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Zeros of partial sums of the Dedekind zeta function of a cyclotomic field

Andrew Ledoan [a], Arindam Roy [b,*], Alexandru Zaharescu [c]

[a] Department of Mathematics, University of Tennessee at Chattanooga, 417F EMCS Building (Department 6956), 615 McCallie Avenue, Chattanooga, TN 37403-2598, United States
[b] Department of Mathematics, University of Illinois at Urbana–Champaign, 165 Altgeld Hall, 1409 W. Green Street (MC-382), Urbana, IL 61801-2975, United States
[c] Department of Mathematics, University of Illinois at Urbana–Champaign, 449 Altgeld Hall, 1409 W. Green Street (MC-382), Urbana, IL 61801-2975, United States

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In this article, we study the zeros of the partial sums of the Dedekind zeta function of a cyclotomic field K defined by the truncated Dirichlet series

$$\zeta_{K,X}(s)=\sum_{\|\mathfrak{a}\|\leqslant X}\frac{1}{\|\mathfrak{a}\|^s},$$

where the sum is to be taken over nonzero integral ideals $\mathfrak{a}$ of K and $\|\mathfrak{a}\|$ denotes the absolute norm of $\mathfrak{a}$. Specifically, we establish the zero-free regions for $\zeta_{K,X}(s)$ and estimate the number of zeros of $\zeta_{K,X}(s)$ up to height T.

* Corresponding author.
E-mail addresses: andrew-ledoan@utc.edu (A. Ledoan), roy22@illinois.edu (A. Roy), zaharesc@illinois.edu (A. Zaharescu).

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1. Introduction

A first generalization of the Riemann zeta function $\zeta(s)$ is provided by the Dirichlet L-functions. Subsequently, Dedekind studied the zeta function $\zeta_K(s)$ of an arbitrary algebraic number field K, defined for $\mathrm{Re}(s) > 1$ by

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{\|\mathfrak{a}\|^s} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

where the first sum is to be taken over all nonzero integral ideals $\mathfrak{a}$ of K and where $\|\mathfrak{a}\|$ denotes the absolute norm of $\mathfrak{a}$. In the second sum, $a(n)$ is used to denote the number of integral ideals $\mathfrak{a}$ with norm $\|\mathfrak{a}\| = n$.

As in the particular case $K = \mathbb{Q}$, where $\zeta(s) = \zeta_{\mathbb{Q}}(s)$, the function $\zeta_K(s)$ is analytic everywhere except solely for a simple pole at $s = 1$. (See Davenport [3] and Neukirch [11].) The residue of this pole is given by the analytic class number formula

$$\operatorname*{Res}_{s=1}\bigl(\zeta_K(s)\bigr) = \frac{2^r \pi^{n_0 - r} R_K h_K}{w_K \sqrt{|d_K|}},$$

where $r = r_1 + r_2$ (with r_1 being the number of real embeddings and r_2 being the number of complex conjugate pairs of complex embeddings of K), $n_0 = [K:\mathbb{Q}]$ denotes the degree of $K/\mathbb{Q}$, R_K denotes the regulator, h_K denotes the class number, w_K denotes the number of roots of unity in K, and d_K denotes the discriminant of K. (See Neukirch [11, p. 467].)

For $\zeta(s)$, Hardy and Littlewood [5] provided the approximate functional equation

$$\zeta(s) = \sum_{n \leqslant X} \frac{1}{n^s} + \pi^{s-1/2} \frac{\Gamma((1-s)/2)}{\Gamma(s/2)} \sum_{n \leqslant Y} \frac{1}{n^{1-s}} + O\bigl(X^{-\sigma}\bigr) + O\bigl(Y^{\sigma-1}|t|^{-\sigma+1/2}\bigr),$$

where $s = \sigma + it$, $0 \leqslant \sigma \leqslant 1$, $X > H > 0$, $Y > H > 0$, and $2\pi XY = |t|$, with the constant implied by the big-O term depending on H only. Such approximate functional equations motivate the study of properties of the partial sums $F_X(s)$ of $\zeta(s)$ defined by

$$F_X(s) = \sum_{n \leqslant X} \frac{1}{n^s}.$$

Gonek and one of the authors [4] studied the distribution of zeros of $F_X(s)$. The authors denote the number of typical zeros $\rho_X = \beta_X + i\gamma_X$ of $F_X(s)$ with ordinates $0 \leqslant \gamma_X \leqslant T$ by $N_X(T)$. In the case that T is the ordinate of a zero, they define $N_X(T)$ as $\lim_{\epsilon \to 0^+} N_X(T + \epsilon)$. In [4], the authors are concerned with results on $N_X(T)$ as both X and T tend to infinity.

Theorem 1 in [4] collects together a number of known results on the zeros of $F_X(s)$ (see Borwein, Fee, Ferguson, and Waall [1], Montgomery [8], and Montgomery and Vaughan [9]), which can be summarized as follows:

The zeros of $F_X(s)$ lie in the strip $\alpha < \sigma < \beta$, where α and β are the unique solutions of the equations $1+2^{-\sigma}+\cdots+(X-1)^{-\sigma} = X^{-\sigma}$ and $2^{-\sigma}+3^{-\sigma}+\cdots+X^{-\sigma} = 1$, respectively. In particular, $\alpha > -X$ and $\beta < 1.72865$. Furthermore, there exists a number X_0 such that if $X \geqslant X_0$, then $F_X(s)$ has no zeros in the half-plane

$$\sigma \geqslant 1 + \left(\frac{4}{\pi} - 1\right)\frac{\log\log X}{\log X}.$$

On the other hand, for any constant C satisfying the inequalities $0 < C < 4/\pi - 1$ there exists a number X_0 depending on C only such that if $X \geqslant X_0$, then $F_X(s)$ has zeros in the half-plane

$$\sigma > 1 + \frac{C\log\log X}{\log X}.$$

Theorem 2 in [4] (see also Langer [7]) can be summarized as follows:

If X and T are both greater than or equal to 2, then one has

$$\left|N_X(T) - \frac{T}{2\pi}\log[X]\right| < \frac{X}{2}.$$

Here and henceforth, $[X]$ denotes the greatest integer less than or equal to X. For $\zeta_K(s)$, Chandrasekharan and Narasimhan [2] gave the approximate functional equation

$$\zeta_K(s) = \sum_{n\leqslant X}\frac{a(n)}{n^s} + B^{2s-1}\frac{A(1-s)}{A(s)}\sum_{n\leqslant Y}\frac{a(n)}{n^{1-s}} + O\big(X^{1-\sigma-1/n_0}\log X\big), \tag{1}$$

where $A(s) = \Gamma^{r_1}(s/2)\Gamma^{r_2}(s)$, $B = 2^{r_2}\pi^{n_0/2}/\sqrt{|d_K|}$, $X > H > 0$, $Y > H > 0$, $XY = |d_K|(|t|/2\pi)^{n_0}$, and $C_1 < X/Y < C_2$ for some constants C_1 and C_2. In the present article, we investigate the distribution of zeros of the partial sums of the function $\zeta_K(s)$ defined by

$$\zeta_{K,X}(s) = \sum_{\|\mathfrak{a}\|\leqslant X}\frac{1}{\|\mathfrak{a}\|^s} = \sum_{n\leqslant X}\frac{a(n)}{n^s},$$

which appears in the approximate functional equation (1). Our purpose is to determine whether $\zeta_{K,X}(s)$ exhibit similar properties. To this end, we denote the number of non-real zeros $\rho_{K,X} = \beta_{K,X} + i\gamma_{K,X}$ of $\zeta_{K,X}(s)$ with ordinates $0 \leqslant \gamma_{K,X} \leqslant T$ by $N_{K,X}(T)$. If T is the ordinate of a zero, then $N_{K,X}(T)$ is to be defined by $\lim_{\epsilon\to 0^+} N_{K,X}(T+\epsilon)$.

We can summarize our first result as follows.

Lemma 1. *Let K be an arbitrary algebraic number field of degree $n_0 = [K\colon\mathbb{Q}]$ over the field $\mathbb{Q}$ of rational numbers, let X be a real number greater than or equal to 2, and* *denote by s the complex variable $\sigma + it$. Then there exist two real numbers α and β, with α depending on n_0 and X only and with β depending on n_0 only, such that the zeros of $\zeta_{K,X}(s)$ all lie within the rectilinear strip of the complex plane given by the inequalities $\alpha < \sigma < \beta$.*

As will be seen in the proof of Lemma 1 in Section 3, for any fixed $\delta_0 > 0$ and any X large enough, an admissible choice for α is $\alpha = -3(\delta_0 + \log 2)n_0 X \log X/\log\log X$. As for β, an admissible choice is of the form $\beta = \log C_{\epsilon_0,n_0} D_{\epsilon_0,n_0}/\log 2$, where ϵ_0 is fixed and satisfies the inequalities $0 < \epsilon_0 < 1/n_0$, $D_{\epsilon_0,n_0} = \sum_{n=2}^{\infty} 4/n^{2-\epsilon_0 n_0}$, and C_{ϵ_0,n_0} is a constant defined in terms of the divisor function.

Furthermore, we provide an asymptotic formula for $N_{K,X}(T)$ when K is a cyclotomic field, which is sharper than the one known in the case of $\zeta(s)$. Let K be any algebraic number field of degree $n_0 = [K:\mathbb{Q}]$ over the field $\mathbb{Q}$ of rational numbers. In a similar fashion to the case of $\zeta(s)$ (see [4] and [7]), it can be shown that

$$\left| N_{K,X}(T) - \frac{T}{2\pi}\log N \right| \leqslant \frac{X}{2}, \tag{2}$$

where T and X both go to infinity together, and N is the largest integer less than or equal to X for which $a(N) \neq 0$. However, if $K = \mathbb{Q}(\zeta_q)$ is a cyclotomic field, we can significantly improve the error term in (2).

Theorem 1. *Let $q \geqslant 2$, let ζ_q be a primitive root of unity of order q, let $K = \mathbb{Q}(\zeta_q)$, and let $T, X \geqslant 3$. Let, further, N be the largest integer less than or equal to X such that $a(N) \neq 0$. We have*

$$N_{K,X}(T) = \frac{T}{2\pi}\log N + O_q\left(X\left(\frac{\log\log X}{\log X}\right)^{1-1/\phi(q)}\right), \tag{3}$$

where ϕ is Euler's totient function.

Finally, we remark that the larger the degree of the cyclotomic field is, the better the asymptotic formula (3) becomes.

2. Preliminary results

To prove Theorem 1, we will make use of two auxiliary lemmas.

Lemma 2. *Fix a positive integer $q \geqslant 2$. We have*

$$\#\{n \leqslant y: \mu(n) \neq 0 \text{ and } p \mid n \text{ imply } p \equiv 1 \pmod q\} = O_q\left(y\left(\frac{\log\log y}{\log y}\right)^{1-1/\phi(q)}\right),$$

where μ denotes the Möbius function.

Proof. Fix a positive integer $q \geqslant 2$ and define

$$\mathcal{B}(q, y) = \bigl\{n \leqslant y : \mu(n) \neq 0 \text{ and } p \mid n \text{ imply } p \equiv 1 \pmod{q}\bigr\}.$$

We apply Brun's pure sieve to estimate the size of the set $\mathcal{B}(q, y)$. (See Murty and Cojocaru [10, p. 86].) Let $\mathcal{A}$ be the set of all positive integers $n \leqslant y$. Let $\mathcal{P}$ be the set of all primes p incongruent to 1 modulo q. Let $\mathcal{A}_p$ be the set of elements of $\mathcal{A}$ which are divisible by p. Let, further, $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{A}_d = \bigcap_{p|d} \mathcal{A}_p$, where d is a square-free positive integer composed of a list of prime factors from $\mathcal{P}$. For any positive real number z, we define

$$S(\mathcal{A}, \mathcal{P}, z) = \mathcal{A} \setminus \bigcup_{p|P(z)} \mathcal{A}_p,$$

where

$$P(z) = \prod_{\substack{p \in \mathcal{P} \\ p < z}} p.$$

We consider the multiplicative function ω defined for all primes p by $\omega(p) = 1$. We have

$$\#\mathcal{A}_d = \#\bigl\{n \leqslant y : n \equiv 0 \pmod{d}\bigr\} = \frac{\omega(d)}{d} y + R_d,$$

where

$$|R_d| \leqslant \omega(d).$$

From Mertens' estimates, we have

$$\sum_{\substack{p \in \mathcal{P} \\ p < z}} \frac{\omega(p)}{p} = \frac{\phi(q) - 1}{\phi(q)} \log\log z + O(1).$$

For the sake of brevity, we let

$$W(z) = \prod_{p|P(z)} \left(1 - \frac{\omega(p)}{p}\right).$$

By Brun's pure sieve, we have

$$\#S(\mathcal{A}, \mathcal{P}, z) = yW(z)\bigl(1 + O\bigl((\log z)^{-A}\bigr)\bigr) + O\bigl(z^{\eta \log\log z}\bigr), \tag{4}$$

where $A = \eta \log \eta$ and, for some $\alpha < 1$,

$$\eta = \frac{\alpha \log y}{\log z \log\log z}.$$

Since $\omega(p) = 1$, Mertens' estimates yield

$$W(z) = O_q\left(\frac{1}{(\log z)^{1-1/\phi(q)}}\right). \tag{5}$$

We now choose $\log z = c \log y/\log\log y$. Then for a suitable positive and sufficiently small constant c and from (4) and (5), we have

$$\#S(\mathcal{A}, \mathcal{P}, z) = O_q\left(y\left(\frac{\log\log y}{\log y}\right)^{1-1/\phi(q)}\right). \tag{6}$$

Since $\mathcal{B}(q, y) \subseteq S(\mathcal{A}, \mathcal{P}, z)$, we have $\#\mathcal{B}(q, z) \leqslant \#S(\mathcal{A}, \mathcal{P}, z)$. Employing this last inequality together with (6), we complete the proof of Lemma 2. $\square$

Lemma 3. *Let $q \geqslant 2$ and let $K = \mathbb{Q}(\zeta_q)$. Let, further,*

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}.$$

We have

$$\#\{n \leqslant x: a(n) \neq 0\} = O_q\left(x\left(\frac{\log\log x}{\log x}\right)^{1-1/\phi(q)}\right).$$

Proof. Let $K = \mathbb{Q}(\zeta_q)$, where ζ_q is a primitive root of unity of order q. We have

$$\zeta_K(s) = \prod_{\mathcal{P}|q}\left(1 - \frac{1}{\|\mathcal{P}\|^s}\right)^{-1} F_q(s),$$

where

$$F_q(s) = \prod_{\chi \ (\mathrm{mod}\ q)} L(s, \chi).$$

(See [11, p. 468].) For $\sigma > 1$, we have

$$F_q(s) = \prod_{\chi \ (\mathrm{mod}\ q)} \prod_{\substack{p \text{ prime} \\ p \nmid q}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Hence, for $\sigma > 1$, we have

$$\log F_q(s) = -\sum_{\chi \ (\mathrm{mod}\ q)} \sum_{\substack{p \text{ prime} \\ p \nmid q}} \log\left(1 - \frac{\chi(p)}{p^s}\right)$$

$$= \sum_{\chi \pmod q} \sum_{\substack{p \text{ prime} \\ p \nmid q}} \sum_{m=1}^{\infty} \frac{\chi(p^m)}{mp^{ms}}$$

$$= \sum_{\substack{p \text{ prime} \\ p \nmid q}} \sum_{m=1}^{\infty} \sum_{\chi \pmod q} \frac{\chi(p^m)}{mp^{ms}},$$

where

$$\sum_{\chi \pmod q} \chi(p^m) = \begin{cases} \phi(q), & \text{if } p^m \equiv 1 \pmod q; \\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$\log F_q(s) = \sum_{\substack{p \text{ prime}, m \geqslant 1 \\ p^m \equiv 1 \pmod q}} \frac{\phi(q)}{mp^{ms}}.$$

Hence, we have

$$F_q(s) = \exp\left(\sum_{\substack{p \text{ prime}, m \geqslant 1 \\ p^m \equiv 1 \pmod q}} \frac{\phi(q)}{mp^{ms}} \right).$$

Now, for $\sigma > 1$,

$$F_q(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s} = \prod_{p \text{ prime}} \left(1 + \frac{c(p)}{p^s} + \frac{c(p^2)}{p^{2s}} + \cdots \right).$$

Thus, we have

$$\log F_q(s) = \sum_{p \text{ prime}} \log\left(1 + \frac{c(p)}{p^s} + \frac{c(p^2)}{p^{2s}} + \cdots \right)$$
$$= \sum_{p \text{ prime}} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \left(\frac{c(p)}{p^s} + \frac{c(p^2)}{p^{2s}} + \cdots \right)^m,$$

and hence

$$c(p) = \begin{cases} \phi(q), & \text{if } p \equiv 1 \pmod q; \\ 0, & \text{if } p \not\equiv 1 \pmod q). \end{cases}$$

For all n such that $c(n) \neq 0$, we have $n = AB$, where A is coprime to B, A is squareful, and B is square-free, that is, $\mu(B) \neq 0$. Furthermore, all the prime factors of B are congruent to 1 modulo q. Letting

$$H(x) = \prod_{\substack{p \leqslant x,\, p \text{ prime} \\ p \equiv 1 \pmod q}} p,$$

we have

$$\begin{aligned}
\#\{n \leqslant x\colon c(n) \neq 0\} &\leqslant \#\{(A,B)\colon A \text{ squareful},\ \mu(B) \neq 0,\ AB \leqslant x,\ B \mid H(x)\} \\
&= \sum_{\substack{A \leqslant x \\ A \text{ squareful}}} \sum_{\substack{B \leqslant x/A \\ B \mid H(x)}} 1 \\
&= \sum_{\substack{A \leqslant x \\ A \text{ squareful}}} \mathcal{B}\left(q, \frac{x}{A}\right) \\
&= \sum_{\substack{A \leqslant \sqrt{x} \log x \\ A \text{ squareful}}} \mathcal{B}\left(q, \frac{x}{A}\right) + \sum_{\substack{\sqrt{x} \log x \leqslant A \leqslant x \\ A \text{ squareful}}} \mathcal{B}\left(q, \frac{x}{A}\right).
\end{aligned}$$

We examine the sums on the far right-hand side separately.

Using Lemma 2, we see that

$$\begin{aligned}
\sum_{\substack{A \leqslant \sqrt{x} \log x \\ A \text{ squareful}}} \mathcal{B}\left(q, \frac{x}{A}\right) &= O_q\left(\sum_{\substack{A \leqslant \sqrt{x} \log x \\ A \text{ squareful}}} \frac{x}{A} \left(\frac{\log\log x}{\log x}\right)^{1-1/\phi(q)} \right) \\
&= O_q\left(x \left(\frac{\log\log x}{\log x}\right)^{1-1/\phi(q)} \sum_{\substack{A \leqslant \sqrt{x} \log x \\ A \text{ squareful}}} \frac{1}{A} \right) \\
&= O_q\left(x \left(\frac{\log\log x}{\log x}\right)^{1-1/\phi(q)} \sum_{a \geqslant 1,\, b \geqslant 1} \frac{1}{a^2 b^3} \right) \\
&= O_q\left(x \left(\frac{\log\log x}{\log x}\right)^{1-1/\phi(q)} \right).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\sum_{\substack{\sqrt{x} \log x \leqslant A \leqslant x \\ A \text{ squareful}}} \mathcal{B}\left(q, \frac{x}{A}\right) &\leqslant \sum_{\substack{\sqrt{x} \log x \leqslant A \leqslant x \\ A \text{ squareful}}} \frac{x}{A} \\
&\leqslant \sum_{\substack{\sqrt{x} \log x \leqslant A \leqslant x \\ A \text{ squareful}}} \frac{x}{\sqrt{x} \log x} \\
&\leqslant \frac{\sqrt{x}}{\log x} \#\{A \leqslant x\colon A \text{ squareful}\} \\
&= O\left(\frac{x}{\log x}\right).
\end{aligned}$$

Suppose that $\mathcal{P}_1, \dots, \mathcal{P}_r$ are the prime ideals in the ring of integers of K lying over the prime factors of q and consider the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \prod_{\mathcal{P}|p} \left(1 - \frac{1}{\|\mathcal{P}\|^s}\right)^{-1}.$$

For all z, we have

$$\begin{aligned} &\#\bigl\{n \leqslant z \colon b(n) \neq 0\bigr\} \\ &\quad \leqslant \#\{n \leqslant z \text{ with all prime factors of } n \text{ in the sets } \mathcal{P}_1, \dots, \mathcal{P}_r\}. \end{aligned} \tag{7}$$

It is well known that the right-hand side of (7) is $O_q((\log z)^r)$. Thus, we have

$$\#\bigl\{n \leqslant z \colon b(n) \neq 0\bigr\} = O_q\bigl((\log z)^r\bigr).$$

For brevity's sake, we let

$$\mathcal{A} = \bigl\{n \colon a(n) \neq 0\bigr\}, \qquad \mathcal{B} = \bigl\{m \colon b(m) \neq 0\bigr\}, \qquad \mathcal{C} = \bigl\{k \colon c(k) \neq 0\bigr\},$$

and denote

$$\mathcal{A}_\omega = \mathcal{A} \cap [1, \omega], \qquad \mathcal{B}_\omega = \mathcal{B} \cap [1, \omega], \qquad \mathcal{C}_\omega = \mathcal{C} \cap [1, \omega].$$

Here, we note that

$$\#\mathcal{B}_\omega = O_q\bigl((\log \omega)^r\bigr)$$

and

$$\#\mathcal{C}_\omega = O_q\left(\omega \left(\frac{\log \log \omega}{\log \omega}\right)^{1 - 1/\phi(q)}\right). \tag{8}$$

Furthermore, we have

$$\zeta_K(s) = \sum_{n \in \mathcal{A}} \frac{a(n)}{n^s} = \sum_{m \in \mathcal{B}} \frac{b(m)}{m^s} \sum_{k \in \mathcal{C}} \frac{c(k)}{k^s}.$$

On noting that $\mathcal{A} \subseteq \mathcal{BC}$, where $\mathcal{BC} = \{bc \colon b \in \mathcal{B},\ c \in \mathcal{C}\}$, we have $\mathcal{A}_x \subset (\mathcal{BC})_x$. It follows that

$$\#\mathcal{A}_x \leqslant \#(\mathcal{BC})_x, \tag{9}$$

where

$$\#(\mathcal{BC})_x = \sum_{\substack{b \leqslant x \\ b \in \mathcal{B}}} \sum_{\substack{c \leqslant x/b \\ c \in \mathcal{C}}} 1 = \sum_{\substack{b \leqslant L \\ b \in \mathcal{B}}} \sum_{\substack{c \leqslant x/b \\ c \in \mathcal{C}}} 1 + \sum_{\substack{L < b \leqslant x \\ b \in \mathcal{B}}} \sum_{\substack{c \leqslant x/b \\ c \in \mathcal{C}}} 1, \tag{10}$$

with $1 \leqslant L \leqslant x$ (to be chosen later). By (8), we have

$$\sum_{\substack{b\leqslant L\\ b\in\mathcal{B}}}\sum_{\substack{c\leqslant x/b\\ c\in\mathcal{C}}} 1 \leqslant \sum_{\substack{b\leqslant L\\ b\in\mathcal{B}}} \#\mathcal{C}_{x/b} = O_q\left(\sum_{\substack{b\leqslant L\\ b\in\mathcal{B}}} \frac{x}{b}\left(\frac{\log\log(x/b)}{\log(x/b)}\right)^{1-1/\phi(q)}\right).$$

Since $b \leqslant L$, we have

$$\left(\log\frac{x}{b}\right)^{1-1/\phi(q)} > \left(\log\frac{x}{L}\right)^{1-1/\phi(q)}.$$

Hence, we have

$$\begin{aligned}\sum_{\substack{b\leqslant L\\ b\in\mathcal{B}}}\sum_{\substack{c\leqslant x/b\\ c\in\mathcal{C}}} 1 &= O_q\left(x\left(\frac{\log\log x}{\log x/L}\right)^{1-1/\phi(q)}\sum_{\substack{b\leqslant L\\ b\in\mathcal{B}}}\frac{1}{b}\right)\\ &= O_q\left(x\left(\frac{\log\log x}{\log(x/L)}\right)^{1-1/\phi(q)}\right),\end{aligned} \tag{11}$$

since

$$\sum_{b\in\mathcal{B}}\frac{1}{b} < \infty.$$

Next, we have

$$\sum_{\substack{L<b\leqslant x\\ b\in\mathcal{B}}}\sum_{\substack{c\leqslant x/b\\ c\in\mathcal{C}}} 1 = \sum_{\substack{L<b\leqslant x\\ b\in\mathcal{B}}} \#\mathcal{C}_{x/b} \leqslant \sum_{\substack{L<b\leqslant x\\ b\in\mathcal{B}}} \frac{x}{b} \leqslant \frac{x}{L}\#\mathcal{B}_x = O_q\left(\frac{x(\log x)^r}{L}\right). \tag{12}$$

In view of (9), we substitute (11) and (12) into (10) to obtain

$$\#\mathcal{A}_x = O_q\left(\frac{x(\log x)^r}{L}\right) + O_q\left(x\left(\frac{\log\log x}{\log(x/L)}\right)^{1-1/\phi(q)}\right).$$

Then choosing $L = (\log x)^{r+1}$, we obtain

$$\#\mathcal{A}_x = O_q\left(x\left(\frac{\log\log x}{\log x}\right)^{1-1/\phi(q)}\right).$$

This finishes the proof of Lemma 3. $\square$

3. Proof of Lemma 1

We show separately that $|\zeta_{K,X}(s)| > 0$ in the right-half plane $\sigma \geqslant \beta$ and in the left-half plane $\sigma \leqslant \alpha$. More specifically, we want to find a β so that

$$1 - \sum_{2 \leqslant n \leqslant X} \frac{a(n)}{n^\sigma} > 0,$$

for $\sigma \geqslant \beta$. Toward this end, we employ the upper bound $a(n) \leqslant d(n)^{n_0 - 1}$, where $d(n)$ denotes the number of divisors of n (see Chandrasekharan and Narasimhan [2, Lemma 9]) and satisfies the upper bound $d(n) \leqslant C_{\epsilon_0} n^{\epsilon_0}$ for all positive ϵ_0 (see Hardy and Wright [6, Chapter XVIII, Theorem 317]). Hence, we have $a(n) \leqslant C_{\epsilon_0, n_0} n^{\epsilon_0 n_0}$.

It is enough to show that

$$C_{\epsilon_0, n_0} \sum_{n=2}^{\infty} \frac{1}{n^{\sigma - \epsilon_0 n_0}} < 1. \tag{13}$$

If we let $\epsilon_0 < 1/n_0$, then for $\sigma \geqslant \beta$ we have

$$\sum_{n=2}^{\infty} \frac{1}{n^{\sigma - \epsilon_0 n_0}} \leqslant \sum_{n=2}^{\infty} \frac{1}{n^{\beta - \epsilon_0 n_0}} \leqslant \frac{1}{2^\beta} D_{\epsilon_0, n_0},$$

where

$$D_{\epsilon_0, n_0} = \sum_{n=2}^{\infty} \frac{4}{n^{2 - \epsilon_0 n_0}}.$$

In order to obtain (13), it is enough to have

$$\beta > \frac{\log C_{\epsilon_0, n_0} D_{\epsilon_0, n_0}}{\log 2}.$$

We have

$$\sum_{n=2}^{\infty} \frac{d(n)^{n_0}}{n^\beta} \leqslant C_{\epsilon_0, n_0} \sum_{n=2}^{\infty} \frac{1}{n^{\beta - \epsilon_0 n_0}} = \frac{1}{2^\beta} C_{\epsilon_0, n_0} D_{\epsilon_0, n_0}.$$

Then for $\sigma \geqslant \beta$, we have

$$\left| \sum_{2 \leqslant n \leqslant X} \frac{a(n)}{n^s} \right| \leqslant \sum_{2 \leqslant n \leqslant X} \frac{d(n)^{n_0}}{n^\beta} < 1, \tag{14}$$

and hence

$$\left| \zeta_{K,X}(s) \right| \geqslant 1 - \left| \sum_{2 \leqslant n \leqslant X} \frac{a(n)}{n^s} \right| > 0.$$

Therefore, $\zeta_{K,X}(s) \neq 0$ on the right-half plane $\sigma \geqslant \beta$.

Next, let N be the largest positive integer less than or equal to X for which $a(N) \neq 0$. Since

$$\left|\zeta_{K,X}(s)\right| \geqslant \frac{a(N)}{N^\sigma} - \left|\sum_{1\leqslant n\leqslant N-1} \frac{a(n)}{n^s}\right|,$$

it is enough to find an α such that

$$\frac{1}{N^\sigma} > \sum_{1\leqslant n\leqslant N-1} \frac{a(n)}{n^\sigma},$$

for $\sigma \leqslant \alpha$.

To this end, let us fix $\delta_0 > 0$. Then there exist constants $C_{\delta_0} > 0$ and $n_{\delta_0} \in \mathbb{Z}^+$ such that for all $1 \leqslant n < n_{\delta_0}$, we have

$$d(n) \leqslant C_{\delta_0} n^{(\delta_0+\log 2)/\log\log n},$$

and that for all $n \geqslant n_{\delta_0}$, we have

$$d(n) \leqslant n^{(\delta_0+\log 2)/\log\log n}.$$

(See Wigert [13].)

It suffices to have

$$\begin{aligned}\frac{1}{N^\sigma} &> C_{\delta_0}^{n_0} \sum_{1\leqslant n\leqslant n_{\delta_0}-1} \frac{n^{(\delta_0+\log 2)n_0/\log\log n}}{n^\sigma} + \sum_{n_{\delta_0}\leqslant n\leqslant N-1} \frac{n^{(\delta_0+\log 2)n_0/\log\log n}}{n^\sigma} \\ &= 1 + C_{\delta_0}^{n_0} S_I(n_0, \delta_0, n_{\delta_0}, \sigma) + S_{II}(n_0, \delta_0, \sigma),\end{aligned}$$

for $\sigma \leqslant \alpha$, where

$$S_I(n_0, \delta_0, n_{\delta_0}, \sigma) = \sum_{2\leqslant n\leqslant n_{\delta_0}-1} \frac{n^{(\delta_0+\log 2)n_0/\log\log n}}{n^\sigma}$$

and

$$S_{II}(n_0, \delta_0, \sigma) = \sum_{n_{\delta_0}\leqslant n\leqslant N-1} \frac{n^{(\delta_0+\log 2)n_0/\log\log n}}{n^\sigma}.$$

This would follow from the inequality

$$\frac{1}{N^\alpha} > 1 + C_{\delta_0}^{n_0} S_I(n_0, \delta_0, n_{\delta_0}, \alpha) + S_{II}(n_0, \delta_0, \alpha),$$

since, for any $\sigma \leqslant \alpha$,

$$\begin{aligned}\frac{1}{N^\sigma} &> \frac{1}{N^{\sigma-\alpha}}\left[1 + C_{\delta_0}^{n_0} S_I(n_0, \delta_0, n_{\delta_0}, \alpha) + S_{II}(n_0, \delta_0, \alpha)\right] \\ &= \frac{1}{N^{\sigma-\alpha}} + C_{\delta_0}^{n_0} \sum_{2\leqslant n\leqslant n_{\delta_0}-1} \frac{n^{(\delta_0+\log 2)n_0/\log\log n}}{N^{\sigma-\alpha} n^\alpha} + \sum_{n_{\delta_0}\leqslant n\leqslant N-1} \frac{n^{(\delta_0+\log 2)n_0/\log\log n}}{N^{\sigma-\alpha} n^\alpha}\end{aligned}$$

$$> 1 + C_{\delta_0}^{n_0} \sum_{2 \leqslant n \leqslant n_{\delta_0}-1} \frac{n^{(\delta_0+\log 2)n_0/\log\log n}}{n^{\sigma-\alpha}n^{\alpha}} + \sum_{2 \leqslant n \leqslant N-1} \frac{n^{(\delta_0+\log 2)n_0/\log\log n}}{n^{\sigma-\alpha}n^{\alpha}}$$
$$= 1 + C_{\delta_0}^{n_0} S_I(n_0, \delta_0, n_{\delta_0}, \sigma) + S_{II}(n_0, \delta_0, \sigma).$$

Thus, it is enough to find α such that

$$\frac{1}{N^{\alpha}} > 2 + 2C_{\delta_0}^{n_0} S_I(n_0, \delta_0, n_{\delta_0}, \alpha) \tag{15}$$

and such that

$$\frac{1}{N^{\alpha}} > 2S_{II}(n_0, \delta_0, \alpha). \tag{16}$$

It is enough to have

$$\frac{1}{N^{\alpha}} > 2 + 2C_{\delta_0}^{n_0} \frac{1}{n_{\delta_0}^{\alpha}} \sum_{2 \leqslant n \leqslant n_{\delta_0}-1} n^{(\delta_0+\log 2)n_0/\log\log n}, \tag{17}$$

since the right-hand side of (17) is greater than the right-hand side of (15).

The inequality in (17) holds for any fixed $\alpha < 0$ and for all N large enough in terms of n_0, δ_0, n_{δ_0}, C_{δ_0}, and α. Therefore, we may take any fixed $\alpha < 0$ as a function of N, n_0, and δ_0 for which (16) holds true. For $n_{\delta_0} \geqslant 16$, we see that

$$\sum_{n_{\delta_0} \leqslant n \leqslant N-1} \frac{n^{(\delta_0+\log 2)n_0/\log\log n}}{n^{\alpha}} \leqslant \sum_{n_{\delta_0} \leqslant n \leqslant N-1} \frac{N^{(\delta_0+\log 2)n_0/\log\log N}}{n^{\alpha}}$$
$$< N^{(\delta_0+\log 2)n_0/\log\log N} \sum_{n_{\delta_0} \leqslant n \leqslant N-1} \frac{1}{n^{\alpha}}. \tag{18}$$

It remains to examine the sum on the far right-hand side of (18).

For $\alpha < 0$, we have

$$\sum_{n_{\delta_0} \leqslant n \leqslant N-1} \frac{1}{n^{\alpha}} \leqslant (N-1)^{-\alpha} + \int_{n_{\delta_0}}^{N-1} \frac{dy}{y^{\alpha}} < (N-1)^{-\alpha}\left(\frac{N-\alpha}{1-\alpha}\right).$$

It follows from (18) that (16) is consequence of

$$N^{-\alpha} > 2N^{(\delta_0+\log 2)n_0/\log\log N}(N-1)^{-\alpha}\left(\frac{N-\alpha}{1-\alpha}\right).$$

One sees that an admissible choice of α is given by

$$\alpha = -3(\delta_0 + \log 2)n_0 \frac{N \log N}{\log\log N}.$$

Then $\zeta_{K,X}(s) \neq 0$ in the left-half plane $\sigma \leqslant \alpha$. This completes the proof of Lemma 1.

4. Proof of Theorem 1

Assuming for simplicity's sake that T does not coincide with the ordinate of any zero, we have

$$N_{K,X}(T) = \frac{1}{2\pi i} \int\limits_R \frac{\zeta'_{K,X}(s)}{\zeta_{K,X}(s)}\, ds,$$

where R is the rectangle with vertices at α, β, $\beta + iT$, and $\alpha + iT$. Thus, we have

$$2\pi N_{K,X}(T) = \int\limits_R \operatorname{Im}\left(\frac{\zeta'_{K,X}(s)}{\zeta_{K,X}(s)}\right) ds = \Delta_R \arg \zeta_{K,X}(s), \tag{19}$$

where Δ_R denotes the change in $\arg \zeta_{K,X}(s)$ as s traverses R in the positive sense.

Since $\zeta_{K,X}(s)$ is real and nonzero on $[\alpha, \beta]$, we have

$$\Delta_{[\alpha,\beta]} \arg \zeta_{K,X}(\sigma) = 0. \tag{20}$$

As s describes the right edge of R, we observe from (14) that

$$\left|\zeta_{K,X}(s) - 1\right| < 1.$$

It follows that $\operatorname{Re} \zeta_{K,X}(\beta + it) > 0$ for $0 \leqslant t \leqslant T$. Hence, we have

$$\Delta_{[0,T]} \arg \zeta_{K,X}(\beta + it) = O(1). \tag{21}$$

Furthermore, along the top edge of R, to estimate the change in $\arg \zeta_{K,X}(s)$ we decompose $\zeta_{K,X}(s)$ into its real part and its imaginary part. We have

$$\zeta_{K,X}(s) = \sum_{n \leqslant [X]} a(n) \exp\bigl\{-(\sigma + it) \log n\bigr\} = \sum_{n \leqslant [X]} \frac{a(n)[\cos(t \log n) - i \sin(t \log n)]}{n^\sigma},$$

so that

$$\operatorname{Im}\bigl(\zeta_{K,X}(\sigma + iT)\bigr) = -\sum_{n \leqslant [X]} \frac{a(n) \sin(T \log n)}{n^\sigma}.$$

By a generalization of Descartes's Rule of Signs (see Pólya and Szegö [12, Part V, Chapter 1, No. 77]), the number of real zeros of $\operatorname{Im}(\zeta_{K,X}(\sigma + iT))$ in the interval $\alpha \leqslant \sigma \leqslant \beta$ is less than or equal to the number of nonzero coefficients $a(n) \sin(T \log n)$. By Lemma 3, the number of nonzero coefficients $a(n)$ is $O_q(X(\log\log X/\log X)^{1-1/\phi(q)})$ at most.

Since the change in argument of $\zeta_{K,X}(\sigma + iT)$ between two consecutive zeros of $\mathrm{Im}(\zeta_{K,X}(\sigma + iT))$ is at most π, it follows that

$$\Delta_{[\alpha,\beta]} \arg \zeta_{K,X}(\sigma + iT) = O_q\left(X \left(\frac{\log \log X}{\log X} \right)^{1-1/\phi(q)} \right). \tag{22}$$

As in the proof of Lemma 1, we let N be the largest integer less than or equal to X so that $a(N) \neq 0$. Along the left edge of R, we have

$$\zeta_{K,X}(\alpha + it) = \left[1 + \frac{1 + a(2)2^{-\alpha-it} + \cdots + a(N-1)(N-1)^{-\alpha-it}}{a(N)N^{-\alpha-it}} \right] a(N)N^{-\alpha-it}.$$

Therefore, we have

$$\begin{aligned} \Delta_{[0,T]} \arg \zeta_{K,X}(\alpha + it) = {} & \Delta_{[0,T]} \arg \left[1 + \frac{1 + a(2)2^{-\alpha-it} + \cdots + a(N-1)(N-1)^{-\alpha-it}}{a(N)N^{-\alpha-it}} \right] \\ & + \Delta_{[0,T]} \arg a(N)N^{-\alpha-it}. \end{aligned} \tag{23}$$

In the proof of Lemma 1, we noticed that

$$\frac{a(N)}{N^\alpha} > \sum_{1 \leqslant n \leqslant N-1} \frac{a(n)}{n^\alpha}.$$

Thus, for any t, we have

$$\left| \frac{1 + a(2)2^{-\alpha-it} + \cdots + a(N-1)(N-1)^{-\alpha-it}}{a(N)N^{-\alpha-it}} \right| < 1,$$

and hence

$$\Delta_{[0,T]} \arg \left[1 + \frac{1 + a(2)2^{-\alpha-it} + \cdots + a(N-1)(N-1)^{-\alpha-it}}{a(N)N^{-\alpha-it}} \right] = O(1). \tag{24}$$

Finally, we have

$$\begin{aligned} \Delta_{[0,T]} \arg a(N)N^{-\alpha-it} &= \Delta_{[0,T]} \arg a(N)N^{-\alpha} \exp\{-it \log N\} \\ &= \Delta_{[0,T]} \arg \exp\{-it \log N\} \\ &= -T \log N. \end{aligned} \tag{25}$$

Then substituting (24) and (25) into (23), we obtain

$$\Delta_{[0,T]} \arg \zeta_{K,X}(\alpha + it) = -T \log N + O(1). \tag{26}$$

Since

$$\Delta_R \arg \zeta_{K,X}(s) = \Delta_{[\alpha,\beta]} \arg \zeta_{K,X}(\sigma) + \Delta_{[0,T]} \arg \zeta_{K,X}(\beta + it) \\ - \Delta_{[\alpha,\beta]} \arg \zeta_{K,X}(\sigma + iT) - \Delta_{[0,T]} \arg \zeta_{K,X}(\alpha + it),$$

we may now substitute (20), (21), (22), and (26) into (19) to obtain Theorem 1.

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