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On the classification of quadratic forms over an integral domain of a global function field ☆

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ABSTRACT

Let C be a smooth projective curve defined over the finite field $\mathbb{F}_q$ (q is odd) and let $K = \mathbb{F}_q(C)$ be its function field. Any finite set S of closed points of C gives rise to an integral domain $\mathcal{O}_S := \mathbb{F}_q[C - S]$ in K. We show that given an $\mathcal{O}_S$-regular quadratic space (V, q) of rank $n \geq 3$, the set of genera in the proper classification of quadratic $\mathcal{O}_S$-spaces isomorphic to (V, q) in the flat or étale topology, is in $1:1$ correspondence with ${}_2\mathrm{Br}(\mathcal{O}_S)$, thus there are $2^{|S|-1}$ genera. If (V, q) is isotropic, then Pic $(\mathcal{O}_S)/2$ classifies the forms in the genus of (V, q). For $n \geq 5$, this is true for all genera, hence the full classification is via the abelian group $H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2)$.

1. Introduction

Let C be a projective algebraic curve defined over a finite field $\mathbb{F}_q$ (with q odd), assumed to be geometrically connected and smooth. Let $K = \mathbb{F}_q(C)$ be its function field and let Ω denote the set of all closed points of C. For any point $\mathfrak{p} \in \Omega$ let $v_{\mathfrak{p}}$ be the induced discrete valuation on K, $\hat{\mathcal{O}}_{\mathfrak{p}}$ the complete discrete valuation ring with respect to

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$v_{\mathfrak{p}}$ and $\hat{K}_{\mathfrak{p}}$ its fraction field. Any *Hasse set* of K, namely a non-empty finite set $S \subset \Omega$, gives rise to an integral domain of K called a *Hasse domain*:

$$\mathcal{O}_S := \{x \in K : v_{\mathfrak{p}}(x) \geq 0 \ \forall \mathfrak{p} \notin S\}.$$

This is a Dedekind domain, regular and one dimensional. Schemes defined over $\operatorname{Spec} \mathcal{O}_S$ are denoted by an underline, being omitted in the notation of their generic fibers.

As 2 is invertible in $\mathcal{O}_S$, the $\mathcal{O}_S$-group $\underline{\mu}_2 := \operatorname{Spec} \mathcal{O}_S[t]/(t^2-1)$ is smooth, whence applying étale cohomology to the Kummer sequence:

$$1 \to \underline{\mu}_2 \to \underline{\mathbb{G}}_m \xrightarrow{x \mapsto x^2} \underline{\mathbb{G}}_m \to 1$$

gives rise to the long exact sequence of abelian groups:

$$\begin{aligned} H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbb{G}}_m) &\xrightarrow{[\mathfrak{L}] \to [\mathfrak{L} \otimes_{\mathcal{O}_S} \mathfrak{L}]} H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbb{G}}_m) \to H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2) \to H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mathbb{G}}_m) \\ &\xrightarrow{[A] \to [A \otimes_{\mathcal{O}_S} A]} H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mathbb{G}}_m). \end{aligned}$$

Identifying $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbb{G}}_m)$ with $\operatorname{Pic}(\mathcal{O}_S)$ by Shapiro's lemma (cf. [SGA3, XXIV, Prop. 8.4]), and $H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mathbb{G}}_m)$ with the Brauer group $\operatorname{Br}(\mathcal{O}_S)$, classifying Azumaya $\mathcal{O}_S$-algebras (cf. [Mil, §2]), we deduce the short exact sequence:

$$1 \to \operatorname{Pic}(\mathcal{O}_S)/2 \xrightarrow{\partial} H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2) \xrightarrow{i^2_*} {}_2\operatorname{Br}(\mathcal{O}_S) \to 1, \tag{1.1}$$

in which the right non-trivial term is the 2-torsion part in $\operatorname{Br}(\mathcal{O}_S)$. We analyze some properties related to this sequence in Section 2, which will be used to classify regular quadratic $\mathcal{O}_S$-spaces.

Let (V, q) (not to be confused with $q = |\mathbb{F}_q|$) be a quadratic $\mathcal{O}_S$-space of rank $n \geq 3$, namely, V is a projective $\mathcal{O}_S$-module of rank n and $q : V \to \mathcal{O}_S$ is a 2-order homogeneous $\mathcal{O}_S$-form. Since 2 is a unit, q corresponds to the symmetric bilinear form $B_q : V \times V \to \mathcal{O}_S$ such that:

$$B_q(u, v) = q(u + v) - q(u) - q(v) \ \ \forall u, v \in V.$$

We assume it to be $\mathcal{O}_S$-*regular*, namely, the induced homomorphism $V \to V^\vee := \operatorname{Hom}(V, \mathcal{O}_S)$ is an isomorphism ([Knu, I. §3, 3.2]). Two quadratic $\mathcal{O}_S$-spaces (V, q) and (V', q') are *isomorphic* over an extension R of $\mathcal{O}_S$, if there exists an R-*isometry* between them, namely, an R-isomorphism $T : V' \otimes_{\mathcal{O}_S} R \cong V \otimes_{\mathcal{O}_S} R$ such that $q \circ T = q'$. The notation of the $\mathcal{O}_S$-isomorphism class $[(V, q)]$ is sometimes, when no ambiguity arises, shortened to $[q]$. The [proper] *genus* of (V, q) is the set of classes of all quadratic $\mathcal{O}_S$-spaces that are [properly, i.e., with det = 1 isomorphisms] isomorphic to (V, q) over K and over

$\hat{O}_{\mathfrak{p}}$ for any prime $\mathfrak{p} \notin S$. This [proper] genus bijects as a pointed-set with the *class set* $[\mathrm{Cl}_S(\underline{\mathbf{SO}}_V)]$, $\mathrm{Cl}_S(\underline{\mathbf{O}}_V)$.

The results generalize the ones in [Bit], in which S is assumed to contain only one (arbitrary) point $\infty \in \Omega$, thus giving rise to an affine curve $C^{\mathrm{af}} = C - \{\infty\}$, for which $\mathrm{Br}(\mathcal{O}_{\{\infty\}}) = 1$ (cf. [Bit, Lemma 3.3]). The quadratic $\mathcal{O}_{\{\infty\}}$-spaces that are locally properly isomorphic to (V, q) for the flat or the étale topology belong all to the same genus, and are classified by the abelian group $H^2_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{\mu}_2) \cong \mathrm{Pic}\ (\mathcal{O}_{\{\infty\}})/2$. Here we show more generally for any finite set S that, as $\mathrm{Cl}_S(\underline{\mathbf{SO}}_V)$ is the kernel of what we call the *relative* Witt-invariant $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V) \xrightarrow{w_V} {}_2\mathrm{Br}(\mathcal{O}_S)$, the latter abelian group bijects to the set of $2^{|S|-1}$ proper genera of (V, q) (Proposition 4.5).

Another consequence of passing to $|S| > 1$ is that $\mathcal{O}_S^\times \neq \mathbb{F}_q^\times$, whence $\mathcal{O}_S$-regularity imposed on (V, q), no longer guarantees its isotropy. Requiring (V, q) to be isotropic, we show that $\mathrm{Pic}\ (\mathcal{O}_S)/2$ still classifies the quadratic spaces in the genus $\mathrm{Cl}_S(\underline{\mathbf{O}}_V)$ being equal to proper genus in this case (Lemma 4.4), for any S (Theorem 4.6). In particular in case $n \geq 5$, in which all classes are isotropic, any proper genus in $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V)$ – corresponding as aforementioned to an element of ${}_2\mathrm{Br}(\mathcal{O}_S)$ – is isomorphic to $\mathrm{Pic}\ (\mathcal{O}_S)/2$, whence their disjoint union $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V)$ is isomorphic to the abelian group $H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2)$ (as for $|S| = 1$ and $n \geq 3$), fitting into the sequence (1.1) (Corollary 4.8).

In Section 5, we refer to the case in which V is split by a hyperbolic plane $H(L_0)$, and provide an isomorphism $\psi_V : \mathrm{Pic}\ (\mathcal{O}_S)/2 \to \mathrm{Cl}_S(\underline{\mathbf{O}}_V)$. In case C is an elliptic curve and $S = \{\infty\}$ where ∞ is $\mathbb{F}_q$-rational, an algorithm, producing explicitly representatives of classes in $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V)$, is given (1).

2. A classification of Azumaya algebras

A faithfully flat projective (right) $\mathcal{O}_S$-module A is an *Azumaya* $\mathcal{O}_S$-algebra if the map

$$A \otimes A^{\mathrm{op}} \to \mathrm{End}_{\mathcal{O}_S}(A) : \ a \otimes b^{\mathrm{op}} \mapsto (x \mapsto axb)$$

is an isomorphism. It is central, separable and finitely generated as an $\mathcal{O}_S$-module. Two Azumaya $\mathcal{O}_S$-algebras A, B are *Brauer equivalent* if there exist faithfully projective modules P, Q such that:

$$A \otimes \mathrm{End}_{\mathcal{O}_S}(P) \cong B \otimes \mathrm{End}_{\mathcal{O}_S}(Q).$$

The tensor product induces the structure of an abelian group $\mathrm{Br}(\mathcal{O}_S)$ on the equivalence classes, in which the neutral element is $[\mathcal{O}_S]$ and the inverse of $[A]$ is $[A^{\mathrm{op}}]$ (cf. [Knu, III. 5.1 and 5.3]).

Let $V \cong \mathcal{O}_S^2$. Consider the following exact and commutative diagram of smooth $\mathcal{O}_S$-groups:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \underline{\mu}_2 \arrow[r] \arrow[d] & \underline{\mathbf{SL}}(V) \arrow[r] \arrow[d] & \underline{\mathbf{PGL}}(V) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \underline{\mathbb{G}}_m \arrow[r] \arrow[d, "x\mapsto x^2"] & \underline{\mathbf{GL}}(V) \arrow[r, "\pi"] \arrow[d, "\det"] & \underline{\mathbf{PGL}}(V) \arrow[r] & 1 \\
 & \underline{\mathbb{G}}_m \arrow[r] \arrow[d] & \underline{\mathbb{G}}_m \arrow[d] & & \\
 & 1 & 1. & &
\end{tikzcd} \tag{2.1}$$

The generalization of the Skolem–Noether Theorem to unital commutative rings, applied to the Azumaya $\mathcal{O}_S$-algebra $A = \mathrm{End}_{\mathcal{O}_S}(V)$, is the exact sequence of groups (see [Knu, III. 5.2.1]):

$$1 \to \mathcal{O}_S^\times \to A^\times \to \mathrm{Aut}_{\mathcal{O}_S}(A) \to \mathrm{Pic}\ (\mathcal{O}_S).$$

This sequence induces by sheafification a short exact sequence of sheaves in the étale topology (cf. [Knu, p. 145]):

$$1 \to \underline{\mathbb{G}}_m \to \underline{\mathbf{GL}}(V) \to \underline{\mathrm{Aut}}(\mathrm{End}_{\mathcal{O}_S}(V)) \to 1$$

from which we see that: $\underline{\mathbf{PGL}}(V) = \underline{\mathrm{Aut}}(\mathrm{End}_{\mathcal{O}_S}(V))$. In this interpretation, étale cohomology applied to the diagram (2.1), plus the sequence (1.1), give rise to the exact and commutative diagram:

$$\begin{tikzcd}
 & & \mathrm{Pic}\ (\mathcal{O}_S) \arrow[d, "\partial"] \\
H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SL}}(V)) \arrow[r] \arrow[d] & H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{PGL}}(V)) \arrow[r, "\partial^1"] \arrow[d, equal] & H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2) \arrow[d] \\
H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathrm{Aut}}(V)) \arrow[r, "\pi_*"] \arrow[d, "\Delta=\det_*"] & H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathrm{Aut}}(\mathrm{End}_{\mathcal{O}_S}(V))) \arrow[r] & \mathrm{Br}(\mathcal{O}_S) \\
\mathrm{Pic}\ (\mathcal{O}_S) & &
\end{tikzcd} \tag{2.2}$$

in which $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathrm{Aut}}(V))$ classifies twisted forms of V in the étale topology, while its image in $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathrm{Aut}}(\mathrm{End}_{\mathcal{O}_S}(V))$ classifies these $\mathcal{O}_S$-modules up to scaling by an $\mathcal{O}_S$-line, i.e., by an invertible $\mathcal{O}_S$-module. Explicitly, $\pi_* : [P] \mapsto [\mathrm{End}_{\mathcal{O}_S}(P)]$ (cf. [Knu, III. 5.2.4]).

Corollary 2.1. *In diagram (2.2): $\partial([L]) = \partial^1([End_{\mathcal{O}_S}(\mathcal{O}_S \oplus L)])$.*

Proof. By chasing diagram (2.2) we may deduce the following reduced one:

$$\begin{tikzcd}
 & \mathrm{Pic}\,(\mathcal{O}_S)/2 \arrow[d, hook, "\partial"] \\
H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathrm{Aut}}(V)) \arrow[ur, "\Delta"] \arrow[r, "\partial^1 \circ \pi_*"] \arrow[dr, "0"'] & H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2) \arrow[d] \\
 & {}_2\mathrm{Br}(\mathcal{O}_S)
\end{tikzcd} \tag{2.3}$$

which shows that: $\partial([L]) = \partial(\Delta([\mathcal{O}_S \oplus L])) = \partial^1(\pi_*([\mathcal{O}_S \oplus L])) = \partial^1([\mathrm{End}_{\mathcal{O}_S}(\mathcal{O}_S \oplus L)])$. $\square$

Lemma 2.2. $|{}_2\mathrm{Br}(\mathcal{O}_S)| = 2^{|S|-1}$.

Proof. Let $r_\mathfrak{p} : \mathrm{Br}(K) \to H^1(k_\mathfrak{p}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$ be the residue map at a prime $\mathfrak{p}$. The ramification map $a := \oplus_\mathfrak{p} r_\mathfrak{p}$ yields the exact sequence from Class Field Theory (see [GS, Theorem 6.5.1]):

$$1 \to \mathrm{Br}(K) \xrightarrow{a} \bigoplus_{\mathfrak{p}} \mathbb{Q}/\mathbb{Z} \xrightarrow{\sum_\mathfrak{p} \mathrm{Cor}_\mathfrak{p}} \mathbb{Q}/\mathbb{Z} \to 1 \tag{2.4}$$

in which the corestriction map $\mathrm{Cor}_\mathfrak{p}$ for any $\mathfrak{p}$ is an isomorphism induced by the Hasse-invariant $\mathrm{Br}(\hat{K}_\mathfrak{p}) \cong \mathbb{Q}/\mathbb{Z}$ (cf. [GS, Proposition 6.3.9]). On the other hand, as all residue fields of K are finite, thus perfect, and $\mathcal{O}_S$ is a one-dimensional regular scheme, it admits due to Grothendieck the following exact sequence (see [Gro, Proposition 2.1] and [Mil, Example 2.22, case (a)]):

$$1 \to \mathrm{Br}(\mathcal{O}_S) \to \mathrm{Br}(K) \xrightarrow{\oplus_{\mathfrak{p} \notin S} r_\mathfrak{p}} \bigoplus_{\mathfrak{p} \notin S} \mathbb{Q}/\mathbb{Z}, \tag{2.5}$$

which means that $\mathrm{Br}(\mathcal{O}_S)$ is the subgroup of $\mathrm{Br}(K)$ of classes that vanish under $r_\mathfrak{p}$ at any $\mathfrak{p} \notin S$. Thus omitting these $r_\mathfrak{p}, \mathfrak{p} \notin S$ in the sequence (2.4), results in $\mathrm{Br}(\mathcal{O}_S) = \ker \left[\bigoplus_{\mathfrak{p} \in S} \mathbb{Q}/\mathbb{Z} \xrightarrow{\sum_{\mathfrak{p} \in S} \mathrm{Cor}_\mathfrak{p}} \mathbb{Q}/\mathbb{Z} \right]$, whence the cardinality of its 2-torsion part is $2^{|S|-1}$. $\square$

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Lemma 2.3. *Let $\underline{G}$ be an affine, flat, connected and smooth $\mathcal{O}_S$-group. Suppose that its generic fiber G is almost simple, simply connected and $\hat{K}_\mathfrak{p}$-isotropic for any $\mathfrak{p} \in S$. Then $H^1_{ét}(\mathcal{O}_S, \underline{G}) = 0$.*

Proof. The proof, basically relying on the strong approximation property related to $\underline{G}$, is similar to that of Lemma 3.2 in [Bit], replacing $\{\infty\}$ by S. □

3. Standard and relative invariants

Let $\underline{\mathbf{O}}_V$ be the *orthogonal group* of (V, q) defined over $\operatorname{Spec} \mathcal{O}_S$, namely, the functor assigning to any $\mathcal{O}_S$-algebra R the group of self-isometries of q over R:

$$\underline{\mathbf{O}}_V(R) = \{A \in \underline{\mathbf{GL}}_n(R) : q \circ A = q\}.$$

Since $2 \in \mathcal{O}_S^\times$ and q is regular, $\underline{\mathbf{O}}_V$ is smooth as well as its connected component, namely, the *special orthogonal group* $\underline{\mathbf{SO}}_V := \ker[\underline{\mathbf{O}}_V \xrightarrow{\det} \underline{\mu}_2]$ (see Definition 1.6, Theorem 1.7 and Corollary 2.5 in [Con]). Thus the pointed set $H^1_{\text{fl}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V)$ – properly (i.e., with $\det = 1$ isomorphisms) classifying $\mathcal{O}_S$-forms that are locally everywhere isomorphic to q in the flat topology – coincides with the classification $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V)$ for the étale topology (see [SGA4, VIII Corollaire 2.3]).

Let $\mathbf{C}(V) := T(V)/(v \otimes v - q(v) \cdot 1 : v \in V)$ be the *Clifford algebra* associated to (V, q) (see [Knu, IV]). The linear map $v \mapsto -v$ on V preserves q, thus extends to an algebra automorphism $\alpha : \mathbf{C}(V) \to \mathbf{C}(V)$. As it is an involution, the graded algebra $\mathbf{C}(V)$ is decomposed into positive and negative eigenspaces: $\mathbf{C}_0(V) \oplus \mathbf{C}_1(V)$ where $\mathbf{C}_i(V) = \{x \in \mathbf{C}(V) : \alpha(x) = (-1)^i x\}$ for $i = 0, 1$. Since (V, q) is projective and $\mathcal{O}_S$-regular, $\mathbf{C}(V)$ is Azumaya over $\mathcal{O}_S$ (cf. [Bas, Theorem, p. 166]).

The *Witt-invariant* of (V, q) is:

$$w(q) = \begin{cases} [\mathbf{C}(V)] \in \operatorname{Br}(\mathcal{O}_S) & n \text{ is even} \\ [\mathbf{C}_0(V)] \in \operatorname{Br}(\mathcal{O}_S) & n \text{ is odd.} \end{cases}$$

As $\mathbf{C}(V)$ and $\mathbf{C}_0(V)$ are algebras with involution, $w(q)$ lies in ${}_2\operatorname{Br}(\mathcal{O}_S)$ ([Knu, IV. 8]).

The *Clifford group* associated to (V, q) is

$$\mathbf{CL}(V) := \{u \in \mathbf{C}(V)^\times \ : \ \alpha(u) v u^{-1} \in V \ \forall v \in V\}.$$

The group $\underline{\mathbf{Pin}}_V(\mathcal{O}_S) := \ker[\mathbf{CL}(V) \xrightarrow{N} \mathcal{O}_S^\times]$ where $N : v \mapsto v\alpha(v)$, admits an underlying $\mathcal{O}_S$-group scheme, called the *Pinor group* denoted by $\underline{\mathbf{Pin}}_V$. It is a double covering of $\underline{\mathbf{O}}_V$ and its center $\underline{\mu}_2$ is smooth. So applying étale cohomology to the *Pinor exact sequence* of smooth $\mathcal{O}_S$-groups:

$$1 \to \underline{\mu}_2 \to \underline{\mathbf{Pin}}_V \to \underline{\mathbf{O}}_V \to 1 \tag{3.1}$$

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gives rise to the coboundary map of pointed-sets

$$\delta_V : H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{O}}_V) \to H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2). \tag{3.2}$$

Let $\underline{\mathbf{O}}_{2n}$ and $\underline{\mathbf{O}}_{2n+1}$ be the orthogonal groups of the hyperbolic spaces $H(\mathcal{O}_S^n)$ and $H(\mathcal{O}_S^n)\perp\langle 1\rangle$, respectively, equipped with the *standard split form* which we denote by q_n (see [Con, Definition 1.1]). The pointed set $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{O}}_n)$ classifies regular quadratic $\mathcal{O}_S$-modules of rank n ([Knu, IV. 5.3.1]). It is identified with the pointed set $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{O}}_V)$ simply obtained by changing the base point to (V, q) (cf. [Knu, IV, Prop. 8.2]). We denote this identification by θ. One has the following commutative diagram of pointed sets (cf. [Gir, IV, Prop. 4.3.4]):

$$\begin{tikzcd} H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{O}}_n) \arrow[r, "\theta", "\cong"'] \arrow[d, "\delta"] & H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{O}}_V) \arrow[d, "\delta_V"'] \\ H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2) \arrow[r, "r_V", "\cong"'] & H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2) \end{tikzcd} \tag{3.3}$$

in which $\delta := \delta_{q_n}$ and $r_V(x) = x - \delta([q])$.

Definition 1. We call the composition of maps of pointed sets:

$$w_V : H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{O}}_V) \xrightarrow{\delta_V} H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2) \xrightarrow{i^2_*} {}_2\mathrm{Br}(\mathcal{O}_S)$$

(see sequence (1.1) for i^2_*) the *relative Witt-invariant*. It is a "shift" of the Witt-invariant $w = i^2_* \circ \delta$, such that the base-point $[(V, q)]$ is mapped to $[0] \in \mathrm{Br}(\mathcal{O}_S)$.

Remark 3.1. The δ-image of a class represented by (V', q'), being a regular $\mathcal{O}_S$-module, is its *second Stiefel–Whitney class*, denoted $w_2(q')$ (cf. [EKV, Definition 1.6 and Corollary 1.19]).

The connected component $\underline{\mathbf{Spin}}_V$ of $\underline{\mathbf{Pin}}_V$ is smooth, and it is the universal covering of $\underline{\mathbf{SO}}_V$:

$$1 \to \underline{\mu}_2 \to \underline{\mathbf{Spin}}_V \xrightarrow{\pi} \underline{\mathbf{SO}}_V \to 1. \tag{3.4}$$

Then étale cohomology gives rise to the exact sequence of pointed sets:

$$H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{Spin}}_V) \to H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V) \xrightarrow{s\delta_V} H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2) \to 1 \tag{3.5}$$

in which the right exactness comes from the fact that $\mathcal{O}_S$ is of Douai-type, thus $H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{Spin}}_V) = 1$ (see Definition 5.2 and Example 5.4(iii) in [Gon]). The inclusion

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$i : \underline{\mathbf{SO}}_V \subset \underline{\mathbf{O}}_V$ with the map i_*^2 from sequence (1.1) induces the commutative diagram (cf. [Knu, IV, 8.3])

$$\begin{tikzcd} H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V) \arrow[r, "i_*"] \arrow[d, "s\delta_V"] & H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{O}}_V) \arrow[d, "w_V"] \arrow[dl, "\delta_V"'] \\ H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2) \arrow[r, "i_*^2"'] & {}_2\mathrm{Br}(\mathcal{O}_S). \end{tikzcd} \tag{3.6}$$

Remark 3.2. The map i_* does not have to be injective, yet any form q', properly isomorphic to q, represents a class in $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{O}}_n)$, so the restriction of w to $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_n)$ is well-defined. Similarly, we may write the restriction $w_V | H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V)$ as $i_*^2 \circ s\delta_v$, being surjective, as both i_*^2 and $s\delta_V$ are such (see sequences (1.1) and (3.5)).

Remark 3.3. Unlike over fields, the Stiefel–Whitney class w_2 for quadratic $\mathcal{O}_S$-spaces, referring to their Clifford algebras not only as Azumaya algebras but as algebras with involution, is richer than the Witt-invariant w lying in ${}_2\mathrm{Br}(\mathcal{O}_S)$. For example, if L is an invertible $\mathcal{O}_S$-module and $H(L) = L \oplus L^*$ is the corresponding hyperbolic plane, then $\mathbf{C}(H(L))$ is isomorphic as a graded algebra to $\mathrm{End}_{\mathcal{O}_S}(\wedge L) = \mathrm{End}_{\mathcal{O}_S}(\mathcal{O}_S \oplus L)$ ([Knu, IV, Prop. 2.1.1]) being Brauer-equivalent to $M_2(\mathcal{O}_S)$, thus $w(H(L)) = [0] \in \mathrm{Br}(\mathcal{O}_S)$, while:

$$\delta([H(L)]) = \partial([L]) \overset{\text{Corollary 2.1}}{=} \partial^1([\mathrm{End}_{\mathcal{O}_S}(\mathcal{O}_S \oplus L)]) \in H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2)$$

(see the left equality in the proof of the Proposition in [EKV, §5.5]), does not have to vanish as we shall see in Proposition 5.1.

4. A classification of quadratic spaces via their genera

Consider the ring of S-integral adèles $\mathbb{A}_S := \prod_{\mathfrak{p} \in S} \hat{K}_{\mathfrak{p}} \times \prod_{\mathfrak{p} \notin S} \hat{\mathcal{O}}_{\mathfrak{p}}$, being a subring of the adèles $\mathbb{A}$. Then the *S-class set* of an $\mathcal{O}_S$-group $\underline{G}$ is the set of double cosets:

$$\mathrm{Cl}_S(\underline{G}) := \underline{G}(\mathbb{A}_S) \backslash \underline{G}(\mathbb{A}) / G(K)$$

(where for any prime $\mathfrak{p}$ the geometric fiber $\underline{G}_{\mathfrak{p}}$ of $\underline{G}$ is taken) and it is finite (cf. [BP, Prop. 3.9]). If $\underline{G}$ is affine and finitely generated over $\mathrm{Spec}\, \mathcal{O}_S$, it admits according to Nisnevich ([Nis, Theorem I.3.5]) the following exact sequence of pointed sets:

$$1 \to \mathrm{Cl}_S(\underline{G}) \to H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}) \to H^1(K, G) \times \prod_{\mathfrak{p} \notin S} H^1_{\text{ét}}(\hat{\mathcal{O}}_{\mathfrak{p}}, (\underline{G})_{\mathfrak{p}}). \tag{4.1}$$

If $\underline{G}$ admits, furthermore, the property:

$$\forall \mathfrak{p} \notin S : \quad H^1_{\text{ét}}(\hat{\mathcal{O}}_{\mathfrak{p}}, \underline{G}_{\mathfrak{p}}) \hookrightarrow H^1(\hat{K}_{\mathfrak{p}}, G_{\mathfrak{p}}), \tag{4.2}$$

then Nisnevich's sequence for $\underline{G}$ reduces to (cf. [GP, Corollary A.8]):

$$1 \to \mathrm{Cl}_S(\underline{G}) \to H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}) \to H^1(K, G). \tag{4.3}$$

Remark 4.1. Since $\operatorname{Spec} \mathcal{O}_S$ is normal, i.e., is integrally closed locally everywhere (due to the smoothness of C), any finite étale covering of $\mathcal{O}_S$ arises by its normalization in some separable unramified extension of K (see [Len, Theorem 6.13]). Consequently, if $\underline{G}$ is a finite $\mathcal{O}_S$-group, then $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G})$ is embedded in $H^1(K, \underline{G})$. This is not true for infinite groups like the multiplicative group $\underline{\mathbb{G}}_m$, for which $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbb{G}}_m) \cong \operatorname{Pic}(\mathcal{O}_S)$ clearly does not have to embed in $H^1(K, \mathbb{G}_m) = 1$.

Remark 4.2. In case $\underline{G} = \underline{\mathbf{O}}_V$, the left exactness of sequence (4.1) reflects the fact that $\mathrm{Cl}_S(\underline{\mathbf{O}}_V)$ is the *genus* of the base point (V, q), namely, the set of classes of quadratic $\mathcal{O}_S$-forms that are K and $\hat{\mathcal{O}}_\mathfrak{p}$-isomorphic to it for all $\mathfrak{p} \notin S$. Furthermore, being connected, $\underline{\mathbf{SO}}_V$ admits property (4.2) by Lang's Theorem (recall that all residue fields are finite), so the *proper genus* can be described as:

$$\mathrm{Cl}_S(\underline{\mathbf{SO}}_V) = \ker[H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V) \to H^1(K, \mathbf{SO}_V)]. \tag{4.4}$$

As $\underline{\mathbf{O}}_V/\underline{\mathbf{SO}}_V$ is the finite representable $\mathcal{O}_S$-group $\underline{\mu}_2$ (cf. [Con, Theorem 1.7]), $\underline{\mathbf{O}}_V$ admits property (4.2) as well (see in the proof of Proposition 3.4 in [CGP]), so we may also write:

$$\mathrm{Cl}_S(\underline{\mathbf{O}}_V) = \ker[H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{O}}_V) \to H^1(K, \mathbf{O}_V)]. \tag{4.5}$$

As a pointed set, $\mathrm{Cl}_S(\underline{\mathbf{SO}}_V)$ is bijective to the first Nisnevich cohomology set $H^1_{\text{Nis}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V)$ (cf. [Nis, Theorem I.2.8] and [Mor, 4.1]), classifying $\underline{\mathbf{SO}}_V$-torsors in the Nisnevich topology. But Nisnevich covers are étale, so it is a subset of $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V)$. Similarly, $\mathrm{Cl}_S(\underline{\mathbf{O}}_V) \subseteq H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{O}}_V)$.

Lemma 4.3. *If (V, q) is isotropic then $\underline{\mathbf{O}}_V(\mathcal{O}_S) \xrightarrow{\det} \underline{\mu}_2(\mathcal{O}_S)$ is surjective.*

Proof. Consider the following exact and commutative diagram that arises by applying étale cohomology to the short exact sequences related to the smooth $\mathcal{O}_S$-groups $\underline{\mathbf{Pin}}_V$ and $\underline{\mathbf{O}}_V$:

$$\begin{tikzcd}
 & & H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{Spin}}_V) \arrow[r] \arrow[d, "s\pi_*"] & H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{Pin}}_V) \arrow[d, "\pi_*"] \\
\underline{\mathbf{O}}_V(\mathcal{O}_S) \arrow[r, "\det"] & \underline{\mu}_2(\mathcal{O}_S) \arrow[r, "\partial_0"] & H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V) \arrow[r, "h"] \arrow[d, "s\delta_V"] & H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{O}}_V) \arrow[d, "\delta_V"] \\
 & & H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2) \arrow[r, equal] & H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2).
\end{tikzcd}$$

Denote $[\gamma] = \partial_0(-1)$. Then $s\delta_V([\gamma]) = \delta_V(h([\gamma]) = [0]) = [0]$, hence $[\gamma] \in \mathrm{Im}(s\pi_*)$. But as q is isotropic, $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{Spin}}_V)$ vanishes by strong approximation (cf. Lemma 2.3), so $[\gamma] = [0]$, which means that ∂_0 is the trivial map and $\det(\mathcal{O}_S)$ surjects on $\underline{\mu}_2(\mathcal{O}_S)$. □

Lemma 4.4. *If n is odd, or (V,q) is isotropic, then $Cl_S(\underline{\mathbf{SO}}_V) = Cl_S(\underline{\mathbf{O}}_V)$.*

Proof. Any representative (V',q') of a class in $\mathrm{Cl}_S(\underline{\mathbf{O}}_V)$, being K isomorphic to q, is regular and isotropic as well, whence $\underline{\mathbf{O}}_{V'}(\mathcal{O}_S) \to \underline{\mu}_2(\mathcal{O}_S)$ is surjective by Lemma 4.3. When n is odd, this surjectivity is retrieved by the fact that $\underline{\mathbf{O}}_{V'} \cong \underline{\mathbf{SO}}_{V'} \times \underline{\mu}_2$ (cf. [Con, Thm. 1.7]), and so applying étale cohomology to the exact sequence of smooth groups

$$1 \to \underline{\mathbf{SO}}_{V'} \to \underline{\mathbf{O}}_{V'} \to \underline{\mu}_2 \to 1$$

we get that $\ker[H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_{V'}) \xrightarrow{\psi'} H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{O}}_{V'})] = 1$ for any $[(V',q')] \in \mathrm{Cl}_S(\underline{\mathbf{O}}_V)$, which means that the restricted map $\mathrm{Cl}_S(\underline{\mathbf{SO}}_V) \xrightarrow{\psi} \mathrm{Cl}_S(\underline{\mathbf{O}}_V)$ is injective. Together with Remark 4.2, this amounts to the existence of the following exact and commutative diagram:

$$\begin{tikzcd}
& \mathrm{Cl}_S(\underline{\mathbf{SO}}_V) \arrow[r, hook, "\psi"] \arrow[d, hook, "i"] & \mathrm{Cl}_S(\underline{\mathbf{O}}_V) \arrow[d, hook, "i'"] & \\
1 \arrow[r] & H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V) \arrow[r, "\psi'"] \arrow[d, "m"] & H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{O}}_V) \arrow[r, "d"] \arrow[d, "m'"] & H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2) \arrow[d, hook, "m''"] \\
1 \arrow[r] & H^1(K, \mathbf{SO}_V) \arrow[r, hook, "h"] & H^1(K, \mathbf{O}_V) \arrow[r, "d'"] & H^1(K, \mu_2)
\end{tikzcd}$$

in which as m'' is injective due to Remark 4.1, ψ is also surjective, thus is the identity. □

Proposition 4.5. *Let (V,q) be a regular quadratic $\mathcal{O}_S$-space of rank $n \geq 3$ with proper genus $Cl_S(\underline{\mathbf{SO}}_V)$. The relative Witt-invariant (cf. Definition 1 and Remark 3.2) induces an exact sequence of pointed sets*

$$1 \to Cl_S(\underline{\mathbf{SO}}_V) \xrightarrow{h} H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V) \xrightarrow{w_V} {}_2\mathrm{Br}(\mathcal{O}_S) \to 1$$

in which h is injective and ${}_2\mathrm{Br}(\mathcal{O}_S)$ bijects to the set of $2^{|S|-1}$ proper genera of q.

Proof. Consider the short exact sequence induced by the double covering of the generic fiber

$$1 \to \mu_2 \to \mathbf{Spin}_V \to \mathbf{SO}_V \to 1.$$

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As $\mathbf{Spin}_V$ is simply connected, we know due to Harder that $H^1(K, \mathbf{Spin}_V) = 1$ (cf. [Hard, Satz A]). This is true for all twisted forms of $\mathbf{Spin}_V$, whence Galois cohomology implies the embedding $H^1(K, \mathbf{SO}_V) \hookrightarrow H^2(K, \mu_2)$. Due to Hilbert's Theorem 90, applying Galois cohomology to the Kummer's exact sequence related to μ_2 over K gives an isomorphism $H^2(K, \mu_2) \cong {}_2\mathrm{Br}(K)$. Moreover, as shown in the sequence (2.5), $\mathrm{Br}(\mathcal{O}_S)$ is a subgroup of $\mathrm{Br}(K)$. All together, the relative Witt-invariant applied to classes in $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V)$ and on their generic fibers, yields the following exact and commutative diagram:

$$\begin{tikzcd} H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V) \arrow[r, "w_V"] \arrow[d] & {}_2\mathrm{Br}(\mathcal{O}_S) \arrow[d, hook] \\ H^1(K, \mathbf{SO}_V) \arrow[r, hook, "w_V"] & {}_2\mathrm{Br}(K) \end{tikzcd} \tag{4.6}$$

which justifies the left exactness in the asserted sequence:

$$\mathrm{Cl}_S(\underline{\mathbf{SO}}_V) \overset{(4.4)}{=} \ker[H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V) \to H^1(K, \mathbf{SO}_V)] = \ker[H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V) \xrightarrow{w_V} {}_2\mathrm{Br}(\mathcal{O}_S)]. \tag{4.7}$$

The surjectivity of $w_V : H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V) \to {}_2\mathrm{Br}(\mathcal{O}_S)$ (cf. Remark 3.2) completes the proof.

The first equality in (4.7) suggests that for any $[q'] \in H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V)$, $q' \in \mathrm{Cl}_S(\underline{\mathbf{SO}}_V)$ if and only if q' is K-isomorphic to q while the second equality claims that this holds if and only if $w_V(q') = w_V(q)$. This is true for any choice of base point, being regular as well. So as $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V)$ is a disjoint union of the proper genera of q, together with the surjectivity of w_V, we deduce that the set of proper genera bijects with ${}_2\mathrm{Br}(\mathcal{O}_S)$, whose cardinality is computed in Lemma 2.2. □

Theorem 4.6. *Let (V, q) be a regular quadratic $\mathcal{O}_S$-space of rank $n \geq 3$. Then there exists a surjection of pointed-sets: $Cl_S(\underline{\mathbf{SO}}_V) \twoheadrightarrow Pic\ (\mathcal{O}_S)/2$. If (V, q) is isotropic, then this is a bijection, so the abelian group $Pic\ (\mathcal{O}_S)/2$ is isomorphic to $Cl_S(\underline{\mathbf{SO}}_V) = Cl_S(\underline{\mathbf{O}}_V)$.*

Proof. Consider the following exact and commutative diagram derived from sequences (1.1) and (3.5):

$$\begin{tikzcd} & 1 \arrow[r] \arrow[d] & H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V) \arrow[r, equal] \arrow[d, "s\delta_V"] & H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V) \arrow[r] \arrow[d, "w_V"] & 1 \\ 1 \arrow[r] & \mathrm{Pic}\ (\mathcal{O}_S)/2 \arrow[r, "\partial"] & H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2) \arrow[r, "i^2_*"] & {}_2\mathrm{Br}(\mathcal{O}_S) \arrow[r] & 1. \end{tikzcd} \tag{4.8}$$

We imitate the Snake Lemma argument (though the diagram terms are not all groups): according to Proposition 4.5 $\mathrm{Cl}_S(\underline{\mathbf{SO}}_V) = \ker(w_V)$, hence for any $[q'] \in \mathrm{Cl}_S(\underline{\mathbf{SO}}_V)$ one has $i^2_*(s\delta_V([q'])) = [0]$, i.e., $[q']$ has a ∂-preimage in $\mathrm{Pic}\ (\mathcal{O}_S)/2$ which is unique as ∂ is a

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monomorphism of groups. Moreover, any element in Pic $(\mathcal{O}_S)/2$ arises in this way, since $s\delta_V$ is surjective. As a result we have an exact sequence of pointed sets:

$$1 \to \mathfrak{K}_1 \to \mathrm{Cl}_S(\underline{\mathbf{SO}}_V) \xrightarrow{s\delta_V} \mathrm{Pic}\ (\mathcal{O}_S)/2 \to 1.$$

If (V,q) is isotropic, then $s\delta_V$ is also injective. Indeed, let $\underline{\mathbf{SO}}_{V'}$ be a twisted form of $\underline{\mathbf{SO}}_V$, properly stabilizing a form q', and let $\underline{\mathbf{Spin}}_{V'}$ be its Spin group. The lower row in the following exact diagram is the one obtained when replacing the base point q by q', as described in [Gir, IV, Proposition 4.3.4]:

$$\begin{tikzcd}
\mathfrak{K}_1 \arrow[r] & \mathrm{Cl}_S(\underline{\mathbf{SO}}_V) \arrow[r, two heads, "s\delta_V"] \arrow[d, phantom, "\subset"] & \mathrm{Pic}\ (\mathcal{O}_S)/2 \arrow[d] \\
H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{Spin}}_V) \arrow[r] & H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V) \arrow[r, two heads, "s\delta_V"] & H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2) \\
H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{Spin}}_{V'}) \arrow[r] & H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}'_V) \arrow[r, two heads, "s\delta_{V'}"] \arrow[u, "\theta", "\cong"'] & H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2) \arrow[u, "r", "\cong"']
\end{tikzcd} \tag{4.9}$$

If $[q'] \in \mathrm{Cl}_S(\underline{\mathbf{SO}}_V)$, then q', being K-isomorphic to q, is also K-isotropic, as well as the generic fiber $\mathbf{Spin}_{V'}$. Then by the Hasse–Minkowsky Theorem (cf. [Lam, VI.3.1]), q' is $\hat{K}_{\mathfrak{p}}$-isotropic everywhere, in particular in S. Hence $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{Spin}}_{V'})$ is trivial by Lemma 2.3 for any class in $\mathrm{Cl}_S(\underline{\mathbf{SO}}_V)$ ($\underline{\mathbf{SO}}_V$ is not commutative for $n \geq 3$ thus $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V)$ does not have to be a group, so the triviality of $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{Spin}}_V)$ does not imply the injectivity of $s\delta_V$, i.e., there still might be distinct anisotropic classes in $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V)$ whose images in $H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2)$ coincide). So $\mathrm{Cl}_S(\underline{\mathbf{SO}}_V)$, being equal in the isotropic case to $\mathrm{Cl}_S(\underline{\mathbf{O}}_V)$ by Lemma 4.4, embeds in Pic $(\mathcal{O}_S)/2$ and the assertion follows. □

Definition 2. We say that the *local-global Hasse principle* holds for a quadratic $\mathcal{O}_S$-space (V,q) if $|\mathrm{Cl}_S(\underline{\mathbf{O}}_V)| = 1$.

Corollary 4.7. *The Hasse principle holds for a regular isotropic quadratic $\mathcal{O}_S$-form of rank ≥ 3 if and only if $|Pic\ (\mathcal{O}_S)|$ is odd.*

Corollary 4.8. *Let (V,q) be an $\mathcal{O}_S$-regular quadratic space of rank ≥ 5. Then the pointed-set $H^1_{ét}(\mathcal{O}_S, \underline{\mathbf{SO}}_V)$ is isomorphic to the abelian group $H^2_{ét}(\mathcal{O}_S, \underline{\mu}_2)$, i.e., any $\mathcal{O}_S$-isomorphism class in the proper classification corresponds to an Azumaya $\mathcal{O}_S$-algebra with involution. There are $2^{|S|-1}$ genera, each of them is isomorphic to Pic $(\mathcal{O}_S)/2$.*

Proof. In rank ≥ 5, any quadratic $\mathcal{O}_S$-space is isotropic; indeed, for any such (V',q'), the generic fiber $q'_K := q' \otimes K$ is isotropic (cf. [OMe, Theorem 66:2]), i.e., there exists a non-zero vector $v_0 \in V' \otimes K$ such that $q'_k(v_0) = 0$. Since K is the fraction field of the Dedekind

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domain $\mathcal{O}_S$, there exists a non-zero vector $\underline{v}_0 \in Kv_0 \cap \mathcal{O}_S$ for which $q(\underline{v}_0) = 0$. Hence according to Theorem 4.6, we deduce that the genus of any $[(V', q')] \in H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V)$ is isomorphic to the abelian group Pic $(\mathcal{O}_S)/2$ and injects into $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V)$. Looking at the obtained exact and commutative diagram:

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Cl}_S(\underline{\mathbf{SO}}_V) \arrow[r] \arrow[d, "\cong"] & H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V) \arrow[r, "w_V"] \arrow[d, twoheadrightarrow, "s\delta_V"] & {}_2\mathrm{Br}(\mathcal{O}_S) \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \text{Pic } (\mathcal{O}_S)/2 \arrow[r, "\partial"] & H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2) \arrow[r, "i^2_*"] & {}_2\mathrm{Br}(\mathcal{O}_S) \arrow[r] & 1 \end{tikzcd}$$

we see that the cardinality of $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V)$, being the disjoint union of its genera, equals the one of its $s\delta_V$-image $H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2)$, hence it is isomorphic to it. We have seen in Proposition 4.5 that ${}_2\mathrm{Br}(\mathcal{O}_S)$ bijects with the set of proper genera of q. Here, as any twisted form of q is isotropic, its proper genus is equal to its genus (Lemma 4.4), thus there are $2^{|S|-1}$ such genera (Lemma 2.2). $\square$

Remark 4.9. Since as we have seen any integral quadratic form of rank ≥ 5 is isotropic, according to Lemma 4.4, $\mathrm{Cl}_S(\underline{\mathbf{SO}}_V)$ might not be equal to $\mathrm{Cl}_S(\underline{\mathbf{O}}_V)$ (for $\mathrm{rank}(V) \geq 3$) only when (V, q) is anisotropic of rank 4.

5. A splitting hyperbolic plane

In this section, we refer to regular quadratic $\mathcal{O}_S$-spaces being split by a hyperbolic plane P (thus being isotropic and so $\mathrm{Cl}_S(\underline{\mathbf{O}}_V) = \mathrm{Cl}_S(\underline{\mathbf{SO}}_V)$). Such P contains a hyperbolic pair $\{v_0, v_1\}$ of V, namely, satisfying: $q(v_0) = q(v_1) = 0$ and $B_q(v_0, v_1) = 1$, and it is of the form $P_\mathfrak{a} = \mathfrak{a}v_0 + \mathfrak{a}^{-1}v_1$ for some fractional ideal $\mathfrak{a}$ of $\mathcal{O}_S$ (cf. [Ger, Proposition 2.1]). L. J. Gerstein established in [Ger, Theorem 4.5] for the ternary case the bijection:

$$\psi : \text{Pic } (\mathcal{O}_S)/2 \xrightarrow{\sim} \mathrm{Cl}_S(\underline{\mathbf{O}}_V) : \quad [\mathfrak{a}] \mapsto [V_\mathfrak{a}] := [P_\mathfrak{a} \perp \mathfrak{b}\lambda]$$

where $\mathfrak{b}$ is again a fractional ideal of $\mathcal{O}_S$ and $\lambda \in \mathcal{O}_S^\times$, for which $V \cong V_\mathfrak{a}$. The existence of ψ can be viewed as a particular case of our Theorem 4.6 (such a splitting does not necessarily exist). The following Lemma suggests an alternative bijection which resembles the one of Gerstein, but does not, however, require finding and multiplying by a hyperbolic pair, and, more important, is valid for any rank $n \geq 3$.

Proposition 5.1. *Suppose V is split by a hyperbolic plane $V = H(L_0) \perp V_0$ where L_0 is an $\mathcal{O}_S$-line. Then $\psi_V : [L] \mapsto [V_L = H(L_0 \otimes L^*) \perp V_0]$ is an isomorphism of groups Pic $(\mathcal{O}_S)/2 \cong \mathrm{Cl}_S(\underline{\mathbf{O}}_V)$.*

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Proof. L^* is locally free, thus V_L obtained from V by tensoring $H(L_0)$ with L^* remains in $\mathrm{Cl}_S(\underline{\mathbf{O}}_V)$. Due to Theorem 4.6, the groups Pic $(\mathcal{O}_S)/2$ and $\mathrm{Cl}_S(\underline{\mathbf{O}}_V)$ are isomorphic through diagram (4.8), so it is sufficient to show by its commutativity that $s\delta_V \circ \psi_V$ coincides with the groups embedding ∂. The i'th *Stiefel–Whitney class* $w_i(E)$ of a regular $\mathcal{O}_S$-module E, as defined in [EKV, §1], gets values in $H^i_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2)$ for $i \geq 1$ and $w_0(E) = 1$. Its basic axioms, namely, $w_i(E) = 0$ for all $i > \mathrm{rank}(E)$ and for any direct sum of regular $\mathcal{O}_S$-modules of finite rank

$$w_k(E \oplus F) = \sum_{i+j=k} w_i(E) \cdot w_j(F),$$

imply that:

$$w_1(E \oplus F) = w_1(E) + w_1(F) \;\text{ and: }\; w_2(E \oplus F) = w_2(E) + w_2(F) + w_1(E) \cdot w_1(F). \tag{5.1}$$

If L is an $\mathcal{O}_S$-line and $H(L) = L \oplus L^*$ is the corresponding hyperbolic plane, this reads:

$$w_1(H(L)) = w_1(L) + w_1(L^*) \;\text{ while }\; w_2(H(L)) = w_1(L) \cdot w_1(L^*).$$

Moreover, w_1 furnishes an isomorphism of abelian groups $\{\mathcal{O}_S\text{-lines}, \otimes\}/\sim \cong H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2)$ by

$$w_1(L_1 \otimes L_2) = w_1(L_1) + w_1(L_2), \quad \text{thus} \;\; w_1(L) = -w_1(L^*),$$

hence

$$\begin{aligned} w_1(H(L_0 \otimes L^*)) - w_1(H(L_0)) &= w_1(L_0 \otimes L^*) + w_1(L_0^* \otimes L) - w_1(L_0) - w_1(L_0^*) \\ &= w_1(L_0) + w_1(L^*) + w_1(L_0^*) + w_1(L) - w_1(L_0) - w_1(L_0^*) \\ &= 0 \end{aligned} \tag{5.2}$$

and:

$$\begin{aligned} & w_2(H(L_0 \otimes L^*)) - w_2(H(L_0)) \\ & \quad = (w_1(L_0) + w_1(L^*)) \cdot (w_1(L_0^*) + w_1(L)) - w_1(L_0) \cdot w_1(L_0^*) \\ & \quad = w_1(L^*) \cdot w_1(L_0^*) + w_1(L_0) \cdot w_1(L) + w_1(L^*) \cdot w_1(L) \\ & \quad = w_1(L^*) \cdot w_1(L_0^*) - w_1(L_0^*) \cdot w_1(L) + w_1(L^*) \cdot w_1(L) \\ & \quad = w_1(L^*) \cdot w_1(L) = w_2(H(L)). \end{aligned} \tag{5.3}$$

In our setting (recall that $\delta = w_2$, see Remark 3.1), we get:

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$$\begin{aligned}\delta([V_L]) - \delta([V]) &\overset{(5.1)}{=} \delta([H(L_0 \otimes L^*)]) + \delta([V_0]) + w_1(H(L_0 \otimes L^*)) \cdot w_1(V_0) \\ &\quad - \Big(\delta([H(L_0)]) + \delta([V_0]) + w_1(H(L_0)) \cdot w_1(V_0)\Big) \\ &\overset{(5.2)}{=} \delta([H(L_0 \otimes L^*)]) - \delta([H(L_0)]) \overset{(5.3)}{=} \delta([H(L)]).\end{aligned} \tag{5.4}$$

Altogether, we may finally conclude that for any $[L] \in \text{Pic}\ (\mathcal{O}_S)/2$:

$$\begin{aligned}(s\delta_V \circ \psi_V)([L]) = s\delta_V([V_L]) &\overset{(3.3)}{=} (r_V \circ \delta \circ \theta^{-1})([V_L]) = (r_V \circ \delta)([V_L]) \\ &= \delta([V_L]) - \delta([V]) \overset{(5.4)}{=} \delta([H(L)]) \overset{\text{Remark 3.3}}{=} \partial([L]). \quad \square\end{aligned}$$

5.1. $|S| = 1$

If $S = \{\infty\}$ where ∞ is an arbitrary closed point, then $C^{\text{af}} := C - \{\infty\}$ is an affine curve whence $\text{Br}(\mathcal{O}_S) = 1$ (cf. Lemma 2.2), i.e., there is a single genus, and any $\mathcal{O}_S$-regular quadratic space (V, q) of rank $n \geq 3$ is isotropic (cf. [Bit]). Suppose furthermore that C is an elliptic curve and ∞ is $\mathbb{F}_q$-rational. For any place P on C^{af} we define the maximal ideal of $\mathcal{O}_S$

$$\mathfrak{m}_P := \{f \in \mathcal{O}_S \ : \ f(P) = 0\}.$$

Then we have an isomorphism of abelian groups (see [Hart, II, Proposition 6.5(c)] for the first map and p. 393 in [Bau] for the second one):

$$\varphi : C(\mathbb{F}_q) \cong \text{Pic}^{\,0}(C) \cong \text{Pic}\ (\mathcal{O}_S) : \quad P \mapsto [P] - [\infty] \mapsto \mathfrak{m}_P.$$

Since $\mathcal{O}_S$ is Dedekind, Steinitz's Theorem (cf. [BK, Corollary 6.1.9]) tells us that for any fractional ideal $\mathfrak{a}$ of $\mathcal{O}_S$ there is an $\mathcal{O}_S$-isomorphism of $\mathcal{O}_S$-modules (though not of quadratic $\mathcal{O}_S$-spaces):

$$\chi : \mathfrak{a} \oplus \mathfrak{a}^{-1} \to \mathcal{O}_S \oplus \mathcal{O}_S.$$

Towards our purpose of finding representatives of twisted K-forms of q, we may choose such a K-isomorphism χ_K to be as in [Cla, Lemma 20.17]:

$$\chi_K : (\alpha, \beta) \mapsto (\alpha, \beta) \cdot A_{\mathfrak{a}}, \quad A_{\mathfrak{a}} = \begin{pmatrix} b_1 & -a_2 \\ b_2 & a_1 \end{pmatrix}$$

where $a_1 b_1 + a_2 b_2 = 1$, $b_1 \in \mathfrak{a}^{-1}$, $b_2 \in \mathfrak{a}$. According to Proposition 5.1, the matrix $A_{\mathfrak{m}_P}^{-t} B_q A_{\mathfrak{m}_P}^{-1}$ represents, for any coset $[P] \in C(\mathbb{F}_q)/2$, a distinct class of quadratic $\mathcal{O}_S$-forms in $\text{Cl}_S(\underline{\mathbf{O}}_V) = H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V)$.

We summarize this procedure by the following algorithm:

Algorithm 1 Generator of classes representatives isomorphic in the flat topology to a regular quadratic space split by a hyperbolic plane, over the coordinate ring of an affine non-singular elliptic curve.

Input: C = elliptic projective $\mathbb{F}_q$-curve, $S = \{\infty \in C(\mathbb{F}_q)\}$, $V = H(L_0)\perp V_0$ = quadratic regular $\mathcal{O}_S$-space of rank $n \geq 3$, $H(L_0)$ is represented by $F_0 \in \mathbf{GL}_2(\mathcal{O}_S)$.

Compute $C(\mathbb{F}_q)$ and $\mathcal{O}_S := C^{\mathrm{af}}(\mathbb{F}_q)$ where $C^{\mathrm{af}} := C - \{\infty\}$.

for each $[P] \in C(\mathbb{F}_q)/2$ **do**

$\mathfrak{m}_P = \{f \in \mathcal{O}_S \ : \ f(P) = 0\}$ for $P \neq \infty$ and $\mathfrak{m}_\infty = \mathcal{O}_S$.

Find $a_1, b_2 \in \mathfrak{m}_P$ and $a_2, b_1 \in \mathfrak{m}_P^{-1}$ such that $a_1 b_1 + a_2 b_2 = 1$.

$A_{\mathfrak{m}_P}^{-1} = \begin{pmatrix} a_1 & a_2 \\ -b_2 & b_1 \end{pmatrix}$ and: $F_P = A_{\mathfrak{m}_P}^{-t} F_0 A_{\mathfrak{m}_P}^{-1}$.

Output: $\mathrm{Cl}_S(\underline{\mathbf{O}}_V) = \{[V_P \perp V_0] : [P] \in C(\mathbb{F}_q)/2\}$, where V_P is the quadratic $\mathcal{O}_S$-form represented by F_P.

Remark 5.2. If C is an elliptic curve and ∞ is $\mathbb{F}_q$-rational, then for any $\mathcal{O}_S$-line L_0, the special orthogonal group of $H(L_0)$, being split, of rank 2 and $\mathcal{O}_S$-regular, is a one dimensional split $\mathcal{O}_S$-torus, i.e., isomorphic to $\underline{\mathbb{G}}_m$ (see in the proof of [Bit, Theorem 4.5]), hence the proper classification is via $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbb{G}}_m) \cong \mathrm{Pic}\ (\mathcal{O}_S) \cong C(\mathbb{F}_q)$. Then the class representatives are obtained by the above algorithm when replacing $C(\mathbb{F}_q)/2$ by $C(\mathbb{F}_q)$. This means that invertible fractional ideals, corresponding to non-trivial squares in $C(\mathbb{F}_q)$, i.e., $[L] \in 2\mathrm{Pic}\ (\mathcal{O}_S)\backslash\{0\}$, and only they, induce spaces $H(L_0 \otimes L^*)$ that are *stably isomorphic* to $H(L_0)$ in the proper classification, namely, become properly isomorphic after being extended by any non-trivial regular orthogonal $\mathcal{O}_S$-space. In other words, the Witt Cancellation Theorem fails over $\mathcal{O}_S$ in this case for the proper classification.

Example 5.3. [1] Let $C = \{Y^2 Z = X^3 + XZ^2\}$ defined over $\mathbb{F}_5$. Then

$$C(\mathbb{F}_5) = \{(0:0:1), (1:0:2), (1:0:3), (0:1:0)\}.$$

Taking $S = \{\infty = (0:1:0)\}$ we get the affine elliptic curve

$$C^{\mathrm{af}} = \{y^2 = x^3 + x\} \ \text{ with: } \ \mathcal{O}_S = \mathbb{F}_5[x,y]/(y^2 - x^3 - x).$$

The affine supports of the points in $C(\mathbb{F}_5) - \{\infty\}$ are: $\{(0,0), (1/2, 0) = (3,0), (1/3, 0) = (2,0)\}$. The y-coordinate of these points vanishes which means that they are of order 2 according to the group law and $C(\mathbb{F}_q) \cong (\mathbb{Z}/2)^2$. We get:

$$\mathfrak{m}_{(0:0:1)} = \langle x, y\rangle, \ \mathfrak{m}_{(1:0:2)} = \langle x-3, y\rangle, \ \mathfrak{m}_{(1:0:3)} = \langle x-2, y\rangle, \ \mathfrak{m}_{(0:1:0)} = \mathcal{O}_S.$$

Now consider the standard ternary quadratic $\mathcal{O}_S$-space $V = H(\mathcal{O}_S)\perp\langle 1\rangle$, i.e., with $B_q = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. For any $[L] \in \mathrm{Pic}\ (\mathcal{O}_S)/2 = \mathrm{Pic}\ (\mathcal{O}_S)$, the quadratic space $V_L = H(L)\perp\langle 1\rangle$

[1] I take this opportunity to correct Example 4.12 in [Bit].

belongs to $\mathrm{Cl}_S(\underline{\mathbf{O}}_V)$ and there are four non-equivalent classes in $\mathrm{Cl}_S(\underline{\mathbf{O}}_V)$. For example, $A^{-1}_{\langle x,y\rangle} = \begin{pmatrix} x & -y/x \\ y & -x \end{pmatrix}$ induces the form represented by $\begin{pmatrix} 2xy & -2x^2-1 & 0 \\ -2x^2-1 & 2y & 0 \\ 0 & 0 & 1 \end{pmatrix}$, being not $\mathcal{O}_S$-isomorphic to V, as $\langle x, y\rangle$ is not principal.

5.2. $|S| > 1$

If $|S| > 1$ then an $\mathcal{O}_S$-regular quadratic space V will possess multiple $(2^{|S|-1})$ genera, and if $\mathrm{rank}(V) = 3, 4$ some of them may be anisotropic, i.e., contain anisotropic representatives only.

Example 5.4. Let C be the projective line defined over $\mathbb{F}_3$, so $K = \mathbb{F}_3(t)$, and let $S = \{t, t^{-1}\}$, so $\mathcal{O}_S = \mathbb{F}_3[t, t^{-1}]$, being the ring of regular functions on the multiplicative group $\underline{\mathbb{G}}_m$, thus it is a PID. The ternary $\mathcal{O}_S$-space $V = \langle 1, -1, -t\rangle$ with $q(x, y, z) = x^2 - y^2 - tz^2$ is isotropic, e.g., $q(1, 1, 0) = 0$. It is properly isomorphic over $\mathcal{O}_S(i)$ (being a scalar extension of $\mathcal{O}_S$ thus an étale one), by $\mathrm{diag}(1, i, -i)$ to the anisotropic form $V' = \langle 1, 1, t\rangle$. Both are $\mathcal{O}_S$-unimodular as $\det(q) = t \in \mathcal{O}_S^\times$, but belong to two distinct genera (there are exactly $2^{|S|-1} = 2$ such, cf. Proposition 4.5). As $\mathcal{O}_S$ is a PID, Pic $(\mathcal{O}_S) = 1$, and so according to Theorem 4.6 there is only one class in $\mathrm{Cl}_S(\underline{\mathbf{O}}_V)$. The hyperbolic plane $\langle 1, -1\rangle$ has trivial Witt invariant, and it is orthogonal to $\langle -t\rangle$ in V, thus $w(q) = 0$ (cf. [Knu, IV, Prop. 8.1.1, 1), 3)]), so $[q]$ corresponds to the trivial element in ${}_2\mathrm{Br}(\mathcal{O}_S)$. To compute the Clifford algebra of (V', q'), we choose the natural basis

$$\{e_1 = (1, 0, 0),\ e_2 = (0, 1, 0),\ e_3 = (0, 0, 1)\}.$$

The embedding $i : V' \hookrightarrow \mathbf{C}(q')$ satisfies the relations $i(v)^2 = q'(v) \cdot 1\ \forall v \in V'$ which imply:

$$i(v)i(u) + i(u)i(v) = B_{q'}(u, v) \quad \forall u, v \in V'.$$

Since $\{e_i\}_{i=1}^3$ are orthogonal, this means that:

$$i(e_i)i(e_j) = -i(e_j)i(e_i) \quad \forall 1 \le i \ne j \le 3,$$

so we may choose (as q' is anisotropic the obtained quaternion algebra is not split):

$$i(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ i(e_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ i(e_3) = \begin{pmatrix} 0 & \sqrt{-t} \\ -\sqrt{-t} & 0 \end{pmatrix}.$$

Then $\{1, i(e_1)i(e_2), i(e_2)i(e_3), i(e_1)i(e_3)\}$ is a basis of $\mathbf{C}_0(q')$ (cf. [Knu, V. §3]):

$$\mathbf{C}_0(q') = \left\langle 1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -\sqrt{-t} & 0 \\ 0 & \sqrt{-t} \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{-t} \\ \sqrt{-t} & 0 \end{pmatrix} \right\rangle$$

and $w(q') = [\mathbf{C}_0(q')]$ is the non-trivial element in ${}_2\mathrm{Br}(\mathcal{O}_S)$.

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References

[SGA4] M. Artin, A. Grothendieck, J.-L. Verdier, in: Théorie des Topos et Cohomologie Étale des Schémas (SGA 4), in: LNM, Springer, 1972/1973.
[Bas] H. Bass, Clifford algebras and spinor norms over a commutative ring, Amer. J. Math. 96 (1974) 156–206.
[Bau] M.L. Bauer, The arithmetic of certain cubic function fields, Math. Comp. 73 (245) (2004) 387–413.
[BK] A.J. Berrick, M.E. Keating, An Introduction to Rings and Modules with K-Theory in View, Cambridge University Press, 2000.
[Bit] R.A. Bitan, The Hasse principle for bilinear symmetric forms over a ring of integers of a global function field, J. Number Theory (2016) 346–359.
[BP] A. Borel, G. Prasad, Finiteness theorems for discrete subgroups of bounded covolume in semi-simple groups, Publ. Math. IHES 69 (1989) 119–171.
[CGP] V. Chernousov, P. Gille, A. Pianzola, A classification of torsors over Laurent polynomial rings, Comment. Math. Helv. 92 (1) (2017) 37–55.
[Cla] P.L. Clark, Commutative algebra, http://math.uga.edu/~pete/integral2015.pdf.
[Con] B. Conrad, Math 252. Properties of orthogonal groups, http://math.stanford.edu/~conrad/252Page/handouts/O(q).pdf.
[SGA3] M. Demazure, A. Grothendieck, in: P. Gille, P. Polo (Eds.), Séminaire de Géométrie Algébrique du Bois Marie – 1962/64 – Schémas en groupes, Tome II, Réédition de SGA3, 2011.
[EKV] H. Esnault, B. Kahn, E. Viehweg, Coverings with odd ramification and Stiefel–Whitney classes, J. Reine Angew. Math. 441 (1993) 145–188.
[Ger] L.J. Gerstein, Unimodular quadratic forms over global function fields, J. Number Theory 11 (1979) 529–541.
[GP] P. Gille, A. Pianzola, Isotriviality and étale cohomology of Laurent polynomial rings, J. Pure Appl. Algebra 212 (2008) 780–800.
[GS] P. Gille, T. Szamueli, Central Simple Algebra and Galois Cohomology, Cambridge Studies in Adv. Math., vol. 101, 2006.
[Gir] J. Giraud, Cohomologie non abélienne, Grundlehren Math. Wiss., Springer-Verlag, Berlin, Heidelberg, New York, 1971.
[Gon] C.D. González-Avilés, Quasi-abelian crossed modules and nonabelian cohomology, J. Algebra 369 (2012) 235–255.
[Gro] A. Grothendieck, Le groupe de Brauer III: Exemples et compléments, in: Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 88–188.
[Hard] G. Harder, Über die Galoiskohomologie halbeinfacher algebraischer Gruppen, III, J. Reine Angew. Math. 274/275 (1975) 125–138.
[Hart] R. Hartshorne, Algebraic Geometry, Springer, 1977.

[Knu] M.A. Knus, Quadratic and Hermitian Forms over Rings, Grundlehren Math. Wiss., vol. 294, Springer, 1991.

[Lam] T.Y. Lam, Introduction to Quadratic Forms over Fields, Graduate Studies in Mathematics, vol. 67, 2005.

[Len] H.W. Lenstra, Galois theory for schemes, http://websites.math.leidenuniv.nl/algebra/GSchemes.pdf.

[Mil] J.S. Milne, Étale Cohomology, Princeton University Press, Princeton, 1980.

[Mor] M. Morishita, On S-class number relations of algebraic tori in Galois extensions of global fields, Nagoya Math. J. 124 (1991) 133–144.

[Nis] Y. Nisnevich, Étale Cohomology and Arithmetic of Semisimple Groups, PhD thesis, Harvard University, 1982.

[OMe] O.T. O'Meara, Introduction to Quadratic Forms, Springer-Verlag, New York, Berlin, 1971.

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