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An elementary bound on Siegel zeroes

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ABSTRACT

We consider Dirichlet L -functions $L(s, \chi)$ where χ is a real, non-principal character modulo q . Using Pintz's refinement of Page's theorem, we prove that for $q \geq 3$ the function $L(s, \chi)$ has at most one real zero β with $1 - 0.933/\log q < \beta < 1$.

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1. Introduction

Let $\chi(n)$ be a real, non-principal Dirichlet character to the modulus q and let $L(s, \chi)$ be the associated Dirichlet L -function, where $s = \sigma + it$. It is known [4, pp. 93–95] that $L(s, \chi)$ has at most one zero with real part larger than $1 - (A \log \max\{q, q|t|\})^{-1}$. Such an exceptional, or Siegel, zero must lie on the real axis. For recent work on the size of Siegel zeroes, one may consult the work of Bennett et al. [1] and Bordignon [2,3]. A classical result of Page [14] is that, given a single character χ modulo q , there can be at most one exceptional zero ‘close’ to unity.

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Theorem 1 (Page). *If $\chi \bmod q$ is a real, non-principal character, and if β_1 and β_2 are real zeroes of $L(s, \chi)$, then there is a positive constant c such that*

$$\min(\beta_1, \beta_2) \leq 1 - \frac{c}{\log q}.$$

McCurley [11] and Kadiri [7] have given values for c : the best is $c = 0.909$ by Kadiri — though note also the work of Louboutin [10] in which Theorem 6 recovers the result of McCurley. Kadiri's method, similar to that in papers on the zero-free region of L -functions (see, e.g. [8] and [13]), uses a special nonnegative trigonometric polynomial, the calculus of variations, and an analysis of the distribution of the imaginary parts of zeroes of $L(s, \chi)$.

Pintz [15, Thm 2] revisits Page's method, which is more elementary. Using the Pólya–Vinogradov inequality, Pintz is able to prove that $c = 2 + o(1)$ as $q \rightarrow \infty$. Indeed, he notes that one can use the Burgess bounds on character sums to improve this to $c = 4 + o(1)$. In making these results explicit there will be some loss in the size of c . We aim to minimise this loss by using the best off-the-shelf explicit estimates. Our main result is the following.

Theorem 2. *If $\chi \bmod q$ is a real, non-principal character, with $q \geq 3$, and if β_1 and β_2 are real zeroes of $L(s, \chi)$, then*

$$\min(\beta_1, \beta_2) \leq 1 - \frac{0.933}{\log q}.$$

Throughout the course of the paper we take χ to be a primitive character. This is no great obstacle, since, as noted by Pintz [15, p. 164], if χ modulo q is induced by a primitive character χ' modulo q' , then if $L(\beta, \chi) = 0$ we have $L(\beta, \chi') = 0$. Since $q \geq q'$, we can therefore extend the result to that in Theorem 2.

The rest of this paper is organised as follows. In §2 we collect some preliminary results. We use these, in §3, to refine Pintz's result for finite ranges of q . We detail, in §3.1, our computations. These prove Theorem 2 for finite ranges of q ; we then use Pintz's original argument for large q . Finally, in §4 we outline some potential improvements to our results.

Throughout the paper ϑ will denote a complex number with modulus at most unity.

2. Preliminary lemmas

In this section we collect some results from the literature. We first note that we need not concern ourselves with small values of q . Watkins [18] showed that there are no Siegel zeroes for $L(s, \chi)$ where χ is odd, and $q \leq 3 \cdot 10^8$; Platt [16] reached the same conclusion for χ even and $q \leq 4 \cdot 10^5$.

We wish to record an explicit version of the Pólya–Vinogradov inequality, given by Lapkova [9], which improves slightly on a result by Frolenkov and Soundararajan [6].

Lemma 1 (*Lapkova*). Let χ be a primitive character with $\chi(-1) = (-1)^i$. We have that

$$\left| \sum_{n=M+1}^{M+N} \chi(n) \right| \leq \begin{cases} \frac{2}{\pi^2} \sqrt{q} \log q + 0.9467 \sqrt{q} + 1.668, & \text{if } i = 0 \\ \frac{1}{2\pi} \sqrt{q} \log q + 0.8204 \sqrt{q} + 1.0286, & \text{if } i = 1. \end{cases}$$

Define

$$A = A(q_0) = A_i(q_0) := \begin{cases} 2/\pi^2 + 0.9467/(\log q_0) + 1.668/(\sqrt{q_0} \log q_0), & \text{if } i = 0 \\ 1/2\pi + 0.8204/(\log q_0) + 1.0286/(\sqrt{q_0} \log q_0), & \text{if } i = 1, \end{cases} \quad (1)$$

so that, for $q \geq q_0 > 1$,

$$\left| \sum_{n=M+1}^{M+N} \chi(n) \right| \leq A \sqrt{q} \log q.$$

We shall make use of the Euler–Maclaurin summation formula — see [12, Thm B.5].

Lemma 2 (*Euler–Maclaurin summation*). Let k be a positive integer and $f(x)$ be $(k+1)$ times differentiable on the interval $[a, b]$, where a and b are real numbers with $a < b$. Then

$$\begin{aligned} \sum_{a < n \leq b} f(n) &= \int_a^b f(t) dt + \sum_{r=0}^k \frac{(-1)^{r+1}}{(r+1)!} \left(B_{r+1}(\{b\}) f^{(r)}(b) - B_{r+1}(\{a\}) f^{(r)}(a) \right) \\ &\quad + \frac{(-1)^k}{(k+1)!} \int_a^b B_{k+1}(x) f^{(k+1)}(x) dx, \end{aligned}$$

where $B_j(x)$ is the j th periodic Bernoulli polynomial and $B_j = B_j(0)$.

Pintz takes $k = 0$ and examines sums of $n^{-\alpha}$ and $(\log n)n^{-\alpha}$. We shall require more precision for our calculations, and, in what follows, we shall choose $k = 2$ in Lemma 2.

Lemma 3. Let $0 < \alpha < 1$. We have

$$\sum_{n \leq x} n^{-\alpha} = C_1(\alpha) + \frac{x^{1-\alpha}}{1-\alpha} + \vartheta \left\{ \frac{1}{2x^\alpha} + \frac{\alpha}{12x^{\alpha+1}} + \frac{\alpha(\alpha+1)}{36\sqrt{3}x^{\alpha+2}} \right\},$$

where

$$C_1(\alpha) = \frac{1}{2} + \frac{\alpha}{12} - \frac{1}{1-\alpha} - \frac{\alpha(\alpha+1)(\alpha+2)}{6} \int_1^\infty \frac{\{t\}^3 - \frac{3}{2}\{t\}^2 + \frac{1}{2}\{t\}}{t^{\alpha+3}} dt.$$

Similarly, we have

$$\begin{aligned} \sum_{n \leq x} (\log n) n^{-\alpha} = & C_2(\alpha) + \frac{x^{1-\alpha} \log x}{1-\alpha} - \frac{x^{1-\alpha}}{(1-\alpha)^2} \\ & + \vartheta \left\{ \frac{\log x}{2x^\alpha} + \frac{1+\alpha \log x}{12x^{1+\alpha}} \right. \\ & \left. + \frac{1}{36\sqrt{3}x^{2+\alpha}} \left(\alpha(\alpha+1) \log x + \frac{3\alpha^2+6\alpha+2}{\alpha+2} \right) \right\}, \end{aligned}$$

where

$$\begin{aligned} C_2(\alpha) = & \frac{1}{(1-\alpha)^2} - \frac{1}{12} \\ & + \frac{1}{6} \int_1^\infty \frac{(2+6\alpha+3\alpha^2 - \alpha(\alpha+1)(\alpha+2) \log t)(\{t\}^3 - \frac{3}{2}\{t\}^2 + \frac{1}{2}\{t\})}{t^{\alpha+3}} dt. \end{aligned}$$

The class-number formula allows one to show [4, p. 95] that $L(1, \chi) \gg q^{-1/2}$. We require an explicit version of this as given by Bennett, Martin, O'Bryant and Reznitz [1, Lem. 6.3 and A.10].

Lemma 4 (Bennett et al.). *If χ modulo q is a real primitive character, then*

$$L(1, \chi) \geq \begin{cases} 79.2q^{-1/2}, & \text{if } 4 \cdot 10^5 \leq q \leq 10^7 \\ 12q^{-1/2}, & \text{if } q > 10^7. \end{cases}$$

We note that Bordignon [3] has improved the constant 12 to 12.53 in Lemma 4: this has minimal impact on our calculations.

Consider

$$\sum_{n \leq x} \frac{g(n) \log n}{n^{1-\tau}}, \quad g(n) = \sum_{d|n} \chi(d), \quad (2)$$

where $0 < \tau < 1$. As in Pintz, we note that

$$g(n) = \prod_{p^e || n} (1 + \chi(p) + \chi(p^2) + \cdots + \chi(p^e)).$$

Since $\chi(n)$ is totally multiplicative we have that $g(m^2) \geq 1$. Pintz uses this to show that the first sum in (2) exceeds $(\log 4)/4$ for $x \geq 4$. We improve this, using partial summation.

Lemma 5.

$$\sum_{n \leq x} \frac{g(n) \log n}{n^{1-\tau}} \geq -2\zeta'(2-2\tau) - f(2\tau-1, x), \quad (3)$$

where

$$f(\alpha, x) = \frac{2(x^{1/2} - 1)^\alpha (1 - \alpha \log(x^{1/2} - 1))}{\alpha^2}, \quad \left(\log(x^{1/2} - 1) > \frac{1}{2\tau - 1} \right).$$

We note that we shall only apply (3) for finite values of x , and, as such, we can avoid the usual irritation about bounding terms such as $x^{1/2} - 1$ from below.

To bound sums of the form $\sum_{z < n} \chi(n) f(n)$, we use two lemmas of Bordignon. The first is a modification of Theorem 2.1 of [3].

Lemma 6. *Let χ be an even primitive Dirichlet character of modulus q . Define $N := \lfloor \sqrt{q} \rfloor - 1$. Let $f : [1, \infty) \rightarrow \mathbb{R}$ be a convex function with $f \searrow 0$. Then,*

$$\begin{aligned} \left| \sum_{n \geq z} \chi(n) f(n) \right| &\leq \left(\sum_{n=\lfloor z \rfloor}^{\lfloor z \rfloor + N} f(n) \right) - \frac{N(N+3)}{2} f(\lfloor z \rfloor + N + 1) \\ &\quad + \frac{N(N+1)}{2} f(\lfloor z \rfloor + N + 2) \\ &\quad + \frac{\sqrt{q}}{2} ((N+1)f(\lfloor z \rfloor + N + 1) - Nf(\lfloor z \rfloor + N + 2)). \end{aligned}$$

For odd characters, we use a more general result from [2].

Lemma 7 (Bordignon). *Let $g(n) : \mathbb{Z}^{\text{nonneg}} \rightarrow \{-1, 0, 1\}$ such that*

$$\max_k \left| \sum_{n=0}^k g(n) \right| = M < \infty.$$

Let f be a positive decreasing function such that

$$\int_0^\infty -f'(x) \, dx < \infty.$$

Then we have

$$\sum_{n=0}^\infty g(n) f(n) \leq \sum_{n=0}^M f(n).$$

Here we take $g = \chi$. Using Lemma 1, and bounding the sums by integrals, we have

$$\begin{aligned} \left| \sum_{z < n} \chi(n) \frac{\log n}{n^{1-\tau}} \right| &\leq \sum_{z < n \leq z+M} \frac{\log n}{n^{1-\tau}} \leq \frac{(z+M+1)^\tau}{\tau} \left(\log(z+M+1) - \frac{1}{\tau} \right) \\ &\quad - \frac{z^\tau}{\tau} \left(\log(z-1) - \frac{1}{\tau} \right), \end{aligned} \quad (4)$$

$$\left| \sum_{z < n} \chi(n) \frac{1}{n^{1-\tau}} \right| \leq \sum_{z < n \leq z+M} \frac{1}{n^{1-\tau}} \leq \frac{(z+M+1)^\tau}{\tau} - \frac{(z-1)^\tau}{\tau}, \quad (5)$$

$$\left| \sum_{z < n} \chi(n) \frac{1}{n} \right| < \sum_{z < n \leq z+M} \frac{1}{n} \leq \log(z+M+1) - \log(z-1), \quad (6)$$

where $M = A\sqrt{q} \log q$.

3. Outline of proof

Following Pintz, we apply our Lemma 3 to obtain

$$\sum_{n \leq x} \frac{g(n) \log n}{n^{1-\tau}} = T_1 + T_2 - T_3 + T_4 + \vartheta W,$$

where

$$\begin{aligned} T_1 &= C_1 \sum_{d \leq z} \chi(d) \frac{\log d}{d^{1-\tau}} \\ T_2 &= C_2 \sum_{d \leq z} \chi(d) \frac{1}{d^{1-\tau}} \\ T_3 &= \frac{x^\tau}{\tau} \left(\frac{1}{\tau} - \log x \right) \sum_{d \leq z} \chi(d) \frac{1}{d} \\ T_4 &= \sum_{z < d \leq x} \chi(d) \sum_{m \leq \frac{x}{d}} \frac{\log dm}{(dm)^{1-\tau}}, \end{aligned}$$

and

$$\begin{aligned} W &= \left(\frac{x^\tau \log x}{2x} + \frac{x^\tau (1 + \alpha \log x)}{24x} \right) \left(\frac{z}{x} + \frac{1}{x} \right) \\ &\quad + x^\tau \frac{\alpha(\alpha+1) \log x + (3\alpha^2 + 6\alpha + 2)/(\alpha+2)}{216\sqrt{3}x} \left(\frac{z}{x} + \frac{1}{x} \right) \left(\frac{2z}{x} + \frac{1}{x} \right) z, \end{aligned}$$

where $\alpha = 1 - \tau$. We bound T_4 using Abel's inequality,

$$\begin{aligned} \sum_{z < d \leq x} \chi(d) \sum_{m \leq \frac{x}{d}} \frac{\log dm}{(dm)^{1-\tau}} &\leq A \sum_{m \leq \frac{x}{z}} \frac{\log zm}{(zm)^{1-\tau}} \\ &\leq Az^{\tau-1} \left(\log z + \frac{1}{\tau} \left(\frac{x}{z} \right)^\tau \left(\log x - \frac{1}{\tau} \right) - \frac{1}{\tau} \left(\log z - \frac{1}{\tau} \right) \right). \end{aligned}$$

Let $D_1 = \frac{1}{2}x^{\tau-1} \log x$ and $D_2 = A/\tau^2$. Choosing $z = \sqrt{D_2/D_1}$ minimizes $D_1 z + D_2/z$.

We bound T_1, T_2, T_3 by using (4), (5), and (6). This produces

$$\sum_{n \leq x} \frac{g(n) \log n}{n^{1-\tau}} = C_1 L'(1-\tau) + C_2 L(1-\tau) - \frac{x}{\tau} \left(\frac{1}{\tau} - \log x \right) L(1) + E(q, \tau, x),$$

where E is an unwieldy, though easily computed, error term. Assume, that there are two zeroes of $L(s, \chi)$ with s in the interval $(1 - c/\log q, 1)$. Hence there is a value of $\tau \in (0, c/\log q)$ for which $L(1-\tau) \leq 0$ and $L'(1-\tau) = 0$. We therefore have

$$\sum_{n \leq x} \frac{g(n) \log n}{n^{1-\tau}} \leq E(q, \tau, x) - \left(\frac{1}{\tau} - \log x \right) \frac{Bx^\tau}{\tau \sqrt{q}}, \quad (\tau^{-1} \geq \log x), \quad (7)$$

where B is either 79.2 or 12, according to Lemma 4. We now invoke Lemma 5, which provides us with a contradiction if $F = F(q, \tau, x) < 0$, where

$$F = E(q, \tau, x) - \left(\frac{1}{\tau} - \log x \right) \frac{Bx^\tau}{\tau \sqrt{q}} + 2\zeta'(2-2\tau) + 2 \left(\frac{1 + (\frac{1}{2} - \tau) \log x_0}{x_0^{1/2-\tau} (1-2\tau)^2} \right) < 0,$$

subject to $x_0 \leq x \leq \exp(\tau^{-1})$.

3.1. Algorithm

To prove Theorem 2, we wish to calculate the best constant c for which $\min(\beta_1, \beta_2) \leq 1 - c/\log q$ on some range $q_0 \leq q \leq q_1$. We calculate an upper bound F^* so that $F \leq F^*$ for all $q \in [q_0, q_1]$. For a fixed c , it suffices to find $0 < x^* \leq \exp(\tau^{-1})$ so that $F^*(q, c/\log q, x) < 0$. The algorithm calculates F^* at test points x^* ; if an x^* is found so that $F^* < 0$, the algorithm increments c and restarts the search. When no admissible x^* values are found, the algorithm terminates and returns the last known c and x^* values for which Theorem 2 is true.

The algorithm is run separately for even and odd characters, using the Pólya–Vinogradov bounds in Lemma 1. The results of this computation are given in Table 1 and Table 2. This computation took 19 minutes to complete running in Sage on a 2.9 GHz processor. The code is available at <https://github.com/tsmorrill/Pintz>.

3.2. Large moduli

For q outside Table 1, we apply Lemma 3 of [15] with $x = A\sqrt{q}(\log q)/\tau^8$, where $A = A(q_0)$ as defined in (1). Suppose $L(s, \chi)$ has two zeroes in the interval $(1 - c/\log q, 1)$. Pintz considers a variation of (7): to obtain a contradiction he requires

$$\frac{6ec}{\log q} < \frac{\log 4}{4} \quad \text{and} \quad \frac{1}{\tau} - \log x > 0. \quad (8)$$

Table 1Values of c and x so that $F(q, \tau, x) < 0$ for $4 \cdot 10^5 \leq q \leq 3 \cdot 10^8$.

q_0	q_1	c_{even}	x_{even}
$4 \cdot 10^5$	$5 \cdot 10^5$	1.019	$10^{5.49}$
$5 \cdot 10^5$	$6 \cdot 10^5$	1.017	$10^{5.6}$
$6 \cdot 10^5$	$7 \cdot 10^5$	1.016	$10^{5.68}$
$7 \cdot 10^5$	$8 \cdot 10^5$	1.015	$10^{5.75}$
$8 \cdot 10^5$	$9 \cdot 10^5$	1.014	$10^{5.82}$
$9 \cdot 10^5$	10^6	1.014	$10^{5.87}$
10^6	$1.8 \cdot 10^6$	0.9933	$10^{6.04}$
$1.8 \cdot 10^6$	$3 \cdot 10^6$	0.9913	$10^{6.31}$
$3 \cdot 10^6$	$4 \cdot 10^6$	0.9964	$10^{6.5}$
$4 \cdot 10^6$	$5.4 \cdot 10^6$	0.9938	$10^{6.64}$
$5.4 \cdot 10^6$	$7.3 \cdot 10^6$	0.9917	$10^{6.78}$
$7.3 \cdot 10^6$	10^7	0.9903	$10^{6.93}$
10^7	$1.1 \cdot 10^7$	0.933	$10^{7.5}$
$1.1 \cdot 10^7$	$1.2 \cdot 10^7$	0.934	$10^{7.53}$
$1.2 \cdot 10^7$	$1.3 \cdot 10^7$	0.9363	$10^{7.56}$
$1.3 \cdot 10^7$	$1.4 \cdot 10^7$	0.9372	$10^{7.59}$
$1.4 \cdot 10^7$	$1.5 \cdot 10^7$	0.939	$10^{7.61}$
$1.5 \cdot 10^7$	$1.6 \cdot 10^7$	0.9392	$10^{7.64}$
$1.6 \cdot 10^7$	$1.7 \cdot 10^7$	0.9404	$10^{7.66}$
$1.7 \cdot 10^7$	$1.8 \cdot 10^7$	0.9414	$10^{7.68}$
$1.8 \cdot 10^7$	$1.9 \cdot 10^7$	0.9424	$10^{7.69}$
$1.9 \cdot 10^7$	$2 \cdot 10^7$	0.944	$10^{7.71}$
$2 \cdot 10^7$	$2.4 \cdot 10^7$	0.9396	$10^{7.77}$
$2.4 \cdot 10^7$	$3 \cdot 10^7$	0.9413	$10^{7.84}$
$3 \cdot 10^7$	$3.5 \cdot 10^7$	0.9476	$10^{7.89}$
$3.5 \cdot 10^7$	$4 \cdot 10^7$	0.9501	$10^{7.94}$
$4 \cdot 10^7$	$4.6 \cdot 10^7$	0.9526	$10^{7.98}$
$4.6 \cdot 10^7$	$5.4 \cdot 10^7$	0.9542	$10^{8.03}$
$5.4 \cdot 10^7$	$7.3 \cdot 10^7$	0.9522	$10^{8.12}$
$7.3 \cdot 10^7$	10^8	0.9566	$10^{8.22}$
10^8	$1.3 \cdot 10^8$	0.9638	$10^{8.3}$
$1.3 \cdot 10^8$	$1.8 \cdot 10^8$	0.9659	$10^{8.4}$
$1.8 \cdot 10^8$	$2.3 \cdot 10^8$	0.9734	$10^{8.48}$
$2.3 \cdot 10^8$	$3 \cdot 10^8$	0.9778	$10^{8.55}$

For $c = 0.933$, a quick check shows that we need $q \geq 9.2 \cdot 10^{18}$ for the first inequality in (8) to hold. Consider the second inequality: for $\tau \leq c/\log q$, we have that

$$1/\tau - \log x = 1/\tau - \log A - \log(\sqrt{q} \log q) - 8 \log 1/\tau. \quad (9)$$

If we have $\tau < 1/8$, then (9) is decreasing – fortunately, this is implied by $6ec/\log q < (\log 4)/4$. Therefore, we have

$$1/\tau - \log x \geq \left(\frac{1}{c} - \frac{1}{2}\right) \log q - \log A - 9 \log \log q + 8 \log c, \quad (10)$$

for $q \geq 9.2 \cdot 10^{18}$. To ensure that the right-side of (10) is positive, we need $q > 1.5 \cdot 10^{30}$ for even characters, and $q > 3.3 \cdot 10^{29}$ for odd characters. Thus, Theorem 2 holds for all $q > 1.5 \cdot 10^{30}$. This, along with Table 1 completes the proof.

Table 2Values of c and x so that $F(q, \tau, x) < 0$ for $3 \cdot 10^8 \leq q \leq 10^{32}$.

q_0	q_1	c_{even}	x_{even}	c_{odd}	x_{odd}
$2.3 \cdot 10^8$	$3.0 \cdot 10^8$	0.9778	$10^{8.55}$	0.9907	$10^{8.44}$
$3.0 \cdot 10^8$	$4.0 \cdot 10^8$	0.9811	$10^{8.64}$	0.9938	$10^{8.53}$
$4.0 \cdot 10^8$	$5.4 \cdot 10^8$	0.9853	$10^{8.73}$	0.999	$10^{8.61}$
$5.4 \cdot 10^8$	$7.3 \cdot 10^8$	0.9911	$10^{8.81}$	1.003	$10^{8.7}$
$7.3 \cdot 10^8$	$1.0 \cdot 10^9$	0.9958	$10^{8.9}$	1.008	$10^{8.79}$
$1.0 \cdot 10^9$	$1.8 \cdot 10^9$	0.9922	$10^{9.07}$	1.005	$10^{8.95}$
$1.8 \cdot 10^9$	$3.0 \cdot 10^9$	1.004	$10^{9.21}$	1.017	$10^{9.1}$
$3.0 \cdot 10^9$	$5.4 \cdot 10^9$	1.011	$10^{9.37}$	1.023	$10^{9.26}$
$5.4 \cdot 10^9$	$1.0 \cdot 10^{10}$	1.02	$10^{9.54}$	1.032	$10^{9.43}$
$1.0 \cdot 10^{10}$	$3.0 \cdot 10^{10}$	1.016	$10^{9.84}$	1.028	$10^{9.72}$
$3.0 \cdot 10^{10}$	$1.0 \cdot 10^{11}$	1.031	$10^{10.16}$	1.042	$10^{10.05}$
$1.0 \cdot 10^{11}$	$1.0 \cdot 10^{12}$	1.022	$10^{10.76}$	1.032	$10^{10.65}$
$1.0 \cdot 10^{12}$	$1.0 \cdot 10^{13}$	1.056	$10^{11.36}$	1.066	$10^{11.25}$
$1.0 \cdot 10^{13}$	$1.0 \cdot 10^{14}$	1.086	$10^{11.97}$	1.097	$10^{11.85}$
$1.0 \cdot 10^{14}$	$1.0 \cdot 10^{15}$	1.116	$10^{12.54}$	1.126	$10^{12.43}$
$1.0 \cdot 10^{15}$	$1.0 \cdot 10^{16}$	1.142	$10^{13.13}$	1.152	$10^{13.02}$
$1.0 \cdot 10^{16}$	$1.0 \cdot 10^{17}$	1.167	$10^{13.7}$	1.177	$10^{13.59}$
$1.0 \cdot 10^{17}$	$1.0 \cdot 10^{18}$	1.19	$10^{14.28}$	1.199	$10^{14.17}$
$1.0 \cdot 10^{18}$	$1.0 \cdot 10^{19}$	1.212	$10^{14.85}$	1.221	$10^{14.74}$
$1.0 \cdot 10^{19}$	$1.0 \cdot 10^{20}$	1.232	$10^{15.42}$	1.241	$10^{15.31}$
$1.0 \cdot 10^{20}$	$1.0 \cdot 10^{21}$	1.251	$10^{15.98}$	1.26	$10^{15.87}$
$1.0 \cdot 10^{21}$	$1.0 \cdot 10^{22}$	1.269	$10^{16.54}$	1.278	$10^{16.43}$
$1.0 \cdot 10^{22}$	$1.0 \cdot 10^{23}$	1.286	$10^{17.1}$	1.294	$10^{17.0}$
$1.0 \cdot 10^{23}$	$1.0 \cdot 10^{24}$	1.302	$10^{17.66}$	1.31	$10^{17.55}$
$1.0 \cdot 10^{24}$	$1.0 \cdot 10^{25}$	1.317	$10^{18.22}$	1.325	$10^{18.1}$
$1.0 \cdot 10^{25}$	$1.0 \cdot 10^{26}$	1.331	$10^{18.78}$	1.339	$10^{18.67}$
$1.0 \cdot 10^{26}$	$1.0 \cdot 10^{27}$	1.345	$10^{19.33}$	1.353	$10^{19.21}$
$1.0 \cdot 10^{27}$	$1.0 \cdot 10^{28}$	1.358	$10^{19.88}$	1.366	$10^{19.76}$
$1.0 \cdot 10^{28}$	$1.0 \cdot 10^{29}$	1.371	$10^{20.42}$	1.378	$10^{20.31}$
$1.0 \cdot 10^{29}$	$1.0 \cdot 10^{30}$	1.383	$10^{20.96}$	1.39	$10^{20.86}$
$1.0 \cdot 10^{30}$	$1.0 \cdot 10^{31}$	1.394	$10^{21.52}$	1.401	$10^{21.4}$
$1.0 \cdot 10^{31}$	$1.0 \cdot 10^{32}$	1.405	$10^{22.06}$	1.412	$10^{21.95}$
$1.0 \cdot 10^{32}$	$1.0 \cdot 10^{33}$	1.416	$10^{22.59}$	1.422	$10^{22.5}$

We note that the argument leading to the contradiction could be improved by replacing the $(\log 4)/4$ bound with the result from Lemma 5. However, the more difficult inequality to satisfy is (10), and retaining the $(\log 4)/4$ eases the computation.

4. Conclusions

Our result can be improved at several places. A marginally better constant in the Pólya–Vinogradov inequality in Lemma 1 gives little overall improvement. Similarly, if one extended the computation done by Bennett et al. [1], and dealt with some small values of q directly, one may improve slightly on the lower bounds on $L(1, \chi)$ in Lemma 4, and thus improve c_{even} for $10^7 \leq q \leq 10^8$, which is the value of c in Theorem 2.

More importantly, the small values of q we are forced to consider impede our calculation of c . If Platt's result were extended to show that there are no Siegel zeroes for $L(s, \chi)$ for χ even and $q \leq Q$ where $Q > 4 \cdot 10^5$, then Theorem 2 may be improved according to Table 1. Note that odd characters above $3 \cdot 10^8$ must also be dealt with to improve $c \geq 0.9845$.

We have essentially ‘lost half’ of Pintz’s $c = 2 + o(1)$ result in obtaining our Theorem 2. This gives hope to using the Burgess bounds (asymptotically giving $c = 4 + o(1)$) to improve further on our results. Explicit versions of the Burgess bounds are available when χ is a character modulo a prime — e.g. [17] and [5]. One could splice these results with the trivial and Pólya–Vinogradov estimates to improve upon our result in this restricted case.

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