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Asymptotic expansions for Barnes G-function

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ABSTRACT

We present two classes of asymptotic expansions for Barnes G-function, and provide the formulas for determining the coefficients of each class of the asymptotic expansions.

The double gamma function Γ_2 and the multiple gamma functions Γ_n were introduced and investigated by Barnes in a series of papers [1–4]. Barnes applied these functions in the theory of elliptic functions and theta functions. Nonetheless, except possibly for the citations of Γ_2 in the exercises by Whittaker and Watson [27, p. 264] and also by Gradshteyn and Ryzhik [13, 13, p. 661, Entry 6.441 (4); p. 937, Entry 8.333], these functions were revived only in about the middle of the 1980s in the study of the determinants of the Laplacians on the n-dimensional unit sphere S^n (see, e.g., [11,16,18,21,25,26]). The

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theory of the double gamma function has indeed found interesting applications in many other recent investigations (see, for details, [23,24]).

We begin by recalling that Barnes G-function ($1/G = \Gamma_2$ being the so-called double gamma function) is defined as the integral function [1]:

$$\begin{aligned}
&\left[\Gamma_2(z+1)\right]^{-1} \\
&\quad = G(z+1) \\
&\quad = (2\pi)^{z/2}\exp\left(-\frac{1}{2}z-\frac{1}{2}(\gamma+1)z^2\right)\prod_{k=1}^{\infty}\left[\left(1+\frac{z}{k}\right)^k\cdot\exp\left(-z+\frac{z^2}{2k}\right)\right], \qquad (1)
\end{aligned}$$

where $\gamma = 0.5772156649\ldots$ denotes the Euler–Mascheroni constant. Barnes G-function satisfies $G(1) = 1$ and $G(z+1) = \Gamma(z)G(z)$, where Γ denotes the gamma function.

The following integral representation for Barnes G-function was established by Ferreira and López [12, Theorem 1]: For $|\operatorname{Arg}(z)| < \pi$,

$$\begin{aligned}
\ln G(z+1) = {}& \frac{1}{4}z^2 + z\ln\Gamma(z+1) - \left(\frac{1}{2}z^2+\frac{1}{2}z+\frac{1}{12}\right)\ln z - \ln A \\
& + \sum_{k=1}^{N-1}\frac{B_{2k+2}}{2k(2k+1)(2k+2)z^{2k}} + R_N(z) \quad (N = 1,2,\ldots), \qquad (2)
\end{aligned}$$

where B_{2k+2} are the Bernoulli numbers and A is the Glaisher–Kinkelin constant defined by

$$\ln A = \lim_{n\to\infty}\left\{\ln\left(\prod_{k=1}^{n}k^k\right) - \left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\ln n + \frac{n^2}{4}\right\}, \qquad (3)$$

the numerical value of A being $1.282427\ldots$. The remainder $R_N(z)$ is for $\Re(z) > 0$ given by

$$R_N(z) = \int_0^{\infty}\left(\frac{t}{e^t-1} - \sum_{k=0}^{2N}\frac{B_k}{k!}t^k\right)\frac{e^{-zt}}{t^3}\,dt. \qquad (4)$$

Estimates for $|R_N(z)|$ are also found by Ferreira and López [12], showing that the expansion is indeed an asymptotic expansion of $\ln G(z+1)$ in sectors of the complex plane cut along the negative axis. Pedersen [19, Theorem 1.1] proved that for any $N \geqslant 1$, the function $x \mapsto (-1)^N R_N(x)$ is completely monotonic on $(0,\infty)$. Other asymptotic relations (avoiding the $\ln\Gamma$ term) have been obtained by Ruijsenaars [22] and investigated by Pedersen [20], Koumandos [14] and Koumandos and Pedersen [15]. Recently, some upper and lower bounds for the double gamma function were derived in terms of the gamma, psi and polygamma functions, see [5–7,10]. Chen [8] and Mortici [17] established the inequalities and asymptotic expansions for $\ln A$ in (3). By using the Bell polynomials, Chen and Lin [9] presented a class of asymptotic expansions related to Glaisher–Kinkelin constant.

Write (2) as

$$G(z+1) \sim A^{-1} z^{-\frac{1}{2}z^2-\frac{1}{2}z-\frac{1}{12}} e^{z^2/4} \big(\Gamma(z+1)\big)^z \cdot \exp\left(\sum_{k=1}^{\infty} \frac{B_{2k+2}}{2k(2k+1)(2k+2)z^{2k}}\right) \tag{5}$$

for $z \to \infty$ and $|\mathrm{Arg}(z)| < \pi$, namely,

$$G(z+1) \sim A^{-1} z^{-\frac{1}{2}z^2-\frac{1}{2}z-\frac{1}{12}} e^{z^2/4} \big(\Gamma(z+1)\big)^z \cdot \exp\left(-\frac{1}{720z^2} + \frac{1}{5040z^4} - \frac{1}{10\,080z^6} + \frac{1}{9504z^8} - \frac{691}{3\,603\,600z^{10}} + \cdots\right) \tag{6}$$

for $z \to \infty$ and $|\mathrm{Arg}(z)| < \pi$.

Using $e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!}$, we deduce from (6) that

$$G(z+1) \sim A^{-1} z^{-\frac{1}{2}z^2-\frac{1}{2}z-\frac{1}{12}} e^{z^2/4} \big(\Gamma(z+1)\big)^z \cdot \left(1 - \frac{1}{720z^2} + \frac{1447}{7\,257\,600z^4} - \frac{1\,559\,527}{15\,676\,416\,000z^6} + \cdots\right). \tag{7}$$

Even though as many coefficients as we please in the right-hand side of (7) can be obtained by using Mathematica, here we aim at giving a formula for determining these coefficients. In fact, Theorem 1 below presents a general asymptotic expansion for $G(z+1)$ which includes (7) as its special case.

Theorem 1. *Let r be a given nonzero real number and $\ell \geqslant 0$ be a given integer. Barnes G-function has the following asymptotic expansion*:

$$G(z+1) \sim A^{-1} z^{-\frac{1}{2}z^2-\frac{1}{2}z-\frac{1}{12}} e^{z^2/4} \big(\Gamma(z+1)\big)^z \left(1 + \sum_{j=1}^{\infty} \frac{b_j}{z^j}\right)^{z^\ell/r} \tag{8}$$

for $z \to \infty$ and $|\mathrm{Arg}(z)| < \pi$, where the coefficients $b_j = b_j(\ell, r)$ $(j \in \mathbb{N})$ are given by

$$b_j = \sum_{(1+\ell)k_1+(2+\ell)k_2+\cdots+(j+\ell)k_j=j} \frac{r^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!} \cdot \left(\frac{B_3}{1\cdot 2\cdot 3}\right)^{k_1} \left(\frac{B_4}{2\cdot 3\cdot 4}\right)^{k_2} \cdots \left(\frac{B_{j+2}}{j(j+1)(j+2)}\right)^{k_j}, \tag{9}$$

summed over all nonnegative integers k_j satisfying the equation

$$(1+\ell)k_1 + (2+\ell)k_2 + \cdots + (j+\ell)k_j = j.$$

Proof. To determine b_j $(j = 1, 2, \ldots)$, we first express (8) as follows:

$$\left(\frac{G(z+1)}{A^{-1}z^{-\frac{1}{2}z^2-\frac{1}{2}z-\frac{1}{12}}e^{z^2/4}(\Gamma(z+1))^z}\right)^{r/z^\ell} = 1 + \sum_{j=1}^{m}\frac{b_j}{z^j} + O\left(z^{-m-1}\right). \tag{10}$$

Write (5) as

$$\frac{G(z+1)}{A^{-1}z^{-\frac{1}{2}z^2-\frac{1}{2}z-\frac{1}{12}}e^{z^2/4}(\Gamma(z+1))^z} = \exp\left(\sum_{k=1}^{m}\frac{B_{k+2}}{k(k+1)(k+2)z^k} + \mathcal{R}_m(z)\right),$$

where $\mathcal{R}_m(z) = O(z^{-m-1})$. Further, we have

$$\begin{aligned}
&\left(\frac{G(z+1)}{A^{-1}z^{-\frac{1}{2}z^2-\frac{1}{2}z-\frac{1}{12}}e^{z^2/4}(\Gamma(z+1))^z}\right)^{r/z^\ell} \\
&= e^{r\mathcal{R}(z)/z^\ell}\exp\left(\sum_{k=1}^{m}\frac{rB_{k+2}}{k(k+1)(k+2)z^{k+\ell}}\right) \\
&= e^{r\mathcal{R}(z)/z^\ell}\prod_{k=1}^{m}\left[1 + \left(\frac{rB_{k+2}}{k(k+1)(k+2)z^{k+\ell}}\right) + \frac{1}{2!}\left(\frac{rB_{k+2}}{k(k+1)(k+2)z^{k+\ell}}\right)^2 + \cdots\right] \\
&= e^{r\mathcal{R}(z)/z^\ell}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}\frac{1}{k_1!k_2!\cdots k_m!}\left(\frac{rB_3}{1\cdot 2\cdot 3}\right)^{k_1}\left(\frac{rB_4}{2\cdot 3\cdot 4}\right)^{k_2} \\
&\quad\cdots\left(\frac{rB_{m+2}}{m(m+1)(m+2)}\right)^{k_m}\cdot\frac{1}{z^{(1+\ell)k_1+(2+\ell)k_2+\cdots+(m+\ell)k_m}}.
\end{aligned} \tag{11}$$

Equating the coefficients by the equal powers of z in (10) and (11), we see that

$$\begin{aligned}
b_j = &\sum_{(1+\ell)k_1+(2+\ell)k_2+\cdots+(j+\ell)k_j=j}\frac{r^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!} \\
&\cdot\left(\frac{B_3}{1\cdot 2\cdot 3}\right)^{k_1}\left(\frac{B_4}{2\cdot 3\cdot 4}\right)^{k_2}\cdots\left(\frac{B_{j+2}}{j(j+1)(j+2)}\right)^{k_j}.
\end{aligned}$$

This completes the proof of Theorem 1. $\square$

Remark 1. Setting $(\ell, r) = (0, 1)$ in (8), yields (7). Here, from (8), we give two new asymptotic expansions:

$$\begin{aligned}
G(z+1) \sim\ &A^{-1}z^{-\frac{1}{2}z^2-\frac{1}{2}z-\frac{1}{12}}e^{z^2/4}\left(\Gamma(z+1)\right)^z \\
&\cdot\left(1 - \frac{1}{360z^2} + \frac{727}{1\,814\,400z^4} - \frac{390\,967}{1\,959\,552\,000z^6} + \cdots\right)^{1/2}
\end{aligned} \tag{12}$$

and

$$G(z+1) \sim A^{-1} z^{-\frac{1}{2}z^2-\frac{1}{2}z-\frac{1}{12}} e^{z^2/4} \big(\Gamma(z+1)\big)^z \left(1 - \frac{1}{720z^3} + \frac{1}{5040z^5} + \cdots\right)^z. \tag{13}$$

Remark 2. Noting that

$$1^1 \cdot 2^2 \cdots n^n = \frac{(n!)^n}{G(n+1)}, \tag{14}$$

we derive from (8) that

$$1^1 2^2 \cdots n^n \sim A \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}} \left(1 + \sum_{j=1}^{\infty} \frac{b_j}{n^j}\right)^{-n^\ell/r}, \tag{15}$$

with the coefficients $b_j = b_j(\ell, r)$ (for $j \in \mathbb{N}$) given by the formula (9). Formula (15) includes theorem in [9] as its special case. In particular, setting $(\ell, r) = (0, -1)$ in (15), we obtain

$$1^1 2^2 \cdots n^n \sim A \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}} \left(1 + \frac{1}{720n^2} - \frac{1433}{7\,257\,600n^4} + \cdots\right), \tag{16}$$

which can be found in [9]. Setting $(\ell, r) = (1, -1)$ in (15), we obtain

$$1^1 2^2 \cdots n^n \sim A \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}} \left(1 + \frac{1}{720n^3} - \frac{1}{5040n^5} + \cdots\right)^n. \tag{17}$$

Theorem 2 presents new asymptotic expansions for Barnes G-function.

Theorem 2. *Let $r \neq 0$ be a given real number and $\ell \geqslant 0$ be a given integer. Barnes G-function has the following asymptotic expansion:*

$$G(z+1) \sim A^{-1} z^{-\frac{1}{2}z^2-\frac{1}{2}z-\frac{1}{12}} e^{z^2/4} \big(\Gamma(z+1)\big)^z \left[1 + \ln\left(1 + \sum_{j=1}^{\infty} \frac{a_j}{z^j}\right)\right]^{z^\ell/r} \tag{18}$$

for $z \to \infty$ and $|\operatorname{Arg}(z)| < \pi$, where the coefficients $a_j = a_j(\ell, r)$ $(j \in \mathbb{N})$ are given by

$$a_j = \sum_{k_1+2k_2+\cdots+jk_j=j} \frac{1}{k_1!k_2!\cdots k_j!} b_1^{k_1} b_2^{k_2} \cdots b_j^{k_j}, \tag{19}$$

and b_j $(j \in \mathbb{N})$ are determined in (9).

Proof. A similar argument as in the proof of Theorem 1 will establish the result in Theorem 2. For completeness, we repeat to prove Theorem 2. To determine real numbers a_j $(j \in \mathbb{N})$, we first express (18) as follows:

$$\exp\left(\left(\frac{G(z+1)}{A^{-1}z^{-\frac{1}{2}z^2-\frac{1}{2}z-\frac{1}{12}}e^{z^2/4}(\Gamma(z+1))^z}\right)^{r/z^\ell}-1\right) = 1+\sum_{j=1}^{m}\frac{a_j}{z^j}+O\left(z^{-m-1}\right). \tag{20}$$

Write (8) as

$$\left(\frac{G(z+1)}{A^{-1}z^{-\frac{1}{2}z^2-\frac{1}{2}z-\frac{1}{12}}e^{z^2/4}(\Gamma(z+1))^z}\right)^{r/z^\ell}-1=\sum_{k=1}^{m}\frac{b_k}{z^k}+R(z),$$

where $R(z)=O(z^{-m-1})$. Further, we have

$$\begin{aligned}
&\exp\left(\left(\frac{G(z+1)}{A^{-1}z^{-\frac{1}{2}z^2-\frac{1}{2}z-\frac{1}{12}}e^{z^2/4}(\Gamma(z+1))^z}\right)^{r/z^\ell}-1\right)\\
&=e^{R(z)}\exp\left(\sum_{k=1}^{m}\frac{b_k}{z^k}\right)\\
&=e^{R(z)}\prod_{k=1}^{m}\left[1+\left(\frac{b_k}{z^k}\right)+\frac{1}{2!}\left(\frac{b_k}{z^k}\right)^2+\cdots\right]\\
&=e^{R(z)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}\frac{1}{k_1!k_2!\cdots k_m!}b_1^{k_1}b_2^{k_2}\cdots b_m^{k_m}\frac{1}{z^{k_1+2k_2+\cdots+mk_m}}.
\end{aligned} \tag{21}$$

Equating the coefficients by the equal powers of z in (20) and (21), we see that

$$a_j=\sum_{k_1+2k_2+\cdots+jk_j=j}\frac{1}{k_1!k_2!\cdots k_j!}b_1^{k_1}b_2^{k_2}\cdots b_j^{k_j},$$

where b_j $(j \in \mathbb{N})$ are determined in (9). This completes the proof of Theorem 2. □

Remark 3. Setting $(r,\ell)=(1,0)$ and $(r,\ell)=(1,1)$ in (18), respectively, we obtain two explicit expressions:

$$\begin{aligned}
G(z+1) &\sim A^{-1}z^{-\frac{1}{2}z^2-\frac{1}{2}z-\frac{1}{12}}e^{z^2/4}\left(\Gamma(z+1)\right)^z\\
&\quad\times\left[1+\ln\left(1-\frac{1}{720z^2}+\frac{727}{3\,628\,800z^4}-\frac{12\,511}{125\,411\,328z^6}+\cdots\right)\right]
\end{aligned} \tag{22}$$

and

$$G(z+1) \sim A^{-1} z^{-\frac{1}{2}z^2 - \frac{1}{2}z - \frac{1}{12}} e^{z^2/4} \big(\Gamma(z+1)\big)^z \times \left[1 + \ln\left(1 - \frac{1}{720z^3} + \frac{1}{5040z^5} + \frac{1}{518\,400z^6} - \cdots\right)\right]^z. \tag{23}$$

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