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Criteria of measure-preserving for p-adic dynamical systems in terms of the van der Put basis

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ABSTRACT

This paper is devoted to (discrete) p-adic dynamical systems, an important domain of algebraic and arithmetic dynamics. We consider the following open problem from theory of p-adic dynamical systems. Given continuous function $f : \mathbb{Z}_p \to \mathbb{Z}_p$. Let us represent it via special convergent series, namely van der Put series. How can one specify whether this function is measure-preserving or not for an arbitrary p? In this paper, for any prime p, we present a complete description of all compatible measure-preserving functions in the additive form representation. In addition we prove the criterion in terms of coefficients with respect to the van der Put basis determining whether a compatible function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ preserves the Haar measure.

1. Introduction

Algebraic and arithmetic dynamics are actively developed fields of general theory of dynamical systems. The bibliography collected by Franco Vivaldi [30] contains 216 articles and books; extended bibliography also can be found in books of Silverman [29] and Anashin and Khrennikov [6]. Theory of dynamical systems in fields of p-adic numbers and their algebraic extensions is an important part of algebraic and arithmetic dynamics, see, e.g., [1–32] (the complete list of references would be very long; hence, we refer to [30,6,21]). As in general theory of dynamical systems, problems of ergodicity and measure-preserving play fundamental roles in theory of p-adic dynamical systems, see [6–9,21, 14]. Traditionally studies in these domains of p-adic dynamics were restricted to analytic (mainly polynomial) or at least smooth maps $f : \mathbf{Q}_p \to \mathbf{Q}_p$, where $\mathbf{Q}_p$ is the field of p-adic numbers. However, the internal mathematical development of theory of p-adic dynamical systems as well as applications

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to cryptography [6] stimulated the interest to nonsmooth dynamical maps. An important class of (in general) nonsmooth maps is given by *Lipschitz one functions.* In cryptographic applications such functions are called *compatible*. We shall use this terminology in this paper.

The main mathematical tool used in this paper is the representation of the function by the *van der Put series* which is actively used in p-adic analysis, see e.g. Mahler [23] and Schikhof [28]. Marius van der Put introduced this series in his dissertation “Algèbres de fonctions continues p-adiques” at Utrecht Universiteit in 1967 [24]. There are numerous results in studies of functions with zero derivatives, antiderivation [28] obtained using van der Put series. Later van der Put basis was adapted to the case of n-times continuously differentiable functions in one and several variables [15]. First results on applications of the van der Put series in theory of p-adic dynamical systems, the problems of ergodicity and measure-preserving, were obtained in [7]. The present paper is the first attempt to use van der Put basis to examine such property as measure-preserving of (discrete) dynamical systems in a space of p-adic integers $\mathbb{Z}_p$ for an arbitrary prime p.

Note that the van der Put basis differs fundamentally from previously used ones (for example, the monomial and Mahler basis [5]) which are related to the algebraic structure of p-adic fields. The van der Put basis is related to the zero dimensional topology of these fields (ultrametric structure), since it consists of characteristic functions of p-adic balls. In other words, the basic point in the construction of this basis is the continuity of the characteristic function of a p-adic ball.

In this paper, we present a description of all compatible measure-preserving functions by using the *additive form representation,* Theorem 4.1. In the additive form criteria, a compatible measure-preserving function is decomposed into a sum of two functions. The first one is an arbitrary compatible function – “free” part, and the second one is a compatible function of a special type. This “special” function is given by the van der Put basis, where the coefficients are defined via an arbitrary set of substitution on the set of nonzero residues modulo p and one substitution modulo p.

The additive form representation, Theorem 4.1, is based on the criterion of measure-preserving in terms of coefficients of the van der Put series, see Theorem 2.1. It was announced in [7,33]. In this paper we give its proof. As an example of its application, we show how known results on the description of certain classes of compatible measure-preserving functions can be obtained from Theorem 3.2. Namely, the classes of compatible 2-adic functions and uniformly differentiable functions modulo p.

2. Criterion of measure-preserving

Let $p > 1$ be an arbitrary prime number. The ring of p-adic integers is denoted by the symbol $\mathbb{Z}_p$. The p-adic valuation is denoted by $|\cdot|_p$. We remind that this valuation satisfies the *strong triangle inequality*:

$$|x+y|_p \leqslant \max\bigl[|x|_p, |y|_p\bigr].$$

This is the main distinguishing property of the p-adic valuation inducing essential departure from the real or complex analysis (and hence essential difference of p-adic dynamical systems from real and complex dynamical systems).

We shall use the terminology of papers [7,33].

Namely, van der Put series are defined in the following way. Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be a continuous function. Then there exists a unique sequence of p-adic coefficients $B_0, B_1, B_2, \ldots$ such that

$$f(x) = \sum_{m=0}^{\infty} B_m \chi(m, x) \tag{1}$$

for all $x \in \mathbb{Z}_p$. Here the *characteristic function* $\chi(m, x)$ is given by

$$\chi(m, x) = \begin{cases} 1, & \text{if } |x-m|_p \leqslant p^{-n}, \\ 0, & \text{otherwise}, \end{cases}$$

where $n = 1$ if $m = 0$, and n is uniquely defined by the inequality $p^{n-1} \leqslant m \leqslant p^n - 1$ otherwise (see Schikhof's book [28] for detailed presentation of theory of van der Put series).

The *van der Put coefficients* B_m are related to the values of f as follows. Let $m = m_0 + \cdots + m_{n-2}p^{n-2} + m_{n-1}p^{n-1}$ be the representation of m in the p-ary number system, i.e., $m_j \in \{0, \ldots, p-1\}$, $j = 0, 1, \ldots, n-1$ and $m_{n-1} \neq 0$. Then

$$B_m = \begin{cases} f(m) - f(m - m_{n-1}p^{n-1}), & \text{if } m \geqslant p, \\ f(m), & \text{otherwise.} \end{cases}$$

Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be a function and let f satisfy the *Lipschitz condition with constant* 1 (with respect to the p-adic valuation $|\cdot|_p$):

$$\left|f(x) - f(y)\right|_p \leqslant |x - y|_p$$

for all $x, y \in \mathbb{Z}_p$.

We state again that a mapping of an algebraic system A to itself is called *compatible* if it preserves all the congruences of A. It is easy to check that a map $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is Lipschitz one iff it is compatible (with respect to mod p^k, $k = 1, 2, \ldots$ congruences).

The space $\mathbb{Z}_p$ is equipped by the natural probability measure, namely, the Haar measure μ_p normalized so that $\mu_p(\mathbb{Z}_p) = 1$.

Recall that a mapping $f : \mathbb{S} \to \mathbb{S}$ of a measurable space $\mathbb{S}$ with a probability measure μ is called measure-preserving if $\mu(f^{-1}(S)) = \mu(S)$ for each measurable subset $S \subset \mathbb{S}$.

We say that a compatible function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is bijective modulo p^k if the induced mapping $x \mapsto f(x) \bmod p^k$ is a permutation on $\mathbb{Z}/p^k\mathbb{Z}$. It was shown in [9] (see also [1, Section 4.4]) that a compatible function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is measure-preserving if and only if it is bijective modulo p^k for all $k = 1, 2, 3, \ldots$.

Theorem 2.1. *Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be a compatible function and*

$$f(x) = \sum_{m=0}^{\infty} p^{\lfloor \log_p m \rfloor} b_m \chi(m, x)$$

be the van der Put representation of this function, where $b_m \in \mathbb{Z}_p, m = 0, 1, 2, \ldots$. Then $f(x)$ preserves the Haar measure iff

1. *$b_0, b_1, \ldots, b_{p-1}$ establish a complete set of residues modulo p, i.e. the function $f(x)$ is bijective modulo p;*
2. $$b_{m+p^k}, b_{m+2p^k}, \ldots, b_{m+(p-1)p^k}$$

   for any $m = 0, \ldots, p^k - 1$ are all nonzero residues modulo p for $k = 2, 3, \ldots$.

Proof. By the induction for $k = 1, 2, \ldots$ we show that the function f is bijective modulo p^k. For $k = 1$ it is true by the first condition of the theorem, i.e. $b_i \equiv f(i) \bmod p$ for $i = 0, 1, \ldots, p-1$. Assume that f is bijective modulo p^k. Let us show that the function f is bijective modulo p^{k+1}. In other words, we should show that the comparison $f(x) \equiv \acute{t} + p^k t \bmod p^{k+1}$ has a unique solution for any $\acute{t} \in \{0, \ldots, p^k - 1\}$ and $t = 0, 1, \ldots, p-1$. By the induction hypothesis the comparison $f(x) \equiv \acute{t} \bmod p^k$ has a unique solution $\acute{x} \in \{0, \ldots, p^k - 1\}$. Then to check bijective property of the function $f \bmod p^{k+1}$ it is enough to show that for a given value $\acute{t} \in \{0, \ldots, p^k - 1\}$ the comparison

$$f\left(\acute{x}+p^k x\right) \equiv \acute{t}+p^k t \bmod p^{k+1} \tag{2}$$

has unique solution with respect to $x \in \{0, \dots, p-1\}$ for any $t \in \{0, \dots, p-1\}$.

For every $\acute{x} \in \{0, \dots, p^k-1\}$ we set a function

$$\varphi_{\acute{x}}(h) = \begin{cases} b_{\acute{x}+p^k h} \bmod p, & h \neq 0, \\ 0, & h = 0, \end{cases}$$

which is defined and valued in the residue ring modulo p.

To calculate values of the function f using the van der Put representation we write a comparison (2) as

$$f\left(\acute{x}+p^k x\right) = f(\acute{x}) + p^k \varphi_{\acute{x}}(x). \tag{3}$$

We take into account that $\acute{x}$ is unique solution of the comparison $f(x) \equiv \acute{t} \bmod p^k$ and assume $f(\acute{x}) \equiv \acute{t}+p^k\xi \bmod p$. Thus we transform the comparison (3) as

$$\varphi_{\acute{x}}(x) \equiv t-\xi \bmod p. \tag{4}$$

Under the second condition of Theorem 2.1 the function $\varphi_{\acute{x}}$ is bijective on $\{0, \dots, p-1\}$. Then for any $t = 0, \dots, p-1$ the comparison (4) has unique solution on $\{0, \dots, p-1\}$. That is (2) has unique solution, and therefore the function f is bijective modulo p^k for any $k = 1, 2, \dots$. Thereby the function f preserves measure by Theorem 1.1 in [4].

Now let us prove the theorem in the opposite direction. Let the function f preserves measure. By [6] f is bijective modulo p^{k+1} for any $k = 1, 2, \dots$. The first condition follows immediately from this result (here $k = 0$). The following comparisons have unique solution $(\acute{x}; x)$, where $\acute{x} \in \{0, \dots, p^k-1\}$ and $x \in \{0, \dots, p-1\}$ for any $\acute{t} \in \{0, \dots, p^k-1\}$ and $t \in \{0, \dots, p-1\}$:

$$f\left(\acute{x}+p^k x\right) \equiv \acute{t}+p^k t \bmod p^{k+1}, \tag{5}$$

$$f(\acute{x}) \equiv \acute{t} \bmod p^k. \tag{6}$$

After transformations presented at the beginning of the proof we can see that the condition of uniqueness of the solution of comparisons (6) is equivalent to unique solvability of the comparison (4) with respect to $x \in \{0, \dots, p-1\}$ for any $t \in \{0, \dots, p-1\}$. It means that the function $\varphi_{\acute{x}}$ is bijective on $\{0, \dots, p-1\}$. And therefore $b_{\acute{x}+p^k}, b_{\acute{x}+2p^k}, \dots, b_{\acute{x}+(p-1)p^k}$ coincide with the set of all nonzero residues modulo p. $\square$

The formulation and the proof of measure-preservation of the locally compatible p-adic functions are similar to the previous reasoning. Remind that locally compatible functions are ones satisfying the p-adic Lipschitz condition with a constant of 1 locally, i.e., in a suitable neighborhood of each point from $\mathbb{Z}_p$, see [7].

Corollary 2.2. *Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be a locally compatible function and*

$$f(x) = \sum_{m=0}^{\infty} p^{\lfloor \log_p m \rfloor} b_m \chi(m, x)$$

be the van der Put representation of this function, where $b_m \in \mathbb{Z}_p$, $m \geqslant N$. Then $f(x)$ preserves the Haar measure iff

1. *the function $f(x)$ is bijective modulo p^N;*
2. $$b_{m+p^k}, b_{m+2p^k}, \ldots, b_{m+(p-1)p^k}$$

   for any $m = 0, \ldots, p^k - 1$ are all nonzero residues modulo p for $k > N$.

3. Connection to known results

Here we show how to use theorems above by proving some known results on description of compatible measure-preserving p-adic functions.

A compatible measure-preserving 2-adic function represented via the van der Put series has been described in papers, see for example [7,33], and state that

Theorem 3.1. *The function $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ is compatible and preserves the measure μ_p if and only if it can be represented as*

$$f(x) = b_0\chi(0, x) + b_1\chi(1, x) + \sum_{m=2}^{\infty} 2^{\lfloor \log_2 m \rfloor} b_m \chi(m, x),$$

where $b_m \in \mathbb{Z}_2$ for $m = 0, 1, 2, \ldots,$ and

1. $b_0 + b_1 \equiv 1 \bmod 2$*,*
2. $|b_m|_2 = 1$*, if* $m \geqslant 2$*.*

It turns out that this result immediately follows from Theorem 2.1. Indeed, from the second condition of this theorem follows that $b_m \equiv 1 \bmod 2$ for $m \geqslant 2$ or, in another words, $|b_m|_2 = 1$. The first condition of Theorem 2.1 means that $b_0 + b_1 \equiv 1 \bmod 2$, which is equivalent to the first condition in the theorem mentioned above.

In papers [6,4] outlined characterization of measure-preserving, uniformly differentiable modulo p compatible p-adic functions $f : \mathbb{Z}_p \to \mathbb{Z}_p$. We remind, see also Definitions 3.27 and 3.28 from [6], that the function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is uniformly differentiable modulo p if for any $u \in \mathbb{Z}_p$ there exist a positive rational integer N and $f_1'(u) \in \mathbb{Q}_p$ such that for every $k \geqslant N$ and $h \in \mathbb{Z}_p$ we have $f(u + p^k h) \equiv f(u) + p^k h \cdot f_1'(u) \bmod p^{k+1}$. Note that a uniformly differentiable modulo p function is locally compatible as soon as $|f_1'(u)|_p \leqslant 1$ $(n = m = 1)$, according to Proposition 3.41 in [6].

Therefore, we can see that Theorem 4.45 in [6] follows from our Corollary 2.2. Indeed, from the definition of the uniformly differentiable function modulo p and periodically function $f_1'(u)$ with period p^N, see Proposition 3.32 from [6], follows that for sufficiently large k there exists the comparison $b_{u+p^k h} \equiv p^k h \cdot f_1'(u) \bmod p^{k+1}$. The set of the coefficients $b_{u+p^k h} \bmod p$ for $h = 1, 2, \ldots, p-1$ coincides with the set of all nonzero residues modulo p iff $f_1'(u) \neq 0 \bmod p$ for any $u \in \mathbb{Z}_p$. Thereby here the function f preserves measure iff the function f is bijective modulo p^k for some $k \geqslant N$ and $f_1'(u) \neq 0 \bmod p$, as well as stated in Theorem 4.45 in [6].

4. Additive representation of compatible, measure-preserving p-adic functions

By Lemma 4.41 from [6] we know that for arbitrary compatible function $g : \mathbb{Z}_p \to \mathbb{Z}_p$ the function $f(x) = d + cx + pg(x)$, $d, c \in \mathbb{Z}_p$, $c \neq 0 \bmod p$ preserves measure. So we see that measure-preserving functions are linear combinations of some “fixed” measure-preserving function and “arbitrary” compatible function. Moreover, in the case $p = 2$ necessary and sufficient conditions for measure-preserving property were obtained [6]. It turns out that we can find a similar representation for all compatible measure-preserving functions for any prime p. Here as a “fixed” part the special class of measure-preserving functions appears.

Theorem 4.1. *Let $h : \mathbb{Z}_p \to \mathbb{Z}_p$ be an arbitrary compatible function. A compatible function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ preserves measure iff it can be represented as*

$$f(x) = \xi(x) + p \cdot h(x),$$

where the function $\xi(x)$ represented via the van der Put series is such that

$$\begin{aligned}\xi(x) &= \sum_{m=0}^{p-1} G(i)\chi(i,x) + \sum_{k=1}^{\infty}\sum_{m=0}^{p^k-1}\sum_{i=1}^{p-1} g_m(i)p^k \cdot \chi\left(m + i \cdot p^k, x\right) \\ &= \sum_{m=0}^{p-1} G(i)\chi(i,x) + \sum_{k=1}^{\infty}\sum_{m=0}^{p^k-1}\sum_{i=1}^{p-1} i \cdot p^k \cdot \chi\left(m + g_m^{-1}(i) \cdot p^k, x\right),\end{aligned} \tag{7}$$

where g_m is a substitution on the set $\{1, \ldots, p-1\}$ and G is a substitution on the set $\{0, 1, \ldots, p-1\}$.

Proof. By the hypothesis of this theorem the function f is compatible. Let us represent f via the following van der Put series. Here the van der Put coefficients are $B_m = p^{\lfloor \log_p m \rfloor}(b_m + p \cdot \tilde{b_m})$, where $b_m, \tilde{b_m} \in \mathbb{Z}_p$. Then

$$f(x) = \sum_{m=0}^{\infty} p^{\lfloor \log_p m \rfloor} b_m \cdot \chi(m,x) + p \cdot \sum_{m=0}^{\infty} p^{\lfloor \log_p m \rfloor} \tilde{b_m} \cdot \chi(m,x).$$

By the theorem about compatibility (i.e. the Lipschitz property with constant 1) of functions represented via the van der Put series, see [28,7,33], the function

$$h(x) = \sum_{m=0}^{\infty} p^{\lfloor \log_p m \rfloor} \tilde{b_m} \cdot \chi(m,x)$$

is compatible. Now we set

$$\xi(x) = \sum_{m=0}^{\infty} p^{\lfloor \log_p m \rfloor} b_m \cdot \chi(m,x).$$

By Theorem 2.1 a compatible function $\xi(x)$ preserves measure iff it is bijective modulo p, i.e. $b_i = G(i)$ for $i = 0, 1, \ldots, p-1$, where

$$b_{m+p^k}, b_{m+2p^k}, \ldots, b_{m+(p-1)p^k}$$

are all nonzero residues modulo p for $k = 1, 2, 3, \ldots$ and $m = 0, \ldots, p^k - 1$. Let g_m be a substitution on the set $1, 2, \ldots, p-1$ such that $g_m(i) = b_{m+i\cdot p^k} \bmod p$ for $k = 1, 2, 3, \ldots$ and $m = 0, \ldots, p^k - 1$. Then

$$\begin{aligned}\xi(x) &= \sum_{m=0}^{p-1} G(i)\chi(i,x) + \sum_{k=1}^{\infty}\sum_{m=0}^{p^k-1}\sum_{i=1}^{p-1} g_m(i)p^k \cdot \chi\left(m + i \cdot p^k, x\right) \\ &= \sum_{m=0}^{p-1} G(i)\chi(i,x) + \sum_{k=1}^{\infty}\sum_{m=0}^{p^k-1}\sum_{i=1}^{p-1} i \cdot p^k \cdot \chi\left(m + g_m^{-1}(i) \cdot p^k, x\right),\end{aligned} \tag{8}$$

where g_m^{-1} is an inverse substitution to g_m. $\square$

4.1. Example of measure-preserving function in additive representation

Now we consider an example of the compatible measure-preserving p-adic function constructed by using the additive representation. For $k = 0, 1, 2, \ldots$ we choose substitutions g_m such that $g_0 = g_1 = \cdots = g_{p^k-1} = h_k$. Denote via $\delta_k(x)$ the value of k-th p-adic digit in a canonical expansion of the number x. Then

$$\sum_{m=0}^{p^k-1} \sum_{i=1}^{p-1} g_m(i) p^k \cdot \chi\left(m + i \cdot p^k, x\right) = \sum_{i=1}^{p-1} h_k(i) p^k \cdot \sum_{m=0}^{p^k-1} \chi\left(m + i \cdot p^k, x\right)$$
$$= \sum_{i=1}^{p-1} h_k(i) p^k \cdot I\left(\delta_k(x) = i\right), \tag{9}$$

where

$$I\left(\delta_k(x) = i\right) = \begin{cases} 1, & \delta_k(x) = i, \\ 0, & \delta_k(x) \neq i. \end{cases}$$

Then let us choose integer s such that $GCD(s, p-1) = 1$. Substitutions h_k defined on the set $\{1, 2, \ldots, p-1\}$ determine by the comparison $h_k(i) = i^s \bmod p$ for $k = 1, 2, \ldots$. Using the equality $\sum_{i=0}^{p-1} G(i)\chi(i, x) = G(\delta_0(x))$ we represent the function $\xi(x)$ as

$$\xi(x) = G\left(\delta_0(x)\right) + \sum_{k=1}^{\infty} p^k \cdot \left(\delta_k(x)\right)^s \bmod p$$

or

$$\xi\left(x_0 + x_1 p + \cdots + x_k p^k + \cdots\right) = G(x_0) + \sum_{k=1}^{\infty} p^k \cdot x_k^s \bmod p.$$

The substitution G we define, let us say, by $G(x_0) = p - 1 - x_0$, where $x_0 = \{0, \ldots, p-1\}$. Then as a function $h(x)$ from Theorem 4.1 we take the pseudo-constant function $h(x) = \sum_{k=0}^{\infty} x_k p^{2k}$ with $x = \sum_{k=0}^{\infty} x_k p^k \in \mathbb{Z}_p$, Example 26.4, p. 74 in [28]. Finally we get that the function

$$\begin{aligned} & f\left(x_0 + x_1 p + \cdots + x_k p^k + \cdots\right) \\ & = (p-1)(1+x_0) + \sum_{k=1}^{\infty} p^k \cdot x_k^s \bmod p + \sum_{k=1}^{\infty} p^{2k+1} \cdot x_k \\ & = (p-1)(1+x_0) + \sum_{k=1}^{\infty} \left(x_{2k+1}^s \bmod p + x_k\right) \cdot p^{2k+1} + \sum_{k=1}^{\infty} p^{2k} \cdot x_{2k}^s \bmod p \end{aligned} \tag{10}$$

preserves measure.

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