - \frac{\chi_1(p)}{p^{iT}}\right| \ll_q \log T. \tag{3.102}$$

Collecting (3.100)–(3.102),

$$\log|L(iT,\chi)| = O_q(\log^2 T) \tag{3.103}$$

follows. Moreover, we have

$$\arg L(iT,\chi) = O(1) + \Im\int_{2+iT}^{iT}\frac{L'(s,\chi_1)}{L(s,\chi_1)}\,ds + \Im\sum_{p|q}\log p\int_{2+iT}^{iT}\frac{\chi_1(p)}{p^s - \chi_1(p)}\,ds \tag{3.104}$$

and using

$$\Im\int_{2+iT}^{iT}\frac{L'(s,\chi_1)}{L(s,\chi_1)}\,ds \ll_q \log T, \quad \Im\sum_{p|q}\log p\int_{2+iT}^{iT}\frac{\chi_1(p)}{p^s - \chi_1(p)}\,ds \ll_q T^{C(\log q)^2\log\log q},$$

together with the $O((T+4)\log q(T+4))$ change in the argument between the two branches, one infers from (3.104) that

$$\arg L(iT,\chi) = O_q(T^{C(\log q)^2\log\log q}). \tag{3.105}$$

From (3.99), (3.103) and (3.105), one sees that

$$|e^{iTz}\log L(iT,\chi)| = e^{x-Ty}|\log L(iT,\chi)| \to 0$$

as $T \to \infty$. Similarly

$$|e^{(1+iT)z} \log L(1+iT,\chi)| \to 0$$

as $T \to \infty$. Let $C^\times$ be any of the indentations on the path of integration. Consider

$$\int\limits_{C^\times} e^{sz} \log L(s,\chi)\, ds = \int\limits_{C^\times} e^{sz} \log |L(s,\chi)|\, ds + i \int\limits_{C^\times} e^{sz} \arg L(s,\chi)\, ds \tag{3.106}$$

and

$$\int\limits_{C^\times} e^{sz} \log |L(s,\chi)|\, ds = \int\limits_{C^\times} e^{sz} \log |L(s,\chi_1)|\, ds + \sum_{p|q} \int\limits_{C^\times} e^{sz} \log \left|1 - \frac{\chi_1(p)}{p^s}\right| ds. \tag{3.107}$$

The center of $C^\times$ is either a $z_n = \frac{2\pi i}{\log p}(k+r_p)$ or one of 0, β, $1-\beta$. Note that $\log |L(s,\chi_1)|$ is continuous and uniformly bounded near z_n (and also near 0, if $L(0,\chi_1) \neq 0$). Otherwise, 0, β, $1-\beta$ are simple zeros of $L(s,\chi_1)$ and one may write

$$L(s,\chi_1) = \begin{cases} sF_1(s) \\ (s-\beta)F_2(s) \\ (s-(1-\beta))F_3(s), \end{cases}$$

where $F_j(s)$ are analytic functions with $F_1(0) \neq 0$, $F_2(\beta) \neq 0$, $F_3(1-\beta) \neq 0$. It follows that

$$\log |L(s,\chi_1)| = \begin{cases} \log |s| + O(1) & \text{if } s \text{ is close to } 0 \\ \log |s-\beta| + O(1) & \text{if } s \text{ is close to } \beta \\ \log |s-(1-\beta)| + O(1) & \text{if } s \text{ is close to } 1-\beta. \end{cases}$$

In all cases, we see that

$$\int\limits_{C^\times} e^{sz} \log |L(s,\chi_1)|\, ds \to 0$$

as radius of $C^\times$ tends to 0. Similarly, each of $1 - \chi_1(p)p^{-s}$ is either continuous and uniformly bounded near the center of $C^\times$ or has a simple zero at z_n so that

$$\sum_{p|q} \int\limits_{C^\times} e^{sz} \log \left|1 - \frac{\chi_1(p)}{p^s}\right| ds \to 0$$

as radius of $C^\times$ tends to 0. Hence by (3.107),

$$\int_{C^\times} e^{sz} \log |L(s,\chi)|\, ds \to 0 \tag{3.108}$$

as radius of $C^\times$ tends to 0. If $s = \sigma + it \in C^\times$, then

$$\arg L(s,\chi) = O(1) + \Im \int_{2+it}^{\sigma+it} \frac{L'(s,\chi_1)}{L(s,\chi_1)}\, ds + \sum_{p|q} \Im \int_{2+it}^{\sigma+it} \frac{\chi_1(p)\log p}{p^s - \chi_1(p)}\, ds.$$

As χ_1 is primitive, we have

$$\Im \int_{2+it}^{\sigma+it} \frac{L'(s,\chi_1)}{L(s,\chi_1)}\, ds = O(\log q(|t|+4)) = O_q(\log(|z|+4)),$$

where z is the center of $C^\times$ and the radius of $C^\times$ is small enough. Writing

$$\frac{\chi_1(p)\log p}{p^s - \chi_1(p)} = \frac{1}{s - z_n} + F(s)$$

if necessary, where $F(s)$ is analytic and bounded near z_n, one easily obtains that $\arg L(s,\chi) = O_q((|z|+4)\log(|z|+4))$ and

$$\int_{C^\times} e^{sz} \arg L(s,\chi)\, ds \to 0 \tag{3.109}$$

as radius of $C^\times$ tends to 0. Gathering (3.106), (3.108) and (3.109), one is justified to assume that the path of integration consists of straight lines. Letting $T \to \infty$,

$$\begin{aligned} 2\pi i \sum_{\gamma>0} e^{\rho_\chi z} = {} & z \int_{CBA} e^{sz} \log L(s,\chi)\, ds - z \int_{CD} e^{sz} \log L(s,\chi)\, ds \\ & - z \int_{DE} e^{sz} \log L(s,\chi)\, ds, \end{aligned} \tag{3.110}$$

where

$$A = +i\infty, \quad B = +i\epsilon, \quad C = \epsilon$$
$$D = 1, \quad E = 1 + i\infty.$$

To make use of symmetry for $L(s,\chi)$, especially for the integral over CBA appearing on the right hand side of (3.110), let us note that by the functional equation for $L(s,\chi_1)$,

$$L(s,\chi) = \epsilon(\chi_1)2^s\pi^{s-1}q_1^{\frac{1}{2}-s}\sin\frac{(s+a)\pi}{2}\Gamma(1-s)L(1-s,\overline{\chi})$$
$$\times\prod_{p|q}\left(1-\frac{\chi_1(p)}{p^s}\right)\left(1-\frac{\overline{\chi_1}(p)}{p^{1-s}}\right)^{-1} \tag{3.111}$$

holds for all s, where $|\epsilon(\chi_1)| = 1$ and $a = 0$ or $a = 1$ in (3.111) according to χ_1 is even or odd, respectively. As we will take the logarithm of both sides of (3.111), the initial difference between the arguments has to be taken into account. For $0 < s < 1$, we know that

$$\arg L(s,\chi_1) = O(\log q), \quad \arg L(1-s,\chi_1) = O(\log q). \tag{3.112}$$

If $\chi_1(p) = 1$, then

$$\arg\left(1-\frac{\chi_1(p)}{p^s}\right) = 0 \tag{3.113}$$

for $0 < s < 1$ and if $\chi_1(p) \neq 1$, then

$$\arg\left(1-\frac{\chi_1(p)}{p^s}\right) = O(1) \tag{3.114}$$

for $0 < s < 1$. As a result of (3.113) and (3.114),

$$\arg\prod_{p|q}\left(1-\frac{\chi_1(p)}{p^s}\right)\left(1-\frac{\overline{\chi_1}(p)}{p^{1-s}}\right)^{-1} = O(\omega(q)) = O\left(\frac{\log q}{\log\log q}\right) \tag{3.115}$$

follows for $0 < s < 1$. Since $\arg\epsilon(\chi_1) = O(1)$ and the arguments of all the other terms on the right hand side of (3.111) are 0 for $0 < s < 1$, we see from (3.111), (3.112) and (3.115) that

$$z\int_{CBA} e^{sz}\log L(s,\chi)\,ds = z\int_{CBA} e^{sz}\big(\log\epsilon(\chi_1) + s\log(2\pi/q_1)$$
$$+\log(\sqrt{q_1}/\pi) + \log\sin\frac{(s+a)\pi}{2} + \log\Gamma(1-s) + \log L(1-s,\overline{\chi})$$
$$+\sum_{p|q}\log\left(1-\frac{\chi_1(p)}{p^s}\right) - \sum_{p|q}\log\left(1-\frac{\overline{\chi_1}(p)}{p^{1-s}}\right) + iA(q,\chi)\big)\,ds, \tag{3.116}$$

where $A(q,\chi) = O(\log q)$ is the initial difference in the arguments of the logarithms of both sides of (3.111) when $s = \epsilon$. Letting $\epsilon \to 0$ in (3.110) and (3.116), one has

$$z\left(\log\epsilon(\chi_1)+\log(\sqrt{q_1}/\pi)+iA(q,\chi)\right)\int_0^{i\infty} e^{sz}\,ds = -\left(\log\epsilon(\chi_1)+\log(\sqrt{q_1}/\pi)+iA(q,\chi)\right) \tag{3.117}$$

and

$$z\log(2\pi/q_1)\int_0^{i\infty} se^{sz}\,ds = \frac{1}{z}\log(2\pi/q_1). \tag{3.118}$$

Moreover, when $a=1$

$$\int_0^{i\infty} e^{sz}\log\sin\frac{(s+1)\pi}{2}\,ds = e^{-z}\int_1^{1+i\infty} e^{sz}\log\sin\left(\frac{s\pi}{2}\right)\,ds. \tag{3.119}$$

For any positive parameter M, one obtains that

$$\begin{aligned}\int_1^{1+iM} e^{sz}\log\sin\left(\frac{s\pi}{2}\right)\,ds &= \int_{iM}^{1+iM} e^{sz}\log\sin\left(\frac{s\pi}{2}\right)\,ds \\ &\quad + \int_0^{iM} e^{sz}\log\sin\left(\frac{s\pi}{2}\right)\,ds - \int_0^{1} e^{sz}\log\sin\left(\frac{s\pi}{2}\right)\,ds. \end{aligned}\tag{3.120}$$

But when $s=\sigma+iM$ with $0\le\sigma\le 1$, the estimate

$$\left|\log\sin\left(\frac{s\pi}{2}\right)\right| = \left|\log\left(\frac{e^{\frac{\pi}{2}(-M+i\sigma)}-e^{\frac{\pi}{2}(M-i\sigma)}}{2i}\right)\right| = O(M) \tag{3.121}$$

holds for all large M. Letting $M\to\infty$ in (3.120) and using (3.121), we see from (3.119) that

$$\begin{aligned}\int_0^{i\infty} e^{sz}\log\sin\frac{(s+1)\pi}{2}\,ds &= \int_0^{i\infty} e^{(s-1)z}\log\sin\left(\frac{s\pi}{2}\right)\,ds \\ &\quad - \int_0^{1} e^{(s-1)z}\log\sin\left(\frac{s\pi}{2}\right)\,ds. \end{aligned}\tag{3.122}$$

Next consider

$$\int_0^{i\infty} e^{sz}\log\left(1-\frac{\chi_1(p)}{p^s}\right)\,ds = \int_0^{iT} e^{sz}\log\left(1-\frac{\chi_1(p)}{p^s}\right)\,ds + O(e^{-yT}). \tag{3.123}$$

Using

$$\log\left(1-\frac{\chi_1(p)}{p^s}\right)=-\sum_{m\geq 1}\frac{\chi_1(p^m)}{mp^{ms}}$$

for all $s=it$ with $t\neq \frac{2\pi i}{\log p}(k+r_p)$ and the facts that

$$\Re\log\left(1-\frac{\chi_1(p)}{p^s}\right)=-\sum_{m\geq 1}\frac{\cos m(t\log p-2\pi r_p)}{m},$$
$$\Im\log\left(1-\frac{\chi_1(p)}{p^s}\right)=\sum_{m\geq 1}\frac{\sin m(t\log p-2\pi r_p)}{m}$$

are both boundedly convergent almost everywhere on $[0,T]$, one justifies that interchange is permissible and

$$\int_0^{iT} e^{sz}\log\left(1-\frac{\chi_1(p)}{p^s}\right)\,ds=\sum_{m\geq 1}\left(\frac{\chi_1(p^m)}{m(z-\log p^m)}-\frac{\chi_1(p^m)e^{iT(z-\log p^m)}}{m(z-\log p^m)}\right) \tag{3.124}$$

holds. Letting $T\to\infty$ in (3.124), one obtains from (3.123) that

$$\int_0^{i\infty} e^{sz}\log\left(1-\frac{\chi_1(p)}{p^s}\right)\,ds=\sum_{m=1}^{\infty}\frac{\chi_1(p^m)}{m(z-\log p^m)}. \tag{3.125}$$

Similarly,

$$\int_0^{i\infty} e^{sz}\log\left(1-\frac{\overline{\chi_1}(p)}{p^{1-s}}\right)\,ds=\sum_{m=1}^{\infty}\frac{\overline{\chi_1}(p^m)}{mp^m(z+\log p^m)} \tag{3.126}$$

follows. Finally, as interchange was shown to be valid above for all characters, one easily derives that

$$-z\int_1^{1+i\infty} e^{sz}\log L(s,\chi)=ze^z\sum_{m,p}\frac{\chi(p^m)}{mp^m(z-\log p^m)} \tag{3.127}$$

and

$$z\int_0^{i\infty} e^{sz}\log L(1-s,\overline{\chi})=-z\sum_{m,p}\frac{\overline{\chi}(p^m)}{mp^m(z+\log p^m)}. \tag{3.128}$$

Therefore, assembling (3.116), (3.117), (3.118), (3.125)–(3.128), the following formula which is analogous to (3.86) is verified.

$$\begin{aligned}2\pi i V(z,\chi) &= ze^{z}\sum_{m,p}\frac{\chi(p^m)}{mp^m(z-\log p^m)} - z\sum_{m,p}\frac{\overline{\chi}(p^m)}{mp^m(z+\log p^m)}\\&\quad + z\sum_{p|q}\sum_{m=1}^{\infty}\frac{\chi_1(p^m)}{m(z-\log p^m)} - z\sum_{p|q}\sum_{m=1}^{\infty}\frac{\overline{\chi_1}(p^m)}{mp^m(z+\log p^m)}\\&\quad - z\int_0^1 e^{sz}\log L(s,\chi)\,ds + z\int_0^{i\infty} e^{sz}\log\sin\frac{(s+a)\pi}{2}\,ds\\&\quad + \frac{1}{z}\left(U+\log\frac{2\pi}{q_1}\right) - \left(\log\epsilon(\chi_1)+\log\frac{\sqrt{q_1}}{\pi}+iA(q,\chi)\right)\\&\quad - \frac{1}{z}\int_0^{\infty}\frac{t}{e^t-1}\cdot\frac{dt}{t+z}. \end{aligned} \tag{3.129}$$

For the fusion of (3.86) and (3.129), some preliminary estimations are necessary. First note that

$$\left|\frac{1}{e^z}\sum_{p|q}\sum_{m=1}^{\infty}\frac{\chi_1(p^m)}{m(z-\log p^m)}\right| \le e^{-x}\sum_{p|q}\sum_{m=1}^{\infty}\frac{1}{m\sqrt{(x-\log p^m)^2+y^2}}. \tag{3.130}$$

To estimate the right hand side of (3.130) further, let m_p be the closest integer to $x/\log p$ for each $p \mid q$. Then the contribution of m_p for $p \mid q$ is

$$e^{-x}\sum_{p|q}\frac{1}{m_p\sqrt{(x-\log p^m)^2+y^2}} \ll \frac{e^{-x}}{xy}\sum_{p|q}\log p \ll \frac{e^{-x}\log q}{xy}. \tag{3.131}$$

Moreover, we have

$$\begin{aligned}&e^{-x}\sum_{p|q}\sum_{\substack{m<\frac{x}{\log p}\\ m\neq m_p}}\frac{1}{m\sqrt{(x-\log p^m)^2+y^2}}\\&\le e^{-x}\sum_{p|q}\log p\sum_{\substack{m<\frac{x}{\log p}\\ m\neq m_p}}\frac{1}{m\log p(x-\log p^m)}\\&= \frac{e^{-x}}{x}\sum_{p|q}\log p\sum_{\substack{m<\frac{x}{\log p}\\ m\neq m_p}}\left(\frac{1}{m\log p}+\frac{1}{x-m\log p}\right)\\&\ll \frac{e^{-x}\log q\log x}{x}. \end{aligned} \tag{3.132}$$

Similarly,

$$e^{-x} \sum_{p|q} \sum_{\substack{m > \frac{x}{-\frac{1}{2} + \log p} \\ m \neq m_p}} \frac{1}{m\sqrt{(x - \log p^m)^2 + y^2}} \ll e^{-x} \log q \tag{3.133}$$

follows as $|x - m \log p| > \frac{1}{2m}$ in the range of summation over m. Lastly, we consider

$$\begin{aligned} & e^{-x} \sum_{p|q} \sum_{\substack{\frac{x}{\log p} < m < \frac{x}{-\frac{1}{2} + \log p} \\ m \neq m_p}} \frac{1}{m\sqrt{(x - \log p^m)^2 + y^2}} \\ & \leq \frac{e^{-x}}{x} \sum_{p|q} \log p \sum_{\substack{\frac{x}{\log p} < m < \frac{x}{-\frac{1}{2} + \log p} \\ m \neq m_p}} \left(\frac{1}{m \log p - x} - \frac{1}{m \log p} \right) \\ & \ll \frac{e^{-x} \log q \log x}{x}. \end{aligned} \tag{3.134}$$

As y will be taken as a very small function of x later, it is safe to infer from (3.130)–(3.134) that

$$\left| \frac{1}{e^z} \sum_{p|q} \sum_{m=1}^{\infty} \frac{\chi_1(p^m)}{m(z - \log p^m)} \right| \ll \frac{e^{-x} \log q}{xy}. \tag{3.135}$$

Clearly,

$$\left| \frac{1}{e^z} \sum_{m,p} \frac{\overline{\chi}(p^m)}{mp^m(z + \log p^m)} \right| \ll e^{-x} \tag{3.136}$$

and

$$\left| \frac{1}{e^z} \sum_{p|q} \sum_{m=1}^{\infty} \frac{\overline{\chi_1}(p^m)}{mp^m(z + \log p^m)} \right| \ll e^{-x} \log q \tag{3.137}$$

hold with absolute implied constants. Using the well known estimate for the digamma function in the form

$$\Psi(z) \ll \log |z|$$

when z is in the first quadrant and $|z|$ tends to infinity, one gets from (3.67) that

$$\left| \frac{1}{e^z} \int_0^{i\infty} e^{sz} \log \sin\left(\frac{s\pi}{2}\right) ds \right| \ll e^{-x} \log |z| \ll e^{-x} \log x. \tag{3.138}$$

Since

$$\left|\int_0^1 e^{(s-1)z}\log\sin\left(\frac{s\pi}{2}\right)ds\right| \ll \int_0^1 |\log s|\,ds \ll 1,$$

(3.122) gives that

$$\left|\frac{1}{e^z}\int_0^{i\infty} e^{sz}\log\sin\frac{(s+1)\pi}{2}\,ds\right| \ll e^{-x}. \tag{3.139}$$

It is also clear that

$$\Bigg|\frac{1}{z^2e^z}\left(U+\log\frac{2\pi}{q_1}\right)-\frac{1}{ze^z}\left(\log\epsilon(\chi_1)+\log\frac{\sqrt{q_1}}{\pi}+iA(q,\chi)\right) - \frac{1}{z^2e^z}\int_0^\infty \frac{t}{e^t-1}\cdot\frac{dt}{t+z}\Bigg| \ll e^{-x}\log q. \tag{3.140}$$

Corresponding terms on the right hand side of (3.86) can be estimated similarly. Combining (3.86), (3.129) and (3.135)–(3.140), one gets

$$\sum_{m,p}\frac{\chi(p^m)}{mp^m(z-\log p^m)} = \frac{2\pi i}{ze^z}\sum_{\gamma>0}e^{\rho_\chi z}+\frac{1}{e^z}\int_0^1 e^{sz}\log L(s,\chi)\,ds+R(z,q,\chi) \tag{3.141}$$

for any character χ modulo q, where

$$R(z,q,\chi)=O\left(\frac{e^{-x}\log q}{xy}\right). \tag{3.142}$$

Using the orthogonality of characters, (3.141) and (3.142) give that

$$\sum_{\substack{m,p\\ p^m\equiv a \pmod q}}\frac{\varphi(q)}{mp^m(z-\log p^m)}$$

$$=\frac{2\pi i}{ze^z}\sum_\chi \chi(\overline{a})\sum_{\gamma>0}e^{\rho_\chi z}+\frac{1}{e^z}\int_0^1 e^{sz}\sum_\chi \chi(\overline{a})\log L(s,\chi)\,ds+R(z,q), \tag{3.143}$$

with

$$R(z,q)=O\left(\frac{e^{-x}\varphi(q)\log q}{xy}\right). \tag{3.144}$$

Comparing imaginary parts of both sides of (3.143), the formula

$$-\sum_{\substack{m,p\\ p^m\equiv a \pmod q}} \frac{\varphi(q)y}{mp^m[(x-\log p^m)^2+y^2]}$$

$$=\Re\frac{2\pi}{ze^z}\sum_{\chi}\chi(\overline{a})\sum_{\gamma>0}e^{\rho_\chi z}+\Im\frac{1}{e^z}\int_0^1 e^{sz}\sum_{\chi}\chi(\overline{a})\log L(s,\chi)\,ds+\Im R(z,q) \qquad (3.145)$$

holds with

$$\Im R(z,q)=O\left(\frac{e^{-x}\varphi(q)\log q}{xy}\right) \qquad (3.146)$$

by (3.144). Let us consider the function

$$F(s,a)=\prod_{\chi}L(s,\chi)^{\chi(\overline{a})}.$$

In particular, $F(s,a)$ can be recovered by the unique branch of its logarithm which is given as

$$\log F(s,a)=\sum_{\chi}\chi(\overline{a})\log L(s,\chi)=\varphi(q)\sum_{\substack{n\geq 2\\ n\equiv a \pmod q}}\frac{\Lambda(n)}{n^s\log n} \qquad (3.147)$$

for $\Re(s)>1$. Thus $F(s,a)$ is analytic in a maximal simply connected open subset of $\mathbb{C}$ and it follows from (3.147) that

$$\frac{F'}{F}(\overline{s},a)=\overline{\frac{F'}{F}(s,a)} \qquad (3.148)$$

for all possible s. Our task is then to get a useful estimate for the quantity

$$\Im\frac{1}{e^z}\int_0^1 e^{sz}(\log|F(s,a)|+i\arg F(s,a))\,ds.$$

To begin with, we have

$$\log|F(s,a)|=\sum_{\chi}\left(\Re(\chi(\overline{a}))\log|L(s,\chi)|-\Im(\chi(\overline{a}))\arg L(s,\chi)\right). \qquad (3.149)$$

Using (3.149), one sees that

$$\Im \frac{1}{e^z} \int_0^1 e^{sz} \log |F(s,a)|\, ds = \sum_\chi \Re(\chi(\overline{a})) \Im \int_0^1 e^{(s-1)z} \log |L(s,\chi)|\, ds$$
$$- \sum_\chi \Im(\chi(\overline{a})) \Im \int_0^1 e^{(s-1)z} \arg L(s,\chi)\, ds. \tag{3.150}$$

Recalling that $z = x + iy$ with $x, y > 0$, one obtains

$$\left| \Im \int_0^1 e^{(s-1)z} \log |L(s,\chi)|\, ds \right| \le \left| \frac{\cos y}{e^x} \int_0^1 e^{sx} \sin(sy) \log |L(s,\chi)|\, ds \right|$$
$$+ \left| \frac{\sin y}{e^x} \int_0^1 e^{sx} \cos(sy) \log |L(s,\chi)|\, ds \right|. \tag{3.151}$$

Let us temporarily assume that χ is primitive. For all $\delta_1 > 0$ small enough, there are no critical zeros of $L(s,\chi)$ with ordinate δ_1 and the set of (finitely many) critical zeros with ordinates in $(\delta_1 - 1, \delta_1 + 1)$ is the same as the set of critical zeros with ordinates in $(-1, 1)$. Suppose now that $0 \le s \le 1$. As χ is primitive, using a well known result (see p. 102 of [6]), we may write

$$\frac{L'(s + i\delta_1, \chi)}{L(s + i\delta_1, \chi)} = \sum_\rho \frac{1}{s + i\delta_1 - \rho} + O(\log q), \tag{3.152}$$

where the finite sum in (3.152) is over the critical zeros with ordinates in $(-1, 1)$. Consequently, we see from (3.152) that

$$\log |L(s + i\delta_1, \chi)| = \log |L(1 + i\delta_1, \chi)| + \Re \int_1^s \frac{L'(t + i\delta_1, \chi)}{L(t + i\delta_1, \chi)}\, dt$$
$$= \log |L(1 + i\delta_1, \chi)| + \sum_\rho \log |s + i\delta_1 - \rho|$$
$$- \sum_\rho \log |1 + i\delta_1 - \rho| + O(\log q). \tag{3.153}$$

Letting $\delta_1 \to 0$ in (3.153), one obtains

$$\log |L(s,\chi)| = \log |L(1,\chi)| + \sum_\rho \log |s - \rho| - \sum_\rho \log |1 - \rho| + O(\log q) \tag{3.154}$$

for all $0 \le s \le 1$ not coinciding with a real zero of $L(s,\chi)$. Armed with (3.154), let us revisit the terms on the right hand side of (3.151). Precisely, one has

$$\begin{aligned}
&\frac{\cos y}{e^x}\int_0^1 e^{sx}\sin(sy)\log|L(s,\chi)|\,ds \\
&\quad = \frac{\cos y}{e^x}\int_0^1 e^{sx}\sin(sy)\sum_\rho(\log|s-\rho|-\log|1-\rho|)\,ds \\
&\qquad + \frac{\cos y}{e^x}\int_0^1 e^{sx}\sin(sy)(\log|L(1,\chi)|+O(\log q))\,ds. \qquad (3.155)
\end{aligned}$$

To control the right hand side of (3.155), note that if $\rho=\beta$ is a Siegel zero, then

$$1-\beta > \frac{c_1}{\sqrt{q}\log^2 q}$$

for some constant $c_1>0$ and for other zeros $\rho=\sigma+it$ with $t\in(-1,1)$, we have

$$1-\sigma > \frac{c_2}{\log q}$$

for some constant $c_2>0$. In any case we see that $\log|1-\rho|\ll\log q$ for all critical zeros with ordinates in $(-1,1)$. As the number of zeros with $t\in(-1,1)$ is $\ll\log q$, one easily obtains

$$\frac{\cos y}{e^x}\sum_\rho \log|1-\rho|\int_0^1 e^{sx}\sin(sy)\,ds \ll \frac{y\log^2 q}{x}. \qquad (3.156)$$

We know that

$$\frac{1}{\sqrt{q}} \ll |L(1,\chi)| \ll \log q. \qquad (3.157)$$

Using (3.157),

$$\frac{\cos y}{e^x}\int_0^1 e^{sx}\sin(sy)\log|L(1,\chi)|\,ds \ll \frac{y\log q}{x} \qquad (3.158)$$

follows. Clearly, we have

$$\frac{\cos y}{e^x}\int_0^1 e^{sx}\sin(sy)\log q\,ds \ll \frac{y\log q}{x}. \qquad (3.159)$$

Finally let us observe that

$$\frac{\cos y}{e^x}\sum_{\rho}\int_0^1 e^{sx}\sin(sy)\log|s-\rho|\,ds \ll \frac{y}{e^x}\sum_{\rho}\int_0^1 e^{sx}|\log|s-\sigma||\,ds$$

$$\ll y\sum_{\rho}\int_0^1 |\log|s-\sigma||\,ds \ll y\log q. \tag{3.160}$$

Gathering (3.155)–(3.160), one gets

$$\frac{\cos y}{e^x}\int_0^1 e^{sx}\sin(sy)\log|L(s,\chi)|\,ds \ll y\log q + \frac{y\log^2 q}{x}. \tag{3.161}$$

We can show similarly that

$$\frac{\sin y}{e^x}\int_0^1 e^{sx}\cos(sy)\log|L(s,\chi)|\,ds \ll y\log q + \frac{y\log^2 q}{x}. \tag{3.162}$$

Thus (3.151), (3.161) and (3.162) give that

$$\left|\Im\int_0^1 e^{(s-1)z}\log|L(s,\chi)|\,ds\right| \ll y\log q + \frac{y\log^2 q}{x}. \tag{3.163}$$

Moreover writing

$$\left|\Im\int_0^1 e^{(s-1)z}\arg L(s,\chi)\,ds\right| \le \left|\frac{\cos y}{e^x}\int_0^1 e^{sx}\sin(sy)\arg L(s,\chi)\,ds\right|$$

$$+\left|\frac{\sin y}{e^x}\int_0^1 e^{sx}\cos(sy)\arg L(s,\chi)\,ds\right| \tag{3.164}$$

and using the fact that $\arg L(s,\chi) \ll \log q$ when χ is primitive, we deduce from (3.164) that

$$\left|\Im\int_0^1 e^{(s-1)z}\arg L(s,\chi)\,ds\right| \ll \frac{y\log q}{x}. \tag{3.165}$$

On the other hand, if χ is not a primitive character modulo q, then using (3.87), (3.113) and (3.114), we can easily see that (3.165) holds for all nonprincipal characters modulo q. Note that for any $p \mid q$,

$$\lim_{s\to 0}\frac{\log(1-p^{-s})}{\log s} = 1.$$

Then we see that

$$\frac{\cos y}{e^x}\int_0^1 e^{sx}\sin(sy)\log\left|1-\frac{\chi_1(p)}{p^s}\right| ds \ll y\int_0^1 \log\left(1-\frac{1}{p^s}\right) ds \ll y$$

and

$$\frac{\sin y}{e^x}\int_0^1 e^{sx}\cos(sy)\log\left|1-\frac{\chi_1(p)}{p^s}\right| ds \ll y,$$

where χ_1 is the primitive character inducing χ. Therefore, (3.163) holds for all nonprincipal characters modulo q as well. In case when $\chi = \chi_0$ is the principal character modulo q, one can similarly show that (3.163) and (3.165) continue to hold (which is even easier as no Siegel zeros enter into the argument) by considering logarithmic singularities of $L(s,\chi_0)$ at $s = 0$ and $s = 1$ arising from the finite product over $p \mid q$ and from the Riemann zeta function, respectively. Assembling (3.150), (3.163) and (3.165), one infers that

$$\Im\frac{1}{e^z}\int_0^1 e^{sz}\log|F(s,a)|\, ds \ll \varphi(q)\left(y\log q+\frac{y\log^2 q}{x}\right). \tag{3.166}$$

If $\beta > 1/2$ is a Siegel zero, then it is easy to see by a calculation as in (3.58) and (3.59) that

$$\arg F(s,a) = \begin{cases} -\pi & \text{if } \beta < s < 1 \\ 0 & \text{if } 1-\beta < s < \beta \\ \pi & \text{if } 0 < s < 1-\beta. \end{cases} \tag{3.167}$$

Consequently from (3.167), we see that

$$\begin{aligned} \Im\frac{i}{e^z}\int_0^1 e^{sz}\arg F(s,a)\, ds &= \Re\frac{1}{e^z}\int_0^1 e^{sz}\arg F(s,a)\, ds \\ &= \Re\frac{\pi}{z}(-1-e^{-z}+e^{-\beta z}+e^{-(1-\beta)z}) \\ &= -\Re\left(\frac{\pi}{z}\right)+O\left(\frac{e^{-x/2}}{x}\right)+O\left(\frac{e^{-(1-\beta)x}}{x}\right). \end{aligned} \tag{3.168}$$

By Siegel's estimate (see p. 127 of [6]), we know for $\theta > 0$ that

$$\frac{C(\theta)}{q^\theta} < 1-\beta,$$

where $C(\theta) > 0$ is an ineffective constant. In this way (3.168) becomes

$$\Im \frac{i}{e^z} \int_0^1 e^{sz} \arg F(s,a)\, ds = -\frac{\pi x}{x^2+y^2} + O\left(\frac{e^{-\frac{C(\theta)x}{q^\theta}}}{x}\right) + O\left(\frac{e^{-x/2}}{x}\right) \tag{3.169}$$

for all $\theta > 0$. Furthermore, if we assume that critical zeros of all Dirichlet L-functions have real part $1/2$, then

$$\Im \frac{i}{e^z} \int_0^1 e^{sz} \arg F(s,a)\, ds = -\frac{\pi x}{x^2+y^2} + O\left(\frac{e^{-x/2}}{x}\right) \tag{3.170}$$

follows. Collecting (3.145), (3.166), (3.169) and (3.170), one justifies the formula

$$\begin{aligned} &\sum_{\substack{m,p \\ p^m \equiv a \pmod q}} \frac{\varphi(q) y}{m p^m [(x-\log p^m)^2 + y^2]} \\ &= -\Re \frac{2\pi}{z e^z} \sum_\chi \chi(\overline{a}) \sum_{\gamma>0} e^{\rho_\chi z} + \frac{\pi x}{x^2+y^2} + O\left(\frac{e^{-\frac{C(\theta)x}{q^\theta}}}{x}\right) + O\left(\frac{e^{-x/2}}{x}\right) \\ &\quad + O\left(\varphi(q)\left(y\log q + \frac{y \log^2 q}{x}\right)\right) + O\left(\frac{e^{-x}\varphi(q)\log q}{xy}\right). \end{aligned} \tag{3.171}$$

But writing $\rho_\chi = 1/2 + i\gamma_\chi$ with $\gamma_\chi > 0$ and noting that

$$\frac{\pi x}{x^2+y^2} = \frac{\pi}{x} + O\left(\frac{y^2}{x^3}\right),$$

the following relation is immediate from (3.171).

$$\begin{aligned} &\left| \sum_{\substack{m,p \\ p^m \equiv a \pmod q}} \frac{y}{m p^m [(x-\log p^m)^2 + y^2]} - \frac{\pi}{\varphi(q) x} \right| \\ &\le \frac{2\pi e^{-x/2}}{\varphi(q)\sqrt{x^2+y^2}} \sum_\chi \sum_{\gamma_\chi>0} e^{-\gamma_\chi y} + O\left(\frac{y^2}{\varphi(q)x^3}\right) + O\left(\frac{e^{-\frac{C(\theta)x}{q^\theta}}}{\varphi(q)x}\right) + O\left(\frac{e^{-x/2}}{\varphi(q)x}\right) \\ &\quad + O\left(y\log q + \frac{y\log^2 q}{x}\right) + O\left(\frac{e^{-x}\log q}{xy}\right). \end{aligned} \tag{3.172}$$

The O-term involving θ in (3.172) has to be discarded if no Siegel zeros exist. To make (3.172) manageable, it suffices to study the behavior of

$$\sum_{\chi}\sum_{\gamma_\chi>0} e^{-\gamma_\chi y}$$

in terms of y. Then a judicious choice of y as a small function of q and x will prepare us for the minimization of the right hand side of (3.172). Let $N(T,\chi)$ be the number of nonreal critical zeros of $L(s,\chi)$ with ordinates in $(-T,T)$. It is known that (see p. 101 of [6])

$$N(T,\chi)=\begin{cases}\frac{T}{\pi}\log T+O(T\log q) & \text{if } T\geq 2\\ O(\log q) & \text{if } 0\leq T\leq 2.\end{cases}$$

In particular when χ is a real character, then the number of critical zeros with ordinates in $(0,T)$ is $\frac{1}{2}N(T,\chi)$. If χ is not a real character, then using the correspondence between the critical zeros of $L(s,\chi)$ and $L(s,\overline{\chi})$, we see that the total number of critical zeros of $L(s,\chi)$ and $L(s,\overline{\chi})$ with ordinates in $(0,T)$ is $N(T,\chi)$. Therefore, if χ is real, then we have

$$\begin{aligned}\sum_{0<\gamma_\chi<T} e^{-\gamma_\chi y} &= \frac{1}{2}\int_0^T e^{-ty}\,d(N(t,\chi))\\ &= \frac{1}{2}e^{-Ty}N(T,\chi)+\frac{y}{2}\int_0^T e^{-ty}N(t,\chi)\,dt. \qquad (3.173)\end{aligned}$$

As $e^{-Ty}N(T,\chi)\to 0$ when $T\to\infty$, one sees from (3.173) that

$$\sum_{\gamma_\chi>0} e^{-\gamma_\chi y}=\frac{y}{2}\int_0^2 e^{-ty}N(t,\chi)\,dt+\frac{y}{2}\int_2^\infty e^{-ty}N(t,\chi)\,dt. \qquad (3.174)$$

To proceed, we need to study the right hand side of (3.174). First we have

$$\frac{y}{2}\int_0^2 e^{-ty}N(t,\chi)\,dt \ll \log q(1-e^{-2y})\ll y\log q. \qquad (3.175)$$

Second we observe that

$$\frac{y}{2}\int_2^\infty e^{-ty}N(t,\chi)\,dt=\frac{y}{2\pi}\int_2^\infty te^{-ty}\log t\,dt+O\left(y\log q\int_2^\infty te^{-ty}\,dt\right). \qquad (3.176)$$

One easily calculates that

$$\frac{y}{2\pi}\int_{2}^{\infty} te^{-ty}\log t\,dt=\frac{\log\left(\frac{1}{y}\right)}{2\pi y}\left(\frac{1}{\log\left(\frac{1}{y}\right)}\int_{2y}^{\infty} te^{-t}\log t\,dt+\int_{2y}^{\infty} te^{-t}\,dt\right). \tag{3.177}$$

Obviously,

$$\frac{1}{\log\left(\frac{1}{y}\right)}\int_{2y}^{\infty} te^{-t}\log t\,dt\to 0 \quad \text{and} \quad \int_{2y}^{\infty} te^{-t}\,dt\to 1$$

as $y\to 0$. It follows from (3.174)–(3.177) that

$$\sum_{\gamma_\chi>0} e^{-\gamma_\chi y}=(1+o(1))\frac{\log\left(\frac{1}{y}\right)}{2\pi y}+O\left(\frac{\log q}{y}\right) \tag{3.178}$$

as $y\to 0$. Similarly,

$$\sum_{\gamma_\chi>0} e^{-\gamma_\chi y}+\sum_{\gamma_{\overline{\chi}}>0} e^{-\gamma_{\overline{\chi}} y}=(1+o(1))\frac{\log\left(\frac{1}{y}\right)}{\pi y}+O\left(\frac{\log q}{y}\right) \tag{3.179}$$

is obtained for each pair of complex characters $\chi\neq\overline{\chi}$. Collecting (3.178) and (3.179), one gets

$$\sum_{\chi}\sum_{\gamma_\chi>0} e^{-\gamma_\chi y}=(1+o(1))\frac{\varphi(q)}{2\pi y}\log\left(\frac{1}{y}\right)+O\left(\frac{\varphi(q)\log q}{y}\right) \tag{3.180}$$

as $y\to 0$. For the judicious choice of y that was alluded to above, we take $y=\varphi(q)xe^{-\frac{x}{2}}$. In the presence of Siegel zeros, let us assume that $q\le x^A$ for any given number $A>0$. Then using a small enough $\theta>0$ (in terms of A), one can satisfy

$$O\left(\frac{e^{-\frac{C(\theta)x}{q^\theta}}}{\varphi(q)x}\right)=o\left(\frac{1}{\varphi(q)x}\right) \quad \text{as } x\to\infty. \tag{3.181}$$

Moreover, we have

$$\begin{aligned}&O\left(\frac{y^2}{\varphi(q)x^3}\right), O\left(\frac{e^{-x/2}}{\varphi(q)x}\right),\\ &\quad O\left(y\log q+\frac{y\log^2 q}{x}\right), O\left(\frac{e^{-x}\log q}{xy}\right)=o\left(\frac{1}{\varphi(q)x}\right) \quad \text{as } x\to\infty\end{aligned} \tag{3.182}$$

whenever

$$q=o\left(\frac{e^{\frac{x}{4}}}{x^{\frac{3}{2}}}\right).$$

Since $\frac{\log q}{\log(1/y)} \to 0$ as $x \to \infty$ when $q \leq x^A$, one obtains from (3.172), (3.180)–(3.182) that

$$\sum_{\substack{m,p \\ p^m \equiv a \pmod q}} \frac{y}{mp^m[(x-\log p^m)^2 + y^2]} = \frac{\pi + \Theta(x)}{\varphi(q)x}, \tag{3.183}$$

where $\Theta(x)$ is a function satisfying $|\Theta(x)| < 1$ for all sufficiently large values of x. On the other hand, if no Siegel zeros exist, then (3.183) still holds when $q \leq e^{\lambda x}$ for some sufficiently small number $\lambda > 0$ (so that $\frac{\log q}{\log(1/y)}$ is small enough for all sufficiently large values of x). Let us remark that λ is an effective constant. It is now immediate from (3.183) that

$$\sum_{\substack{m,p \\ p^m \equiv a \pmod q}} \frac{y}{mp^m[(x-\log p^m)^2 + y^2]} > \frac{2}{\varphi(q)x} \tag{3.184}$$

for all sufficiently large values of x. Let us estimate the contribution of terms with $|x - \log p^m| \geq 1$ on the left hand side of (3.184). Precisely, consider

$$\sum_{\substack{m,p \\ p^m \equiv a \pmod q \\ |x-\log p^m| \geq 1}} \frac{y}{mp^m[(x-\log p^m)^2 + y^2]} \leq y \sum_{\substack{m,p \\ p^m \equiv a \pmod q \\ |x-\log p^m| \geq 1}} \frac{1}{mp^m|x-\log p^m|}. \tag{3.185}$$

Clearly, we have

$$\sum_{\substack{m,p \\ m \geq 2 \\ p^m \equiv a \pmod q \\ |x-\log p^m| \geq 1}} \frac{1}{mp^m|x-\log p^m|} = O(1). \tag{3.186}$$

Hence it suffices to look at

$$\sum_{\substack{p \\ |x-\log p| \geq 1}} \frac{1}{p|x-\log p|} \tag{3.187}$$

as a function of all large x. When $|x - \log p| \geq 1$, it is easy to see that

$$\frac{1}{|x-\log p|} \ll \frac{x}{\log p}.$$

Consequently, (3.187) is $\ll x$ and one obtains from (3.185), (3.186) that

$$\sum_{\substack{m,p \\ p^m \equiv a \pmod q \\ |x-\log p^m| \geq 1}} \frac{y}{mp^m[(x-\log p^m)^2 + y^2]} \ll yx. \tag{3.188}$$

Since $q \leq e^{\lambda x}$ for some sufficiently small number $\lambda > 0$, one gets $yx = o(1/\varphi(q)x)$ and (3.184), (3.188) give that

$$\sum_{\substack{m,p \\ p^m \equiv a \pmod{q} \\ |x - \log p^m| < 1}} \frac{y}{mp^m[(x - \log p^m)^2 + y^2]} > \frac{1.9}{\varphi(q)x} \tag{3.189}$$

for all sufficiently large x. To analyze the left hand side of (3.189) further, we decompose the interval $(x-1, x+1)$ to tiny subintervals of length y starting from the point x (the rightmost and the leftmost subintervals being possibly of length $< y$). We then look at the contribution of each such subinterval. For a nonnegative integer ν with $\nu y < 1$, let us first study sums of the type

$$\begin{aligned}\sum_{\substack{m,p \\ p^m \equiv a \pmod{q} \\ \nu y \leq |x - \log p^m| \leq (\nu+1)y}} \frac{1}{m} &= \sum_{\substack{p \equiv a \pmod{q} \\ e^{x+\nu y} \leq p \leq e^{x+(\nu+1)y}}} 1 + \sum_{\substack{p \equiv a \pmod{q} \\ e^{x-(\nu+1)y} \leq p \leq e^{x-\nu y}}} 1 \\ &+ \sum_{m \geq 2} \frac{1}{m} \left(\sum_{\substack{p^m \equiv a \pmod{q} \\ e^{\frac{x+\nu y}{m}} \leq p \leq e^{\frac{x+(\nu+1)y}{m}}}} 1 + \sum_{\substack{p^m \equiv a \pmod{q} \\ e^{\frac{x-(\nu+1)y}{m}} \leq p \leq e^{\frac{x-\nu y}{m}}}} 1 \right),\end{aligned} \tag{3.190}$$

where the sum over m in (3.190) is actually finite with $m \ll x$. Estimating trivially,

$$\sum_{m \geq 2} \frac{1}{m} \left(\sum_{\substack{p^m \equiv a \pmod{q} \\ e^{\frac{x+\nu y}{m}} \leq p \leq e^{\frac{x+(\nu+1)y}{m}}}} 1 + \sum_{\substack{p^m \equiv a \pmod{q} \\ e^{\frac{x-(\nu+1)y}{m}} \leq p \leq e^{\frac{x-\nu y}{m}}}} 1 \right) \ll e^{\frac{x}{2}} \tag{3.191}$$

holds. Moreover, by the Brun–Titchmarsh theorem, one has

$$\sum_{\substack{p \equiv a \pmod{q} \\ e^{x+\nu y} \leq p \leq e^{x+(\nu+1)y}}} 1 \leq 3 \frac{e^{x+\nu y}(e^y - 1)}{\varphi(q) \log\left(\frac{e^{x+\nu y}(e^y-1)}{q}\right)} \ll e^{\frac{x}{2}} \tag{3.192}$$

for all sufficiently large x. Similarly,

$$\sum_{\substack{p \equiv a \pmod{q} \\ e^{x-(\nu+1)y} \leq p \leq e^{x-\nu y}}} 1 \ll e^{\frac{x}{2}} \tag{3.193}$$

follows for all sufficiently large x. Gathering (3.190)–(3.193), one infers that

$$\sum_{\substack{m,p \\ p^m \equiv a \pmod{q} \\ \nu y \le |x - \log p^m| \le (\nu+1)y}} \frac{1}{m} \ll e^{\frac{x}{2}} \tag{3.194}$$

with an absolute and computable implied constant. Using now (3.194), it is easy to see that

$$\sum_{\substack{m,p \\ p^m \equiv a \pmod{q} \\ \nu y \le |x - \log p^m| < 1}} \frac{y}{mp^m[(x - \log p^m)^2 + y^2]} \ll \frac{1}{\varphi(q)x} \left(\frac{1}{\nu^2} + \frac{1}{(\nu+1)^2} + \dots \right), \tag{3.195}$$

where the implied constant is absolute. Therefore, fixing a large enough value of ν in (3.195), we obtain from (3.189) that

$$\sum_{\substack{m,p \\ p^m \equiv a \pmod{q} \\ |x - \log p^m| < \nu y}} \frac{y}{mp^m[(x - \log p^m)^2 + y^2]} > \frac{1.8}{\varphi(q)x} \tag{3.196}$$

for all sufficiently large x. As a result of (3.196),

$$\sum_{\substack{m,p \\ p^m \equiv a \pmod{q} \\ |x - \log p^m| < \nu y}} \frac{1}{m} > 1.8 \left(\frac{ye^{x-\nu y}}{\varphi(q)x} \right) > 1.7e^{\frac{x}{2}} \tag{3.197}$$

follows for all sufficiently large x. Let us define the function

$$f(x, q, a) := \sum_{\substack{m,p \\ p^m \equiv a \pmod{q} \\ p^m \le x}} \frac{1}{m}.$$

Then (3.197) can be reformulated as

$$f(e^{x+\nu y}, q, a) - f(e^{x-\nu y}, q, a) > 1.7e^{\frac{x}{2}} \tag{3.198}$$

for all sufficiently large x. Let us rewrite (3.198) in terms of a more customary way of representing short intervals. To this end, it is enough to replace x by $\log x$ in (3.198). Note that the conditions on q should be scaled in the same way. Namely, if there are Siegel zeros, then $q \le (\log x)^A$ for any given $A > 0$ and otherwise $q \le x^\lambda$ for some small enough $\lambda > 0$. In this way the interval $(e^{x-\nu y}, e^{x+\nu y}]$ is mapped to the interval

$$\left(xe^{-\frac{\nu\varphi(q)\log x}{\sqrt{x}}}, xe^{\frac{\nu\varphi(q)\log x}{\sqrt{x}}} \right]. \tag{3.199}$$

Next we show that the interval in (3.199) is a subset of the interval

$$(x - 2\nu\varphi(q)\sqrt{x}\log x, x + 2\nu\varphi(q)\sqrt{x}\log x]$$

for all sufficiently large x. For the right end points of the intervals, this amounts to checking the inequality

$$e^{\frac{\nu\varphi(q)\log x}{\sqrt{x}}} < 1 + 2\frac{\nu\varphi(q)\log x}{\sqrt{x}}$$

which is obviously true since $\frac{\nu\varphi(q)\log x}{\sqrt{x}} \to 0$ as $x \to \infty$. The verification for the left end points of the intervals can be done in the same way. Therefore, (3.198) becomes

$$f(x + 2\nu\varphi(q)\sqrt{x}\log x, q, a) - f(x - 2\nu\varphi(q)\sqrt{x}\log x, q, a) > 1.7\sqrt{x} \tag{3.200}$$

for all sufficiently large x. Using (3.200), we can find an absolute constant $c > 0$ (in terms of ν) such that

$$f(x + c\,\varphi(q)\sqrt{x}\log x, q, a) - f(x, q, a) > 1.7\sqrt{x}.$$

Finally observing

$$f(x, q, a) - \pi(x, q, a) \ll \frac{\sqrt{x}}{\log x},$$

one easily obtains

$$\pi(x + c\,\varphi(q)\sqrt{x}\log x, q, a) - \pi(x, q, a) > \sqrt{x} \tag{3.201}$$

for all sufficiently large x. It is possible to arrange an effective constant c that works for all x and q subject to $q \le x^{\lambda}$ but in the presence of Siegel zeros and $q \le (\log x)^A$, the choice of c_A in place of c is ineffective since we can not know precisely how large x should be to have (3.201). This completes the proof of Theorem 2.

Acknowledgments

The authors are grateful to the referee for a careful reading of the paper and constructive remarks which improved the presentation.

References

[1] E. Alkan, Nonvanishing of Fourier coefficients of modular forms, Proc. Amer. Math. Soc. 131 (2003) 1673–1680.
[2] E. Alkan, On the sizes of gaps in the Fourier expansion of modular forms, Canad. J. Math. 57 (2005) 449–470.
[3] A. Baker, Transcendental Number Theory, Cambridge Univ. Press, 1975.
[4] A. Baker, The theory of linear forms in logarithms, in: Transcendence Theory: Advances and Applications, Proc. Conf., Univ. Cambridge, Cambridge, 1976, Academic Press, London, 1977, pp. 1–27.

[5] H. Cramér, On the order of magnitude of the difference between consecutive prime numbers, Acta Arith. 2 (1936) 23–46.
[6] H. Davenport, Multiplicative Number Theory, third ed., Grad. Stud. Math., vol. 74, Springer-Verlag, New York, 2000.
[7] A.W. Dudek, On the Riemann hypothesis and the difference between primes, Int. J. Number Theory 3 (2015) 771–778.
[8] H.M. Edwards, Riemann's Zeta Function, Academic Press, New York, 1974.
[9] P.X. Gallagher, On the distribution of primes in short intervals, Mathematika 23 (1976) 4–9.
[10] E. Landau, Neuer Beweis des Primzahlsatzes und Beweis des Primidealsatzes, Math. Ann. 56 (1903) 645–670.
[11] A. Languasco, A. Perelli, A. Zaccagnini, Explicit relations between pair correlation of zeros and primes in short intervals, J. Math. Anal. Appl. 394 (2012) 761–771.
[12] A. Ledoan, A. Roy, A. Zaharescu, Zeros of partial sums of the Dedekind zeta function of a cyclotomic field, J. Number Theory 136 (2014) 118–133.
[13] K. Matomäki, On the distribution of $\mathfrak{B}$-free numbers and non-vanishing Fourier coefficients of cusp forms, Glasg. Math. J. 54 (2012) 381–397.
[14] J. Neukirch, Algebraic Number Theory, Grundlehren Math. Wiss. (A Series of Comprehensive Studies in Mathematics), vol. 322, Springer-Verlag, Berlin, 1999.
[15] B. Rousselet, Inégalités de type Brun–Titchmarsh en moyenne, in: Groupe de travail en théorie analytique et élémentaire des nombres, in: Publ. Math. Orsay, vol. 88/01, Univ. Paris XI, Orsay, 1988, pp. 91–123.
[16] A. Selberg, Old and new conjectures and results about a class of Dirichlet series, in: Collected Papers, vol. 2, Springer-Verlag, 1991, pp. 47–64.
[17] M.-T. Tsai, A. Zaharescu, On the distribution of algebraic primes in small regions, Manuscripta Math. 145 (2014) 111–123.
[18] M. Waldschmidt, A lower bound for linear forms in logarithms, Acta Arith. 37 (1980) 257–283.