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Differential modular forms on Shimura curves over totally real fields

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We study the theory of differential modular forms for compact Shimura curves over totally real fields and construct differential modular forms, which are generalizations of the fundamental differential modular forms. We also construct the Serre–Tate expansions of such differential modular forms as a possible alternative to the Fourier expansion maps and calculate the Serre–Tate expansions of some of these differential modular forms.

1. Introduction

The differential modular forms are invented by Buium and his collaborators in a beautiful series of papers [8–10,12,11]. These are the modular forms obtained by applying the arithmetic p-jet space functor (adjoint to the p-typical Witt vector functor) to the ring of modular forms. We wish to understand the differential modular forms obtained by applying arithmetic jet space functor to the ring of modular forms on Shimura curves over totally real fields [19]. This paper is a modest attempt to study the differential modular forms for Shimura curves over totally real fields extending the results for the Shimura curves over $\mathbb{Q}$ [9,10]. We note that the present paper is the first initiative to investigate the differential modular forms for fields different from $\mathbb{Q}$. It is expected that the study of the differential modular forms on the Shimura curves over totally real fields should be

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useful in an "effective" proof of the André–Oort conjecture for the Shimura curves over totally real fields. The aim of the present paper is to study the following questions.

Question 1. Describe the Shimura curves over totally real fields modulo Hecke correspondences.

Question 2. Describe the quotient of Shimura curves over totally real fields modulo isogeny.

Question 3. Describe the "test objects" of the Shimura curves over totally real fields which have lifts of Frobenius.

Question 4. Describe explicit lifts of Hasse invariants for the Shimura curves over totally real fields.

We observe that isogeny covariance is stronger condition than being Hecke equivariance. Inspired by Kolchin's theory for differential algebras, Buium introduced δ-geometry and the theory of differential modular forms to answer these questions. δ is an analogue of the differentials for number fields. We enlarge the algebraic geometry to δ-geometry of Buium to study the questions. As in [11], we replace polynomials with arithmetic differential equations to describe the *categorical* quotients of Questions 1, 2 and the geometrically significant class of abelian schemes inside the unitary PEL Shimura curves as in Questions 3 and 4.

Some of the differential modular forms over totally real fields also have certain symmetries, namely isogeny covariance. This is one of the motivation of the study/construction of the differential modular forms. We closely follow the constructions of basic differential modular forms of Buium to show that there exist differential modular forms over totally real fields, whose zero sets are the solutions to the questions.

Let $f = \sum_n a_n q^n$ be a classical elliptic newform of weight 2 with respect to the congruence subgroup $\Gamma_1(N)$. Let $K_f = \mathbb{Q}(\{a_n\}_n)$ be the coefficient field of this newform with $g_f = [K_f : \mathbb{Q}]$. Buium attached g_f differential eigenforms of order 2, weight 0 to such a classical newform in [12]. We fix a totally real number field F with ring of integers O_F and let N be an ideal of O_F. We assume that the field F and the ideal N satisfy the following *Jacquet–Langlands* condition: either $[F : \mathbb{Q}]$ is odd or $\mathrm{ord}_v(N) = 1$ for all $v \mid N$. Let $\overline{T}$ denote $T \otimes \prod_p \mathbb{Z}_p$ for any abelian group T and let $K_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\overline{O_F}) | \overline{N} \mid c\}$ be a subgroup of $\mathrm{GL}_2(\overline{F})$.

Let p be an odd prime not dividing the discriminant of the totally real field F and let $\mathfrak{P}$ be a prime ideal of O_F over p. We also assume that a fixed quaternion algebra over F is split at $\mathfrak{P}$ (cf. Section 2). In this paper, we will study differential modular forms w.r.t. this fixed prime ideal.

Let $\mathbf{f}$ be a Hilbert modular newform over F of parallel weight 2, level $K_0(N)$ and trivial central character with the coefficient field $K_{\mathbf{f}}$, a number field of degree h. Then there is an abelian variety of dimension h with a motivic L-function. This abelian variety is a quotient of the Jacobian of a suitably defined Shimura curve. We now state one of the main theorems of this paper, which gives a partial answer to Question 1.

Theorem 5. *Let* $\mathbf{f}$ *be a Hilbert modular newform as above. There exist* h *non-zero linearly independent differential modular forms on Shimura curves over totally real fields (cf. Section 3) of weight* 0*, which are eigenforms of the Hecke operators for all ideals* $\mathfrak{m}$ *coprime to the ideal* $\mathfrak{P}N$*.*

Notice that the above theorem is a generalization of Theorem 2.6 in [12] for totally real number fields. This theorem is a partial result towards the "Main Conjecture" 2.54 of [11].

Theorem 6. *There exist* $16d^2$ *isogeny covariant full cotangent differential modular forms (cf. Section 3) of order* 1 *and weight* $1+\phi$ *on any affine, open subscheme of the Shimura curves over totally real fields of degree* $d>1$ *such that the quaternionic abelian schemes (cf. Section 2) on this affine scheme have lifts of the Frobenius at* $\mathfrak{P}$ *if and only if they belong to the zero set of these forms. One of these full cotangent differential modular forms is a differential modular form of weight* $1+\phi$*.*

Proposition 7. *For all* r*, there exist non-zero isogeny covariant differential modular forms of order* r *and weight* $1+\phi^r$*.*

The points of these Shimura curves are abelian schemes with some extra structure coming from endomorphism, polarization and level. Recall that Hasse invariants for the quaternionic Shimura curves over totally real fields are defined in [19, Section 5].

Theorem 8. *Let* p *be an odd prime that splits completely in* F*. There exist isogeny covariant differential modular forms of weight* $1-\phi$ *and order* 1 *on the Shimura curves over totally real fields such that on the set of ordinary points of the special fiber they coincide with the Hasse invariants of the Shimura curves over totally real fields.*

Since differential modular forms are global sections of certain line bundles on compact Shimura curves, they do not have Fourier expansions. We define the Serre–Tate expansions for differential modular forms on Shimura curves over totally real fields in Section 6 as a special case of the general expansion principle in [11].

2. Modular forms on quaternionic Shimura curves over totally real fields

We start by recalling few basic facts about Shimura curves over totally real fields mentioned above. Let F be a totally real field of degree $d>1$ with $\tau_i : F \to \mathbb{R}$ for $1 \leqslant i \leqslant d$ its embeddings in $\mathbb{R}$. We denote τ_1 simply by τ. Let O_F be the ring of integers of F and let N be an ideal of O_F. Fix a prime p as in the introduction and we denote the primes of F lying above p by $\mathfrak{P}_1, \ldots, \mathfrak{P}_m$. We call $\mathfrak{P}_1$ simply $\mathfrak{P}$. Let $F_{\mathfrak{P}}$ denote the completion of F at $\mathfrak{P}$. Let $O_{\mathfrak{P}}$ be the ring of integers of $F_{\mathfrak{P}}$ with residue field κ of order $q = p^f$. Without loss of generality, we may assume p is the uniformizer of $O_{\mathfrak{P}}$.

Let $O_{(\mathfrak{P})}$ be the localization of O_F at $\mathfrak{P}$ with completion $\widehat{O_{(\mathfrak{P})}}$. Let $\widehat{O_{(\mathfrak{P})}}^{nr}$ denote the maximal unramified extension of $\widehat{O_{(\mathfrak{P})}}$. This is a discrete valuation ring with maximal ideal generated by p. Let R denote the completion of $\widehat{O_{(\mathfrak{P})}}^{nr}$.

Let B be a quaternion algebra over F which is split at τ and ramified at all other infinite places. We also assume that B is split at $\mathfrak{P}$. Let N be a non-zero ideal of O_F and E be a totally imaginary quadratic extension of F whose relative discriminant is prime to N. Let $D = B \otimes_F E$ be the quaternion algebra with center E. We have an inclusion,

$$E \to E \otimes \mathbb{Q}_p = F_p \oplus F_p = F_{\mathfrak{P}_1} \oplus \cdots \oplus F_{\mathfrak{P}_m} \oplus F_{\mathfrak{P}_1} \oplus \cdots \oplus F_{\mathfrak{P}_m}.$$

The above decomposition of $E \otimes \mathbb{Q}_p$ induces a decomposition $O_D \otimes \mathbb{Z}_p = (O_{D_1} \oplus O_{D_2} \cdots O_{D_m}) \oplus (O_{D_1} \oplus O_{D_2} \cdots O_{D_m})$. Hence every $O_D \otimes \mathbb{Z}_p$ module Λ decomposes as

$$\Lambda = \left(\Lambda_1^1 \oplus \Lambda_2^1 \oplus \cdots \oplus \Lambda_m^1\right) \oplus \left(\Lambda_1^2 \oplus \Lambda_2^2 \oplus \cdots \oplus \Lambda_m^2\right).$$

Since B is unramified at the prime $\mathfrak{P}_1$, we have $O_{D_1} = M_2(O_{\mathfrak{P}})$. The matrix algebra $M_2(O_{\mathfrak{P}})$ has two idempotents $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $1 - e = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. There is a further decomposition of Λ_1^2 as two isomorphic projective R module $e\Lambda_1^2$ and $(1-e)\Lambda_1^2$ with an action of $O_{\mathfrak{P}}$. We denote them by $\Lambda_1^{2,1}$ and $\Lambda_1^{2,2}$ respectively. Let Γ be the restricted direct product of $(B \otimes F_\nu)^*$ for all places ν but $\mathfrak{P}$. We are interested in subgroups of Γ of the form $\mathrm{GL}_2(O_{\mathfrak{P}}) \times H$ and we denote them by $\{0, H\}$.

For any abelian scheme A, let $A^t = \mathrm{Pic}^0(A)$ be the dual abelian scheme of A and $\widehat{A}$ be the formal completion of A along the closed fiber p. Let $G = Res_{F/\mathbb{Q}}(B^*)$ be an algebraic group such that $G(\mathbb{Q}) = B^*$ acts on the complex manifold $\mathbb{H}^{\pm} = \mathbb{C} - \mathbb{R}$. Let $A^f = \prod_p \mathbb{Z}_p$ denote the ring of finite adeles of $\mathbb{Q}$. We fix an open, compact subgroup $K \subset G(A^f)$ and now consider the Shimura curve, whose $\mathbb{C}$ valued points are $M_K(\mathbb{C}) = G(\mathbb{Q})\backslash\mathbb{H}^{\pm} \times G(A^f)/K$. By the work of Shimura, there is a model of this Shimura curve over F, though it won't have any modular interpretation. According to Jarvis (cf. [18, p. 4, Thm. 3.1]), this Shimura curve has a model over $O_{\mathfrak{P}} \cap F$. Carayol introduced an auxiliary Shimura curve in [16] with a model over E. We fix an embedding of E in $F_{\mathfrak{P}}$ and base change this new unitary PEL Shimura curve to $F_{\mathfrak{P}}$. The model of this Shimura curve has a modular interpretation.

We now define these PEL Shimura curves. Let V be the underlying $\mathbb{Q}$ vector space of D with a fixed trace form (cf. [19, p. 4]). For any $F_{\mathfrak{P}}$-algebra R, let $G'(R)$ be the group of symplectic similitudes w.r.t. this trace form [19]. Let $\Gamma' = G'(A^{f,p}) \times (B \otimes F_{\mathfrak{P}_1})^* \times \cdots \times (B \otimes F_{\mathfrak{P}_m})^*$ be an adelic group with $G'(A^f) = \mathbb{Q}_p^* \times \mathrm{GL}_2(F_{\mathfrak{P}}) \times \Gamma'$. Let $K' = \mathbb{Z}_p \times K_{\mathfrak{P}} \times H'$ be an open compact subgroup of Γ'. We consider the Shimura curve with $\mathbb{C}$ valued points $M'_{K'}(\mathbb{C}) = G'(\mathbb{Q})\backslash X' \times G'(A^f)/K'$. This Shimura curve has a smooth, canonical proper model over E. The base change of this curve to $F_{\mathfrak{P}}$ is denoted by $M'_{K'}$.

Theorem 9. *(See Carayol [16].) $M'_{K'}$ represents the functor*

$$(F_{\mathfrak{P}}\text{-}algebra)^{op} \to \mathbf{Sets}$$

that sends any $F_{\mathfrak{P}}$-algebra S to the set of all isomorphism classes of $(A, i, \theta, \alpha_{\mathfrak{P}})$ such that

- *A is an abelian scheme over S of relative dimension $4d$ equipped with an action of O_D given by $i : O_D \to \mathrm{End}_S(A)$ such that*
  (1) *the projective S module $Lie_1^{2,1}(A)$ has rank one and $O_{\mathfrak{P}}$ acts on it via $O_{\mathfrak{P}} \to S$,*
  (2) *for $j \geqslant 2$, we have $Lie_2^j(A) = 0$,*
- *θ is a polarization of A of degree prime to p such that the corresponding Rosati involution sends $i(l)$ to $i(l^*)$,*
- *$\alpha_{\mathfrak{P}}$ is a K' level structure, so a class modulo K' of symplectic O_D linear isomorphisms $\alpha_{\mathfrak{P}} : \widehat{T}(A) \simeq V \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$.*

If we choose the subgroup of the form $\{0, H'\}$, then there is a specific choice of the level structure (last condition in the above theorem) [5]. For H' sufficiently small, there is a smooth curve $\mathbb{M}'_{0,H'}$ over $O_{\mathfrak{P}}$ such that $M'_{0,H'} = \mathbb{M}'_{0,H'} \otimes F_{\mathfrak{P}}$ [16, Prop. 5.3, p. 191]. By [5, p. 38], the smooth curve $\mathbb{M}'_{0,H'}$ solves the same moduli problem but now for $O_{\mathfrak{P}}$-algebras.

In this paper, we denote the points of this quaternionic Shimura curve by "quaternionic abelian schemes". The above moduli problem is fine. There is a universal object $(\mathcal{A}'_{K'} = \mathcal{A}'_{0,H'}, i, \theta, \bar{\alpha}_{\mathfrak{P}})$ over $\mathbb{M}'_{0,H'}$ such that any test object over an $O_{\mathfrak{P}}$-algebra R is obtained by pulling back the universal quadruple via the corresponding morphism $Spec(R) \to \mathbb{M}'_{0,H'}$. Let $\epsilon : \mathcal{A}'_{0,H'} \to \mathbb{M}'_{0,H'}$ denote the structure map. The $\mathcal{O}_{\mathbb{M}'_{0,H'}}$ module $\epsilon_*(\Omega^1_{\mathcal{A}'_{0,H'}/\mathbb{M}'_{0,H'}})$ is an $O_D \otimes \mathbb{Z}_p$ module and $\underline{\omega} = (\epsilon_*(\Omega^1_{\mathcal{A}'_{0,H'}/\mathbb{M}'_{0,H'}}))_1^{2,1}$ is a line bundle on $\mathbb{M}'_{0,H'}$. Let $\underline{\omega}_{A/R}$ be the corresponding line bundle on the quaternionic abelian scheme A/R.

For any $O_{\mathfrak{P}}$-algebra R, the space of modular forms over R of level K' and weight k is defined by

$$S^D(K', k, R) = \mathrm{H}^0(M'_{0,H'} \otimes R, \underline{\omega}^{\otimes k}).$$

To study the Shimura curves $M_{0,H}$, the following theorem of Carayol [16] shows that it is enough to study the Shimura curves $M'_{0,H'}$.

Theorem 10 *(Carayol). Let $H \subset \Gamma$ be a small enough open compact subgroup and N_H a connected component of $M_{0,H} \otimes F_{\mathfrak{P}}^{nr}$. There exists an open compact subgroup $H' \subset \Gamma'$ and a connected component $N'_{H'}$ of $M'_{0,H'} \otimes F_{\mathfrak{P}}^{nr}$, such that N_H and $N'_{H'}$ are isomorphic over $F_{\mathfrak{P}}^{nr}$.*

2.1. Arithmetic jet spaces and Witt vectors

In this section, we briefly recall the concept of p-typical Witt vectors and arithmetic jet spaces by closely following [4] and [6]. For any ring A, we denote by $\widehat{A}$ its completion in the p-adic topology. By p-adic ring, we mean a ring A such that $A = \widehat{A} = \varprojlim_n A/p^n A$. Any such ring has a natural structure of $\mathbb{Z}_p$-algebra. For any p-adic ring S, let $Spf(S)$ be the formal scheme obtained by completing $Spec(S)$ along the closed subscheme defined by the ideal pS. By a p-formal scheme, we shall understand a formal scheme locally isomorphic to formal schemes $Spf(S)$. For a scheme X over $O_{\mathfrak{P}}$, let $\widehat{X}$ be the completion of X along the closed subscheme defined by the ideal (p). Let $\mathcal{C}$ be the category of p-formal schemes.

Let **CRings** be the category of p-adic rings. For any $A \in$ **CRings**, we may consider the ghost maps $w_i : A^{m+1} \to A$, $0 \leqslant i \leqslant m$ by $w_i(a_0, a_1, \ldots, a_m) = \sum_j p^j a_j^{p^{i-j}}$. Let $\circ^n$: **CRings** $\to$ **CRings** be the functor $\circ^n(A) = A^n$. Recall that the Witt vector is a functor W_n : **CRings** $\to$ **CRings** whose underlying functor **CRings** $\to$ **Sets** is $W_n(A) = A^{n+1}$ and for which the ghost maps $w : W_n(A) \to A^{n+1}$; $(x_n)_n \to (w_n)_n$ define a natural transformation of functor of rings $W_n \to \circ^{n+1}$. The ring $W(A)$ is an inverse limit of the rings $W_n(A)$.

Following [4], we recall that $W(A)$ is an example of a ring B with a map $\psi : B \to A$ and a Frobenius map $F : B \to B$ which reduces to the map $x \to x^p$ mod pB. The ring $W(A)$ is a universal object in the category of rings with Frobenius map and with a map to A. In other words, if B' is another ring with a map $\psi : B' \to A$ and a Frobenius map $F : B' \to B'$ which reduces to the map $x \to x^p$ mod pB' then there exists a map $\tilde{\psi} : B' \to W(R)$ which commutes with Frobenius and satisfies the following diagram:

$$\begin{tikzcd} B' \arrow[r, "\tilde{\psi}"] \arrow[dr, "\psi"'] & W(A) \arrow[d, "w_0"] \\ & A. \end{tikzcd}$$

Let $\varphi : A \to B$ be a ring homomorphism.

Definition 1. A map $\delta : A \to B$ is called a p-derivation of φ if it satisfies the following equalities:

- $\delta(1) = 0$.
- $\delta(xy) = (\varphi(x))^p \delta(y) + (\varphi(y))^p \delta(x) + p\delta(x)\delta(y)$.
- $\delta(x + y) = \delta(x) + \delta(y) + C_p(\varphi(x), \varphi(y))$.

Here, $C_p(x, y) = \frac{x^p + y^p - (x+y)^p}{p}$ is a polynomial with integer coefficients.

In other words, δ is such that the map $(\varphi, \delta) : A \to W_1(B)$ (where $W_1(B)$ is the ring of "Witt vectors" of length 2 on B) is a ring homomorphism. For a p-derivation δ of φ, let $\phi : A \to B$ denote the ring homomorphism $\phi(a) = \varphi(a)^p + p\delta(a)$. If $R = B$ then ϕ lifts the p-power Frobenius of R/pR. A δ-ring is a ring $A \in$ **CRings** equipped with a p-derivation $A \to A$. The category of δ-rings is the category **DCRings** whose objects are δ-rings and whose morphisms are the ring homomorphisms that commute with δ. The *arithmetic jet space functor* is a left adjoint functor to the Witt vector functor. In other words, we have $\mathrm{Hom}_{\mathbf{CRings}}(J^n(C), R) \simeq \mathrm{Hom}_{\mathbf{DCRings}}(C, W_n(R))$.

By a prolongation sequence, we mean a sequence $S^0 \to^{\varphi^0} S^1 \to^{\varphi^1} S^2 \cdots$ of ring homomorphisms which are equipped with p-derivations of φ^n's $S^0 \to^{\delta^0} S^1 \to^{\delta^1} S^2 \cdots$ such that $\varphi_n \circ \delta_{n+1} = \delta_n \circ \varphi_{n+1}$. For any ring C, $J^*(C) = (J^n(C))$ is naturally a prolongation sequence. We fix an embedding $O_{\mathfrak{P}} \to R$. Let $\phi : R \to R$ be the unique ring homomorphism that lifts the p-power Frobenius endomorphism. We endow the ring R with the derivation of identity $\delta(x) = \frac{\phi(x)-x^p}{p}$. Let R^* be the prolongation sequence obtained by letting all $R^n = R$ and all p-derivations be equal to δ. In this paper, we only consider prolongation sequences S^* over R^* [8, p. 103] with each S^n p-adically complete, Noetherian and flat over R.

We recall the following fundamental proposition of Buium. The arithmetic jet space functor induces a contravariant functor from the category of affine schemes to itself. This functor induces by gluing a functor from the category $\mathcal{C}$ to itself [6].

Proposition 11. *For $X, Y \in \mathcal{C}$, giving a morphism $f \in \mathrm{Hom}(J^n(X), Y)$ is equivalent to attaching any prolongation sequence S^* a map $f_{S^*} : X(S^0) \to Y(S^n)$ which is functorial in S^*.*

Proof. Follows from [8, Prop. 1.9, p. 107]. □

2.2. δ weight

If G and G' are two groups, by a δ-homomorphism of order $\leqslant n$ we mean a group homomorphism $\chi : J^n(G) \to G'$. Let $\mathbb{Z}[\phi]$ denote the polynomial ring in ϕ with coefficients in $\mathbb{Z}$. If $w = \sum_0^n a_i\phi^i$ with $a_i \in \mathbb{Z}$, we get a δ-homomorphism $\chi_w : \widehat{\mathbb{G}_m} \to \widehat{\mathbb{G}_a}$ by $\chi_w(\lambda) = \lambda^{a_0}\phi(\lambda)^{a_1} \cdots \phi^n(\lambda)^{a_n}$. By a δ weight, we mean group homomorphisms $J^r(\mathbb{G}_m) \to \widehat{\mathbb{G}_a}$. For a δ weight $w = \sum_0^n a_i\phi^i \in \mathbb{Z}[\phi]$, let $\deg(w) = \sum_0^n a_i$ be the degree of the weight w. By Proposition 11, giving δ weights are equivalent to giving homomorphisms $(S^0)^* \to S^n$.

3. Differential modular forms on the quaternionic Shimura curve

Let H' be an open compact subgroup of Γ' and let X be an affine, open subscheme of $M'_{0,H'}$. Let $V = Spec(\bigoplus_{n\in\mathbb{Z}} \underline{\omega}^{\otimes n}) \to X$ be the physical line bundle attached to the line bundle $\underline{\omega}$ with zero section removed. The space of modular forms $M = M(X)$ on X are the global sections of V on X [19]. The space of differential modular forms are global sections of $J^n(V)$.

We now make it more explicit following [14]. A differential modular form of weight $w \in \mathbb{Z}[\phi]$, order $\leqslant n$ over X is a rule which assigns to any "test object" $(A, i, \theta, \alpha_{\mathfrak{P}}, \omega, S^*)$ over S^0, where

- $(A, i, \theta, \alpha_{\mathfrak{P}}) \in X(S^0)$,
- ω is a basis of $\underline{\omega}_{A/S^0}$,
- prolongation sequence S^* over R^*,

an element $f(A, i, \theta, \alpha_{\mathfrak{P}}, \omega, S^*) \in S^n$ such that

- $f(A, i, \theta, \alpha_{\mathfrak{P}}, \omega, S^*)$ depends on S^* and the isomorphism classes of $(A, i, \theta, \alpha_{\mathfrak{P}}, \omega)$,
- the formation of $f(A, i, \theta, \alpha_{\mathfrak{P}}, \omega, S^*)$ is functorial in S^*,
- for any $\lambda \in S^0$, we have $f(A, i, \theta, \alpha_{\mathfrak{P}}, \lambda\omega, S^*) = \chi_w(\lambda)^{-1} f(A, i, \theta, \alpha_{\mathfrak{P}}, \omega, S^*)$.

A *full cotangent differential modular form* of order $\leqslant n$ and weight w is a rule f that takes "test object" $(A, i, \theta, \alpha_{\mathfrak{P}}, \varpi, S^*)$ consisting of a prolongation sequence S^*, a quaternionic abelian scheme $(A, i, \theta, \alpha_{\mathfrak{P}})$ and a basis ϖ of the projective $O_{\mathfrak{P}}$ module $\mathrm{H}^0(A, \Omega_A)$ of dimension $4d$, an element $f(A, i, \theta, \alpha_{\mathfrak{P}}, \varpi, S^*) \in S^n$ such that

$$f\big(A, i, \theta, \alpha_{\mathfrak{P}}, \lambda\varpi, S^*\big) = \chi_w(\lambda)^{-1} f\big(A, i, \theta, \alpha_{\mathfrak{P}}, \varpi, S^*\big)$$

for $\lambda \in S^0$. Observe that the information about the 1-dimensional projective $O_{\mathfrak{P}}$ module $\mathrm{H}^0(A, \Omega_A)_1^{2,1}$ is clearly not sufficient to completely determine when a "test object" consisting of a $4d$-dimensional abelian scheme has a lift of Frobenius (Theorem 6). To salvage the situation, we introduce *full cotangent differential modular forms* which may be thought of as a generalization of the differential modular forms in our setting. This is inspired by the concept of matrix valued Siegel differential modular forms (cf. [2, p. 1461]). Let M^n be the space of all differential modular forms over X of all weights and order n.

Lemma 12. $\widehat{M^n} = J^n(M)$.

Proof. Since $M = \mathcal{O}(V)$ and X is affine, hence we have $J^n(M) = \mathcal{O}(J^n(V))$. The points of $X(S^0)$ are the isomorphism classes of quaternionic abelian schemes over S^0 with certain structures coming from endomorphism, polarization and level structures. By Proposition 11, giving a morphism $J^n(V) \to S^0$ is equivalent to giving a morphism $V(S^0) \to S^n$. Hence, we deduce the desired equality. $\square$

For an isogeny $u : A_1 \to A_2$ of degree prime to p between two quaternionic abelian schemes over S^0, let $[u]$ be the $4d \times 4d$ matrix corresponding to the natural linear map between the cotangent spaces. A *full cotangent differential modular form* f of weight k is said to be isogeny covariant if for any isogeny $u : A_1 \to A_2$ with $[u] = Id$,

$$f\big(A_1, i_1, \theta_1, \omega_1, S^*\big) = \deg(u)^{-\frac{\deg(k)}{2}} f\big(A_2, i_2, \theta_2, \omega_2, S^*\big).$$

We note that the isogenies are not required to commute with polarizations or endomorphism structures.

4. *d*-Dimensional formal group attached to abelian schemes

In this section, we recall some basic facts about d-dimensional formal group. Let A be an abelian scheme of relative dimension d over R. Since the abelian schemes are commutative, it is enough to study the behavior of the tangent space at the origin. Let $e : Spec(R) \to A$ be the identity section and let $\mathcal{O}_{A,e}$ be the regular local ring at the origin with maximal ideal $m_{A,e}$. Finally let $\mathcal{O}^{for}_{A,e}$ be the completion of $\mathcal{O}_{A,e}$ w.r.t. $m_{A,e}$-adic topology. Since A is smooth of relative dimension d, there exist indeterminates $X_1, X_2, \dots, X_d$ such that $\mathcal{O}^{for}_{A,e} = R[\![X_1, \dots, X_d]\!]$. Similarly, $\mathcal{O}^{for}_{A\times A, e\times e} = R[\![Y_1, \dots, Y_d, Z_1, \dots, Z_d]\!]$ with $X_i \circ \pi_1 = Y_i$ and $X_i \circ \pi_2 = Z_i$. The multiplication map $m : A \times A \to A$ induces morphism $m^* : \mathcal{O}^{for}_{A,e} \to \mathcal{O}^{for}_{A\times A, e\times e}$ with $\mathcal{F}_i(Y_i, Z_i) = m^*(X_i)$. We get a d-tuple of power series $\mathcal{F} = (\mathcal{F}_1, \dots, \mathcal{F}_d) \in R[\![X, Y]\!]^d$, which satisfies the following properties:

Proposition 13. *If* $\underline{Y} = (Y_1, \dots, Y_d)$ *and* $\underline{Z} = (Z_1, \dots, Y_d)$, *then*

- $\mathcal{F}(\underline{Y}, \underline{Z}) = \underline{Y} + \underline{Z} \pmod{m_{A,e}}$,
- $\mathcal{F}(\underline{Y}, 0) = \underline{Y}$, $\mathcal{F}(0, \underline{Y}) = \underline{Y}$,
- $\mathcal{F}(\underline{X}, \mathcal{F}(\underline{Y}, \underline{Z})) = \mathcal{F}(\mathcal{F}(\underline{X}, \underline{Y}), \underline{Z})$.

If in addition, $\mathcal{F}_i(X, Y) = \mathcal{F}_i(Y, X)$ for all i then the formal group law is said to be commutative. Once we choose a basis ω_i of $\mathrm{H}^0(A, \Omega_A)$, by duality it will determine a basis of the tangent space $\mathrm{H}^0(A, T_{A/S})$. Now, let $\mathcal{F}(X, Y) \in R[\![X, Y]\!]^d$ be a d-dimensional formal group and n be a positive integer. We can construct a formal group with a formal group law $p^{-n}\mathcal{F}(pX, pY)$. This gives a p-formal group structure $\mathcal{F}\{n\}$ on the affine space $\widehat{\mathbb{G}_a^d}$ and we call them the n-twists of $\mathcal{F}$. According to [6, Prop. 3.3], we have a complete description of the kernel $J^n(G) \to J^{n-1}(G)$.

Proposition 14. *Let G/R be a smooth group scheme of finite type and let $\mathcal{F}$ be any formal group law defining the associated local formal group. The kernel of $J^n(G) \to J^{n-1}(G)$ is isomorphic as a p-formal group to the twist $\mathcal{F}^{\phi^n}\{n\}$ of $\mathcal{F}^{\phi^n}$. Here, $\mathcal{F}^{\phi^n}$ is the formal group law obtained by applying ϕ^n to the coefficients of the formal group law $\mathcal{F}$.*

4.1. The Hasse invariant

Let R_0 be an $\mathbb{F}_p$-algebra and (A, i) be a quaternionic abelian scheme over an R_0-algebra R. Consider the abelian scheme $A^{(q)}$ obtained from base change by the

q-power Frobenius map on R. Finally, let $F : A \to A^{(q)}$ be the Frobenius map with $V : A^{(q)} \to A$ the dual isogeny (Verschiebung) and let ω be a basis of $\mathrm{H}^0(A, \Omega_{A/R})_1^{2,1}$. To prove Theorem 8, it is useful to recall the definition of the Hasse invariant for Shimura curves over totally real field [19, Prop. 3.3].

Let $W = \mathit{Spec}(R)$ be an affine open subset of $M'_{0,H'} \otimes k$. We choose a coordinate of $(\mathcal{O}_{A,e})_1^{2,1}$ such that $\omega = (1 + a_1 X + a_2 X^2 \cdots)dx$. In this coordinate, the action of the uniformizer takes the form $[p](X) = pX + aX^q + \sum_j c_j X^{j(q-1)}$. We now define the Hasse invariant by $H|W = a\omega^{\otimes(q-1)}$. By suitably modifying [21, Lemma 3.6.1], we observe that $C(\omega) = H\omega^{(q)}$ for the Cartier operator C on the cotangent complex.

4.2. Frobenius operator and unit root subspace of the De Rham cohomology of quaternionic abelian schemes

Let $(A/S^0, i, \theta, \alpha, \omega, S^*)$ be a test object as in Section 3. Let $\varphi : S^0 \to S^1$ be a ring homomorphism and $\delta : S^0 \to S^1$ be the p-derivation of φ. Let $\phi : S^0 \to S^1$ be the ring homomorphism $\phi(x) = \varphi(x)^p + p\delta(x)$. Let A^φ/S^1 and A^ϕ/S^1 be the pullback of A/S^0 by φ and ϕ.

Let $pr : A^\phi \to A$ be the projection map. A quaternionic abelian scheme A/S^0 is said to have a lift of Frobenius if there is a morphism $F_{A/S^*} : A^\varphi \to A^\phi$ whose reduction modulo p is the p-power Frobenius map and which satisfies the following diagram:

$$\begin{tikzcd}
A^\varphi \arrow[dr, dotted, "F_{A/S^*}"] \arrow[drr, bend left, "F_A"] \arrow[ddr, bend right, "u^\varphi"'] & & \\
& A^\phi \arrow[r, "pr"] \arrow[d, "u^\phi"] & A \arrow[d, "u"'] \\
& S^1 \arrow[r, "\phi^*"] & S^0.
\end{tikzcd}$$

For an affine scheme of the form $S = \mathit{Spec}(R)$, the De Rham cohomology H^n_{dR} is a contravariant functor from the category of S-schemes to the category of R modules. Let $i_\phi : \mathrm{H}^1_{dR}(A/S^0) \to \mathrm{H}^1_{dR}(A/S^0) \otimes_\phi S^1$ be the natural inclusion of the De Rham cohomology groups and let $\Phi = \mathrm{H}^1_{cry}(F_{\varphi,\phi})$ be the Frobenius endomorphism on the De Rham cohomology [8, p. 135].

Recall, the Frobenius Φ has the property $\Phi(i_\phi(\lambda x)) = \phi(\lambda)\Phi(i_\phi(x))$ [1, p. 244]. Let $C \subset A$ be the canonical subgroup [19, Thm. 9.1, p. 19] with $A' = A/C$ the quotient abelian scheme and $\pi : A \to A'$ the projection map. Let M^0 be the ring of p-adic ordinary modular forms and let $\mathcal{A}$ be the universal "ordinary" quaternionic abelian scheme. Let $\mathfrak{C}$ be the canonical subgroup of $\mathcal{A}$ and let $\mathcal{A}' = \mathcal{A}/\mathfrak{C}$. Again by [19, Thm. 9.1, p. 19], this quaternionic abelian scheme is ordinary. Hence, there is a unique homomorphism $\phi : M^0 \to M^0$ such that $\mathcal{A}' = \mathcal{A}^\phi$ and ϕ is the Frobenius homomorphism of M^0. Let $\Phi = \pi^* \circ \phi^{-1} : \mathrm{H}^1_{dR}(\mathcal{A}/M^0) \to \mathrm{H}^1_{dR}(\mathcal{A}/M^0)$ be the Frobenius morphism on the

De Rham cohomology group. Note that the Frobenius Φ respects the Hodge filtration and hence by an argument of successive approximation, there is a subspace of the De Rham cohomology invariant under the Frobenius. In other words, there exists $U \subset \mathrm{H}^1_{dR}(\mathcal{A})$ such that $\Phi(u) = u$ for all $u \in U$ and $\mathrm{H}^1_{dR}(\mathcal{A}) = \mathrm{H}^0(\mathcal{A}, \Omega_{\mathcal{A}}) \oplus U$. For any quaternionic ordinary abelian scheme, we get the unit root subspace as a pullback in the De Rham cohomology of the unit root subspace U (cf. [1, p. 248]).

We now prove a lemma about the De Rham cohomology of an abelian scheme in characteristic $p > 0$:

Lemma 15. *For a "test object" on the special fiber of the Shimura curve $\mathbb{M}'_{0,H'}$, the Frobenius endomorphism commutes with the Hodge filtration of the De Rham cohomology groups. In other words, we have a commutative diagram in the De Rham cohomology groups*:

$$\begin{array}{ccc} \mathrm{H}^1_{dR}(\overline{A^{\phi}}) & \xrightarrow{\pi} & \mathrm{H}^1(\overline{A^{\phi}}, \mathcal{O}_{\overline{A^{\phi}}}) \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ \mathrm{H}^1_{dR}(\overline{A^{\varphi}}) & \xrightarrow{\pi} & \mathrm{H}^1(\overline{A^{\varphi}}, \mathcal{O}_{\overline{A^{\varphi}}}). \end{array}$$

Proof. Since the Hodge–De Rham spectral sequence degenerates at $\mathbb{E}_1$, we have a short exact sequence for any abelian scheme A/S

$$0 \to \Omega^1_{A/S} \to \mathrm{H}^1_{dR}(A/S) \to \mathrm{H}^1(A, O_A) \to 0.$$

Since the formation of this short exact sequence is functorial, so the Frobenius map of the abelian scheme $\overline{A^{\varphi}/S^1}$ gives rise to a commutative diagram as in the statement of the lemma. $\square$

According to [3], there exists a canonical perfect bilinear pairing

$$\langle\ \rangle_d : \mathrm{H}^1_{dR}(A) \times \mathrm{H}^1_{dR}\big(A^t\big) \to \mathcal{O}_S.$$

We also have an evaluation bilinear pairing $\langle\ \rangle_e : \mathrm{H}^0(A, \Omega_A) \times \mathit{Lie}(A) \to \mathcal{O}_S$. There is a natural isomorphism $\mathit{Lie}(A) \to \mathrm{H}^1(A^t, \mathcal{O}_{A^t})$ [15, p. 114]. Recall that the Cartier operator is dual to the absolute Frobenius w.r.t. the pairing $\langle\ \rangle_e$. In other words, we have an equality $\langle \eta, C(\omega)\rangle_e = \langle F^*\eta, \omega\rangle_e$.

For any $\mathcal{O}_S$ module T, we set $\widehat{T} = \mathrm{Hom}_{\mathcal{O}_S}(T, \mathcal{O}_S)$. The above two bilinear pairings are induced by functorial homomorphisms $\psi_A : \widehat{\mathrm{H}^1_{dR}(A^t)} \to \mathrm{H}^1_{dR}(A)$ and $\psi^0_A : \mathrm{H}^0\widehat{(A^t, \Omega_{A^t})} \to \mathrm{H}^1(A, \mathcal{O}_A)$. By [15, Thm. 5.1.6], these homomorphisms are actually isomorphisms. We can write

$$\langle \eta_1, \eta_2\rangle_d = \psi_A^{-1}(\eta_1)(\eta_2)$$

for η_1 and η_2 in $\mathrm{H}^1_{dR}(A)$ and $\mathrm{H}^1_{dR}(A^t)$ respectively. Similarly, we conclude that $\langle \omega, \eta\rangle_e = {\psi^0_A}^{-1}(\eta)(\omega)$ for η and ω in $\mathrm{H}^1(A, \mathcal{O}_A)$ and $\mathrm{H}^0(A^t, \Omega_{A^t})$.

Lemma 16. *The evaluation pairing is compatible with the pairing* $\langle\ \rangle_d$ *in the De Rham cohomology groups.*

Proof. Let $i : \mathrm{H}^0(A^t, \Omega_{A^t}) \to \mathrm{H}^1_{dR}(A^t)$ be the natural inclusion map. Recall [15, Prop. 7.2.1, p. 115], we have the following commutative diagram in the De Rham cohomology group:

$$\begin{tikzcd} \widehat{\mathrm{H}^1_{dR}(A^t)} \arrow[r, "\widehat{i}"] \arrow[d, "\psi_A"] & \mathrm{H}^0\widehat{(A^t, \Omega_{A^t})} \arrow[d, "\psi^0_A"] \\ \mathrm{H}^1_{dR}(A) \arrow[r, "\pi_A"] & \mathrm{H}^1(A, \mathcal{O}_A). \end{tikzcd}$$

Using this, we get

$$\begin{aligned} \langle \pi_A(\eta), \omega^t\rangle_e &= {\psi^0_A}^{-1}\pi_A(\eta)(\omega^t) = \widehat{i}\psi_A^{-1}(\eta)(\omega^t) \\ &:= \psi_A^{-1}(\eta)(i(\omega^t)) = \langle \eta, i(\omega^t)\rangle_d. \qquad \square \end{aligned}$$

Lemma 17. *If* $u : A \to B$ *be an isogeny between two abelian schemes over* S*, then we have a commutative diagram in the De Rham cohomology*:

$$\begin{tikzcd} \mathrm{H}^1_{dR}(B/S) \times \mathrm{H}^1_{dR}(B^t/S) \arrow[r] \arrow[d, shift right, "u^*"'] & S \\ \mathrm{H}^1_{dR}(A/S) \times \mathrm{H}^1_{dR}(A^t/S) \arrow[r] \arrow[u, shift right, "u^{t*}"'] & S. \arrow[u] \end{tikzcd}$$

Proof. By [15, Prop. 7.2.2, p. 116], we have a commutative diagram

$$\begin{tikzcd} \mathrm{H}^1_{dR}\widehat{(B^t/S)} \arrow[r, "\psi_B"] \arrow[d, "\widehat{u^{t*}}"] & \mathrm{H}^1_{dR}(B/S) \arrow[d, "u^*"] \\ \mathrm{H}^1_{dR}\widehat{(A^t/S)} \arrow[r, "\psi_A"] & \mathrm{H}^1_{dR}(A/S). \end{tikzcd}$$

We obtain

$$\begin{aligned} \langle u^*(\phi_2), \omega_1^t\rangle_A &= \psi_A^{-1}u^*(\phi_2)(\omega_1^t) = \widehat{u^{t*}}\psi_B^{-1}(\phi_2)(\omega_1^t) \\ &= \psi_B^{-1}(\phi_2)(u^{t*}\omega_1^t) = \langle \phi_2, u^{t*}\omega_1^t\rangle_B. \qquad \square \end{aligned}$$

The following two important lemmas of Buium will be very useful in the proof of Theorem 6. We fix a prolongation sequence S^* (cf. Section 2.1).

Lemma 18 *(Buium). An abelian scheme A over S^0 has a lift of Frobenius if and only if the projection $J^1(A) \to \widehat{A}$ has a section.*

Lemma 19 *(Buium). For any abelian scheme A of relative dimension g over S^0, the following are equivalent:*

- *A has a lift of Frobenius,*
- *the S^1 module $\mathrm{Hom}(J^1(A), \widehat{\mathbb{G}_a})$ has rank g.*

The proofs are given in [6, Prop. 3.2, p. 328] and [11, p. 74, Cor. 3.7].

5. Explicit differential modular forms for totally real fields

In this section, we prove Theorems 5, 6 and 8 by closely following the fundamental constructions of Buium and Barcau.

5.1. Proof of Theorem 5

Proof of Theorem 5. Let the field F and the ideal N satisfy the Jacquet–Langlands condition. We use the converse of the Jacquet–Langlands correspondence due to Zhang [24, Thm. 5.4, p. 32] to identify automorphic representations of $\mathrm{PGL}_2(\mathbb{A}_F)$ of parallel weight 2, level $K_0(N)$ and holomorphic 2-forms on the Shimura curve $M_{0,H}$, which are common eigenforms for all the Hecke operators. We also denote by $\mathbf{f}$ the holomorphic 2-form on the Shimura curve corresponding to $\mathbf{f}$. Let $\alpha_{\mathbf{f}}$ be the character of $\mathbb{T}$ induced by $\mathbf{f}$ and let $A_{\mathbf{f}}$ be the maximal abelian subscheme of $Jac(M_{0,H})$ killed by the kernel of $\alpha_{\mathbf{f}}$. According to [23, Lemma 3.4.5], we have a decomposition $Jac(M_{0,H}) \simeq \bigoplus_{\mathbf{f}} A_{\mathbf{f}}$ with a projection map $\Pi : Jac(M_{0,H}) \to A_{\mathbf{f}}$.

Recall that the Hodge class $[\zeta]$ is a degree 1 line bundle obtained by a suitable normalization of the canonical bundle. In [23], Zhang defined a map $\beta : M_{0,H} \to Jac(M_{0,H})$ by $\beta(x) = [x] - [\zeta]$. We use Theorem 10 and fix an isomorphism s between the connected components $M^0_{0,H}$ and $T = M'^0_{0,H'}$ of two Shimura curves. By composing the above morphism with s, we get a morphism $T \to Jac(M_{0,H})$ over $\mathbb{C}$ and now the Neron mapping property allows one to get a map $\Pi \circ \beta \circ s : T \to A_{\mathbf{f}}$ over R. We apply the arithmetic p-jet space functor to the above mentioned homomorphism and obtain $J^n(\Pi \circ \beta \circ s) : J^n(T) \to J^n(A_{\mathbf{f}})$ for all n. One knows that $\mathrm{Hom}(J^\infty(A_{\mathbf{f}}), \widehat{\mathbb{G}_a})$ is of rank at least g [6, Thm. A, p. 311].

Since each connected component of $M_{0,H}$ is isomorphic to one of the connected components of $M'_{0,H'}$, we deduce that each of these non-zero δ-homomorphisms gives a global section of $J^\infty(M'_{0,H'})$ and hence differential Hilbert modular forms of weight 0. Let $\mathfrak{P} \mid p$

be a prime such that $A_{\mathbf{f}}$ has a lift of Frobenius at $\mathfrak{P}$. By Lemma 19, there exist at least g linearly independent δ-characters of order 1. For all $1 \leqslant j \leqslant g$, let $\psi_j : J^1(A_{\mathbf{f}}) \to \widehat{\mathbb{G}_a}$ be the δ-characters. We get induced homomorphisms

$$\mathbf{f}_j^\sharp = \psi_j \circ J^1(\beta \circ \Pi \circ s) : J^1(T) \to \widehat{\mathbb{G}_a}.$$

On the other hand if $\mathrm{Hom}(J^1(A_{\mathbf{f}}), \widehat{\mathbb{G}_a}) = 0$, then $\mathrm{Hom}(J^2(A_{\mathbf{f}}), \widehat{\mathbb{G}_a})$ is of rank g. There exist g linearly independent δ-characters $\psi_i : J^2(A_{\mathbf{f}}) \to \widehat{\mathbb{G}_a}$. By composing these δ-characters with induced maps on arithmetic jet spaces, we obtain the differential Hilbert modular forms of weight 0 and order 2

$$\mathbf{f}_j^\sharp = \psi_j \circ J^2(\beta \circ \Pi \circ s) : J^2(T) \to \widehat{\mathbb{G}_a}.$$

For ideals m of O_F coprime to the level N, we consider the Hecke operators $T(m)$'s as in [23, Section 3.2]. Recall that $\mathbf{f}$ is a common Hecke eigenform. We prove that each $\mathbf{f}_j^\sharp$ is also a common Hecke eigenform. Say $P = s(P')$ for $P \in M_{0,H}^0$ and $P' \in M'^{\,0}_{0,H'}$. The Hecke operators have the properties that $T(m)\zeta = \deg(T(m))\zeta$. We now have

$$\begin{aligned}\psi_j \Pi T(m)\bigl(\beta(P)\bigr) &= \psi_j \Pi T(m)(P-\zeta) = \psi_j \Pi \sum_i (P_i - \zeta) = \psi_j \sum_i \Pi\bigl(\beta(P_i)\bigr) \\ &= \sum_i \psi_j \Pi\bigl(\beta(P_i)\bigr) = T(m)\psi_j \circ \Pi \circ \beta(P) = T(m)\mathbf{f}_j^\sharp(P).\end{aligned}$$

Let $O_{\mathbf{f}}$ be the ring of integers of $\mathbb{Q}(a_m(\mathbf{f}))$, we fix an embedding $i : O_{\mathbf{f}} \to \mathrm{End}(A_{\mathbf{f}})$. Applying [12, Prop. 4.5], we observe that $\psi_j \Pi T(m)(\beta(P)) = \chi_j(i(a_m)) \cdot \mathbf{f}_j^\sharp(P)$ and hence $T(m)\mathbf{f}_j^\sharp(P) = \chi_j(i(a_m))\mathbf{f}_j^\sharp(P)$. We deduce that the differential Hilbert modular forms $\mathbf{f}_j^\sharp$'s are eigenforms of Hecke operators $T(m)$'s for all ideals m coprime to $\mathfrak{P}N$. □

Remark 20. We proved the above theorem without using analogue of the Manin–Drinfeld theorem [12]. In [12], the embedding of modular curve to it's Jacobian is obtained by associating to any P the point of the Jacobian $[P] - [P_0]$ for any cusp P_0. For the cusp $P_0 = \infty$, the proof may be simplified using $T_l(\infty) = (1+l)\infty$.

5.2. Frobenius lifting of an abelian scheme

The aim of the present section is to construct *full cotangent differential modular forms*, which are generalization in the totally real field setting of the differential modular form f^1_{jet} in [8]. In other words, we prove Theorem 6.

Proof of Theorem 6. Let $(A, i, \theta, \alpha, \bar{\omega}, S^*)$ be a quaternionic test object with $\bar{\omega}$ basis of the full cotangent space $\mathrm{H}^0(A, \Omega_A)$. Since the quaternionic abelian schemes are of relative dimension $4d$, one knows that the formal completion of A along the identity section is given by $\mathcal{O}_{A,e} = R[\![X_1, X_2, \ldots, X_{4d}]\!]$. The kernel of the S^1 schemes $J^1(A) \to \widehat{A}$ is given by the twisted formal scheme $\mathfrak{F}^{\phi}\{1\} = p^{-1}\mathfrak{F}^{\phi}(pX, pY)$ (cf. [6, Prop. 2.2]). According to Grothendieck's existence theorem, an abelian scheme A has a lift of Frobenius if and only if $\widehat{A}$ has a lift of Frobenius. By Lemma 19, the abelian scheme $\widehat{A}$ has a lift of Frobenius if and only if $\mathrm{Hom}(J^1(A), \widehat{\mathbb{G}}_a/S^1)$ is of rank $4d$. We choose a basis of $Lie(A)$ and $N^1(A)$ corresponding to the basis $\bar{\varpi}$ of $\mathrm{H}^0(A, \Omega_A)$ and let L^i be a basis of $\mathrm{Hom}(N^1(A), \widehat{\mathbb{G}_a})$.

Recall that the short exact sequence of p-formal abelian schemes $0 \to N^1 \to J^1(A) \to \widehat{A} \to 0$ gives rise to a long exact sequence,

$$0 \to \mathrm{Hom}\big(J^1(A), \widehat{\mathbb{G}}_a\big) \to \mathrm{Hom}\big(N^1, \widehat{\mathbb{G}}_a\big) \xrightarrow{\partial} \mathrm{H}^1(A, \mathcal{O}_A).$$

An abelian scheme of dimension $4d$ has a lift of Frobenius if and only if each L^i lifts to elements of $\mathrm{Hom}(J^1(A), \widehat{\mathbb{G}}_a)$. In other words, the existence of lifting is equivalent to $\partial(L^i) = 0$ in the above short exact sequence.

Since the polarization θ of A is of degree prime to p, we obtain that θ^* is invertible and hence $\mathrm{H}^0(A, \Omega_A) \cong \mathrm{H}^0(A^t, \Omega_{A^t})$. Corresponding to the basis $\bar{\varpi}$ of $\mathrm{H}^0(A, \Omega_A)$, we choose a basis $\overline{\omega^t} = (\theta^*)^{-1}(\bar{\varpi})$ of $\mathrm{H}^0(A^t, \Omega_{A^t})$. We now define the $16d^2$ *full cotangent differential modular forms* by

$$\mathbf{F}_{ij}\big(A, i, \theta, \alpha, \bar{\varpi}, S^*\big) = \big\langle \partial\big(L^i\big), \overline{\omega^t_j} \big\rangle_e.$$

That $\mathbf{F}_{ij}(A, i, \theta, \omega, \alpha, S^*) = 0$ if and only if $\partial(L^i) = 0$ for all i follows from the fact that the evaluation pairing is a non-degenerate bilinear pairing. The last assertion is equivalent to the fact that $\mathrm{Hom}(J^1(A), \widehat{\mathbb{G}_a})$ has rank $4d$. Clearly, $\mathbf{F}_{11}$ is a differential modular form corresponding to the basis $\underline{\omega}$ of $\mathrm{H}^0(A, \Omega_A)^{2,1}_1$.

We now prove they are of weight $1 + \phi$. Under the identification of $\mathcal{O}_{A,e} = R[\![X_1, \ldots, X_{4d}]\!]$, a choice of basis $\lambda\omega_i$ instead of ω_i of $\mathrm{H}^0(A, \Omega_A)$ is equivalent to a choice λX_i of the dual basis instead of X_i and hence $\mathcal{F}_i$ will be replaced by $\lambda\mathcal{F}_i$ (cf. Section 4). Since the kernel of the group scheme $J^1(A) \to \widehat{A}$ is $p^{-1}\mathcal{F}_i^{\phi}(pX, pY)$, hence we get the formal group law $\phi(\lambda)L_i$ instead of L_i and the weight of the *full cotangent differential modular forms* as $1 + \phi$.

We now prove that these forms are isogeny covariant. Let $u : A_1 \to A_2$ be an isogeny of degree prime to p such that $\overline{\omega^{1,t}}$, $\overline{\omega^{2,t}}$ be the basis of $\mathrm{H}^0(A_1^t, \Omega_{A_1^t})$ and $\mathrm{H}^0(A_2^t, \Omega_{A_2^t})$ respectively and T^1, T^2 be the basis of $\mathrm{H}^1(A_1, \mathcal{O}_{A_1})$ and $\mathrm{H}^1(A_2, \mathcal{O}_{A_2})$ dual to the above basis. Recall, we have $[u \circ u^t] = [\deg(u)]$ and $[u \circ u^t] = [u] \circ [u^t]$ (cf. [2, p. 1463]). If $u^{t*}\overline{\omega^{1,t}} = \deg(u)\overline{\omega^{2,t}}$ then by Lemma 17 $u^*(T^2) = \deg(u)T^1$ and hence we get $u^*(\partial(L^2)) = \deg(u)\partial(L^1)$. Consider now the commutative diagram:

$$\begin{array}{ccc}
\mathrm{H}^1_{dR}(A_2) & \xrightarrow{\pi_{A_2}} & \mathrm{H}^1(A_2, \mathcal{O}_{A_2}) \\
\downarrow{\scriptstyle u^*} & & \downarrow{\scriptstyle u^*} \\
\mathrm{H}^1_{dR}(A_1) & \xrightarrow{\pi_{A_1}} & \mathrm{H}^1(A_1, \mathcal{O}_{A_1}).
\end{array}$$

Say $\partial L_i^2 = \pi_{A_2}(\mathcal{T}_i)$, then Lemma 17 shows that

$$\begin{aligned}\langle\partial(L_i^1),\omega_j^{1,t}\rangle_{A_1,e} &= \frac{1}{\deg(u)}\langle u^*(\partial(L_i^2)),\omega_j^{1,t}\rangle_{A_1,e} = \frac{1}{\deg(u)}\langle u^*(\pi_{A_2}(\mathcal{T}_i)),\omega_j^{1,t}\rangle_{A_1,e}\\ &= \frac{1}{\deg(u)}\langle\pi_{A_1}u^*(\mathcal{T}_i),\omega_j^{1,t}\rangle_{A_1,e} = \frac{1}{\deg(u)}\langle u^*(\mathcal{T}_i),\omega_j^{1,t}\rangle_{A_1,d}\\ &= \frac{1}{\deg(u)}\langle\mathcal{T}_i,u^{t*}\omega_j^{1,t}\rangle_{A_2,d} = \frac{1}{\deg(u)}\langle\pi_{A_2}(\mathcal{T}_i),u^{t*}\omega_j^{1,t}\rangle_{A_2,e}\\ &= \frac{1}{\deg(u)}\langle\partial(L_i^2),u^{t*}\omega_j^{1,t}\rangle_{A_2,e} = \deg(u)^{-1}\langle\partial(L_i^2),\omega_j^{2,t}\rangle_{A_2,e}.\end{aligned}$$

Hence, the *full cotangent differential modular forms* $\mathbf{F}_{ij}$'s are isogeny covariant. □

We now prove Proposition 7.

Proof of Proposition 7. Since the degree of polarization of the "test object" is prime to p, θ^* induces an isomorphism $\mathrm{H}^0(A,\Omega_A)\to\mathrm{H}^0(A^t,\Omega_{A^t})$. Let $\overline{\omega^t}=(\theta^*)^{-1}\overline{(\omega)}$ with ω and ω^t basis of $\mathrm{H}^0(A,\Omega_A)_1^{2,1}$ and $\mathrm{H}^0(A^t,\Omega_{A^t})_1^{2,1}$ respectively. Let Φ be the Frobenius morphism on the De Rham cohomology groups (cf. Section 4.2). We define the differential modular forms on the Shimura curves over totally real fields for all $r\geqslant 1$ by

$$\mathbf{G}^{\mathbf{r}}_{\mathbf{crys}}(A,i,\theta,\alpha,\omega,S^*) = \frac{1}{p}\langle\Phi^r(\omega),\omega^t\rangle_d.$$

That they are isogeny covariant follows again from $[u]=Id$ and hence $[u^t]=[\deg(u)]$.

Let $\tilde{A}$ be a quaternionic abelian scheme of the form Jacobian $Jac(C)$ of a curve C, then the pairing $\langle\,,\rangle_d$ coincides with the usual cup product [20, p. 192] pairing of the De Rham cohomology group. In this case, we note that the p-adic valuation of $\mathbf{G}^{\mathbf{r}}_{\mathbf{crys}}(\tilde{A},\tilde{i},\tilde{\theta},\tilde{\alpha},\tilde{\omega},S^*)$ is finite (cf. [8, Section 5, p. 137]). Hence, we proved that the differential modular forms $\mathbf{G}^{\mathbf{r}}_{\mathbf{crys}}$'s are non-zero. □

5.3. Construction of lifts of Hasse invariants

We now construct differential modular forms which are lifts of Hasse invariants on the ordinary locus of the unitary PEL Shimura curves over totally real fields. In [19], Kassaei proved that the lifts of the Hasse invariants exist as $\mathfrak{P}$-adic modular forms on Shimura curves. In this section, we also assume that the inertia degree $f=1$. We construct differential modular forms on the ordinary points of Shimura curves, which are lifts of Hasse invariants. The weights of these differential modular forms are $\phi-1$.

Recall that Kassaei proved the existence of lifts of Hasse invariants by showing $\mathrm{H}^1(M'_{K'},\underline{\omega}^{\otimes k})=0$. We consider the formal $O_{\mathfrak{P}}$ modules attached to quaternionic abelian schemes. According to [19], a quaternionic formal abelian scheme is supersingular if the corresponding 1-dimensional formal $O_{\mathfrak{P}}$ module is of height two. By [19, Prop. 5.1], the

Hasse invariant vanishes at the supersingular points of the quaternionic Shimura curve. We now prove Theorem 8.

Proof of Theorem 8. By Lemma 16, the pairings $\langle\ \rangle_d$ and $\langle\ \rangle_e$ are compatible. Since the polarization is of degree prime to p, hence the polarization induces an isomorphism of corresponding local rings. Recall that ordinarity of the quaternionic abelian scheme is determined by the rank of the corresponding formal $O_{\mathfrak{P}}$ module. This allows one to conclude that the quaternionic abelian scheme A is ordinary if and only if the dual quaternionic scheme A^t is ordinary. Let U be the unit root subspace inside the De Rham cohomology group $\mathrm{H}^1_{dR}(A^t)$ as in Section 4.2 with u a basis of $U_1^{2,1}$ and let $\bar{x} = \overline{(A, i, \theta, \alpha)}$ be a geometric point of the special fiber $\mathbb{M}'_{0,H'} \otimes k$. Let u be a basis of the 1-dimensional unit root subspace $U_1^{2,1} \cong \mathrm{H}^1(A^t, \mathcal{O}_{A^t})_1^{2,1}$. Let Φ be the Frobenius endomorphism on $\mathrm{H}^1_{dR}(A^t)$ as in Section 4.2.

We define the order 1 differential modular form by

$$\mathbf{f}^{\partial}\bigl(A, i, \theta, \alpha, \omega, S^*\bigr) = \frac{\langle \Phi(u), i_{\phi}\omega\rangle_d}{\phi(\langle u, \omega\rangle_d)}.$$

Again since $\Phi(\lambda u) = \phi(\lambda)\Phi(u)$, so $\mathbf{f}^{\partial}$ is independent of the choice of basis $u \in U_1^{2,1}$ and hence they are well-defined. The Frobenius induces an endomorphism F^* on the 1-dimensional $O_{\mathfrak{P}}$ module $\mathrm{H}^1(\overline{A^t}, \mathcal{O}_{\overline{A^t}})_1^{2,1}$ and Section 4.1 says that the Cartier operator C is dual to F^* for the pairing $\langle\ \rangle_e$ with $C(\omega) = H\omega$.

We now prove that $\mathbf{f}^{\partial}$ is actually a lift of the Hasse invariant. Since the definition of the differential modular form is independent of the choice of basis of unit root subspace, without loss of generality we may assume that $\langle u_1, \omega\rangle_d = 1$. Let Π be the natural projection $\mathrm{H}^1_{dR}(A^t) \to \mathrm{H}^1(A^t, \mathcal{O}_{A^t})$. The map Π commutes with the Frobenius endomorphism of the De Rham cohomology. By [10, Prop. 6.1], we have

$$\begin{aligned}
\overline{\mathbf{f}^{\partial}} &= \overline{\bigl\langle \Phi(u_1), \omega\bigr\rangle_d} = \overline{\bigl\langle \Pi\bigl(\Phi(u_1)\bigr), \omega\bigr\rangle_e} = \overline{\bigl\langle F^*\Pi(u_1), \omega\bigr\rangle_e} \\
&= \overline{\bigl\langle \Pi(u_1), C(\omega)\bigr\rangle_e} = \overline{\bigl\langle \Pi(u_1), H\omega\bigr\rangle_e} = \overline{H\langle u_1, \omega\rangle_d} = \overline{H}.
\end{aligned}$$

That the differential modular forms $\mathbf{f}^{\partial}$'s are isogeny covariant, follows from [1, Lemma 5.1, p. 253] and Lemma 17. $\square$

6. Serre–Tate expansions of differential modular forms

We study the Serre–Tate expansions of differential modular forms for Shimura curves over totally real fields. The Serre–Tate expansions of differential modular forms are the expansions induced by the Serre–Tate's deep theorem in deformation theory about lifting of characteristic p ordinary abelian schemes to characteristic zero. We first show that Serre–Tate expansions exist for differential modular forms in this setting. We note that the special fiber $M'_{0,H'}$ is not geometrically irreducible. By [23, p. 43], the set of geometric connected component of the special fiber is the same as that of the generic fiber. We restrict our attention to one of the connected components of the geometric special fiber.

Let now C be the category of complete, Noetherian local $\widehat{O_{\mathfrak{P}}^{nr}}$-algebras with residue field $\bar{k}$ and let $x_0 = (A_0, \theta_0, i_0, \alpha_0)$ be a geometric point on the connected components of the special fiber of the quaternionic Shimura curve. Denote by M_0' the projective limit of $M_{0,H'}'$ for all compact, open subgroups H', providing an étale cover $M_0' \to M_{0,H'}'$. Finally, let y be a pullback of x_0 w.r.t. this map giving an isomorphism of local rings $\widehat{O_{M_0',y}} \simeq \widehat{O_{M_{0,H'}',x_0}}$.

According to [16, Section 5.4], there is a bijection between the isomorphism classes of quaternionic abelian schemes A with an isomorphism $A \otimes \mathit{Spec}(R_0) = A_0$ and the isomorphism classes of p-divisible groups with an action of O_D and this bijection is given by $A \to A[p^\infty]$. The deformation of a point $x_0 = (A_0, i_0, \theta_0)$ is determined by the corresponding p-divisible group $A_0[p^\infty]$. To study this p-divisible group, it is enough to study the formal $O_{\mathfrak{P}}$ module $\mathcal{E}_0 = A_0[p^\infty]_1^{2,1}$ of height 2 [18].

Let $\mathcal{E}$ be the pullback of $\mathcal{E}_0$ by the map $y : \mathit{Spec}(\bar{k}) \to M_0'$. This module $\mathcal{E}$ is independent of the choice of the pullback y and it is a formal $O_{\mathfrak{P}}$ module $f : O_{\mathfrak{P}} \to \mathcal{E}$ of height two. For $h = 1, 2$, let Σ_h be the unique formal divisible $O_{\mathfrak{P}}$ modulo of height h. Drinfeld's classification of formal $O_{\mathfrak{P}}$ module of height 2 says that there are only two possibilities for $\mathcal{E}$, namely $\mathcal{E} = \Sigma_1 \times F_{\mathfrak{P}}/O_{\mathfrak{P}}$ or $\mathcal{E} = \Sigma_2$.

By Appendices 7 and 8 of [16], the functor which to each $S \in C$ associates the set of isomorphism classes of deformations of $(\mathcal{E}, f)$ to S is represented by a ring isomorphic to $W(\bar{k})[\![T]\!]$. By the well-known fact, the deformation functor is isomorphic to the completed local ring at the point x_0. Since $O^{for}_{M_{0,H'}',x_0} \simeq W(\bar{k})[\![T]\!]$, we have a map $E : O(M_{0,H'}') \to W(\bar{k})[\![T]\!]$ and hence an induced map $E : O(J^n(M_{0,H'}')) \to W(\bar{k})[\![T]\!]\widehat{[T', T'', \ldots, T^n]}$ (cf. Proposition 11). If we start with a line bundle L, locally this line bundle is trivial. Hence, we have a Serre–Tate expansion map

$$E_{x_0} : M^n \to R[\![T]\!]\widehat{[T', T'', \ldots, T^n]}.$$

Since we have chosen the geometric point in one of the connected component of the special fiber, hence the Serre–Tate expansion maps are injective on these components. If the strict class number of F is one, then the Serre–Tate expansion maps are injective.

6.1. Serre–Tate expansion of $\mathbf{f}_1^\sharp$

In this section, we again assume that the inertia degree $f = 1$. Recall [22], an elliptic curve E over $\widehat{\mathbb{Z}_p^{nr}}$ is called CL (canonical lifting) at the prime p if there exists a morphism $F_E : E \to E$ that satisfies the commutative diagram of Section 4.2 with $S^0 = S^1 = \widehat{\mathbb{Z}_p^{nr}}$ and $\varphi = Id$. According to Theorem 4.4 of [13], the CL and CM elliptic curves are closely related.

Theorem 21. *Let* $\mathbf{f}$ *be a Hilbert modular newform of parallel weight* 2*, level* $K_0(N)$ *and trivial central character with* $K_{\mathbf{f}} = \mathbb{Q}$*. Suppose* $\mathbf{f}_{\mathbf{1}}^{\sharp}$ *be a differential modular form as in* Theorem 5*. There exists a power series* $G(S) \in R[\![S]\!]$ *such that the Serre–Tate expansion of* $\mathbf{f}_{\mathbf{1}}^{\sharp}$ *is*

- $E(\mathbf{f}_{\mathbf{1}}^{\sharp}) = \frac{1}{p}(\phi^2 - a_p\phi + p)\sum_n \frac{\beta_n}{n} G(S)^n$,
- $E(\mathbf{f}_{\mathbf{1}}^{\sharp}) = \frac{1}{p}(\phi - up)\sum_n \frac{\beta_n}{n} G(S)^n$

depending on whether the elliptic curve $E_{\mathbf{f}}$ *is non-CL or CL. Here,* $G(S)$ *and* $f(X) = \sum_n \frac{\beta(n)}{n} T^n$ *are two well-defined power series related to the Hilbert modular newform* $\mathbf{f}$.

Proof. Following [23], let $\Pi : M'_{0,H'} \to E_{\mathbf{f}} = E$ be the Shimura curve parametrization of the elliptic curve E with $\Pi(x) = e$. Let $T = -\frac{X}{Y}$ be a local parameter at the zero element on the global minimal model of the modular elliptic curve E. We consider the formal group (cf. Section 4) attached to the elliptic curve E. Let ω_E be the invariant differential on the global minimal (Neron) model of E. We develop ω_E locally in terms of T to find an expression of the form $\omega_E = (\sum_{n=1}^{\infty} \beta(n)T^n)dT$, $\beta(1) = 1$. The formal minimal model of $E_{\mathbf{f}}$ is now $G_E(X,Y) = f_E^{-1}(f_E(X) + f_E(Y))$ for $f_E(X) = \sum_{n=1}^{\infty} \beta(n)X^n$. If l is the logarithm of the formal group attached to the elliptic curve E, then we have $\omega_E = \frac{dl(T)}{dT} dT$. The induced map $\pi^* : O^{for}_{\overline{E},\bar{e}} = R[\![T]\!] \to O^{for}_{M'_{0,H'},\bar{x}} = R[\![S]\!]$ satisfies $\pi^*(T) = G(S)$.

Since $e = f = 1$, we view $E_{\mathbf{f}}$ as an elliptic curve over $\mathbb{Z}_p$. Let $E_{\mathbf{f}}$ be a non-CL elliptic curve for a prime p and let ψ be the non-zero δ-character on the elliptic curve $E_{\mathbf{f}}$ over $\mathbb{Z}_p$. By [7, Thm. 1.10], we have $E(\psi) = \frac{1}{p}(\phi^2 - a_p\phi + p)l_E(T)$. On the other hand, let $E_{\mathbf{f}}$ be a CL elliptic curve at p. Let u be the unique root in $p\mathbb{Z}_p$ of the polynomial $\phi^2 - a_p\phi + p$. Again by [7, Thm. 1.10], observe that $E(\psi) = \frac{1}{p}(\phi - up)l_E(T)$. Since the formation of the differential modular form is functorial in S^*, we deduce that the Serre–Tate expansion of $\mathbf{f}_{\mathbf{1}}^{\sharp} = \Pi^*(\psi)$ has the form as in the statement of the theorem. $\square$

We end this paper with some computation. Using the software SAGE, we calculate $\beta(n)$'s for some modular elliptic curves. Consider the totally real number field $F = \mathbb{Q}(\sqrt{5}) = \mathbb{Q}(a)$. By [17, p. 17], A: $y^2 + x * y + (1/2 * a + 1/2) * y = x^3 + (-1/2 * a - 3/2) * x^2$ is a modular elliptic curve over F. Since $l_A(T) = \int \omega_A \, dt$, we use the command sage : A.formalgroup().differential(50) to compute ω_A and hence the coefficients $\beta(n)$'s. We calculate the coefficients up to $n = 10$ as

$$\begin{aligned} l_A(t) = {} & t + (-1/4 * a - 1/4) * t^2 - \frac{1}{3}t^3 + \frac{(3 * a + 3)}{4} * t^4 + \frac{(5/2 * a - 1/2)}{5} * t^5 \\ & + \frac{(-17/2 * a - 47/2)}{6} * t^6 + \frac{(-4 * a - 3)}{7} * t^7 + \frac{(81/2 * a + 235/2)}{8} * t^8 \\ & + \frac{(-7 * a + 33)}{9} * t^9 + \frac{(-475/2 * a - 979/2)}{10} * t^{10} + O(t^{10}). \end{aligned}$$

From the Hilbert modular form database (cf. http://www.lmfdb.org/ModularForm/GL2/), we deduce that the corresponding Hilbert modular form is a non-CM newform. The prime 19 splits into two different prime ideals in $\mathbb{Q}(\sqrt{5})$ with the corresponding eigenvalues of the Hecke operators are 4 and -4 respectively. The elliptic curve A is not a CL elliptic curve over $\widehat{\mathbb{Z}}_{19}^{nr}$. If $\pi^*(T) = G(S)$, then we have $E(\mathbf{f}_\mathbf{1}^\sharp) = \frac{1}{19}(\phi^2 - 4\phi + 19)\sum_n \frac{\beta(n)}{n} G(S)^n$ for coefficients $\beta(n)$'s as in the previous paragraph.

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