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## General Section

## Subspace theorem for moving hypersurfaces and semi-decomposable form inequalities ☆

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## ABSTRACT

In this paper, a Schmidt's subspace type theorem is given for moving hypersurfaces. As the applications, we give some finiteness criteria for the solutions of the sequence of semi-decomposable form equations and inequalities.

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## 1. Introduction

As one of the most important results in Diophantine approximation, Schmidt's subspace theorem and its generalizations made it possible to give some finiteness criteria for some equations and inequalities in number theory.

In recent years, there has been some significant progress in generalizing Schmidt's subspace theorem. For example, Corvaja and Zannier [3,4] extended Schmidt's subspace theorem from hyperplanes to hypersurfaces. Evertse and Ferretti [7] established the quantitative version of the subspace theorem with hypersurfaces.

The purpose of this paper is two-fold: to give a Schmidt's subspace type theorem for moving hypersurfaces, which is motivated by the relationship between Diophantine approximation and Nevanlinna theory, and to give an application of this Schmidt's subspace type theorem in the study of the number of solutions of semi-decomposable form equations and inequalities.

Now, we recall some basic statements in number theory.

Let  $k$  be a number field of degree  $[k : \mathbb{Q}]$ . Denote by  $M_k$  the set of places (i.e., equivalence classes of absolute values) of  $k$  and write  $M_k^\infty$  for the set of archimedean places. In an archimedean class  $v \in M_k^\infty$ , we choose the absolute value  $|\cdot|_v$  such that  $|\cdot|_v = |\cdot|$  on  $\mathbb{Q}$  (the standard absolute value). For a non-archimedean class  $v \in M_k \setminus M_k^\infty$ , we let  $|p|_v = p^{-1}$  if  $v$  lies above the rational prime  $p$ . Denote by  $k_v$  the completion of  $k$  with respect to  $v$  and let  $[k_v : \mathbb{Q}_v]$  be the local degree. Let  $\|\cdot\|_v = |\cdot|_v^{[k_v : \mathbb{Q}_v]/[k : \mathbb{Q}]}$ . The norm  $\|\cdot\|_v$  satisfies the following properties:

- (i)  $\|x\|_v \geq 0$ , with equality if and only if  $x = 0$ ;
  - (ii)  $\|xy\|_v = \|x\|_v \|y\|_v$  for all  $x, y \in k$ ;
  - (iii)  $\|x_1 + \cdots + x_n\|_v \leq n^{N_v} \max\{\|x_1\|_v, \dots, \|x_n\|_v\}$  for all  $x_1, \dots, x_n \in k$ ,  $n \in \mathbb{N}$ ,
- where

$$N_v = \begin{cases} 0 & \text{if } v \text{ is non-archimedean,} \\ 1/[k : \mathbb{Q}] & \text{if } v \text{ is a real place,} \\ 2/[k : \mathbb{Q}] & \text{if } v \text{ is a complex place;} \end{cases}$$

- (iv) for each  $x \in k \setminus \{0\}$ , we have the **product formula**:

$$\prod_{v \in M_k} \|x\|_v = 1.$$

For  $v \in M_k$ , we also extend  $\|\cdot\|_v$  to an absolute value on the algebraic closure  $\bar{k}_v$ .

Let  $S$  be a finite subset of  $M_k$  containing  $M_k^\infty$ . An element  $x \in k$  is said to be an  **$S$ -integer** if  $\|x\|_v \leq 1$  for each  $v \in M_k \setminus S$ . Denote by  $\mathcal{O}_S$  the set of  $S$ -integers. The unit of  $\mathcal{O}_S$  is called  **$S$ -unit**. The set of all  $S$ -units forms a multiplicative group which is denoted by  $\mathcal{O}_S^*$ .

For  $x \in k$ , define the **logarithmic height of  $x$**  by  $h(x) = \sum_{v \in M_k} \log^+ \|x\|_v$ , where  $\log^+ \|x\|_v = \log \max\{\|x\|_v, 1\}$ . For  $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}^n(k)$ , let  $\|\mathbf{x}\|_v = \max_{0 \leq i \leq n} \|x_i\|_v$ . Define the **logarithmic height of  $\mathbf{x}$**  by

$$h(\mathbf{x}) = \sum_{v \in M_k} \log \|\mathbf{x}\|_v.$$

By the product formula, its definition is independent of the choice of the representations.

First, we recall the notion of hypersurface in  $\mathbb{P}^n$  over  $k$ . For a positive integer  $d$ , we set

$$\mathcal{I}_d := \{I = (i_0, \dots, i_n) \in \mathbb{Z}_{\geq 0}^{n+1} \mid i_0 + \cdots + i_n = d\},$$

and

$$n_d = \#\mathcal{I}_d = \binom{d+n}{n}.$$

Let  $Q(x_0, \dots, x_n) = \sum_{I \in \mathcal{I}_d} a_I \mathbf{x}^I$  be a homogeneous polynomial in  $k[x_0, \dots, x_n]$  of degree  $d$ , where  $a_I \in k$  and  $\mathbf{x}^I = x_0^{i_0} \cdots x_n^{i_n}$ . Associated with this homogeneous polynomial, a (fixed) hypersurface  $D$  of degree  $d$  in  $\mathbb{P}^n$  is defined as the zero set of  $Q$ , i.e.,

$$D := \left\{ [x_0 : \cdots : x_n] \in \mathbb{P}^n : \sum_{I \in \mathcal{I}_d} a_I \mathbf{x}^I = 0 \right\}.$$

(If  $d = 1$ , then we call  $D$  a hyperplane in  $\mathbb{P}^n$ .)

This (fixed) hypersurface  $D$  corresponds to a point  $[\cdots : a_I : \cdots]_{I \in \mathcal{I}_d} \in \mathbb{P}^{n_d-1}(k)$ , and we define the **height of  $D$**  as the height of this corresponding point, i.e.,

$$h(D) := \sum_{v \in M_k} \log \max_{I \in \mathcal{I}_d} \{ \|a_I\|_v \}.$$

For any  $v \in M_k$  and  $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}^n(k) \setminus D$ , the **Weil function with respect to  $D$**  is defined by

$$\lambda_{D,v}(\mathbf{x}) := \log \frac{\|\mathbf{x}\|_v^d \cdot \|Q\|_v}{\left\| \sum_{I \in \mathcal{I}_d} a_I \mathbf{x}^I \right\|_v},$$

where  $\|Q\|_v := \max_{I \in \mathcal{I}_d} \|a_I\|_v$ . Let  $S \subset M_k$  be a finite set containing all archimedean places. Define

$$m_S(D, \mathbf{x}) := \sum_{v \in S} \lambda_{D,v}(\mathbf{x}).$$

Hypersurfaces  $D_1, \dots, D_q$  ( $q \geq n+1$ ) are said to be **in general position** if, for any  $1 \leq j_0 < \cdots < j_n \leq q$ , the system of equations

$$Q_{j_i}(x_0, \dots, x_n) = 0, \quad 0 \leq i \leq n,$$

has only the trivial solution  $(x_0, \dots, x_n) = (0, \dots, 0)$  in  $\bar{k}^{n+1}$ , where  $\bar{k}$  is the algebraic closure of  $k$ .

We now turn the attention to moving hypersurfaces. Let  $\Lambda$  be an infinite index set. A **moving hypersurface**  $D$  indexed by  $\Lambda$  assigns, for every  $\alpha \in \Lambda$ , a hypersurface  $D(\alpha)$  in  $\mathbb{P}^n$  over  $k$ . Write

$$D(\alpha) = \left\{ [x_0 : \dots : x_n] \in \mathbb{P}^n : \sum_{I \in \mathcal{I}_d} a_I(\alpha) \mathbf{x}^I = 0 \text{ with } a_I(\alpha) \in k \right\}.$$

Then a moving hypersurface  $D$  indexed by  $\Lambda$  can be regarded as a map  $D : \Lambda \rightarrow \mathbb{P}^{n_d-1}(k)$  given by  $\alpha \mapsto [\dots : a_I(\alpha) : \dots]_{I \in \mathcal{I}_d}$ .

Due to the work of C. Osgood, S. Lang and P. Vojta (see [14,10,24,25] etc.), the closed relationship between Diophantine approximation and Nevanlinna theory is discovered. Especially, Vojta compiled a dictionary (see [24]) providing correspondences for the concepts fundamental to the two theories. In 1997, as the counterpart of the results of Ru and Stoll [18,19], Ru and Vojta [20] gave the following Schmidt's subspace theorem with moving hyperplanes.

**Theorem A.** *Let  $k$  be a number field and let  $S \subset M_k$  be a finite set containing all archimedean places. Let  $\Lambda$  be an infinite index set and let  $\mathcal{H} := \{H_1, \dots, H_q\}$  be a set of moving hyperplanes indexed by  $\Lambda$  in  $\mathbb{P}^n$ . Let  $\mathbf{x} = [x_0 : \dots : x_n] : \Lambda \rightarrow \mathbb{P}^n(k)$  be a sequence of points. Assume that*

*(i) for all  $\alpha \in \Lambda$ ,  $H_1(\alpha), \dots, H_q(\alpha)$  are in general position;*

*(ii)  $h(H_j(\alpha)) = o(h(\mathbf{x}(\alpha)))$  for all  $j = 1, \dots, q$  (that is, for all  $\delta > 0$ ,  $h(H_j(\alpha)) \leq \delta h(\mathbf{x}(\alpha))$  for all but finitely many  $\alpha \in \Lambda$ ).*

*Then, for any  $\varepsilon > 0$ ,*

*(1) there exists an infinite index subset  $A \subseteq \Lambda$  such that*

$$\sum_{j=1}^q m_S(H_j(\alpha), \mathbf{x}(\alpha)) \leq (2n + \varepsilon)h(\mathbf{x}(\alpha))$$

*holds for all  $\alpha \in A$ ;*

*(2) under an additional assumption that  $\mathbf{x}$  is linearly non-degenerate with respect to  $\mathcal{H}$  (see (i) of Definition 2.3), there exists an infinite index subset  $A \subseteq \Lambda$  such that*

$$\sum_{j=1}^q m_S(H_j(\alpha), \mathbf{x}(\alpha)) \leq (n + 1 + \varepsilon)h(\mathbf{x}(\alpha))$$

*holds for all  $\alpha \in A$ .*

As an application of (1) in Theorem A, Györy and Ru [9] studied the solutions of the following inequality.

Let  $F(x_0, \dots, x_m)$  be a homogeneous polynomial in  $m+1 (\geq 2)$  variables with coefficients in  $k$ . For each finite set  $S$  of places of  $k$  containing  $M_k^\infty$ , and for given two positive real numbers  $c$  and  $\lambda$ , we consider the solutions of the inequality

$$0 < \prod_{v \in S} \|F(x_0, \dots, x_m)\|_v \leq cH_S^\lambda(x_0, \dots, x_m) \text{ for } (x_0, \dots, x_m) \in \mathcal{O}_S^{m+1},$$

where we define the  $S$ -height as

$$H_S(x_0, \dots, x_m) = \prod_{v \in S} \max_{0 \leq i \leq m} \|x_i\|_v.$$

For simplicity, we also denote by  $\mathbf{x} := (x_0, \dots, x_m) \in k^{m+1}$  and  $H_S(\mathbf{x}) := H_S(x_0, \dots, x_m)$ . We note that, if  $\mathbf{x} \in \mathcal{O}_S^{m+1} \setminus \{\mathbf{0}\}$ , then  $H_S(\mathbf{x}) \geq 1$  and  $H_S(\eta\mathbf{x}) = H_S(\mathbf{x})$  for all  $\eta \in \mathcal{O}_S^*$ . If  $\mathbf{x}$  is a solution of  $0 < \prod_{v \in S} \|F(\mathbf{x})\|_v \leq cH_S^\lambda(\mathbf{x})$ , then so is  $\mathbf{x}' = \eta\mathbf{x}$  for each  $\eta \in \mathcal{O}_S^*$ . Such solutions  $\mathbf{x}', \mathbf{x}$  are called  $\mathcal{O}_S^*$ -proportional.

**Definition 1.1.** Let  $F(x_0, \dots, x_m)$  be a form (homogeneous polynomial) in  $m+1 (\geq 2)$  variables with coefficients in  $k$ .  $F$  is **decomposable** if it can be factorized into linear factors over some finite extension of  $k$ .

In [9], Györy and Ru gave the following result for a sequence of decomposable form inequalities.

**Theorem B.** Let  $q, m$  be positive integers. Let  $c, \lambda$  be real numbers with  $c > 0, \lambda < q - 2m$  and let  $k'$  be a finite extension of  $k$ . For  $n = 1, 2, \dots$ , let  $F_n(\mathbf{x}) = F_n(x_0, \dots, x_m) \in \mathcal{O}_S[\mathbf{x}]$  denote a decomposable form of degree  $q$  which can be factorized into linear factors over  $k'$ , and suppose that these factors are in general position for each  $n$ , which means if we consider the hyperplanes in  $\mathbb{P}^m$  defined by these linear factors, they are in general position in  $\mathbb{P}^m$ . Then there does not exist an infinite sequence of  $\mathcal{O}_S^*$ -nonproportional  $\mathbf{x}_n \in \mathcal{O}_S^{m+1}$ ,  $n = 1, 2, \dots$ , for which

$$0 < \prod_{v \in S} \|F_n(\mathbf{x}_n)\|_v \leq cH_S^\lambda(\mathbf{x}_n)$$

and

$$h(F_n) = o(h(\mathbf{x}_n)) \text{ if } h(\mathbf{x}_n) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

In 2015, as the counterpart of the result in [5], Chen-Ru-Yan [1] and Le [11] gave the following theorem, which generalized (2) of Theorem A and Corvaja-Zannier's result in [3] to the case of moving hypersurfaces.

**Theorem C.** Let  $k$  be a number field and let  $S \subset M_k$  be a finite set containing all archimedean places. Let  $\Lambda$  be an infinite index set and let  $D_j$  be moving hypersurface in  $\mathbb{P}^n$  of degree  $d_j, j = 1, \dots, q$ , indexed by  $\Lambda$ . Let  $\mathbf{x} : \Lambda \rightarrow \mathbb{P}^n(k)$  be a sequence of points. Assume that

(i)  $D_1(\alpha), \dots, D_q(\alpha)$  are in general position for each  $\alpha \in \Lambda$ ,

(ii)  $\mathbf{x}$  is algebraically non-degenerate with respect to  $\mathcal{D} := \{D_1, \dots, D_q\}$  (see (ii) of Definition 2.3),

(iii)  $h(D_j(\alpha)) = o(h(\mathbf{x}(\alpha)))$  for all  $j = 1, \dots, q$ .

Then, for any  $\varepsilon > 0$ , there exists an infinite index subset  $A \subseteq \Lambda$  such that

$$\sum_{j=1}^q d_j^{-1} m_S(D_j(\alpha), \mathbf{x}(\alpha)) \leq (n+1+\varepsilon)h(\mathbf{x}(\alpha)) \quad (1)$$

holds for all  $\alpha \in A$ .

Recently, Si [21] showed that the above inequality (1) can be replaced by

$$\sum_{j=1}^q d_j^{-1} m_S(D_j(\alpha), \mathbf{x}(\alpha)) \leq ((m-n+1)(n+1)+\varepsilon)h(\mathbf{x}(\alpha)),$$

under a weaker assumption that  $D_1(\alpha), \dots, D_q(\alpha)$  are in  $m$ -subgeneral position ( $m > n$ ) for each  $\alpha \in \Lambda$ . And Nguyen-Tran-Nguyen [13] generalized Theorem C with  $\mathbb{P}^n$  replaced by an algebraic variety of dimension  $n$ , which is also a generalization of Evertse-Ferretti's result in [7]. We note that the main theorems in [1, 11, 21, 13] are obtained under the assumption that  $\mathbf{x}(\alpha)$  is algebraically non-degenerate. Actually, this assumption is difficult to check and those results with this assumption have difficulty in application. In this paper, motivated by the recent progress in Nevanlinna theory (see [26]), we will give a Schmidt's subspace type theorem, like (1) of Theorem A, without this assumption.

**Theorem 1.1.** Let  $k$  be a number field and let  $S \subset M_k$  be a finite set containing all archimedean places. Let  $\Lambda$  be an infinite index set and let  $D_j$  be moving hypersurface in  $\mathbb{P}^n$  of degree  $d_j, j = 1, \dots, q$ , indexed by  $\Lambda$ . Let  $\mathbf{x} : \Lambda \rightarrow \mathbb{P}^n(k)$  be a sequence of points. Assume that

(i)  $D_1(\alpha), \dots, D_q(\alpha)$  are in general position for each  $\alpha \in \Lambda$ ,

(ii)  $h(D_j(\alpha)) = o(h(\mathbf{x}(\alpha)))$  for all  $j = 1, \dots, q$ .

Then, for any  $\varepsilon > 0$ , there exists an infinite index subset  $A \subseteq \Lambda$  such that

$$\sum_{j=1}^q d_j^{-1} m_S(D_j(\alpha), \mathbf{x}(\alpha)) \leq \left( \left( \frac{n}{2} + 1 \right)^2 + \varepsilon \right) h(\mathbf{x}(\alpha)) \quad (2)$$

holds for all  $\alpha \in A$ .

As a consequence, we can generalize Györy and Ru's results to the case of semi-decomposable form which was introduced in [2,17] as follows.

**Definition 1.2.** Let  $k$  be a number field. Let  $q$  be a positive integer with  $q > 1$ . A homogeneous polynomial  $F(x_0, \dots, x_m) \in k[x_0, \dots, x_m]$  ( $m$  is assumed to be  $m \geq 1$ ) is said to be **semi-decomposable** if  $F$  can be factorized into a product of homogeneous polynomials  $Q_1, \dots, Q_q$  over  $k$ .

Now, we consider the integer solutions of a sequence of semi-decomposable form inequalities.

**Theorem 1.2.** Let  $l, m$  be positive integers. Let  $k'$  be a finite extension of  $k$  and  $S \subset M_k$  be a finite set containing all archimedean places. For  $n = 1, 2, \dots$ , let  $F_n(\mathbf{x}) = F_n(x_0, \dots, x_m) \in \mathcal{O}_S[\mathbf{x}]$  denote a sequence of semi-decomposable forms of degree  $l$  for each  $n$ . Assume that  $F_n = Q_{1,n} \cdots Q_{q,n}$  over  $k'$  with  $\deg Q_{j,n} = d_j$ ,  $1 \leq j \leq q$ , for each  $n$ . Let  $d = \max_{1 \leq j \leq q} d_j$  and  $\{D_{1,n}, \dots, D_{q,n}\}$  be the moving hypersurfaces indexed by  $\mathbb{N}$  defined by  $Q_{1,n}, \dots, Q_{q,n}$ . Assume that  $l > d(\frac{m}{2} + 1)^2$  and  $D_{1,n}, \dots, D_{q,n}$  are in general position for each  $n$ . Let  $c, \lambda$  be real numbers with  $c > 0$ ,  $\lambda < l - d(\frac{m}{2} + 1)^2$ . Then, there does not exist an infinite sequence of  $\mathcal{O}_S^*$ -nonproportional  $\mathbf{x}_n \in \mathcal{O}_S^{m+1}$ ,  $n = 1, 2, \dots$ , for which

$$0 < \prod_{v \in S} \|F_n(\mathbf{x}_n)\|_v \leq c H_S^\lambda(\mathbf{x}_n) \quad (3)$$

and

$$h(F_n) = o(h(\mathbf{x}_n)) \text{ if } h(\mathbf{x}_n) \rightarrow \infty \text{ as } n \rightarrow \infty. \quad (4)$$

**Corollary 1.3.** Let  $k$  be a number field and let  $F(x_0, \dots, x_m) \in \mathcal{O}_S[\mathbf{x}]$  be a semi-decomposable form. Write  $F = Q_1 \cdots Q_q$  over  $\bar{k}$ . Assume that  $Q_1, \dots, Q_q$  are in general position which means the hypersurfaces defined by them are in general position in  $\mathbb{P}^m$ . Let  $d_j = \deg Q_j$  and  $d = \max_{1 \leq j \leq q} d_j$ . Assume that  $\deg F > d(\frac{m}{2} + 1)^2$ . Then, for every finite set  $S$  of places of  $k$  containing the archimedean places of  $k$ , for each positive number  $\lambda < \deg F - d(\frac{m}{2} + 1)^2$  and for each constant  $c > 0$ , the inequality

$$0 < \prod_{v \in S} \|F(x_0, \dots, x_m)\|_v \leq c H_S^\lambda(x_0, \dots, x_m) \text{ for } (x_0, \dots, x_m) \in \mathcal{O}_S^{m+1}$$

has only finitely many  $\mathcal{O}_S^*$ -nonproportional solutions.

It can be regarded as an improvement of Theorem 4.2 in [2].

## 2. Schmidt's subspace theorem for moving hypersurfaces

To prove Theorem 1.1, we need some preparations.

In [20], Ru and Vojta introduced the so-called coherent subsets with respect to moving targets, by which they established the counterparts of some notations in higher dimensional Nevanlinna theory. Hence, it is possible to discuss Schmidt's subspace type theorem with moving targets. Here, we introduce the coherent subsets with respect to both the moving targets and the sequence of points  $\mathbf{x}$  so that the method shown in [26] can be used in number theory.

**Definition 2.1.** Let  $D_1, \dots, D_q$  be moving hypersurfaces indexed by  $\Lambda$  of degree  $d_1, \dots, d_q$  given by, for each  $j = 1, \dots, q$  and  $\alpha \in \Lambda$ ,

$$D_j(\alpha) = \left\{ [x_0 : \dots : x_n] \in \mathbb{P}^n : \sum_{I \in \mathcal{I}_{d_j}} a_{j,I}(\alpha) \mathbf{x}^I = 0 \text{ with } a_{j,I}(\alpha) \in k \right\}. \quad (5)$$

Write  $\mathcal{I}_{d_j} = \{I_{j,1}, \dots, I_{j,n_{d_j}}\}$ . Let  $\mathbf{x} = [x_0 : \dots : x_n] : \Lambda \rightarrow \mathbb{P}^n(k)$  be a sequence of points.

(i) A subset  $A \subseteq \Lambda$  is said to be **coherent with respect to**  $\mathcal{D} = \{D_1, \dots, D_q\}$  if for every polynomial

$$P \in k[x_{1,I_{1,1}}, \dots, x_{1,I_{1,n_{d_1}}}, \dots, x_{q,I_{q,1}}, \dots, x_{q,I_{q,n_{d_q}}}]$$

that is homogeneous in  $x_{j,I_{j,1}}, \dots, x_{j,I_{j,n_{d_j}}}$  for each  $j = 1, \dots, q$ , either

$$P(a_{1,I_{1,1}}(\alpha), \dots, a_{1,I_{1,n_{d_1}}}(\alpha), \dots, a_{q,I_{q,1}}(\alpha), \dots, a_{q,I_{q,n_{d_q}}}(\alpha))$$

vanishes for all  $\alpha \in A$  or it vanishes for only finitely many  $\alpha \in A$ .

(ii) A subset  $A \subseteq \Lambda$  is said to be **coherent with respect to**  $\mathcal{D}$  and  $\mathbf{x}$  if for every polynomial

$$P \in k[x_{1,I_{1,1}}, \dots, x_{1,I_{1,n_{d_1}}}, \dots, x_{q,I_{q,1}}, \dots, x_{q,I_{q,n_{d_q}}}, x_0, \dots, x_n]$$

that is homogeneous in  $x_{j,I_{j,1}}, \dots, x_{j,I_{j,n_{d_j}}}$  for each  $j = 1, \dots, q$  and  $x_0, \dots, x_n$ , either

$$P(a_{1,I_{1,1}}(\alpha), \dots, a_{1,I_{1,n_{d_1}}}(\alpha), \dots, a_{q,I_{q,1}}(\alpha), \dots, a_{q,I_{q,n_{d_q}}}(\alpha), x_0(\alpha), \dots, x_n(\alpha))$$

vanishes for all  $\alpha \in A$  or it vanishes for only finitely many  $\alpha \in A$ .

**Remark 2.1.** By this definition, if  $A \subseteq \Lambda$  is coherent with respect to  $\mathcal{D}$  and  $\mathbf{x}$ , then  $A$  is coherent with respect to  $\mathcal{D}$ . For the existence of an infinite coherent subset with respect to  $\mathcal{D}$  (or  $\mathcal{D}$  and  $\mathbf{x}$ ), see Lemma 2.1 in [1].



For a subset  $A \subseteq \Lambda$ , we define  $\mathcal{R}_A^0$  to be the set of equivalence classes of pairs  $(C, a)$ , where  $C \subseteq A$  is a subset with finite complement;  $a : C \rightarrow k$  is a map; and the equivalence relation is defined by  $(C, a) \sim (C', a')$  if there exists  $C'' \subseteq C \cap C'$  such that  $C''$  has finite complement in  $A$  and  $a|_{C''} = a'|_{C''}$ . This is a ring containing  $k$  as a subring.

Now, let's consider the quotients of the coefficients of  $Q_j$  and the quotients of the coordinates of  $\mathbf{x}$ .

**Definition 2.2.** Let  $\Lambda$  be an infinite index set, let  $\mathcal{D} = \{D_1, \dots, D_q\}$  be a set of moving hypersurfaces indexed by  $\Lambda$  and let  $\mathbf{x}$  be a sequence of points in  $\mathbb{P}^n(k)$  indexed by  $\Lambda$ . For each  $j = 1, \dots, q$ , denote by  $Q_j(\alpha) = \sum_{I \in \mathcal{I}_{d_j}} a_{j,I}(\alpha) \mathbf{x}^I$  the defining homogeneous polynomial of  $D_j(\alpha)$ .

(i) If  $A$  is coherent with respect to  $\mathcal{D}$ , and if  $a_{j,I_t}(\alpha) \neq 0$  for all but finitely many  $\alpha \in A$  with some  $I_t \in \mathcal{I}_{d_j}$ , then

$$a_{j,I_s}/a_{j,I_t} : \{\alpha \in A : a_{j,I_t}(\alpha) \neq 0\} \rightarrow k \quad \text{with } \alpha \mapsto \frac{a_{j,I_s}(\alpha)}{a_{j,I_t}(\alpha)}$$

is a map. The equivalence class with respect to  $a_{j,I_s}/a_{j,I_t}$  defines an element of  $\mathcal{R}_A^0$ , denoted by  $a_{j,s,t}$ . Moreover, by coherence, the subring of  $\mathcal{R}_A^0$  generated over  $k$  by all such elements  $a_{j,s,t}$  is entire. We define

$$\mathcal{R}_{A,\mathcal{D}} := \left\{ \frac{P(\dots, a_{j,s,t}, \dots)}{Q(\dots, a_{j,s,t}, \dots)} : P, Q \in k[\dots, y_{j,s,t}, \dots] \right\}$$

to be the field of fractions of this subring.

(ii) If  $A$  is coherent with respect to  $\mathcal{D}$  and  $\mathbf{x}$ , and if  $x_{t'}(\alpha) \neq 0$  for all but finitely many  $\alpha \in A$  with some  $0 \leq t' \leq n$ , then

$$x_{s'}/x_{t'} : \{\alpha \in A : x_{t'}(\alpha) \neq 0\} \rightarrow k \quad \text{with } \alpha \mapsto \frac{x_{s'}(\alpha)}{x_{t'}(\alpha)}$$

is a map. Denoted by  $x_{s',t'}$  the equivalence class with respect to  $x_{s'}/x_{t'}$  in  $\mathcal{R}_A^0$ . The subring of  $\mathcal{R}_A^0$  generated over  $k$  by all such elements  $a_{j,s,t}$  and  $x_{s',t'}$  is entire. We define

$$\mathcal{R}_{A,(\mathcal{D},\mathbf{x})} := \left\{ \frac{P(\dots, a_{j,s,t}, \dots, x_{s',t'}, \dots)}{Q(\dots, a_{j,s,t}, \dots, x_{s',t'}, \dots)} : P, Q \in k[\dots, y_{j,s,t}, \dots, y_{s',t'}, \dots] \right\}$$

be the field of fractions of this subring.

**Remark 2.2.** Let  $B \subseteq A \subseteq \Lambda$  be two infinite index subsets. (i) It is clear that if  $A$  is coherent with respect to  $\mathcal{D}$  (or  $\mathcal{D}$  and  $\mathbf{x}$ ) then so is  $B$ . We have  $\mathcal{R}_{B,\mathcal{D}} = \mathcal{R}_{A,\mathcal{D}}$  ( $\mathcal{R}_{B,(\mathcal{D},\mathbf{x})} = \mathcal{R}_{A,(\mathcal{D},\mathbf{x})}$ ). (ii) If  $A$  is coherent with respect to  $\mathcal{D}$  and  $\mathbf{x}$ , then  $\mathcal{R}_{A,\mathcal{D}} \subseteq \mathcal{R}_{A,(\mathcal{D},\mathbf{x})}$ .

It is clear that for any two special representatives  $\hat{a}_1, \hat{a}_2$  of the same element  $a \in \mathcal{R}_{A,\mathcal{D}}$ , we have  $\hat{a}_1(\alpha) = \hat{a}_2(\alpha)$  for all but finitely many  $\alpha \in A$ . In this paper, for

$$a = \frac{P(\dots, a_{j,s,t}, \dots)}{Q(\dots, a_{j,s,t}, \dots)} \in \mathcal{R}_{A,\mathcal{D}}, \text{ we take the special representative of } a \text{ as } \hat{a} = \frac{P\left(\dots, \frac{a_{j,I_s}}{a_{j,I_t}}(\alpha), \dots\right)}{Q\left(\dots, \frac{a_{j,I_s}}{a_{j,I_t}}(\alpha), \dots\right)}.$$

For a polynomial  $P = \sum_I a_I \mathbf{x}^I \in \mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]$ , assume that  $\hat{a}_I$  is the special representative of  $a_I$ . Then  $\hat{P} := \sum_I \hat{a}_I \mathbf{x}^I$  is called the special representative of  $P$ . Note that such  $\hat{P}$  is well defined at all but finitely many  $\alpha \in A$ .

**Definition 2.3.** Let  $\mathbf{x}$  be a sequence of points in  $\mathbb{P}^n(k)$  indexed by  $\Lambda$ .

(i) We say that  $\mathbf{x}$  is **linearly non-degenerate with respect to  $\mathcal{D}$**  if for each infinite subset  $A \subseteq \Lambda$  that is coherent with respect to  $\mathcal{D}$ , there does not exist a linear form  $L \in \mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]$  such that  $\hat{L}(x_0(\alpha), \dots, x_n(\alpha)) = 0$  for all but finitely many  $\alpha \in A$ .

(ii) We say that  $\mathbf{x}$  is **algebraically non-degenerate with respect to  $\mathcal{D}$**  if for each infinite subset  $A \subseteq \Lambda$  that is coherent with respect to  $\mathcal{D}$ , there does not exist a non-constant homogeneous polynomial  $P \in \mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]$  such that  $\hat{P}(x_0(\alpha), \dots, x_n(\alpha)) = 0$  for all but finitely many  $\alpha \in A$ .

**Remark 2.3.** Note that when  $\mathbf{x}$  is algebraically degenerate with respect to  $\mathcal{D}$ , there exists a coherent subset  $A$  and a homogeneous polynomial  $P \in \mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]$  such that  $\hat{P}(\mathbf{x}(\alpha))$  has infinitely many zeros on  $A$ . Thus all such homogeneous polynomials may not constitute a homogeneous ideal. That is why we consider the notion of coherent subset with respect to  $\mathcal{D}$  and  $\mathbf{x}$  in the proof of Theorem 1.1.

We call  $\mathcal{D} = \{D_1, \dots, D_q\}$  in general position if and only if  $D_1(\alpha), \dots, D_q(\alpha)$  are in general position for each  $\alpha \in \Lambda$ , i.e., for each  $1 \leq j_0 < \dots < j_n \leq q$ , the system of equations

$$Q_{j_i}(\alpha)(x_0, \dots, x_n) = 0, \quad 0 \leq i \leq n$$

has no nontrivial solution in  $\bar{k}^{n+1}$ .

**Proof of Theorem 1.1.** Replacing  $Q_j$  by  $Q_j^{d/d_j}$  if necessary, where  $d$  is the least common multiple of  $d_j$ 's, we can assume that  $Q_1, \dots, Q_q$  have the same degree  $d$ . Set

$$Q_j(\alpha) = \sum_{I \in \mathcal{I}_d} a_{j,I}(\alpha) \mathbf{x}^I, \quad j = 1, \dots, q.$$

There exists an infinite index subset  $A \subseteq \Lambda$  which is coherent with respect to  $\mathcal{D}$  and  $\mathbf{x}$  and  $\mathcal{R}_{A,\mathcal{D}}$  ( $\mathcal{R}_{A,(\mathcal{D},\mathbf{x})}$ ) is the field given in Definition 2.2. By coherence of  $A$ , for each  $j$ , there exists  $a_{j,I_j}(\alpha)$ , one of the coefficients in  $Q_j(\alpha)$ , such that  $a_{j,I_j}(\alpha) \neq 0$  for all but finitely many  $\alpha \in A$ . Fix this  $a_{j,I_j}$  and set

$$\tilde{Q}_j(\alpha) = \sum_{I \in \mathcal{I}_d} \frac{a_{j,I}(\alpha)}{a_{j,I^j}(\alpha)} \mathbf{x}^I. \quad (6)$$

Note that  $\frac{a_{j,I}}{a_{j,I^j}} := \left( \{\alpha \in A \mid a_{j,I^j}(\alpha) \neq 0\}, \alpha \mapsto \frac{a_{j,I}(\alpha)}{a_{j,I^j}(\alpha)} \right)$  defines an element  $a_{j,I,I^j}$  of  $\mathcal{R}_{A,\mathcal{D}}$ , so

$$\tilde{Q}_j := \sum_{I \in \mathcal{I}_d} a_{j,I,I^j} \mathbf{x}^I \quad (7)$$

is a homogeneous polynomial with coefficients in  $\mathcal{R}_{A,\mathcal{D}}$ , i.e.,  $\tilde{Q}_j \in \mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]$ .

In addition, by Remark 2.2, if  $B$  is any infinite subset of  $A$ , then  $B$  is still coherent. Therefore, in the proof, we may freely pass to infinite subsets. For simplicity, we still denote these infinite subsets by  $A$ . Finally, we note that, for each  $a \in \mathcal{R}_{A,\mathcal{D}}$  and  $v \in M_k$ , from the assumption (ii) in Theorem 1.1, we have, for all  $\alpha \in A$ ,

$$\log \|\hat{a}(\alpha)\|_v \leq \sum_{v \in M_k} \log^+ \|\hat{a}(\alpha)\|_v = h(\hat{a}(\alpha)) = o(h(\mathbf{x}(\alpha))). \quad (8)$$

Let  $\hat{\tilde{Q}}_j(\alpha)$  be the special representative of  $\tilde{Q}_j$ , which is well defined at all but finitely many  $\alpha \in A$ . For each given  $v \in S$  and  $\alpha \in A$ , there exists a renumbering  $l^{(0)}(v, \alpha), \dots, l^{(q-1)}(v, \alpha)$  of the indices  $1, \dots, q$  such that

$$0 < \|\hat{\tilde{Q}}_{l^{(0)}(v, \alpha)}(\alpha)(\mathbf{x}(\alpha))\|_v \leq \|\hat{\tilde{Q}}_{l^{(1)}(v, \alpha)}(\alpha)(\mathbf{x}(\alpha))\|_v \leq \dots \leq \|\hat{\tilde{Q}}_{l^{(q-1)}(v, \alpha)}(\alpha)(\mathbf{x}(\alpha))\|_v.$$

By the assumption that  $D_1, \dots, D_q$  are in general position, by Lemma 2.1 in [21] (or Lemma 2.2 in [13]), we have

$$\log \prod_{j=1}^q \frac{\|\mathbf{x}(\alpha)\|_v^d \|\hat{\tilde{Q}}_j(\alpha)\|_v}{\|\hat{\tilde{Q}}_j(\alpha)(\mathbf{x}(\alpha))\|_v} \leq \log \prod_{j=0}^{n-1} \frac{\|\mathbf{x}(\alpha)\|_v^d \|\hat{\tilde{Q}}_{l^{(j)}(v, \alpha)}(\alpha)\|_v}{\|\hat{\tilde{Q}}_{l^{(j)}(v, \alpha)}(\alpha)(\mathbf{x}(\alpha))\|_v} + \log h \quad (9)$$

with  $h \in C_{\mathbf{x}}$ , where  $C_{\mathbf{x}}$  is the set of all positive functions  $g$  defined over  $A$  such that

$$\log^+(g(\alpha)) = o(h(\mathbf{x}(\alpha))).$$

If  $\mathbf{x} : \Lambda \rightarrow \mathbb{P}^n(k)$  is algebraically non-degenerate with respect to  $\mathcal{D}$ , then (2) follows directly from Theorem C. If  $\mathbf{x}$  is algebraically degenerate, then there exists  $P \in \mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]$  such that  $\hat{P}(\mathbf{x}(\alpha))$  has infinitely many zeros on  $A$ . In fact,  $\hat{P}(\mathbf{x}(\alpha)) = 0$  for all but finitely many  $\alpha \in A$ . Since  $A$  is coherent with respect to  $\mathcal{D}$  and  $\mathbf{x}$ . Assume that  $x_0(\alpha) \neq 0$  for all but finitely many  $\alpha \in A$ . Consider

$$\hat{P}(\mathbf{x}(\alpha)) = x_0(\alpha)^{\deg P} \cdot \hat{P}\left(1, \frac{x_1(\alpha)}{x_0(\alpha)}, \dots, \frac{x_n(\alpha)}{x_0(\alpha)}\right).$$

Here  $\hat{P}(1, \frac{x_1(\alpha)}{x_0(\alpha)}, \dots, \frac{x_n(\alpha)}{x_0(\alpha)})$  corresponds to an element in  $\mathcal{R}_{A,(\mathcal{D},\mathbf{x})}$ , so  $\hat{P}(1, \frac{x_1(\alpha)}{x_0(\alpha)}, \dots, \frac{x_n(\alpha)}{x_0(\alpha)})$  either has finitely many zero for  $\alpha \in A$  or is zero for all but finitely many  $\alpha \in A$ . Hence  $\hat{P}(\mathbf{x}(\alpha)) = 0$  for all but finitely many  $\alpha \in A$ . We have, for all  $P \in \mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]$ , that either  $\hat{P}(\mathbf{x}(\alpha))$  has only finitely many zero for  $\alpha \in A$ , or  $\hat{P}(\mathbf{x}(\alpha)) = 0$  for all but finitely many  $\alpha \in A$ .

We can construct a homogeneous ideal  $I_{\mathcal{R}_{A,\mathcal{D}}} \subset \mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]$  generated by all homogeneous polynomials  $P \in \mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]$  such that  $\hat{P}(\mathbf{x}(\alpha)) = 0$  for all but finitely many  $\alpha \in A$ . Obviously, for all  $P \in I_{\mathcal{R}_{A,\mathcal{D}}}$ , we have  $\hat{P}(\mathbf{x}(\alpha)) = 0$  for all but finitely many  $\alpha \in A$ . Moreover, there is no homogeneous polynomial  $P \in \mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n] \setminus I_{\mathcal{R}_{A,\mathcal{D}}}$  such that  $\hat{P}(\mathbf{x}(\alpha)) = 0$  for all but finitely many  $\alpha \in A$ . Since  $\mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]$  is a Noetherian ring,  $I_{\mathcal{R}_{A,\mathcal{D}}}$  is finitely generated. Assume that  $I_{\mathcal{R}_{A,\mathcal{D}}}$  is generated by homogeneous polynomials  $P_1, \dots, P_s$ .

Let  $\Omega$  be an algebraically closed extension of  $\mathcal{R}_{A,\mathcal{D}}$  containing  $\mathcal{R}_{A,(\mathcal{D},\mathbf{x})}$ . We now use the language of the algebraic geometry, with base field  $\mathcal{R}_{A,\mathcal{D}}$  and with the coordinates of the points taken from  $\Omega$  (see [27] or Section 2 in [26]).

Consider the projective variety  $V \subset \mathbb{P}^n(\Omega)$  (with the base field  $\mathcal{R}_{A,\mathcal{D}}$ ) constructed by  $I_{\mathcal{R}_{A,\mathcal{D}}}$ . Let  $\deg V = \Delta$  and  $\dim V = \ell$  where  $0 \leq \ell \leq n$ . We first show that  $\ell > 0$ . Otherwise, if we assume that  $x_0(\alpha)$  has only finitely many zeros in  $A$ , then  $(x_{1,0}, \dots, x_{n,0})$  lies in a zero-dimensional affine variety in the affine space  $\mathbb{A}^n(\Omega)$ . (For the notion of  $x_{j,0}$ ,  $1 \leq j \leq n$ , see (ii) of Definition 2.2.) It follows that each coordinate  $x_{j,0}$  is algebraic over  $\mathcal{R}_{A,\mathcal{D}}$  (see 16.4 in [23]), i.e.,  $x_{j,0}$  satisfies some algebraic equation  $A_t \xi^t + A_{t-1} \xi^{t-1} + \dots + A_0 \equiv 0$  with  $A_0, \dots, A_t \in \mathcal{R}_{A,\mathcal{D}}$ . Let  $P(\alpha) = [\hat{A}_0(\alpha) : \dots : \hat{A}_t(\alpha)]$ . We have

$$h\left(\frac{x_j(\alpha)}{x_0(\alpha)}\right) = h(P(\alpha)) + O(1) = o(h(\mathbf{x}(\alpha)))$$

(see (17) and (18) in [20]), which is a contradiction. Hence,  $\ell > 0$ .

For a positive integer  $N$ , let  $\mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]_N$  be the vector space of homogeneous polynomials of degree  $N$ , and let  $I_{\mathcal{R}_{A,\mathcal{D}},N} := I_{\mathcal{R}_{A,\mathcal{D}}} \cap \mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]_N$ . Denote by  $W_N := \mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]_N / I_{\mathcal{R}_{A,\mathcal{D}},N}$ . For any  $g \in \mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]_N$ , let  $[g]$  be the projection of  $g$  in  $W_N$ . We need the following basic fact of Hilbert polynomials.

**Lemma 2.1.** *There exists a positive integer  $N_0$  such that*

$$M := \dim_{\mathcal{R}_{A,\mathcal{D}}} W_N = \frac{\Delta N^\ell}{\ell!} + O(N^{\ell-1})$$

*is a polynomial of  $N$  for  $N \geq N_0$ .*

Consider  $P_1, \dots, P_s$  the generating homogeneous polynomials of  $I_{\mathcal{R}_{A,\mathcal{D}}}$ . Let  $\alpha \in A$  such that all  $\hat{P}_1, \dots, \hat{P}_s$  are well defined at  $\alpha$ . Denote by  $I(\alpha)$  the homogeneous ideal in  $k[x_0, \dots, x_n]$  generated by  $\hat{P}_1(\alpha), \dots, \hat{P}_s(\alpha)$ . Let  $V(\alpha)$  be the variety in  $\mathbb{P}^n$  defined by  $I(\alpha)$ . Then we have the following fact.

**Lemma 2.2.**  $\dim V(\alpha) = \ell$  for all but finitely many  $\alpha \in A$ .

**Proof.**  $\hat{P}_1, \dots, \hat{P}_s$  are well defined for all but finitely many  $\alpha \in A$ .

Denote by  $I(\alpha)_N := I(\alpha) \cap k[x_0, \dots, x_n]_N$ , and define

$$I_{\mathcal{R}_{A,\mathcal{D}},N}(\alpha) := \{\hat{P}(\alpha)(x_0, \dots, x_n) : P \in I_{\mathcal{R}_{A,\mathcal{D}},N} \text{ with } \hat{P} \text{ well-defined at } \alpha\}$$

which is a  $k$ -vector subspace of  $k[x_0, \dots, x_n]_N$ .

**Claim.**  $I(\alpha)_N = I_{\mathcal{R}_{A,\mathcal{D}},N}(\alpha)$  for all but finitely many  $\alpha \in A$ .

On the one hand, for all  $P \in I(\alpha)_N$ , i.e.,  $P = \sum_{j=1}^s R_j \hat{P}_j(\alpha)$  with  $R_j \in k[x_0, \dots, x_n]$ , let  $\tilde{P} = \sum_{j=1}^s R_j P_j \in I_{\mathcal{R}_{A,\mathcal{D}},N}$ , then  $\hat{\tilde{P}}$  is well defined at  $\alpha$  and  $\hat{\tilde{P}}(\alpha) = P$ , so  $P \in I_{\mathcal{R}_{A,\mathcal{D}},N}(\alpha)$  which implies  $I(\alpha)_N \subset I_{\mathcal{R}_{A,\mathcal{D}},N}(\alpha)$ .

On the other hand, assume that  $\{h_l\}_{l=1}^L$  is a basis of  $I_{\mathcal{R}_{A,\mathcal{D}},N}$ , let  $h_l = \sum_{j=1}^s R_{jl} P_j$  with  $R_{jl} \in \mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]$ . Since  $\hat{R}_{jl}$  is well defined for all but finitely many  $\alpha \in A$ , we have  $\{\hat{h}_l(\alpha)\}_{l=1}^L \subset I(\alpha)_N$  for all but finitely many  $\alpha \in A$ . By Lemma 2.3 in [21],  $\{\hat{h}_l(\alpha)\}_{l=1}^L$  is a basis of  $I_{\mathcal{R}_{A,\mathcal{D}},N}(\alpha)$  for all but finitely many  $\alpha \in A$ . From the above inclusion relation “ $I(\alpha)_N \subset I_{\mathcal{R}_{A,\mathcal{D}},N}(\alpha)$ ”, we know that  $\{\hat{h}_l(\alpha)\}_{l=1}^L$  is a basis of  $I(\alpha)_N$  for all but finitely many  $\alpha \in A$ , which completes the proof of the claim.

Combining this claim and Lemma 2.3 in [21], we have

$$\dim_k I(\alpha)_N = \dim_k I_{\mathcal{R}_{A,\mathcal{D}},N}(\alpha) = \dim_{\mathcal{R}_{A,\mathcal{D}}} I_{\mathcal{R}_{A,\mathcal{D}},N} \quad (10)$$

for all but finitely many  $\alpha \in A$ . For such an  $\alpha \in A$ , by (10) and Lemma 2.1,

$$\begin{aligned} \dim_k \frac{k[x_0, \dots, x_n]_N}{I(\alpha)_N} &= \dim_k \frac{k[x_0, \dots, x_n]_N}{I_{\mathcal{R}_{A,\mathcal{D}},N}(\alpha)} = \dim_{\mathcal{R}_{A,\mathcal{D}}} \frac{\mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]_N}{I_{\mathcal{R}_{A,\mathcal{D}},N}} \\ &= \frac{\Delta N^\ell}{\ell!} + O(N^{\ell-1}) \text{ for } N \gg 0. \end{aligned}$$

By the theory of Hilbert polynomials, we know that  $\dim V(\alpha) = \ell$ , which completes the proof.  $\square$

**Lemma 2.3.** For some  $\alpha \in A$  satisfying the following conditions:

- (i)  $\hat{P}_1, \dots, \hat{P}_s, \hat{\tilde{Q}}_{l^{(0)}(v,\alpha)}, \dots, \hat{\tilde{Q}}_{l^{(n)}(v,\alpha)}$  are well defined at  $\alpha$ ,
- (ii)  $\dim V(\alpha) = \ell$ , there exist polynomials  $\tilde{P}_1(\alpha) = \hat{\tilde{Q}}_{l^{(0)}(v,\alpha)}(\alpha), \tilde{P}_2(\alpha), \dots, \tilde{P}_{\ell+1}(\alpha) \in k[x_0, \dots, x_n]$  with

$$\tilde{P}_t(\alpha) = \sum_{i=1}^{n-\ell+t-1} c_{ti} \hat{\tilde{Q}}_{l^{(i)}(v,\alpha)}(\alpha), \quad c_{ti} \in k, \quad t \geq 2,$$

such that

$$\left\{ \begin{array}{l} \hat{P}_1(\alpha)(x_0, \dots, x_n) = 0 \\ \vdots \\ \hat{P}_s(\alpha)(x_0, \dots, x_n) = 0 \\ \tilde{P}_1(\alpha)(x_0, \dots, x_n) = 0 \\ \vdots \\ \tilde{P}_{\ell+1}(\alpha)(x_0, \dots, x_n) = 0 \end{array} \right. \quad (11)$$

have no nontrivial solution in  $\bar{k}^{n+1}$ .

(We note that the proof of Lemma 2.3 is similar to that of Lemma 3.1 in [22], which is omitted here. Moreover, (i) and (ii) are satisfied for all but finitely many  $\alpha \in A$ .)

Let

$$P_1 := \sum_{I \in \mathcal{I}_{\deg P_1}} a_{P_1, I} \mathbf{x}^I, \dots, P_s := \sum_{I \in \mathcal{I}_{\deg P_s}} a_{P_s, I} \mathbf{x}^I$$

and

$$\tilde{P}_1 = \tilde{Q}_{l^{(0)}(v, \alpha)} := \sum_{I \in \mathcal{I}_d} a_{\tilde{P}_1, I} \mathbf{x}^I, \tilde{P}_t = \sum_{i=1}^{n-\ell+t-1} c_{ti} \tilde{Q}_{l^{(i)}(v, \alpha)} := \sum_{I \in \mathcal{I}_d} a_{\tilde{P}_t, I} \mathbf{x}^I \quad (2 \leq t \leq \ell + 1)$$

be homogeneous polynomials in  $\mathcal{R}_{A, \mathcal{D}}[x_0, \dots, x_n]$ .

Set  $\tilde{P}_\beta := \sum_{I \in \mathcal{I}_{\deg P_\beta}} t_{P_\beta, I} \mathbf{x}^I$  and  $\tilde{\tilde{P}}_\gamma = \sum_{I \in \mathcal{I}_d} t_{\tilde{\tilde{P}}_\gamma, I} \mathbf{x}^I \in k[\mathbf{t}, \mathbf{x}]$ , where

$$\mathbf{t} = (\dots, t_{P_\beta, I}, \dots, t_{\tilde{\tilde{P}}_\gamma, I}, \dots),$$

$\beta = 1, \dots, s$ ,  $\gamma = 1, \dots, \ell + 1$ . There exists a finite system of homogeneous polynomials in  $\mathbf{t}$  with integer coefficients:

$$R_1(\mathbf{t}), \dots, R_m(\mathbf{t}) \text{ (called a resultant system),}$$

which are all zero if and only if  $\tilde{P}_1, \dots, \tilde{P}_s, \tilde{\tilde{P}}_1, \dots, \tilde{\tilde{P}}_{\ell+1}$  have a nontrivial common solution in  $x_0, \dots, x_n$ . Obviously

$$\hat{P}_\beta(x_0, \dots, x_n) = \tilde{P}_\beta(\dots, \hat{a}_{P_\beta, I}, \dots, 0, \dots, 0, x_0, \dots, x_n), \quad \beta = 1, \dots, s,$$

$$\hat{\tilde{P}}_\gamma(x_0, \dots, x_n) = \tilde{\tilde{P}}_\gamma(0, \dots, 0, \dots, \hat{a}_{\tilde{\tilde{P}}_\gamma, I}, \dots, x_0, \dots, x_n), \quad \gamma = 1, \dots, \ell + 1,$$

are well defined at all but finitely many  $\alpha \in A$ .

We note that, by (11), there exists some  $\alpha_0$ ,

$$\left\{ \begin{array}{l} \hat{P}_1(\alpha_0)(x_0, \dots, x_n) = 0 \\ \vdots \\ \hat{P}_s(\alpha_0)(x_0, \dots, x_n) = 0 \\ \hat{\hat{P}}_1(\alpha_0)(x_0, \dots, x_n) = 0 \\ \vdots \\ \hat{\hat{P}}_{\ell+1}(\alpha_0)(x_0, \dots, x_n) = 0 \end{array} \right.$$

have no nontrivial solution in  $\bar{k}^{n+1}$ . There exists some  $R_j(\mathbf{t})$  with

$$R_j(\dots, \hat{a}_{P_{j\beta, I}}(\alpha_0), \dots, \hat{a}_{\tilde{P}_{j\gamma, I}}(\alpha_0), \dots) \neq 0.$$

Hence, by the coherence of  $A$ ,  $R_j(\dots, \hat{a}_{P_{j\beta, I}}(\alpha), \dots, \hat{a}_{\tilde{P}_{j\gamma, I}}(\alpha), \dots) \neq 0$  for all  $\alpha \in A$  except for a finite set. That means

$$\left\{ \begin{array}{l} \hat{P}_1(\alpha)(x_0, \dots, x_n) = 0 \\ \vdots \\ \hat{P}_s(\alpha)(x_0, \dots, x_n) = 0 \\ \hat{\hat{P}}_1(\alpha)(x_0, \dots, x_n) = 0 \\ \vdots \\ \hat{\hat{P}}_{\ell+1}(\alpha)(x_0, \dots, x_n) = 0 \end{array} \right. \quad (12)$$

have no nontrivial solution in  $\bar{k}^{n+1}$  for all but finitely many  $\alpha \in A$ .

For  $\tilde{P}_1, \dots, \tilde{P}_{\ell+1}$ , we have

$$\begin{aligned} \|\hat{P}_t(\alpha)(\mathbf{x}(\alpha))\|_v &\leq C \max_{1 \leq i \leq n-\ell+t-1} \|\hat{\hat{Q}}_{l^{(i)}(v, \alpha)}(\alpha)(\mathbf{x}(\alpha))\|_v \\ &= C \|\hat{\hat{Q}}_{l^{(n-\ell+t-1)}(v, \alpha)}(\alpha)(\mathbf{x}(\alpha))\|_v \end{aligned}$$

for  $2 \leq t \leq \ell + 1$ , where  $C$  is a positive constant. Hence

$$\log \frac{\|\mathbf{x}(\alpha)\|_v^d \|\hat{\hat{Q}}_{l^{(n-\ell+t-1)}(v, \alpha)}(\alpha)\|_v}{\|\hat{\hat{Q}}_{l^{(n-\ell+t-1)}(v, \alpha)}(\alpha)(\mathbf{x}(\alpha))\|_v} \leq \log \frac{\|\mathbf{x}(\alpha)\|_v^d \|\hat{P}_t(\alpha)\|_v}{\|\hat{P}_t(\alpha)(\mathbf{x}(\alpha))\|_v} + \log h' \text{ with } h' \in C_{\mathbf{x}}$$

for  $2 \leq t \leq \ell + 1$ .

Combining with (9), we have

$$\log \prod_{j=1}^q \frac{\|\mathbf{x}(\alpha)\|_v^d \|\hat{\hat{Q}}_j(\alpha)\|_v}{\|\hat{\hat{Q}}_j(\alpha)(\mathbf{x}(\alpha))\|_v} \leq (n - \ell + 1) \log \prod_{t=1}^{\ell} \frac{\|\mathbf{x}(\alpha)\|_v^d \|\hat{P}_t(\alpha)\|_v}{\|\hat{P}_t(\alpha)(\mathbf{x}(\alpha))\|_v} + \log h'' \quad (13)$$

with  $h'' \in C_{\mathbf{x}}$ .

For every positive integer  $N$  with  $d|N$ , we use the following filtration of the vector space  $W_N$  with respect to  $\tilde{P}_1, \dots, \tilde{P}_\ell$ , which is given in [6,13] as a generalization of Corvaja-Zannier's filtration in [3].

Arrange, by the lexicographic order, the  $\ell$ -tuples  $\mathbf{i} = (i_1, \dots, i_\ell)$  of nonnegative integers and set  $\|\mathbf{i}\| = \sum_j i_j$ .

**Definition 2.4.** (i) For each  $\mathbf{i} \in \mathbb{Z}_{\geq 0}^\ell$  and nonnegative integer  $N$  with  $N \geq d\|\mathbf{i}\|$ , denote by  $I_N^{\mathbf{i}}$  the subspace of  $\mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]_{N-d\|\mathbf{i}\|}$  consisting of all  $\gamma \in \mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]_{N-d\|\mathbf{i}\|}$  such that

$$\tilde{P}_1^{i_1} \cdots \tilde{P}_\ell^{i_\ell} \gamma - \sum_{\mathbf{e}=(e_1, \dots, e_\ell) > \mathbf{i}} \tilde{P}_1^{e_1} \cdots \tilde{P}_\ell^{e_\ell} \gamma_{\mathbf{e}} \in I_{\mathcal{R}_{A,\mathcal{D}},N}$$

(or  $[\tilde{P}_1^{i_1} \cdots \tilde{P}_\ell^{i_\ell} \gamma] = [\sum_{\mathbf{e} > \mathbf{i}} \tilde{P}_1^{e_1} \cdots \tilde{P}_\ell^{e_\ell} \gamma_{\mathbf{e}}]$  on  $W_N$ ) for some  $\gamma_{\mathbf{e}} \in \mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]_{N-d\|\mathbf{e}\|}$ .

(ii) Denote by  $I^{\mathbf{i}}$  the homogeneous ideal in  $\mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]$  generated by  $\bigcup_{N \geq d\|\mathbf{i}\|} I_N^{\mathbf{i}}$ .

**Remark 2.4.** From this definition, we have the following properties.

(i)

$$(I_{\mathcal{R}_{A,\mathcal{D}}}, \tilde{P}_1, \dots, \tilde{P}_\ell)_{N-d\|\mathbf{i}\|} \subset I_N^{\mathbf{i}} \subset \mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]_{N-d\|\mathbf{i}\|},$$

where we denote by  $(I_{\mathcal{R}_{A,\mathcal{D}}}, \tilde{P}_1, \dots, \tilde{P}_\ell)$  the ideal in  $\mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]$  generated by  $I_{\mathcal{R}_{A,\mathcal{D}}} \cup \{\tilde{P}_1, \dots, \tilde{P}_\ell\}$ .

(ii)  $I^{\mathbf{i}} \cap \mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]_{N-d\|\mathbf{i}\|} = I_N^{\mathbf{i}}$ .

(iii)  $\frac{\mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]}{I^{\mathbf{i}}}$  is a graded module over  $\mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]$ .

(iv) If  $\mathbf{i}_1 - \mathbf{i}_2 := (i_{1,1} - i_{2,1}, \dots, i_{1,\ell} - i_{2,\ell}) \in \mathbb{Z}_{\geq 0}^\ell$ , then  $I_N^{\mathbf{i}_2} \subset I_{N+d\|\mathbf{i}_1\|-d\|\mathbf{i}_2\|}^{\mathbf{i}_1}$ . Hence  $I^{\mathbf{i}_2} \subset I^{\mathbf{i}_1}$ . Moreover  $\{I^{\mathbf{i}} \mid \mathbf{i} \in \mathbb{Z}_{\geq 0}^\ell\}$  is a finite set (see Lemma 2.6 in [13] or Lemma 3.6 in [26]).

Now, for an integer  $N$  big enough, divisible by  $d$ , we construct the following filtration of  $W_N$  with respect to  $\{\tilde{P}_1, \dots, \tilde{P}_\ell\}$ .

Denote by  $\tau_N$  the set of  $\mathbf{i} \in \mathbb{Z}_{\geq 0}^\ell$  with  $N - d\|\mathbf{i}\| \geq 0$ , arranged by the lexicographic order.

Define

$$W_{\mathbf{i}} = \sum_{\mathbf{e} \geq \mathbf{i}} \tilde{P}_1^{e_1} \cdots \tilde{P}_\ell^{e_\ell} \cdot \mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]_{N-d\|\mathbf{e}\|}.$$

Plainly  $W_{(0,\dots,0)} = \mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]_N$  and  $W_{\mathbf{i}} \supset W_{\mathbf{i}'}$  if  $\mathbf{i}' > \mathbf{i}$ , so  $\{W_{\mathbf{i}}\}$  is a filtration of  $\mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]_N$ . Set  $W_{\mathbf{i}}^* = \{[g] \in W_N \mid g \in W_{\mathbf{i}}\}$ , where  $[g] = g \bmod I_{\mathcal{R}_{A,\mathcal{D}},N}$ . Hence,  $\{W_{\mathbf{i}}^*\}$  is a filtration of  $W_N$ . Suppose that  $\mathbf{i}'$  follows  $\mathbf{i}$  in the lexicographic order, then



$$\frac{W_{\mathbf{i}'}^*}{W_{\mathbf{i}'}^*} \cong \frac{\mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]_{N-d\|\mathbf{i}\|}}{I_N^{\mathbf{i}}}. \quad (\text{See Lemma 3.8 in [26].})$$

Denote by

$$\Delta_N^{\mathbf{i}} := \dim_{\mathcal{R}_{A,\mathcal{D}}} \frac{W_{\mathbf{i}'}^*}{W_{\mathbf{i}'}^*} = \dim_{\mathcal{R}_{A,\mathcal{D}}} \frac{\mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]_{N-d\|\mathbf{i}\|}}{I_N^{\mathbf{i}}}.$$

**Lemma 2.4.** (i) *There exists a positive integer  $N_0$  such that, for each  $\mathbf{i} \in \mathbb{Z}_{\geq 0}^\ell$ ,  $\Delta_N^{\mathbf{i}}$  is independent of  $N$  for all  $N$  satisfying  $N - d\|\mathbf{i}\| > N_0$ .*

(ii) *There is an integer  $\overline{\Delta}$  such that  $\Delta_N^{\mathbf{i}} \leq \overline{\Delta}$  for all  $\mathbf{i} \in \mathbb{Z}_{\geq 0}^\ell$  and  $N$  satisfying  $N - d\|\mathbf{i}\| \geq 0$ .*

**Proof.** (i) For each  $\mathbf{i} \in \mathbb{Z}_{\geq 0}^\ell$ , by (iii) of Remark 2.4,  $\Delta_N^{\mathbf{i}}$  is a polynomial of  $N$  for  $N$  big enough. (See Theorem 14 in [12].)

Similar to the proof of Lemma 2.2, for all  $\alpha \in A$  except for a finite subset,

$$(I(\alpha), \hat{P}_1(\alpha), \dots, \hat{P}_\ell(\alpha))_{N-d\|\mathbf{i}\|} = (I_{\mathcal{R}_{A,\mathcal{D}}}, \tilde{P}_1, \dots, \tilde{P}_\ell)_{N-d\|\mathbf{i}\|}(\alpha)$$

and

$$\begin{aligned} \dim_k(I(\alpha), \hat{P}_1(\alpha), \dots, \hat{P}_\ell(\alpha))_{N-d\|\mathbf{i}\|} &= \dim_k(I_{\mathcal{R}_{A,\mathcal{D}}}, \tilde{P}_1, \dots, \tilde{P}_\ell)_{N-d\|\mathbf{i}\|}(\alpha) \\ &= \dim_{\mathcal{R}_{A,\mathcal{D}}}(I_{\mathcal{R}_{A,\mathcal{D}}}, \tilde{P}_1, \dots, \tilde{P}_\ell)_{N-d\|\mathbf{i}\|}. \end{aligned}$$

Thus, we can find an  $\alpha \in A$  such that

$$\begin{aligned} (1) \quad & \dim_{\mathcal{R}_{A,\mathcal{D}}} \frac{\mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]_{N-d\|\mathbf{i}\|}}{(I_{\mathcal{R}_{A,\mathcal{D}}}, \tilde{P}_1, \dots, \tilde{P}_\ell)_{N-d\|\mathbf{i}\|}} \\ &= \dim_k \frac{k[x_0, \dots, x_n]_{N-d\|\mathbf{i}\|}}{(I_{\mathcal{R}_{A,\mathcal{D}}}, \tilde{P}_1, \dots, \tilde{P}_\ell)_{N-d\|\mathbf{i}\|}(\alpha)} \\ &= \dim_k \frac{k[x_0, \dots, x_n]_{N-d\|\mathbf{i}\|}}{(I(\alpha), \hat{P}_1(\alpha), \dots, \hat{P}_\ell(\alpha))_{N-d\|\mathbf{i}\|}} \\ &= \dim_{\overline{k}} \overline{k} \otimes_k \frac{k[x_0, \dots, x_n]_{N-d\|\mathbf{i}\|}}{(I(\alpha), \hat{P}_1(\alpha), \dots, \hat{P}_\ell(\alpha))_{N-d\|\mathbf{i}\|}} \\ &= \dim_{\overline{k}} \frac{\overline{k}[x_0, \dots, x_n]_{N-d\|\mathbf{i}\|}}{(I(\alpha), \hat{P}_1(\alpha), \dots, \hat{P}_\ell(\alpha))_{N-d\|\mathbf{i}\|}^e}, \end{aligned}$$

where  $(I(\alpha), \hat{P}_1(\alpha), \dots, \hat{P}_\ell(\alpha))^e \subseteq \overline{k}[x_0, \dots, x_n]$  is the extended ideal.

(2)  $\hat{P}_1(\alpha), \dots, \hat{P}_\ell(\alpha)$  are in general position in  $V(\alpha)$  (see Lemma 2.3 and (12)).

By the theory of Hilbert functions, there exists an integer  $N_1 > 0$  such that

$$\dim_{\bar{k}} \frac{\bar{k}[x_0, \dots, x_n]_{N-d\|\mathbf{i}\|}}{(I(\alpha), \hat{\tilde{P}}_1(\alpha), \dots, \hat{\tilde{P}}_\ell(\alpha))_{N-d\|\mathbf{i}\|}^e}$$

is a constant for all  $\mathbf{i} \in \mathbb{Z}_{\geq 0}^\ell$  and  $N$  with  $N - d\|\mathbf{i}\| > N_1$ .

By (i) of Remark 2.4,

$$\dim_{\mathcal{R}_{A,\mathcal{D}}} \frac{\mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]_{N-d\|\mathbf{i}\|}}{I_N^{\mathbf{i}}} \leq \dim_{\mathcal{R}_{A,\mathcal{D}}} \frac{\mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]_{N-d\|\mathbf{i}\|}}{(I_{\mathcal{R}_{A,\mathcal{D}}}, \tilde{P}_1, \dots, \tilde{P}_\ell)_{N-d\|\mathbf{i}\|}}. \quad (14)$$

Hence, there is an integer  $N_2^{\mathbf{i}} (> N_1)$  such that  $\Delta_N^{\mathbf{i}}$  is also a constant for all  $N$  satisfying  $N - d\|\mathbf{i}\| > N_2^{\mathbf{i}}$ . Set  $\Delta^{\mathbf{i}}$  to be this constant. We note that  $N_2^{\mathbf{i}}$  depends on  $I^{\mathbf{i}}$  and  $\{I^{\mathbf{i}} | \mathbf{i} \in \mathbb{Z}_{\geq 0}^\ell\}$  is a finite set (by (iv) of Remark 2.4). Take  $N_0 = \max\{N_2^{\mathbf{i}} | \mathbf{i} \in \mathbb{Z}_{\geq 0}^\ell\}$ , we have  $\Delta^{\mathbf{i}} = \Delta_N^{\mathbf{i}}$  for all  $\mathbf{i} \in \mathbb{Z}_{\geq 0}^\ell$  and  $N$  satisfying  $N - d\|\mathbf{i}\| > N_0$ .

(ii) By (14), we have  $\Delta_N^{\mathbf{i}} \leq \dim_{\mathcal{R}_{A,\mathcal{D}}} \frac{\mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]_{N-d\|\mathbf{i}\|}}{(I_{\mathcal{R}_{A,\mathcal{D}}}, \tilde{P}_1, \dots, \tilde{P}_\ell)_{N-d\|\mathbf{i}\|}}$ . Hence, taking

$$\bar{\Delta} := \max \left\{ \dim_{\mathcal{R}_{A,\mathcal{D}}} \frac{\mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]_N}{(I_{\mathcal{R}_{A,\mathcal{D}}}, \tilde{P}_1, \dots, \tilde{P}_\ell)_N} : N = 0, 1, \dots, N_1 + 1 \right\},$$

we get (ii) of Lemma 2.4.  $\square$

Set  $\Delta_0 := \min_{\mathbf{i} \in \mathbb{Z}_{\geq 0}^\ell} \Delta^{\mathbf{i}}$ , then  $\Delta_0 = \Delta^{\mathbf{i}_0}$  for some  $\mathbf{i}_0 \in \mathbb{Z}_{\geq 0}^\ell$ .

**Remark 2.5.** By (iv) of Remark 2.4, if  $\mathbf{i} - \mathbf{i}_0 \in \mathbb{Z}_{\geq 0}^\ell$ , then  $\Delta^{\mathbf{i}} \leq \Delta^{\mathbf{i}_0}$ .

Set

$$\tau_N^0 = \{\mathbf{i} \in \tau_N | N - d\|\mathbf{i}\| > N_0 \text{ and } \mathbf{i} - \mathbf{i}_0 \in \mathbb{Z}_{\geq 0}^\ell\}.$$

We have the following properties: (i)  $\Delta_0 = \Delta^{\mathbf{i}}$  for all  $\mathbf{i} \in \tau_N^0$ . (ii)  $\#\tau_N^0 = \frac{1}{d^\ell} \frac{N^\ell}{\ell!} + O(N^{\ell-1})$ . Hence,  $\Delta_N^{\mathbf{i}} = \Delta^{\mathbf{i}}$  for all  $\mathbf{i} \in \tau_N^0$  (see Lemma 3.9 in [26]).

We choose a basis  $\mathcal{B} = \{[\psi_1], \dots, [\psi_M]\}$  of  $W_N$  with respect to the above filtration. Let  $[\psi]$  be an element of the basis, which lies in  $W_{\mathbf{i}}^*/W_{\mathbf{i}'}^*$ , we may write  $\psi = \tilde{P}_1^{i_1} \dots \tilde{P}_\ell^{i_\ell} \gamma$ , where  $\gamma \in \mathcal{R}_{A,\mathcal{D}}[x_0, \dots, x_n]_{N-d\|\mathbf{i}\|}$ . For every  $1 \leq j \leq \ell$ , we have

$$\sum_{\mathbf{i} \in \tau_N} \Delta_N^{\mathbf{i}} i_j = \frac{\Delta N^{\ell+1}}{(\ell+1)!d} + O(N^\ell). \quad (15)$$

(The proof of (15) is similar to (3.6) in [16].) Hence

$$\log \prod_{t=1}^M \frac{\|\mathbf{x}(\alpha)\|_v^N \|\hat{\psi}_t(\alpha)\|_v}{\|\hat{\psi}_t(\alpha)(\mathbf{x}(\alpha))\|_v} \geq \left( \frac{\Delta N^{\ell+1}}{(\ell+1)!d} + O(N^\ell) \right) \cdot \log \prod_{j=1}^{\ell} \frac{\|\mathbf{x}(\alpha)\|_v^d \|\hat{P}_j(\alpha)\|_v}{\|\hat{P}_j(\alpha)(\mathbf{x}(\alpha))\|_v} + \log h''' \quad (16)$$

with  $h''' \in C_{\mathbf{x}}$ .

Fix a basis  $[\phi_1], \dots, [\phi_M]$  of  $W_N$  with  $\phi_1, \dots, \phi_M \in \mathcal{R}_{A, \mathcal{D}}[x_0, \dots, x_n]$ , and let

$$\Phi(\alpha) := [\hat{\phi}_1(\alpha)(\mathbf{x}(\alpha)) : \dots : \hat{\phi}_M(\alpha)(\mathbf{x}(\alpha))] : A \rightarrow \mathbb{P}^{M-1}(k).$$

Obviously,  $h(\Phi(\alpha)) = Nh(\mathbf{x}(\alpha)) + o(h(\mathbf{x}(\alpha)))$ .

Write, for  $1 \leq t \leq M$ ,

$$\psi_t(x_0, \dots, x_n) = L_t(\phi_1(x_0, \dots, x_n), \dots, \phi_M(x_0, \dots, x_n)), \quad (17)$$

where  $L_1, \dots, L_M$  are linear forms with coefficients in  $\mathcal{R}_{A, \mathcal{D}}$  of  $M$  variables linearly independent over  $\mathcal{R}_{A, \mathcal{D}}$ . (By passing to an infinite subset if necessary, we can assume that  $\hat{L}_1(\alpha), \dots, \hat{L}_M(\alpha)$  are linearly independent over  $k$  for all  $\alpha \in A$ .) By (17), we have

$$\hat{\psi}_t(\alpha)(\mathbf{x}(\alpha)) = \hat{L}_t(\alpha)(\Phi(\alpha)) \quad \text{for } t = 1, \dots, M.$$

Since there are only finitely many choices of  $\{\tilde{Q}_{l^{(0)}(v, \alpha)}, \dots, \tilde{Q}_{l^{(n)}(v, \alpha)}\}$ , the collection of all possible linear forms  $L_t$  ( $1 \leq t \leq M$ ) is a finite set; let us denote it by  $\{L_1, \dots, L_\mu\}$ . Let  $\mathcal{H} := \{H_1, \dots, H_\mu\}$  be the set of moving hyperplanes in  $\mathbb{P}^{M-1}$  indexed by  $A$  such that  $H_j(\alpha)$  is defined by the linear forms  $\hat{L}_j(\alpha)$  for  $1 \leq j \leq \mu$ . We claim that the sequence of points  $\Phi(\alpha) = [\hat{\phi}_1(\alpha)(\mathbf{x}(\alpha)) : \dots : \hat{\phi}_M(\alpha)(\mathbf{x}(\alpha))] : A \rightarrow \mathbb{P}^{M-1}(k)$  is linearly non-degenerate with respect to  $\mathcal{H}$ . Indeed, if not, then there exists a linear form  $L \in \mathcal{R}_{B, \mathcal{H}}[x_1, \dots, x_M]$  for some infinite coherent subset index set  $B \subseteq A$  such that  $\hat{L}(\alpha)(\Phi(\alpha)) = 0$  for all but finitely many  $\alpha \in B$ , which contradicts the assumption that there is no homogeneous polynomial  $P \in \mathcal{R}_{A, \mathcal{D}}[x_0, \dots, x_n] \setminus I_{\mathcal{R}_{A, \mathcal{D}}}$  such that  $\hat{P}(\alpha)(\mathbf{x}(\alpha)) = 0$  for all but finitely many  $\alpha \in B$ . (Note that  $\mathcal{R}_{A, \mathcal{H}} \subseteq \mathcal{R}_{A, \mathcal{D}}$ .)

By (13) and (16),

$$\begin{aligned} & \frac{\Delta N^{\ell+1}}{(\ell+1)!d} (1 + o(1)) \sum_{v \in S} \log \prod_{j=1}^q \frac{\|\mathbf{x}(\alpha)\|_v^d \|\hat{Q}_j(\alpha)\|_v}{\|\hat{Q}_j(\alpha)(\mathbf{x}(\alpha))\|_v} \\ & \leq (n - \ell + 1) \sum_{v \in S} \max_K \sum_{j \in K} \log \frac{\|\Phi(\alpha)\|_v \|\hat{L}_j(\alpha)\|_v}{\|\hat{L}_j(\alpha)(\Phi(\alpha))\|_v} \\ & \quad - (n - \ell + 1) M \sum_{v \in S} \log \|\Phi(\alpha)\|_v \\ & \quad + (n - \ell + 1) MN \sum_{v \in S} \log \|\mathbf{x}(\alpha)\|_v + o(h(\mathbf{x}(\alpha))), \end{aligned} \quad (18)$$

where  $\max_K$  is taken over all subsets  $K$  of  $\{1, \dots, \mu\}$  with  $\#K = M$  such that  $\hat{L}_j(\alpha)$ ,  $j \in K$ , are linearly independent over  $k$  for all  $\alpha \in A$ . Since the above inequality is independent of the choice of components of  $\mathbf{x}(\alpha)$ , we can choose the components of  $\mathbf{x}(\alpha)$  being  $S$ -integers so that

$$\begin{aligned} \sum_{v \in S} \log \|\mathbf{x}(\alpha)\|_v &= h(\mathbf{x}(\alpha)) \text{ and } \sum_{v \in S} \log \|\Phi(\alpha)\|_v \\ &= h(\Phi(\alpha)) = Nh(\mathbf{x}(\alpha)) + o(h(\mathbf{x}(\alpha))). \end{aligned} \quad (19)$$

Combining (18) and (19), we have

$$\begin{aligned} &\frac{\Delta N^{\ell+1}}{(\ell+1)!d} (1+o(1)) \sum_{v \in S} \log \prod_{j=1}^q \frac{\|\mathbf{x}(\alpha)\|_v^d \|\hat{\tilde{Q}}_j(\alpha)\|_v}{\|\hat{\tilde{Q}}_j(\alpha)(\mathbf{x}(\alpha))\|_v} \\ &\leq (n-\ell+1) \sum_{v \in S} \max_K \sum_{j \in K} \log \frac{\|\Phi(\alpha)\|_v \|\hat{L}_j(\alpha)\|_v}{\|\hat{L}_j(\alpha)(\Phi(\alpha))\|_v} + o(h(\mathbf{x}(\alpha))). \end{aligned} \quad (20)$$

By Theorem B4.1.6 in [15] with  $\varepsilon = \frac{1}{2}$ , we have

$$\sum_{v \in S} \max_K \sum_{j \in K} \log \frac{\|\Phi(\alpha)\|_v \|\hat{L}_j(\alpha)\|_v}{\|\hat{L}_j(\alpha)(\Phi(\alpha))\|_v} \leq \left(M + \frac{1}{2}\right) h(\Phi(\alpha)) \quad (21)$$

for all  $\alpha \in A$ . Hence, by (20) and (21),

$$\begin{aligned} \frac{\Delta N^{\ell+1}}{(\ell+1)!d} (1+o(1)) \sum_{v \in S} \log \prod_{j=1}^q \frac{\|\mathbf{x}(\alpha)\|_v^d \|\hat{\tilde{Q}}_j(\alpha)\|_v}{\|\hat{\tilde{Q}}_j(\alpha)(\mathbf{x}(\alpha))\|_v} &\leq (n-\ell+1) \left(M + \frac{1}{2}\right) Nh(\mathbf{x}(\alpha)) \\ &\quad + o(h(\mathbf{x}(\alpha))), \end{aligned}$$

i.e.,

$$\frac{1}{d} \sum_{v \in S} \log \prod_{j=1}^q \frac{\|\mathbf{x}(\alpha)\|_v^d \|\hat{\tilde{Q}}_j(\alpha)\|_v}{\|\hat{\tilde{Q}}_j(\alpha)(\mathbf{x}(\alpha))\|_v} \leq (n-\ell+1)(\ell+1+o(1))h(\mathbf{x}(\alpha)).$$

For  $\varepsilon > 0$  given in Theorem 1.1, take  $N$  large enough such that  $(n-\ell+1)o(1) < \varepsilon$ . Then

$$\frac{1}{d} \sum_{v \in S} \log \prod_{j=1}^q \frac{\|\mathbf{x}(\alpha)\|_v^d \|\hat{\tilde{Q}}_j(\alpha)\|_v}{\|\hat{\tilde{Q}}_j(\alpha)(\mathbf{x}(\alpha))\|_v} \leq ((n-\ell+1)(\ell+1) + \varepsilon) h(\mathbf{x}(\alpha))$$

for all  $\alpha \in A$ .

Note that  $(n-\ell+1)(\ell+1) \leq (\frac{n}{2}+1)^2$ , which completes the proof of Theorem 1.1.  $\square$

**Remark 2.6.** Consider an algebraic variety  $V \subset \mathbb{P}^M$  with dimension  $n$ . Let  $\mathbf{x}(\alpha) : \Lambda \rightarrow V(k)$  be a sequence of points and  $D_1(\alpha), \dots, D_q(\alpha)$  be the moving hypersurfaces in  $\mathbb{P}^M$  located in  $m$ -subgeneral position ( $m > n$ ) in  $V$  for all  $\alpha \in \Lambda$ . Assume that  $h(D_j(\alpha)) = o(h(\mathbf{x}(\alpha)))$ ,  $j = 1, \dots, q$ . Then, for any  $\varepsilon > 0$ , there exists an infinite index subset  $A \subseteq \Lambda$  such that

$$\sum_{j=1}^q d_j^{-1} m_S(D_j(\alpha), \mathbf{x}(\alpha)) \leq ((m - \min\{n, m/2\} + 1)(\min\{n, m/2\} + 1) + \varepsilon) h(\mathbf{x}(\alpha)) \quad (22)$$

holds for all  $\alpha \in A$ . The proof of (22) is similar to that of Theorem 1.1. Thus it is omitted here.

### 3. Semi-decomposable form equations and inequalities

**Proof of Theorem 1.2.** Let  $S' \subset M_{k'}$  consists of the extension of the places of  $S$  to  $k'$ , then every  $S$ -integer in  $k$  is also an  $S'$ -integer in  $k'$ . Moreover, we have  $H_S(\mathbf{x}_n) = H_{S'}(\mathbf{x}_n)$  and

$$\prod_{v \in S} \|F_n(\mathbf{x}_n)\|_v = \prod_{w \in S'} \|F_n(\mathbf{x}_n)\|_w \quad \text{for } \mathbf{x}_n \in \mathcal{O}_S^{m+1}.$$

So (3) is preserved when we work on  $k'$ . Therefore, for simplicity, we assume that  $k' = k$ .

Assume that there is an infinite sequence  $\mathbf{x}_n = (x_{0,n}, \dots, x_{m,n}) \in \mathcal{O}_S^{m+1}$  which satisfies (3).

First consider the case when the values  $h(\mathbf{x}_n)$  are bounded. We may assume without loss of generality that  $x_{0,n} \neq 0$  for each  $n$ . Then the  $h(\mathbf{x}_n/x_{0,n})$  are bounded and this implies that  $\mathbf{x}_n/x_{0,n}$  may assume only finitely many values in  $k^{m+1}$ . Hence there are infinitely many  $n$  such that  $\mathbf{x}_n = x_{0,n}\mathbf{x}_0$  for some  $\mathbf{x}_0 \in k^{m+1}$ . For these  $n$ 's we deduce from (3) that

$$0 < \left( \prod_{v \in S} \|x_{0,n}\|_v \right)^l \prod_{v \in S} \|F_n(\mathbf{x}_0)\|_v \leq c \left( \prod_{v \in S} \|x_{0,n}\|_v \right)^\lambda H_S^\lambda(\mathbf{x}_0)$$

and hence  $\prod_{v \in S} \|x_{0,n}\|_v$  are bounded. Since  $x_{0,n} \in \mathcal{O}_S$ , it follows that there are infinitely many  $n$ 's for which  $x_{0,n} = \eta_n x'_0$  with some  $\eta_n \in \mathcal{O}_S^*$  and fixed  $x'_0 \in \mathcal{O}_S$  (see [8]). This implies that for these  $n$ 's the  $\mathbf{x}_n$  considered above are  $\mathcal{O}_S^*$ -proportional, which is a contradiction.

Next consider the case when  $h(\mathbf{x}_n)$  are not bounded. We may assume that  $h(\mathbf{x}_n) \rightarrow \infty$  as  $n \rightarrow \infty$ . Then, by assumption, (4) also holds. Further it follows that  $H_S(\mathbf{x}_n) \rightarrow \infty$  as  $n \rightarrow \infty$ . By  $\max_j h(Q_{j,n}) \leq h(F_n) + O(1)$  and (4), this gives

$$\max_j h(Q_{j,n}) = o(h(\mathbf{x}_n)) \text{ as } n \rightarrow \infty.$$

By Theorem 1.1, there is an infinite subsequence  $\mathbf{x}_{n_k} \in \mathcal{O}_S^{m+1}$ ,  $k = 1, 2, \dots$ , of  $\{\mathbf{x}_n\}$ , which without loss of generality we may assume to be  $\{\mathbf{x}_n\}$  itself, such that

$$\sum_{v \in S} \sum_{j=1}^q \log \frac{\|\mathbf{x}_n\|_v^{d_j} \cdot \|Q_{j,n}\|_v}{\|Q_{j,n}(\mathbf{x}_n)\|_v} \leq d \left( \left( \frac{m}{2} + 1 \right)^2 + \varepsilon \right) h(\mathbf{x}_n).$$

On the other hand, since  $\mathbf{x}_n \in \mathcal{O}_S^{m+1}$ , we have

$$h(\mathbf{x}_n) \leq \log H_S(\mathbf{x}_n).$$

Hence,

$$\prod_{v \in S} \frac{\|\mathbf{x}_n\|_v^l \cdot \prod_{j=1}^q \|Q_{j,n}\|_v}{\|F_n(\mathbf{x}_n)\|_v} \leq (H_S(\mathbf{x}_n))^{d((\frac{m}{2}+1)^2+\varepsilon)},$$

whence

$$\frac{H_S^l(\mathbf{x}_n) \cdot \prod_{v \in S} \prod_{j=1}^q \|Q_{j,n}\|_v}{\prod_{v \in S} \|F_n(\mathbf{x}_n)\|_v} \leq (H_S(\mathbf{x}_n))^{d((\frac{m}{2}+1)^2+\varepsilon)},$$

where  $\prod_{v \in S} \prod_{j=1}^q \|Q_{j,n}\|_v \geq c' \prod_{v \in S} \|F_n\|_v \geq c'$  for some positive constant  $c'$ .

Furthermore, it follows from (3) that, for  $n = 1, 2, \dots$ ,

$$\begin{aligned} H_S^l(\mathbf{x}_n) &\leq \frac{1}{c'} \prod_{v \in S} \|F_n(\mathbf{x}_n)\|_v \cdot (H_S(\mathbf{x}_n))^{d((\frac{m}{2}+1)^2+\varepsilon)} \\ &\leq \frac{c}{c'} (H_S(\mathbf{x}_n))^{d((\frac{m}{2}+1)^2+\varepsilon)+\lambda}. \end{aligned}$$

Since  $H_S(\mathbf{x}_n) \rightarrow \infty$  as  $n \rightarrow \infty$ , and  $l > \lambda + d((\frac{m}{2} + 1)^2 + \varepsilon)$ , this is a contradiction. This completes our proof of Theorem 1.2.  $\square$

We now give an application of our result on semi-decomposable form equation. Consider the  $S$ -integer solutions of a sequence of semi-decomposable form equations of the form

$$F_n(\mathbf{x}) = G_n(\mathbf{x}), \quad (23)$$

where  $G_n(\mathbf{x})$  are nonzero polynomials.

Similar to the proof of Theorem 3 in [9], Theorem 1.2 implies the following result. Let  $\mathbf{x}_n$  be the  $\mathcal{O}_S^*$ -nonproportional solutions of (23). It follows that

$$0 < \prod_{v \in S} \|F_n(\mathbf{x}_n)\|_v = \prod_{v \in S} \|G_n(\mathbf{x}_n)\|_v \leq c \prod_{v \in S} \|G_n\|_v (H_S(\mathbf{x}_n))^{\deg G_n}$$

with some positive constant  $c$ . Applying Theorem 1.2 for this  $c$  and  $l - d(\frac{m}{2} + 1)^2 - 1 < \lambda < l - d(\frac{m}{2} + 1)^2$ , we have the following theorem.

**Theorem 3.1.** *Let  $l, m$  be positive integers. Let  $k'$  be a finite extension of  $k$  and  $S \subset M_k$  be a finite set containing all archimedean places. For  $n = 1, 2, \dots$ , let  $F_n(\mathbf{x}) = F_n(x_0, \dots, x_m) \in \mathcal{O}_S[\mathbf{x}]$  denote a sequence of semi-decomposable forms of degree  $l$  for each  $n$ . Assume that  $F_n = Q_{1,n} \cdots Q_{q,n}$  over  $k'$  with  $\deg Q_{j,n} = d_j$ ,  $1 \leq j \leq q$ , for each  $n$ . Let  $d = \max_{1 \leq j \leq q} d_j$  and  $\{D_{1,n}, \dots, D_{q,n}\}$  be the moving hypersurfaces indexed by  $\mathbb{N}$  defined by  $Q_{1,n}, \dots, Q_{q,n}$ . Assume that  $l > d(\frac{m}{2} + 1)^2$  and  $D_{1,n}, \dots, D_{q,n}$  are in general position for each  $n$ . Let  $G_n(\mathbf{x}) \in \mathcal{O}_S[\mathbf{x}]$  such that  $\deg G_n < l - d(\frac{m}{2} + 1)^2$  for each  $n$ . Then there does not exist an infinite sequence of  $\mathcal{O}_S^*$ -nonproportional  $\mathbf{x}_n \in \mathcal{O}_S^{m+1}$  for which*

$$F_n(\mathbf{x}_n) = G_n(\mathbf{x}_n) \neq 0, \quad n = 1, 2, \dots,$$

$$\log \prod_{v \in S} \|G_n\|_v = o(\log H_S(\mathbf{x}_n)) \quad \text{if } H_S(\mathbf{x}_n) \rightarrow \infty \text{ as } n \rightarrow \infty$$

and

$$h(F_n) = o(h(\mathbf{x}_n)) \quad \text{if } h(\mathbf{x}_n) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

In particular, for each nonzero  $S$ -integer  $b$ , the equation

$$F(x_0, \dots, x_m) = b$$

(under the given assumption in Corollary 1.3 for  $F$ ) has finitely many  $\mathcal{O}_S^*$ -nonproportional solutions, if  $\deg F > d(\frac{m}{2} + 1)^2$  with  $d = \max_{1 \leq j \leq q} d_j$ .

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