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On the distribution properties of Niederreiter–Halton sequences

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ABSTRACT

We study the distribution properties of sequences which are a generalization of the well-known van der Corput–Halton sequence on the one hand, and digital $(\mathbf{T}, s)$-sequences in the sense of Niederreiter on the other. In this paper we completely answer the question under which conditions such a sequence is uniformly distributed in the s-dimensional unit cube, by using methods based on the q-additive property of the weighted q-ary sum-of-digits function.

1. Introduction

A sequence $(\boldsymbol{x}_n)_{n\geqslant 0}$ in the s-dimensional unit cube $[0,1)^s$, is said to be *uniformly distributed modulo one* if for all intervals $[\boldsymbol{a},\boldsymbol{b}) \subseteq [0,1)^s$ we have

$$\lim_{N\to\infty} \frac{\#\{n\colon\ 0\leqslant n<N,\ \boldsymbol{x}_n\in[\boldsymbol{a},\boldsymbol{b})\}}{N} = \lambda\big([\boldsymbol{a},\boldsymbol{b})\big),$$

where λ denotes the s-dimensional Lebesgue measure.

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The *discrepancy* D_N, which is one of the most important measures for the quality of the uniformity of a finite point set $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}$ in $[0,1)^s$ is defined by

$$D_N = D_N(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}) := \sup_{B \subseteq [0,1)^s} \left| \frac{A_N(B)}{N} - \lambda(B) \right|,$$

where $A_N(B)$ denotes $\#\{n\colon\ 0 \leqslant n < N,\ \boldsymbol{x}_n \in B\}$ and the supremum is extended over all sub-boxes B of $[0,1)^s$ of the form $B = \prod_{i=1}^{s} [a_i, b_i)$ with $0 \leqslant a_i < b_i \leqslant 1$ for $i \in \{1, \ldots, s\}$.

For an infinite sequence $\omega = (\boldsymbol{x}_n)_{n \geqslant 0}$ in $[0,1)^s$, $D_N(\omega)$ denotes the discrepancy of the first N elements of the sequence.

It is easy to check that a sequence $(\boldsymbol{x}_n)_{n \geqslant 0}$ in $[0,1)^s$ is uniformly distributed if and only if D_N tends to zero as N increases.

Excellent introductions to these and related topics can be found in the book of Kuipers and Niederreiter [9] or in the book of Drmota and Tichy [2].

The two most important concepts for the construction of uniformly distributed low-discrepancy sequences are the concept of Halton (van der Corput–Halton sequences) and the concept of Niederreiter (digital $(\mathbf{T}, s)$-sequences).

In [5] it was pointed out that these two concepts are special examples of a more general concept producing sequences which we would like to call Niederreiter–Halton sequences.

These sequences are defined in the following:

Definition 1. Let $q_1, q_2, \cdots, q_v$ be different primes and let $v, w_1, \ldots, w_v$ be positive integers. For $l \in \{1, \ldots, v\}$ and $j \in \{1, \ldots, w_l\}$ we have given $\mathbb{N} \times \mathbb{N}$-matrices over $\mathbb{Z}_{q_l}$ (i.e. the finite field of residue classes modulo q_l), of the following form:

$$C^{(l,j)} := \begin{pmatrix} \gamma_{1,0}^{(l,j)} & \gamma_{1,1}^{(l,j)} & \gamma_{1,2}^{(l,j)} & \gamma_{1,3}^{(l,j)} & \cdots \\ \gamma_{2,0}^{(l,j)} & \gamma_{2,1}^{(l,j)} & \gamma_{2,2}^{(l,j)} & \gamma_{2,3}^{(l,j)} & \cdots \\ \gamma_{3,0}^{(l,j)} & \gamma_{3,1}^{(l,j)} & \gamma_{3,2}^{(l,j)} & \gamma_{3,3}^{(l,j)} & \cdots \\ \gamma_{4,0}^{(l,j)} & \gamma_{4,1}^{(l,j)} & \gamma_{4,2}^{(l,j)} & \gamma_{4,3}^{(l,j)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathbb{Z}_{q_l}^{\mathbb{N} \times \mathbb{N}}.$$

We denote row r of the matrix $C^{(l,j)}$ by $\gamma_r^{(l,j)} = (\gamma_{r,t}^{(l,j)})_{t \geqslant 0}$ in $\mathbb{Z}_{q_l}$. We define the s-dimensional sequence $(\boldsymbol{x}_n)_{n \geqslant 0}$ in $[0,1)^s$, where $s := w_1 + \cdots + w_v$, by

$$\boldsymbol{x}_n := \left(x_n^{(1,1)}, \ldots, x_n^{(1,w_1)}, x_n^{(2,1)}, \ldots, x_n^{(2,w_2)}, \ldots, x_n^{(v,1)}, \ldots, x_n^{(v,w_v)} \right).$$

The component $x_n^{(l,j)}$, for $j \in \{1, \ldots w_l\}$, $l \in \{1, \ldots, v\}$, is generated as follows.

Let $n = n_0^{(l)} + n_1^{(l)} q_l + n_2^{(l)} q_l^2 + \cdots$ be the q_l-ary representation of n for $l \in \{1, \ldots, v\}$. Then we set (by using matrix multiplication in $\mathbb{Z}_{q_l}$)

$$C^{(l,j)} \cdot \left(n_0^{(l)}, n_1^{(l)}, \ldots \right)^\top =: \left(y_1^{(l,j)}, y_2^{(l,j)}, \ldots \right)^\top$$

and

$$x_n^{(l,j)} := \frac{y_1^{(l,j)}}{q_l} + \frac{y_2^{(l,j)}}{q_l^2} + \cdots.$$

In the case of $w_l = 1$ and $C^{(l,1)}$ is the unit matrix for all $l \in \{1, \ldots, v\}$, we get the van der Corput–Halton sequence. If we set $v = 1$ Definition 1 is consistent with the definition of digital (t, s)-sequences over $\mathbb{Z}_{q_1}$ as introduced by Niederreiter (see [12,13]) or more generally digital $(\mathbf{T}, s)$-sequences in the sense of Larcher and Niederreiter (see [11]), which for special chosen matrices provide sequences with excellent distribution properties.

The sequences defined in Definition 1 represent a hybrid of these well-known in some cases low-discrepancy sequences and it is interesting and reasonable to ask under which condition such a hybrid is uniformly distributed in the s-dimensional unit cube. We will answer this question completely in Theorem 4 of this paper.

For the statement of Theorem 4 we need the following definitions:

Definition 2. For each $l \in \{1, \ldots, v\}$ and for each choice of non-negative integers $d^{(l,1)}, \ldots, d^{(l,w_l)}$ let $F^{(l)}(d^{(l,1)}, \ldots, d^{(l,w_l)})$ be minimal such that the $(d^{(l,1)} + \cdots + d^{(l,w_l)}) \times F^{(l)}(d^{(l,1)}, \ldots, d^{(l,w_l)})$-matrix formed by

the left upper $d^{(l,1)} \times F^{(l)}(d^{(l,1)}, \ldots, d^{(l,w_l)})$-submatrix of $C^{(l,1)}$ together with
the left upper $d^{(l,2)} \times F^{(l)}(d^{(l,1)}, \ldots, d^{(l,w_l)})$-submatrix of $C^{(l,2)}$ together with
$\vdots$
the left upper $d^{(l,w_l)} \times F^{(l)}(d^{(l,1)}, \ldots, d^{(l,w_l)})$-submatrix of $C^{(l,w_l)}$

has rank $d^{(l,1)} + \cdots + d^{(l,w_l)}$ over $\mathbb{Z}_{q_l}$. If this minimum does not exist (i.e. the matrix above never has rank $d^{(l,1)} + \cdots + d^{(l,w_l)}$ over $\mathbb{Z}_{q_l}$ no matter where the rows are truncated), we set $F^{(l)}(d^{(l,1)}, \ldots, d^{(l,w_l)}) := \infty$.

Definition 3. We denote sequences as introduced in Definition 1 by digital $(\mathbf{F}, s)$-sequences in bases $((q_1, w_1), \ldots, (q_v, w_v))$, where $\mathbf{F} := (F^{(1)}, \ldots, F^{(v)})$ and the $F^{(l)} : \mathbb{N}_0^{w_l} \to \mathbb{N}_0 \cup \{\infty\}$ are given by Definition 2.

Theorem 4. *A digital $(\mathbf{F}, s)$-sequence is uniformly distributed in $[0, 1)^s$ if and only if for all non-negative $d^{(l,j)}$, $j \in \{1, \ldots, w_l\}$ and $l \in \{1, \ldots, v\}$, all $F^{(l)}(d^{(l,1)}, \ldots, d^{(l,w_l)})$ are finite.*

Digital $(\mathbf{F}, s)$-sequences were already introduced by Hofer, Kritzer, Larcher and Pillichshammer [5], but they restricted their investigation in large part to $(\mathbf{F}, s)$-sequences produced by matrices consisting of rows of finite length exclusively, which are much easier to handle than the general ones. We say a row $(\gamma_{r,t}^{(l,j)})_{t \geqslant 0}$ has finite length if there exists $t_0 \geqslant 0$, such that $\gamma_{r,t}^{(l,j)} = 0$ for all $t \geqslant t_0$. For these special kinds of $(\mathbf{F}, s)$-sequences they proved the following result [5, Theorem 1].

A digital $(\mathbf{F}, s)$-sequence produced by matrices consisting of rows of finite length exclusively is uniformly distributed in $[0, 1)^s$ if and only if for all non-negative $d^{(l,j)}$, $j \in \{1, \ldots, w_l\}$ and $l \in \{1, \ldots, v\}$, all $F^{(l)}(d^{(l,1)}, \ldots, d^{(l,w_l)})$ are finite.

The trick of their proof is to reduce the question of uniform distribution to the number of solutions of systems of congruences. By applying the Chinese remainder theorem, the number of solutions can be computed. If a matrix contains rows of infinite length, application of the Chinese remainder theorem does not succeed any more.

In this paper we develop methods for the investigation of general digital $(\mathbf{F}, s)$-sequences using several properties of the integer weighted q-ary sum-of-digits function.

Definition 5. We define the weighted q-ary sum-of-digits function of a non-negative integer n as

$$s_{q,\gamma}(n) := \sum_{r=0}^{p} n_r \gamma_r,$$

where $(\gamma_r)_{r\geqslant 0}$ is a given weight sequence in $\mathbb{R}$, $n = n_0 + n_1 q + \cdots + n_p q^p$ is the q-ary representation of n with $0 \leqslant n_i \leqslant q-1$, $p = [\log_q(n)]$ and $n_p > 0$. Especially, if $(\gamma_r)_{r\geqslant 0}$ is a given weight sequence in $\mathbb{Z}$ then we call $s_{q,\gamma}(n)$ the integer-weighted q-ary sum-of-digits function.

The distribution modulo one of multidimensional sequences based on weighted q-ary sum-of-digits function and some generalizations were investigated in large part by Hofer, Larcher and Pillichshammer (see for example [4,6,14]). Note that $y_r^{(l,j)}$ for a given $n \geqslant 0$ in Definition 1 can be interpreted as $s_{q_l,\gamma_r^{(l,j)}}(n)$ modulo q_l, where $(\gamma_r^{(l,j)})_{r\geqslant 0}$ in $\mathbb{Z}_{q_l}$ is given by row r of the matrix $C^{(l,j)}$.

Our investigation of the distribution of sequences given in Definition 1 is based on the q-additive property of the weighted q-ary sum-of-digits function. Let $q \geqslant 2$ be an integer. A function $f : \mathbb{N}_0 \to \mathbb{R}$ is called q-additive if $f(0) = 0$ and if for any non-negative integers a, b, j with $0 \leqslant b \leqslant q^j - 1$ the relation $f(aq^j + b) = f(aq^j) + f(b)$ holds. f is called completely q-additive if $f(aq^j) = f(a)$ is true for all non-negative integers a, j in addition.

Note that the (integer)-weighted q-ary sum-of-digits function is q-additive.

Hence the basic result for the proof of Theorem 4 will be the following proposition. For the proof we will heavily use techniques developed by Kim [7] also by Drmota and Larcher [3] and by Hofer [4].

Proposition 6. *Let v be a positive integer, $q_1, q_2, \ldots, q_v$ be different primes and for each $l \in \{1, \ldots, v\}$ we have given sequences $(\gamma_t^{(l)})_{t\geqslant 0}$ in $\mathbb{Z}$. The following two assertions are equivalent*:

- *there exists at least one $l \in \{1, \ldots, v\}$ such that the sequence $(\gamma_t^{(l)})_{t\geqslant 0}$ contains at least one element not congruent zero modulo q_l,*
- *the following equation holds*

$$\frac{1}{N}\sum_{n=0}^{N-1}\prod_{l=1}^{v} e\left(\frac{s_{q_l,\gamma^{(l)}}(n)}{q_l}\right) = o(1). \tag{1}$$

Here and later on $e(x)$ denotes $e^{2\pi i x}$ for any real x. There for we use techniques elaborated by D.-H. Kim, improved by Drmota and Larcher and by H.

We already mentioned that in the case $v = 1$ our Definition 1 reduces to the definition of a digital $(\mathbf{T}, w_1)$-sequence. We define $\mathbf{T} : \mathbb{N} \to \mathbb{N}_0$ as follows and call the corresponding $(\mathbf{T}, w_1)$-sequence *strict*.

Let $C_1, \ldots, C_{w_1}$ be $\mathbb{N} \times \mathbb{N}$-matrices over $\mathbb{Z}_{q_1}$. For any $m \in \mathbb{N}$ by $C_j(m)$ we define the left-upper $(m \times m)$-matrix of C_j. We set ρ_m maximal, such that for any choice of non-negative integers $d_1, \ldots, d_{w_1}$, whose sum is ρ_m, the following holds:

the first d_1 row-vectors of $C_1(m)$ together with
the first d_2 row-vectors of $C_2(m)$ together with
$\vdots$
the first d_{w_1} row-vectors of $C_{w_1}(m)$

are linearly independent over $\mathbb{Z}_{q_1}$.

We set $\mathbf{T}(m) := m - \rho_m$ for all $m \geqslant 1$. We have the following well-known result (see [11]).

A strict $(\mathbf{T}, w_1)$-sequence over $\mathbb{Z}_{q_1}$ is uniformly distributed in $[0,1)^{w_1}$ if and only if

$$\lim_{m\to\infty}\bigl(m - \mathbf{T}(m)\bigr) = +\infty.$$

It is easy to check, that the condition $(m - \mathbf{T}(m))$ *diverges as m increases* is equivalent to $F^{(1)}(d^{(1,1)}, \ldots, d^{(1,w_1)})$ *is finite for all non-negative integers* $d^{(1,1)}, \ldots, d^{(1,w_1)}$.

For digital $(\mathbf{F}, s)$-sequences we establish the notion strict digital $((\mathbf{T}^{(1)}, \ldots, \mathbf{T}^{(v)}), (w_1, \ldots, w_v))$-sequence, where for $l \in \{1, \ldots, v\}$ the $\mathbf{T}^{(l)} : \mathbb{N} \to \mathbb{N}_0$ are defined componentwise as for strict digital $(\mathbf{T}^{(l)}, w_l)$-sequences above and deduce the following corollary.

Corollary 7. *A strict digital* $((\mathbf{T}^{(1)},\dots,\mathbf{T}^{(v)}),(w_1,\dots,w_v))$*-sequence is uniformly distributed in* $[0,1)^s$ *if and only if for all* $l \in \{1,\dots,v\}$,

$$\lim_{m\to\infty}\big(m-\mathbf{T}^{(l)}(m)\big)=+\infty.$$

Proof. The statement follows by Theorem 4 and by the equivalence of the following conditions for an arbitrary $l \in \{1,\dots,v\}$: $(m-\mathbf{T}^{(l)}(m)) \to +\infty$ and $F^{(l)}(d^{(l,1)},\dots,d^{(l,w_l)})$ *is finite for all non-negative* $d^{(l,1)},\dots,d^{(l,w_l)}$. □

We summarize: To prove uniform distribution of a strict digital $((\mathbf{T}^{(1)},\dots,\mathbf{T}^{(v)}),(w_1,\dots,w_v))$-sequence it suffices to show uniform distribution for the v different projections of this digital sequence in the corresponding w_l-dimensional unit cube and it seems the smaller the $\mathbf{T}^{(l)}$ are the better the strict digital $((\mathbf{T}^{(1)},\dots,\mathbf{T}^{(v)}),(w_1,\dots,w_v))$-sequence is distributed in $[0,1)^s$. Discrepancy estimates are until now only known for some special cases, as for strict digital $(\mathbf{T},s)$-sequences (see for example [10,12,13]), for the van der Corput–Halton sequences and related ones (see for example [1,8]) and for digital $(\mathbf{L},\mathbf{F},s)$-sequences as considered in [5]. Discrepancy estimates will be the topic of forthcoming work of the author and the coauthors of [5].

The paper is organized as follows: In Section 2 we will deduce Theorem 4 from Proposition 6, which will be proved in Section 3.

Throughout the paper we fix the positive integers $v, w_1,\dots,w_v$, set $s=w_1+\cdots+w_v$ and fix the different primes $q_1,\dots,q_v$.

2. Proof of Theorem 4

We have given the $(\mathbf{F},s)$-sequence $(\boldsymbol{x}_n)_{n\geqslant 0}$ produced by the following matrices:

$$C^{(l,j)} := \begin{pmatrix} \gamma_{1,0}^{(l,j)} & \gamma_{1,1}^{(l,j)} & \gamma_{1,2}^{(l,j)} & \gamma_{1,3}^{(l,j)} & \cdots \\ \gamma_{2,0}^{(l,j)} & \gamma_{2,1}^{(l,j)} & \gamma_{2,2}^{(l,j)} & \gamma_{2,3}^{(l,j)} & \cdots \\ \gamma_{3,0}^{(l,j)} & \gamma_{3,1}^{(l,j)} & \gamma_{3,2}^{(l,j)} & \gamma_{3,3}^{(l,j)} & \cdots \\ \gamma_{4,0}^{(l,j)} & \gamma_{4,1}^{(l,j)} & \gamma_{4,2}^{(l,j)} & \gamma_{4,3}^{(l,j)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathbb{Z}_{q_l}^{\mathbb{N}\times\mathbb{N}},$$

where $j \in \{1,\dots,w_l\}$ and $l \in \{1,\dots,v\}$. We assume that for each $l \in \{1,\dots,v\}$ we can set $F^{(l)}(d^{(l,1)},\dots,d^{(l,w_l)}) < \infty$ for all possible non-negative integers $d^{(l,1)},\dots,d^{(l,w_l)}$.

We consider elementary intervals of the following form:

$$I=\prod_{l=1}^{v}\prod_{j=1}^{w_l}\left[\frac{a^{(l,j)}}{q_l^{d^{(l,j)}}},\frac{a^{(l,j)}+1}{q_l^{d^{(l,j)}}}\right),$$

where $d^{(l,j)}$ are arbitrary non-negative integers and $a^{(l,j)} \in \{0,1,\dots,q_l^{d^{(l,j)}}-1\}$ for $j \in \{1,\dots,w_l\}$ and $l \in \{1,\dots,v\}$. In order to show uniform distribution of the sequence $(\boldsymbol{x}_n)_{n\geqslant 0}$, it suffices to show that the following relation holds for each such interval

$$\lim_{N\to\infty}\frac{1}{N}\#\{0\leqslant n<N\colon\ \boldsymbol{x}_n\in I\}=\lambda(I), \tag{2}$$

where $\lambda(I)=\prod_{l=1}^{v}\prod_{j=1}^{w_l}1/(q_l^{d^{(l,j)}})$.

In the following we will compute the number of elements $\boldsymbol{x}_n$ for $0 \leqslant n < N$, which are contained in such an arbitrary interval. Just to avoid non-uniqueness in the q_l-ary representation of the components of $\boldsymbol{x}_n$ we restrict to matrices $C^{(l,j)}$ with finite columns.

If $d^{(l,j)} = 0$ for all $j \in \{0, \dots, w_l\}$ then we have $I = [0,1)^s$ and (2) is trivially fulfilled. In the following we assume at least one $d^{(l,j)} > 0$. Let $d^{(l,j)} \in \mathbb{N}_0$ and $a^{(l,j)} \in \{0, 1, \dots, q_l^{d^{(l,j)}} - 1\}$ be arbitrary but fixed for $j = 1, \dots, w_l$ and $l = 1, \dots, v$.

We regard the q_l-ary representation of $a^{(l,j)}/(q_l^{d^{(l,j)}}) = (0, a_1^{(l,j)} a_2^{(l,j)} \dots a_{d^{(l,j)}}^{(l,j)})_{q_l}$ and observe that the following condition is equivalent to $\boldsymbol{x}_n \in I$:

$$\bigwedge_{l=1}^{v} \bigwedge_{j=1}^{w_l} \bigwedge_{r=1}^{d^{(l,j)}} \left(s_{q_l, \gamma_r^{(l,j)}}(n) \equiv a_r^{(l,j)} \pmod{q_l} \right). \tag{3}$$

The empty conjunction is considered to be a tautology.

We compute $\#\{0 \leqslant n < N\colon\ \boldsymbol{x}_n \in I\}/N$ for given N by using exponential sums.

We can calculate the number of $0 \leqslant n < N$, which solve the congruence

$$s_{q_l, \gamma_r^{(l,j)}}(n) \equiv a_r^{(l,j)} \pmod{q_l}$$

using the following exponential sum

$$\sum_{n=0}^{N-1} \frac{1}{q_l} \sum_{z_r^{(l,j)}=0}^{q_l-1} e\left(z_r^{(l,j)} \left(s_{q_l, \gamma_r^{(l,j)}}(n) - a_r^{(l,j)} \right) / (q_l) \right).$$

Analogously we can compute the ratio, $R(N)$, of solutions of the system of congruences given in (3) for $0 \leqslant n < N$ by the following

$$R(N) = \frac{1}{q_1} \sum_{z_1^{(1,1)}=0}^{q_1-1} \cdots \frac{1}{q_1} \sum_{z_{d^{(1,1)}}^{(1,1)}=0}^{q_1-1} \cdots \frac{1}{q_v} \sum_{z_{d^{(v,1)}}^{(v,1)}=0}^{q_v-1} \cdots \frac{1}{q_v} \sum_{z_{d^{(v,w_v)}}^{(v,w_v)}=0}^{q_v-1} \frac{1}{N}$$
$$\times \sum_{n=0}^{N-1} \prod_{l=1}^{v} e\left(\sum_{j=1}^{w_l} \sum_{r=1}^{d^{(l,j)}} \left(\left(s_{q_l, \gamma_r^{(l,j)}}(n) - a_r^{(l,j)} \right) z_r^{(l,j)} \right) / q_l \right).$$

We have $\sum_{l=1}^{v} \sum_{j=1}^{w_l} d^{(l,j)}$ parameters $z_r^{(l,j)}$ and $\prod_{l=1}^{v} \prod_{j=1}^{w_l} q_l^{d^{(l,j)}}$ different settings of them (note that we have finitely many settings) in the term

$$\frac{1}{N} \sum_{n=0}^{N-1} \prod_{l=1}^{v} e\left(\sum_{j=1}^{w_l} \sum_{r=1}^{d^{(l,j)}} \left(\left(s_{q_l, \gamma_r^{(l,j)}}(n) - a_r^{(l,j)} \right) z_r^{(l,j)} \right) / q_l \right). \tag{4}$$

We compute (or estimate respectively) the term above for the different settings of the parameters.

If we set $z_r^{(l,j)} = 0$ for all $1 \leqslant r \leqslant d^{(l,j)}$, $1 \leqslant j \leqslant w_l$, $1 \leqslant l \leqslant v$, the term given in (4) equals 1. Together with the previous factors in the formula of the ratio, $R(N)$, we get the desired term $\lambda(I) = \prod_{l=1}^{v} \prod_{j=1}^{w_l} 1/(q_l^{d^{(l,j)}})$.

In the following we show that (4) tends to zero as N increases for each other setting of the parameters, and get after applying the triangular inequality

$$R(N) = \lambda(I) + \left(\prod_{l=1}^{v} \prod_{j=1}^{w_l} q_l^{d^{(l,j)}} - 1 \right) \cdot o(1),$$

which implies the desired relation (2).

Bearing in mind that we have at least one parameter $z_r^{(l,j)} \neq 0$ for any l, j, r in the corresponding range, we define for each l, $\overline{\gamma}^{(l)} = (\overline{\gamma}_t^{(l)})_{t \geqslant 0}$ by

$$\overline{\gamma}_t^{(l)} := \sum_{j=1}^{w_j} \sum_{r=1}^{d^{(l,j)}} \gamma_{r,t}^{(l,j)} z_r^{(l,j)}.$$

We can omit the terms in (4) which are independent of N, since the corresponding constant factor ($e^{\cdots}$) has absolute value one. By the definition of the sequences $\overline{\gamma}^{(l)}$ it remains to estimate the following term:

$$\frac{1}{N} \sum_{n=0}^{N-1} \prod_{l=1}^{v} e\left(\frac{s_{q_l, \overline{\gamma}^{(l)}}(n)}{q_l} \right).$$

In the following we observe that the condition in Proposition 6 is fulfilled. We know that at least one $z_r^{(l,j)} \neq 0$, where the corresponding $d^{(l,j)} \geqslant 1$. Let this l be fixed. We consider the string $(\overline{\gamma}_0^{(l)}, \ldots, \overline{\gamma}_{F^{(l)}-1}^{(l)})$. By our assumptions we have $0 < F^{(l)} = F^{(l)}(d^{(l,1)}, \ldots, d^{(l,w_l)}) < \infty$. We consider the matrix formed by

the left upper $d^{(l,1)} \times F^{(l)}(d^{(l,1)}, \ldots, d^{(l,w_l)})$-submatrix of $C^{(l,1)}$ together with
the left upper $d^{(l,2)} \times F^{(l)}(d^{(l,1)}, \ldots, d^{(l,w_l)})$-submatrix of $C^{(l,2)}$ together with
$\vdots$
the left upper $d^{(l,w_l)} \times F^{(l)}(d^{(l,1)}, \ldots, d^{(l,w_l)})$-submatrix of $C^{(l,w_l)}$.

We know that the rows of this matrix are linearly independent over $\mathbb{Z}_{q_l}$ by the definition of $F^{(l)}(d^{(l,1)}, \ldots, d^{(l,w_l)})$, which implies that our string $(\overline{\gamma}_0^{(l)}, \ldots, \overline{\gamma}_{F^{(l)}-1}^{(l)})$ has at least one entry not congruent zero modulo q_l, since it is a linear combination over $\mathbb{Z}_{q_l}$ of the rows of the matrix above, where at least one factor is not congruent zero modulo q_l. By applying Proposition 6 we get the term given in (4) is $o(1)$ as N increases if at least one $z_r^{(l,j)} \neq 0$. This concludes the proof of sufficiency of finite $F^{(l)}(d^{(l,1)}, \ldots, d^{(l,w_l)})$ for all non-negative integers $d^{(l,1)}, \ldots, d^{(l,w_l)}$.

Conversely: We assume that there exist l and corresponding $d^{(l,j)}$ such that $F^{(l)}(d^{(l,1)}, \ldots, d^{(l,w_l)})$ is infinite. Let this $d^{(l,j)}$ and l be fixed. We consider the projection of our sequence to the corresponding w_l-dimensional unit cube and show that it is not uniformly distributed in $[0,1)^{w_l}$, which immediately implies that $(\boldsymbol{x}_n)_{n \geqslant 0}$ is not uniformly distributed in the s-dimensional unit cube. Let $z \geqslant d$ be an arbitrary positive integer, where $d = \sum_{j=1}^{w_l} d^{(l,j)}$. We consider $0 \leqslant n < q_l^z$. We define the matrix $C_z^{(l)} \in \mathbb{Z}_{q_l}^{d \times z}$, formed by

the left upper $d^{(l,1)} \times z$-submatrix of $C^{(l,1)}$ together with
the left upper $d^{(l,2)} \times z$-submatrix of $C^{(l,2)}$ together with
$\vdots$
the left upper $d^{(l,w_l)} \times z$-submatrix of $C^{(l,w_l)}$,

and consider the equation system over $\mathbb{Z}_{q_l}$,

$$C_z^{(l)} \cdot \left(n_0^{(l)}, n_1^{(l)}, \ldots, n_{z-1}^{(l)}\right)^\top =: \left(a_1^{(l)}, a_2^{(l)}, \ldots, a_d^{(l)}\right)^\top.$$

Since the matrix $C_z^{(l)}$ has rank lower than d over $\mathbb{Z}_{q_l}$ ($F^{(l)}$ is infinite) the system is not solvable for all $(a_1^{(l)}, a_2^{(l)}, \ldots, a_d^{(l)}) \in \mathbb{Z}_{q_l}^d$. Hence, we can find an elementary interval of the following form:

$$I = \prod_{l=1}^{v} \prod_{j=1}^{w_l} \left[\frac{a^{(l,j)}}{q_l^{d^{(l,j)}}}, \frac{a^{(l,j)}+1}{q_l^{d^{(l,j)}}} \right),$$

where $d^{(l,j)}$ are the given non-negative integers and $a^{(l,j)} \in \{0, 1, \ldots, q_l^{d^{(l,j)}} - 1\}$, $j \in \{1, \ldots, w_l\}$ and $l \in \{1, \ldots, v\}$, that remains empty if we consider the first q_l^z points of the sequence. Actually we can find an elementary interval of the above form, that remains empty, no matter how many points we consider. This yields

$$D_N(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}) \geqslant \left| \frac{A_N(I)}{N} - \lambda(I) \right| = \prod_{l=1}^{v} \prod_{j=1}^{w_l} 1/\left(q_l^{d^{(l,j)}}\right)$$

for all positive integers N. Hence $\lim_{N\to\infty} D_N \neq 0$ and the sequence $(\boldsymbol{x}_n)_{n\geqslant 0}$ is not uniformly distributed in $[0,1)^s$.

This concludes the proof of Theorem 4.

3. Proof of Proposition 6

We have v a positive integer, $q_1, q_2, \ldots, q_v$ different primes and for each $l \in \{1, \ldots, v\}$ let $(\gamma_t^{(l)})_{t\geqslant 0}$ be given sequences in $\mathbb{Z}$. If $\gamma_t^{(l)} \equiv 0 \pmod{q_l}$ for all l and $t \geqslant 0$ we get

$$\frac{1}{N} \sum_{n=0}^{N-1} \prod_{l=1}^{v} e\left(\frac{s_{q_l,\gamma^{(l)}}(n)}{q_l} \right) = 1$$

for all positive integers N. Hence it remains to show relation (1) if there exists at least one $l \in \{1, \ldots, v\}$ such that $(\gamma_t^{(l)})_{t\geqslant 0}$ contains at least one element not a multiple of q_l. Without loss of generality we can assume that every $(\gamma_t^{(l)})_{t\geqslant 0}$ contains at least one such element, because otherwise we omit the corresponding factors all equal to one. We distinguish two cases: In the first case we have that all $(\gamma_t^{(l)})_{t\geqslant 0}$ are finite with respect to residue classes, i.e. for all $l \in \{1, \ldots, v\}$ there exists t_l such that $\gamma_t^{(l)} \equiv 0 \pmod{q_l}$ for all $t \geqslant t_l$ (see Section 3.1). In the second case, there exists at least one $l \in \{1, \ldots, v\}$ so that $\gamma_t^{(l)} \not\equiv 0 \pmod{q_l}$ for infinitely many $t \geqslant 0$ (see Section 3.2) and we say $(\gamma_t^{(l)})_{t\geqslant 0}$ is infinite with respect to residue classes. It turns out that the second case is much harder to handle than the first one.

3.1. Finite weight-sequences with respect to residue classes for all $l \in \{1, \ldots, v\}$

For all $l \in \{1, \ldots, v\}$ we set t_l minimal such that $\gamma_t^{(l)} \equiv 0 \pmod{q_l}$ for all $t \geqslant t_l$ and deduce the following relation:

$$\sum_{n=0}^{Q-1} \prod_{l=1}^{v} e\left(\frac{s_{q_l,\gamma^{(l)}}(n)}{q_l} \right) = 0, \tag{5}$$

where $Q = \prod_{l=1}^{v} q_l^{t_l}$. For all $l \in \{1, \ldots, v\}$, we have because of the features of $\gamma^{(l)}$ the following

$$s_{q_l,\gamma^{(l)}}(n + zQ) = s_{q_l,\gamma^{(l)}}(n) \tag{6}$$

for all $n \geqslant 0$ and $z \in \mathbb{N}$.

In order to show Eq. (5) we prove for all arbitrary $0 \leqslant a_l < q_l, 1 \leqslant l \leqslant v$, the following relation:

$$\#\left\{0 \leqslant n < Q\colon \bigwedge_{l=1}^{v} \bigl(s_{q_l,\gamma^{(l)}}(n) \equiv a_l \pmod{q_l}\bigr)\right\} = Q/(q_1 \cdots q_v).$$

Let $n = n_0^{(l)} + n_1^{(l)} q_l + n_2^{(l)} q_l^2 + \cdots$ be the q_l-ary representation of n. We consider the following equation over $\mathbb{Z}_{q_l}$ for an arbitrary $l \in \{1, \dots, v\}$:

$$\bigl(\gamma_0^{(l)}, \dots, \gamma_{t_l-1}^{(l)}\bigr) \cdot \bigl(n_0^{(l)}, n_1^{(l)}, \dots, n_{t_l-1}^{(l)}\bigr)^\top = (a_l)$$

which is equivalent to

$$s_{q_l,\gamma^{(l)}}(n) \equiv a_l \pmod{q_l}. \tag{7}$$

The equation above (and therefore (7)) has exactly $q_l^{t_l-1}$ solutions in $\{0, 1, \dots, q_l^{t_l} - 1\}$, because $(\gamma_0^{(l)} \dots \gamma_{t_l-1}^{(l)})$ has rank 1 over $\mathbb{Z}_{q_l}$, since one entry is not congruent zero modulo q_l. Thus (7) is equivalent to the following condition:

$$\bigvee_{i=1}^{q_l^{t_l-1}} \bigl(n \equiv \alpha_i^{(l)} \pmod{q_l^{t_l}}\bigr),$$

where the $\alpha_i^{(l)}$ are the $q_l^{t_l-1}$ different solutions of (7) in $\{0, 1, \dots, q_l^{t_l} - 1\}$. Since the $q_1, \dots, q_v$ are different primes, by the Chinese Remainder Theorem the following condition:

$$\bigwedge_{l=1}^{v} \bigvee_{i=1}^{q_l^{t_l-1}} \bigl(n \equiv \alpha_i^{(l)} \pmod{q_l^{t_l}}\bigr)$$

has exactly $Q/(q_1 \cdots q_v)$ solutions in $\{0, 1, \dots, Q-1\}$ (note that we have $\prod_{l=1}^{v} q_l^{t_l-1}$ different systems of congruences with exactly one solution in $\{0, 1, \dots, Q-1\}$ and that all these solutions are pairwise different). Thus (5) follows. The periodic property (6) yields

$$\left|\frac{1}{N}\sum_{n=0}^{N-1}\prod_{l=1}^{v} e\left(\frac{s_{q_l,\gamma^{(l)}}(n)}{q_l}\right)\right| = \left|\frac{1}{N}\sum_{n=[N/Q]Q}^{N-1}\prod_{l=1}^{v} e\left(\frac{s_{q_l,\gamma^{(l)}}(n)}{q_l}\right)\right| \leqslant \frac{Q}{N},$$

which concludes the proof of Proposition 6 in case of finite weight sequences $(\gamma^{(l)})_{t \geqslant 0}$ with respect to residue classes.

3.2. *At least one weight sequence is infinite with respect to residue classes*

We know that there exists at least one $l \in \{1, \dots, v\}$ with $\gamma_t^{(l)} \not\equiv 0 \pmod{q_l}$ for infinitely many $t \geqslant 0$. We have to estimate the following term:

$$\frac{1}{N}\sum_{n=0}^{N-1}\prod_{l=1}^{v} e\left(\frac{s_{q_l,\gamma^{(l)}}(n)}{q_l}\right).$$

We use methods based on q-additive properties similar as in [3,4,7].

Because a product of q_l-multiplicative functions (the notion *q_l-multiplicative function* is defined in the obvious way), where the q_l are different integers, is difficult to handle, we exchange product and summation using the following lemma.

Lemma 8. *Let $q_1, \dots, q_v \geqslant 2$ be pairwise coprime integers. For $1 \leqslant l \leqslant v$ let $f_l : \mathbb{N}_0 \to \mathbb{R}$ be q_l-additive functions. We set $g_l(n) := e(f_l(n))$ and $g(n) := \prod_{l=1}^{v} g_l(n)$, then the following inequality holds*

$$\left|\sum_{n=0}^{N-1} g(n)\right|^2 \leqslant 4N^2 \prod_{l=1}^{v}\left(\frac{1}{K}\sum_{k=1}^{K}\left|\frac{1}{Q_l}\sum_{r_l=0}^{Q_l-1}\overline{g_l(r_l)}g_l(r_l+k)\right|^2\right)^{\frac{1}{v+1}} + O\left(\frac{2N^2}{K}\right), \tag{8}$$

where $K = [N^{1/(3v)}]$, $Q_l = q_l^{t_l}$ with $t_l = [2\log_{q_l}(K)]$ for $l \in \{1, \dots, v\}$.

Proof. We refer to [4, p. 38] and the references therein. □

Since $s_{q_l,\gamma^{(l)}}(n)/q_l$ is a q_l-additive function and the q_l are different primes, we can apply Lemma 8 and we show that the lth factor tends to zero as K increases under our assumption that we have infinitely many $t \geqslant 0$ such that $\gamma_t^{(l)} \not\equiv 0 \pmod{q_l}$. Note that each factor is trivially bounded by 1 in (8). For simplicity of notation we omit the index l and the superscript (l) and fix the integer $q \geqslant 2$ in the following.

For arbitrary positive integers N, K and $r \in \{0, 1\}$ we define the correlation functions

$$\Phi_N(k) = \frac{1}{N}\sum_{n=0}^{N-1}\overline{g(n)}g(n+k),$$

$$\Phi_{K,N}(r) = \frac{1}{K}\sum_{k=0}^{K-1}\overline{\Phi_N(k)}\Phi_N(k+r)$$

for $g(n) = e(s_{q,\gamma}(n)/q)$.

Note that the difference between the lth factor in (8), apart from the exponent, and $\Phi_{K,Q_l}(0)$, where $g(n) = e(s_{q_l,\gamma^{(l)}}(n)/q_l)$ is bounded by $2/K$. So it suffices to show that $\Phi_{K,Q_l}(0)$ tends to zero as K increases in order to show this asymptotic behavior for the corresponding factor. In the following we will estimate $\Phi_{K,N}(0)$ under the assumption that we have infinitely many elements not congruent zero modulo q in the given sequence $(\gamma_t)_{t\geqslant 0}$ in $\mathbb{Z}$.

We need some notation and several lemmata.

We define the superscript (b) for $b \in \mathbb{N}_0$, which changes the argument in an arithmetic function from n to $q^b n$. Note that if $f(n) = s_{q,\gamma}(n)$, $f^{(b)}(n)$ remains a weighted q-ary sum-of-digits function $s_{q,\gamma'}(n)$ with the new weight sequence $(\gamma'_t)_{t\geqslant 0} = (\gamma_{t+b})_{t\geqslant 0}$ for all $t \in \mathbb{N}_0$. For the correlation functions the superscript (b) denotes

$$\Phi_N^{(b)}(k) = \frac{1}{N}\sum_{n=0}^{N-1}\overline{g^{(b)}(n)}g^{(b)}(n+k),$$

$$\Phi_{K,N}^{(b)}(r) = \frac{1}{K}\sum_{k=0}^{K-1}\overline{\Phi_N^{(b)}(k)}\Phi_N^{(b)}(k+r),$$

where $g^{(b)}(n) = g(q^b n)$, since $g : \mathbb{N}_0 \to \mathbb{C}$ is an arithmetic function.

If $b = 0$ we will omit the superscript (b) for simplicity.

We reduce the estimate of $\Phi_{K,N}(0)$ to the one of $\Phi_{q^z L, q^z M}(0)$ for some integers z, L, M big enough, using the following lemma.

Lemma 9. *Let $\sqrt{N} \leqslant K \leqslant N$ and we set $0 \leqslant R, S < q^z$, $N = q^z M + R$, $K = q^z L + S$. Then we have*

$$\Phi_{K,N}(0) = \Phi_{q^z L, q^z M}(0) + O\big(q^z/\sqrt{N}\big). \tag{9}$$

Proof. We estimate for any non-negative integer k the following

$$\begin{aligned}
\big|\Phi_N(k) - \Phi_{q^z M}(k)\big| &= \left|\frac{1}{N}\sum_{n=0}^{N-1} \overline{g(n)} g(n+k) - \frac{1}{q^z M}\sum_{n=0}^{q^z M-1} \overline{g(n)} g(n+k)\right| \\
&= \left|\left(\frac{1}{N} - \frac{1}{q^z M}\right)\sum_{n=0}^{q^z M-1} \overline{g(n)} g(n+k) + \frac{1}{N}\sum_{n=q^z M}^{N-1} \overline{g(n)} g(n+k)\right| \\
&\leqslant 2\left(\frac{N - q^z M}{N}\right) \leqslant \frac{2q^z}{N}.
\end{aligned}$$

Since $|\Phi_N(k)| \leqslant 1$, $|\Phi_{q^z M}(k)| \leqslant 1$ and $|q^z/N| \leqslant 1$ it is easy to deduce

$$\overline{\Phi_N(k)}\Phi_N(k) = \overline{\Phi_{q^z M}(k)}\Phi_{q^z M}(k) + O\left(\frac{q^z}{N}\right),$$

where the O-constant is absolute.

We use this relation to compute $\Phi_{K,N}(0)$,

$$\begin{aligned}
\Phi_{K,N}(0) &= \frac{1}{K}\sum_{k=0}^{K-1} \overline{\Phi_N(k)}\Phi_N(k) \\
&= \frac{1}{K}\sum_{k=0}^{K-1} \overline{\Phi_{q^z M}(k)}\Phi_{q^z M}(k) + O\left(\frac{q^z}{N}\right) \\
&= \Phi_{K, q^z M}(0) + O\left(\frac{q^z}{N}\right).
\end{aligned}$$

Replacing K by $q^z L$ analogously as N by $q^z M$ yields

$$\big|\Phi_{K,q^z M}(0) - \Phi_{q^z L, q^z M}(0)\big| = O\left(\frac{q^z}{K}\right).$$

Altogether we obtain

$$\Phi_{K,N}(0) = \Phi_{q^z L, q^z M}(0) + O\left(\frac{q^z}{N}\right) + O\left(\frac{q^z}{K}\right).$$

Since $\sqrt{N} \leqslant K \leqslant N$ the result follows. $\square$

We now consider the correlation function according to the lth factor on the right-hand side of (8), $\Phi_{K,Q_l}(0)$, and deduce the desired asymptotic behavior. For simplicity the index l and superscript (l) are omitted.

We assume $Q \geqslant q^{10}$ and set $t := [\log_q(Q)/5]$ then we have in both cases for $z = 2t$, $Q = q^{2t}M + R$, $K = q^{2t}L + S$ and for $z = 2t+1$, $Q = q^{2t+1}M' + R'$, $K = q^{2t+1}L' + S'$ that the error term in (9) tends to zero as Q increases and further $L, L' \geqslant 1$, $M, M' \geqslant 1$, note that by the definitions in Lemma 8 we have $K \leqslant Q \leqslant K^2$ for K big enough. We are now in a position to apply Lemma 9 for $z = 2t$ and for $z = 2t + 1$, respectively, to estimate $\Phi_{K,Q}(0)$.

In the following we estimate $\Phi_{q^{2t}L,q^{2t}M}(0)$. In [4, p. 43] we proved for $g(n) = e(hs_{q,\overline{\gamma}}(n))$ where h is a non-zero integer and $(\overline{\gamma}_i)_{i\geqslant 0}$ is a given sequence in $\mathbb{R}$ and $s_{q,\overline{\gamma}}(n)$ denotes the weighted q-ary sum-of-digits function that for $r \in \{0, 1\}$ we have

$$\Phi_{q^{2t}L,q^{2t}M}(r) \leqslant e^{-\sum_{i=0}^{t-1}\|h(\overline{\gamma}_{2i+1}-q\overline{\gamma}_{2i})\|^2/4}\left(1 + \frac{7q^2}{L}\right), \tag{10}$$

where for a real x, $\|x\|$ here and later on denotes the distance to the nearest integer. Since $L \geqslant 1$ we have $1 + 7q^2/L \leqslant 1 + 7q^2$. If we set $h = 1$ and $\overline{\gamma}_i = \gamma_i/q$ we obtain the following statement.

If

$$\sum_{i=0}^{\infty} \|\gamma_{2i+1}/q\|^2 = \infty$$

then $\Phi_{q^{2t}L,q^{2t}M}(r)$ tends to zero as t (and therefore K, Q and N) increases. Furthermore under this condition we have

$$\frac{1}{N}\sum_{n=0}^{N-1}\prod_{l=1}^{\nu} e\left(\frac{s_{q_l,\gamma^{(l)}}(n)}{q_l}\right) = o(1)$$

as N increases.

We know by our assumption that infinitely many $\gamma_i \not\equiv 0 \pmod q$. If

$$\sum_{i=0}^{\infty} \|\gamma_{2i+1}/q\|^2$$

is convergent, then

$$\sum_{i=1}^{\infty} \|\gamma_{2i}/q\|^2$$

has to be divergent. In the following we show, that if this holds instead, then $\Phi_{q^{2t+1}L',q^{2t+1}M'}(0)$ tends to zero as t increases (and therefore K, Q, N), which will conclude the proof of Proposition 6. In [4, p. 40] we proved the following result.

Lemma 10. *For $r \in \{0, 1\}$ we have*

$$\Phi^{(b)}_{qK,qN}(r) = \lambda_r^{(b)}\Phi^{(b+1)}_{K,N}(0) + \mu_r^{(b)}\Phi^{(b+1)}_{K,N}(1) + \nu_r^{(b)}\overline{\Phi^{(b+1)}_{K,N}(0)} + E^{(b+1)}_{K,N}(r), \tag{11}$$

where $|E^{(b+1)}_{K,N}(r)| \leqslant 2/K$ and with certain $\lambda_r^{(b)}, \mu_r^{(b)}, \nu_r^{(b)}$ satisfying

$$\left|\lambda_r^{(b)}\right| + \left|\mu_r^{(b)}\right| + \left|\nu_r^{(b)}\right| \leqslant 1.$$

Let us use (11) for $\Phi_{q^{2t+1}L',q^{2t+1}M'}(0)$ and $b = 0$. We obtain after applying the triangular inequality

$$\begin{aligned}\left|\Phi_{q^{2t+1}L',q^{2t+1}M'}(0)\right| \leqslant{}& |\lambda_0|\left|\Phi^{(1)}_{q^{2t}L',q^{2t}M'}(0)\right| + |\mu_0|\left|\Phi^{(1)}_{q^{2t}L',q^{2t}M'}(1)\right| \\ &+ |\nu_0|\left|\Phi^{(1)}_{q^{2t}L',q^{2t}M'}(0)\right| + \left|E^{(1)}_{q^{2t}L',q^{2t}M'}(0)\right|.\end{aligned}$$

We have

- $|\lambda_0|, |\mu_0|, |\nu_0|$ are trivially bounded by 1.
- $|E^{(1)}_{q^{2t}L',q^{2t}M'}(0)| \leqslant 2/(q^{2t}L')$, which tends to zero as t increases since $L' \geqslant 1$.
- $|\Phi^{(1)}_{q^{2t}L',q^{2t}M'}(r)| = O(e^{-\sum_{i=1}^{t}\|\gamma_{2i}/q\|^2})$.

In the last item we used (10) for $h = 1$ and $\overline{\gamma}_i = \gamma_{i+1}/q$ for all $i \geqslant 0$ (we raise the index i because of the superscript (1)). Altogether we arrive at

$$\Phi_{q^{2t+1}L',q^{2t+1}M'}(0) = o(1)$$

as t (and therefore K, Q, N) increases, if $\sum_{i=1}^{\infty}\|\gamma_{2i}/q\|^2 = \infty$. This leads to the conclusion that

$$\frac{1}{N}\sum_{n=0}^{N-1}\prod_{l=1}^{\nu} e\left(\frac{s_{q_l,\gamma^{(l)}}(n)}{q_l}\right) = o(1)$$

as N increases if

$$\sum_{i=1}^{\infty}\|\gamma_{2i}/q\|^2 = \infty \quad \text{or} \quad \sum_{i=0}^{\infty}\|\gamma_{2i-1}/q\|^2 = \infty.$$

The condition is equivalent to there exist infinitely many $\gamma_i \not\equiv 0 \pmod q$, which is fulfilled by our assumption. Hence the proof of Proposition 6 is complete.

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