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Ramanujan sums are nearly orthogonal to powers

Emre Alkan

Department of Mathematics, Koç University, 34450, Sarıyer, Istanbul, Turkey

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ABSTRACT

Sign of averages of Ramanujan sums is studied and it is shown that these averages have a curious tendency to be positive. This in turn gives that Ramanujan sums are nearly orthogonal to a family of vectors whose entries are powers of consecutive integers. Further applications are given to the limit points of these averages along semigroups of integers, the peak size of partial sums of Ramanujan sum and an optimization problem on weighted exponential sums supported on reduced residue systems. Exact evaluations of trigonometric sums having combinatorially significant coefficients and subject to divisibility constraints are obtained in terms of Bernoulli numbers.

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1. Introduction

For any positive integer k and complex number z, the Ramanujan sum is

$$c_k(z) := \sum_{\substack{1 \leqslant q \leqslant k \\ (q,k)=1}} e^{\frac{2\pi i q z}{k}}.$$

E-mail address: ealkan@ku.edu.tr.

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Ramanujan [21] originally introduced these sums to obtain remarkable trigonometric series representations of certain normalized arithmetical functions. Today there is a profound theory on the representation of arithmetical functions as infinite orthogonal expansions involving Ramanujan sums (see [16,19,22,23]). Carmichael [11] was the first to obtain specific orthogonality properties such as

$$\sum_{j=1}^{N} c_{k_1}(j)c_{k_2}(j) = 0$$

whenever $k_1 \neq k_2$ and k_1, k_2 both divide N, and

$$\sum_{j=1}^{N} c_k(j)^2 = \varphi(k)N$$

whenever k divides N and φ is Euler's totient function. It would be of interest to search for additional orthogonality relations or obstructions to such relations, specifically, a family of vectors that are not orthogonal but arbitrarily close to being orthogonal to vectors arising from values of Ramanujan sums. Asymptotic orthogonality of the closely related Möbius function to nilsequences is recently established by Green and Tao [14] (see [25] and [7] for other types of cancellation results on partial sums of the Möbius function along semigroups of integers). For any positive integer r and modulus k, consider the sum

$$\sum_{j=1}^{k} j^r c_k(j).$$

Making use of properties of such sums, the author [1,6] derived exact formulas involving Euler's and Jordan's totient functions for averages of special values of L-functions under parity conditions imposed on the characters. The author [3] further studied the distribution of values of a normalization in the form

$$\frac{1}{k^{r+1}} \sum_{j=1}^{k} j^r c_k(j)$$

and showed that average value of this normalization over k is

$$\frac{3}{\pi^2} + \frac{1}{(r+1)} \sum_{m=1}^{[\frac{r}{2}]} \binom{r+1}{2m} \frac{B_{2m}}{\zeta(2m+1)},$$

where ζ is Riemann zeta function and B_{2m}'s are Bernoulli numbers. Connections between Ramanujan sums and Burgess type zeta functions were investigated in [4]. For similar average type results concerning values of a special multiplicative function known as the restrictive factor, the reader is referred to the recent work of Ledoan and Zaharescu [17]. The author [5] further showed that weighted averages of Gauss sums in the form

$$\frac{1}{k}\sum_{j=1}^{k} w\left(\frac{j}{k}\right)G(j,\chi),$$

where χ is a non-principal character modulo k, G is the corresponding Gauss sum and w is a Lipschitz continuous function, can be well approximated by certain linear combinations using algebraic parts of special values of L-functions. Note that when χ is the principal character modulo k and $w(x) = x^r$ for $r \geqslant 1$, one obtains the above normalization involving values of Ramanujan sum and this gives further motivation for the study of its distribution. Precisely, we focus on the sign of such a normalization as a real number and show that these sums have more tendency to be positive. In terms of the standard inner product on $\mathbb{R}^k$, our results below confirm that for many values of k and r, the inner product of the vectors

$$\frac{1}{k^{r+1}}\langle 1^r, 2^r, \ldots, k^r\rangle \quad \text{and} \quad \langle c_k(1), c_k(2), \ldots, c_k(k)\rangle$$

is positive and they are therefore not orthogonal but somehow arbitrarily close to being so (see Theorem 2 below). Whether this holds for all values of k and r remains an open problem.

Theorem 1. *The average value over k of*

$$\frac{1}{k^{r+1}}\sum_{j=1}^{k} j^r c_k(j)$$

is positive for all $r \geqslant 1$. For any given k and r large enough only in terms of k,

$$\frac{1}{k^{r+1}}\sum_{j=1}^{k} j^r c_k(j)$$

is positive.

A natural question is about the positivity of this normalization for all values of k and r. For fixed r, the limit points of such a normalization as k ranges over certain semigroups of positive integers were studied in [3]. If r is allowed to vary as well, then we are able to get more precise results on the set of limit points. In particular, our next result shows that verifying positivity for all values of k and r is delicate.

Theorem 2. *Let P be a set of primes having positive lower density such that for some constant $0 < \theta < 1$ and for all large enough x, there exists $p \in P$ with $x \leqslant p \leqslant x + x^\theta$.*

If M is an infinite set of positive integers and (P) is the semigroup of positive integers generated by the primes in P, then any number in $[0,1]$ is a limit point of the set

$$U=\left\{\frac{1}{k^{r+1}}\sum_{j=1}^{k} j^r c_k(j):\ k\in(P),\ r\in M\right\}.$$

We remark that P in Theorem 2 can be taken as the set of all primes in a fixed arithmetic progression containing infinitely many primes. To have a more general construction, assume that for all large enough integers n, there exist primes in $[n,n+n^\theta]$. We may then choose a prime p in $[n,n+n^\theta]$ (note that for any $n\leqslant j\leqslant p$, it is not necessary to choose a new prime in $[j,j+j^\theta]$ since $p\in[j,j+j^\theta]$, hence the most efficient way is to choose the largest prime in $[n,n+n^\theta]$). Doing this for all large enough n, one obtains a set T consisting of all the primes that are chosen. Then any set of primes $P\supseteq T$ with positive lower density clearly satisfies the conditions of Theorem 2. Let

$$J_n(k)=k^n\prod_{p|k}\left(1-\frac{1}{p^n}\right)$$

be Jordan's totient function of order $n\geqslant 1$. For fixed r, it is possible to produce explicit positive values of the normalization for certain values of k. Indeed we have

Corollary 1. *Let*

$$a_r(k):=\frac{1}{k^{r+1}}\sum_{j=1}^{k} j^r c_k(j).$$

Assume that $r\geqslant 2$ is fixed and $\{k_n\}$ is an increasing sequence of integers satisfying

$$\lim_{n\to\infty}\frac{J_2(k_n)}{k_n^2}=1.$$

Then we have

$$\liminf_{n\to\infty} a_r(k_n)\geqslant\frac{1}{2}-\frac{1}{r+1}.$$

Moreover, for any given $r\geqslant 1$, there exists a positive density of k's, where the density depends only on r, such that $a_r(k)$ is positive.

It is true that for small values of r, $a_r(k)$ is always positive regardless of the value of k. Precisely, the following result holds.

Corollary 2. *For any $k\geqslant 1$, we have $a_2(k)>\frac{1}{\pi^2}$, $a_3(k)>\frac{3}{2\pi^2}$, $a_4(k)>\frac{2}{\pi^2}-\frac{1}{30}$, $a_5(k)>\frac{5}{2\pi^2}-\frac{1}{12}$, $a_6(k)>\frac{3}{\pi^2}-\frac{1}{6}+\frac{45}{2\pi^6}$.*

Determination of the size of peak value concerning partial sums of a non-principal character χ modulo k is an important problem. As a consequence of the Polya–Vinogradov inequality, this peak value is $< \sqrt{k}\log k$ and it is also $> \frac{\sqrt{k}}{2\pi}$ (see [13]). Assuming the Riemann hypothesis for Dirichlet L-functions, Montgomery and Vaughan [18] showed that

$$\max_N \left| \sum_{n \leqslant N} \chi(n) \right| \ll \sqrt{k} \log\log k,$$

which is essentially best possible as Paley [20] showed that

$$\max_N \left| \sum_{n \leqslant N} \left(\frac{d}{n}\right) \right| > \frac{1}{7}\sqrt{d} \log\log d$$

holds for infinitely many quadratic discriminants $d > 0$, where $(\frac{d}{n})$ is the Kronecker symbol for the real quadratic field $\mathbb{Q}(\sqrt{d})$. Cancellation type results for convolution sums of two characters over short ranges were obtained by Güloğlu [15]. Here we offer to study this peak value for partial sums of Ramanujan sum.

Theorem 3. *For any $k \geqslant 1$, the estimate*

$$\max_N \left| \sum_{j=1}^{N} c_k(j) \right| \geqslant \frac{J_2(k)}{4k} + \frac{\varphi(k)}{2}$$

holds.

Consider the simple but elegant inequality

$$\frac{k}{\varphi(k)} = \prod_{p|k} \left(1 + \frac{1}{p-1}\right) \leqslant 2^{\omega(k)},$$

where ω is the number of distinct prime divisors. One has the following curious inequality involving these arithmetic functions.

Corollary 3. *For integers $r \geqslant 1$ and $k \geqslant 1$, the inequality*

$$\left| \frac{1}{2} + \left(\frac{1}{(r+1)} \sum_{m=1}^{[\frac{r}{2}]} \binom{r+1}{2m} B_{2m} \frac{J_{2m}(k)}{k^{2m}} \right) \frac{k}{\varphi(k)} \right| \leqslant 2^{\omega(k)}$$

holds.

Our next result is inspired by the work of Bachman [9] who showed uniqueness of Ramanujan sums as solution of a certain minimization problem. We see below that Ramanujan sums are unique solutions of a more general optimization problem with nonnegative weight functions and subject to the constraints $|b_n| \leqslant 1$ for all $n \geqslant 1$. In this connection, nontrivial lower bounds are produced concerning weighted exponential sums supported on reduced residue systems.

Theorem 4. *Let w be a nonnegative function. For any sequence b_n of complex numbers and real number $q \geqslant 1$, the inequality*

$$\sum_{j=1}^{k} w(j)\left|\sum_n b_n e^{\frac{2\pi i n j}{k}}\right|^q \geqslant \left(\frac{|\sum_n b_n|}{\varphi(k)}\right)^q \sum_{j=1}^{k} w(j)\left|c_k(j)\right|^q$$

holds, where the sum over n is for $1 \leqslant n \leqslant k$ and $(n,k)=1$. In particular, if $w(j)=j^r$ for some integer $r \geqslant 2$ and $q=1$, then we have

$$\sum_{j=1}^{k} j^r\left|\sum_n b_n e^{\frac{2\pi i n j}{k}}\right| \geqslant \left|\sum_n b_n\right|\left|\frac{1}{2}+\left(\frac{1}{(r+1)}\sum_{m=1}^{[\frac{r}{2}]}\binom{r+1}{2m}B_{2m}\frac{J_{2m}(k)}{k^{2m}}\right)\frac{k}{\varphi(k)}\right|k^r.$$

If $w(j)=j$ and $q=1$, then we have

$$\sum_{j=1}^{k} j\left|\sum_n b_n e^{\frac{2\pi i n j}{k}}\right| \geqslant \frac{1}{2}\left|\sum_n b_n\right|\left(2^{\omega(k)}+1\right)k.$$

Define the constants $c_2=\frac{1}{\pi^2}$, $c_3=\frac{3}{2\pi^2}$, $c_4=\frac{2}{\pi^2}-\frac{1}{30}$, $c_5=\frac{5}{2\pi^2}-\frac{1}{12}$, $c_6=\frac{3}{\pi^2}-\frac{1}{6}+\frac{45}{2\pi^6}$. Then for $r \in \{2,3,4,5,6\}$, we have

$$\sum_{j=1}^{k} j^r\left|\sum_n b_n e^{\frac{2\pi i n j}{k}}\right| > c_r\left(\frac{|\sum_n b_n|}{\varphi(k)}\right)k^{r+1}.$$

For any given $r \geqslant 2$, there exists a positive constant M_r depending only on r and infinitely many k such that

$$\sum_{j=1}^{k} j^r\left|\sum_n b_n e^{\frac{2\pi i n j}{k}}\right| > M_r\left|\sum_n b_n\right|k^r \log\log k.$$

Finally, for $r \geqslant 1$, we have

$$\sum_{j=1}^{k} j^r\left|\sum_n b_n e^{\frac{2\pi i n j}{k}}\right| \ll_r \left(\sum_n |b_n|^2\right)^{\frac{1}{2}} k^{r+1},$$

where the implied constant depends only on r.

As a remark note the trivial bounds

$$\left|\sum_n b_n\right| k^r \leqslant \sum_{j=1}^{k} j^r \left|\sum_n b_n e^{\frac{2\pi i n j}{k}}\right| \leqslant \left(\sum_n |b_n|\right) k^{r+1}.$$

To have an application of Theorem 4, let χ be a non-principal character modulo k and take $b_n = 1 - \chi(n)$. Then since

$$\sum_n b_n = \varphi(k) - \sum_n \chi(n) = \varphi(k),$$

we see that

$$\sum_{j=1}^{k} j\left|G(j,\chi) - c_k(j)\right| \geqslant \frac{1}{2}\left(2^{\omega(k)} + 1\right)\varphi(k)k,$$

where

$$G(z,\chi) = \sum_n \chi(n) e^{\frac{2\pi i n z}{k}}$$

is the Gauss sum. Moreover, by a result of Hardy and Ramanujan, $\omega(k)$ is about $\log\log k$ for almost all k and since $\varphi(k) \gg \frac{k}{\log\log k}$ for all $k \geqslant 3$, the estimate

$$\sum_{j=1}^{k} j\left|G(j,\chi) - c_k(j)\right| \gg \frac{k^2 (\log k)^{(1+o(1))\log 2}}{\log\log k}$$

holds for almost all k with an absolute implied constant. Relations between special values of L-functions, class numbers and finite trigonometric sums were studied by Berndt and Zaharescu [10], Chan [12] and the author [2]. As a further contribution to this topic, we give the following evaluation of trigonometric sums subject to divisibility constraints as a natural consequence of our study of Ramanujan sums.

Theorem 5. *For integers $k \geqslant 2$ and $r \geqslant 1$, we have*

$$\sum_{s=1}^{r}\left(\sum_{j=1}^{\min(r-1,r-s+1)} \binom{r}{j} A_{r-j,s} k^j\right)\left(\sum_{\substack{1\leqslant m\leqslant k\\ (m,k)=1}} \frac{g(k,m,s)}{(2\sin(\frac{\pi m}{k}))^s}\right)$$
$$= \frac{k^{r+1}}{(r+1)} \sum_{m=1}^{[\frac{r}{2}]} \binom{r+1}{2m} B_{2m} \prod_{p\mid k}\left(1 - \frac{1}{p^{2m}}\right),$$

where both sides are interpreted to be zero when $r = 1$*, the coefficients* $A_{q,j}$ *for* $q, j \geqslant 1$ *are uniquely determined by the relations* $A_{q,1} = (-1)^q$*,* $A_{q,q+1} = (-1)^q q!$*,*

$$A_{q,j} = -(j-1)A_{q-1,j-1} - jA_{q-1,j}$$

for $1 < j < q+1$ *and*

$$g(k,m,s) = \begin{cases} (-1)^{\frac{s+1}{2}} \sin(\frac{\pi ms}{k}) & \text{if } s \equiv \pm 1 \pmod 4 \\ (-1)^{\frac{s}{2}} \cos(\frac{\pi ms}{k}) & \text{if } s \equiv 0, 2 \pmod 4). \end{cases}$$

Combinatorial properties of $A_{q,j}$ are studied in [2] and it is shown that they can be written in terms of Stirling numbers of the second kind together with the formula

$$A_{q,j} = \sum_{v=0}^{j-1} (-1)^{v+q} \binom{j-1}{v} (j-v)^q.$$

2. Proof of Theorem 1

Let us start by giving an alternative representation of the normalization of weighted averages of Ramanujan sums. Indeed using an evaluation of Ramanujan sum in the form

$$c_k(j) = \frac{\varphi(k)\mu(\frac{k}{(k,j)})}{\varphi(\frac{k}{(k,j)})},$$

where μ is the Möbius function and a recent result of Singh [24] given as

$$Q_r(n) := \sum_{\substack{1 \leqslant j \leqslant n \\ (j,n)=1}} j^r = \frac{n^{r+1}}{(r+1)} \sum_{m=0}^{[\frac{r}{2}]} \binom{r+1}{2m} \frac{B_{2m}}{n^{2m}} \prod_{p|n} (1 - p^{2m-1})$$

for $r \geqslant 1$ and $n \geqslant 2$ together with some inspired calculations, one obtains the formula (see [3] for details)

$$\frac{1}{k^{r+1}} \sum_{j=1}^{k} j^r c_k(j) = \frac{\varphi(k)}{2k} + \frac{1}{(r+1)} \sum_{m=1}^{[\frac{r}{2}]} \binom{r+1}{2m} B_{2m} \prod_{p|k} \left(1 - \frac{1}{p^{2m}}\right) \tag{1}$$

for $r \geqslant 1$ and $k \geqslant 2$ with the convention that the sum over m in (1) is taken to be zero when $r = 1$. A more useful representation of the right side of (1) is required. Such a representation was also needed in [3] and for convenience let us reproduce it here. Assuming $r \geqslant 2$, one has from (1) that

$$\begin{aligned}\frac{1}{k^{r+1}}\sum_{j=1}^{k} j^r c_k(j) &= \frac{\varphi(k)}{2k} + \frac{1}{(r+1)}\sum_{m=1}^{[\frac{r}{2}]}\binom{r+1}{2m}B_{2m}\sum_{d|k}\frac{\mu(d)}{d^{2m}} \\ &= \frac{\varphi(k)}{2k} + \frac{1}{(r+1)}\sum_{d|k}\mu(d)\sum_{m=1}^{[\frac{r}{2}]}\binom{r+1}{2m}\frac{B_{2m}}{d^{2m}}. \end{aligned} \tag{2}$$

If r is even, then the monic Bernoulli polynomial of degree $r+1$ is given by

$$B_{r+1}(x) = \sum_{s=0}^{r+1}\binom{r+1}{s}B_s x^{r+1-s} = \sum_{s=0}^{r}\binom{r+1}{s}B_s x^{r+1-s} \tag{3}$$

since $B_{r+1} = 0$. Rewriting (3), one obtains

$$B_{r+1}(x) = x^{r+1} - \frac{(r+1)}{2}x^r + x^{r+1}\sum_{m=1}^{\frac{r}{2}}\binom{r+1}{2m}\frac{B_{2m}}{x^{2m}}. \tag{4}$$

Taking $x = d$ in (4), we see that

$$\sum_{m=1}^{\frac{r}{2}}\binom{r+1}{2m}\frac{B_{2m}}{d^{2m}} = \frac{1}{d^{r+1}}\left(B_{r+1}(d) - d^{r+1} + \frac{(r+1)}{2}d^r\right). \tag{5}$$

Consequently from (5), it follows that

$$\begin{aligned}\sum_{d|k}\mu(d)\sum_{m=1}^{[\frac{r}{2}]}\binom{r+1}{2m}\frac{B_{2m}}{d^{2m}} &= \sum_{d|k}\mu(d)\left(\frac{B_{r+1}(d)}{d^{r+1}} - 1 + \frac{r+1}{2d}\right) \\ &= \sum_{d|k}\frac{\mu(d)B_{r+1}(d)}{d^{r+1}} - \sum_{d|k}\mu(d) + \frac{(r+1)}{2}\sum_{d|k}\frac{\mu(d)}{d} \\ &= \sum_{d|k}\frac{\mu(d)B_{r+1}(d)}{d^{r+1}} + \frac{(r+1)\varphi(k)}{2k} \end{aligned} \tag{6}$$

since the Möbius sum is zero for $k \geqslant 2$. As a result of (2) and (6), we see that

$$\begin{aligned}\frac{1}{k^{r+1}}\sum_{j=1}^{k} j^r c_k(j) &= \frac{\varphi(k)}{k} + \frac{1}{(r+1)}\sum_{d|k}\frac{\mu(d)B_{r+1}(d)}{d^{r+1}} \\ &= \frac{\varphi(k)}{k} + \frac{1}{(r+1)}\sum_{\substack{d|k \\ d>1}}\frac{\mu(d)B_{r+1}(d)}{d^{r+1}}, \end{aligned} \tag{7}$$

where we used the fact that $B_{r+1}(1) = 0$. For $n > 1$, define

$$S_r(n) := \sum_{j=1}^{n-1} j^r.$$

Using the well-known evaluation of $S_r(n)$ in the form

$$(r+1)S_r(n) = \sum_{s=0}^{r} \binom{r+1}{s} B_s n^{r+1-s},$$

one obtains

$$S_r(d) = \frac{B_{r+1}(d)}{r+1} \tag{8}$$

for any $d > 1$. Combining (7) and (8), the representation

$$\frac{1}{k^{r+1}} \sum_{j=1}^{k} j^r c_k(j) = \frac{\varphi(k)}{k} + \sum_{\substack{d|k \\ d>1}} \frac{\mu(d)S_r(d)}{d^{r+1}} \tag{9}$$

follows for even $r \geqslant 2$. A similar argument shows that (9) holds for odd $r \geqslant 2$ as well. Next observe that for $d > 1$,

$$L_{d,r} := \frac{S_r(d)}{d^{r+1}} = \frac{1}{d^{r+1}} \sum_{j=1}^{d-1} j^r = \frac{1}{d} \sum_{j=1}^{d-1} \left(\frac{j}{d}\right)^r \tag{10}$$

is a lower Riemann sum using the equally spaced partitioning points $\{\frac{1}{d}, \frac{2}{d}, \ldots, \frac{d-1}{d}\}$ corresponding to the integral $\int_0^1 x^r\,dx = \frac{1}{r+1}$. For any $d > 1$, $L_{d,r} < \frac{1}{r+1}$ and this motivates to define the numbers

$$\delta_{d,r} := \frac{1}{r+1} - \frac{S_r(d)}{d^{r+1}} > 0 \tag{11}$$

as an error in the approximation of this integral by lower Riemann sums. Also considering the upper Riemann sum for the same integral defined as

$$U_{d,r} := \frac{1}{d} \sum_{j=1}^{d} \left(\frac{j}{d}\right)^r, \tag{12}$$

one obtains that $L_{d,r} < \frac{1}{r+1} < U_{d,r}$ and $U_{d,r} - L_{d,r} = \frac{1}{d}$. It follows from (10)–(12) that $0 < \delta_{d,r} < \frac{1}{d}$ for any $d > 1$ and $r \geqslant 2$. As a result of (9) and (11), the representation

$$\frac{1}{k^{r+1}}\sum_{j=1}^{k} j^r c_k(j) = \frac{\varphi(k)}{k} + \sum_{\substack{d\mid k\\ d>1}} \mu(d)\left(\frac{1}{r+1} - \delta_{d,r}\right)$$
$$= \frac{\varphi(k)}{k} - \frac{1}{r+1} - \sum_{\substack{d\mid k\\ d>1}} \mu(d)\delta_{d,r} \tag{13}$$

follows for $k \geqslant 2$. Using (13), we have that

$$\sum_{k\leqslant x}\left(\frac{1}{k^{r+1}}\sum_{j=1}^{k} j^r c_k(j)\right) = 1 + \sum_{2\leqslant k\leqslant x}\left(\frac{1}{k^{r+1}}\sum_{j=1}^{k} j^r c_k(j)\right)$$
$$= \sum_{k\leqslant x}\frac{\varphi(k)}{k} - \frac{1}{r+1}\sum_{2\leqslant k\leqslant x} 1 - \sum_{k\leqslant x}\sum_{\substack{d\mid k\\ d>1}} \mu(d)\delta_{d,r}. \tag{14}$$

Using the well-known asymptotic relation

$$\sum_{k\leqslant x}\frac{\varphi(k)}{k} = \frac{6}{\pi^2}x + O(\log x),$$

we see that (14) further equals

$$= \left(\frac{6}{\pi^2} - \frac{1}{r+1}\right)x - \sum_{k\leqslant x}\sum_{\substack{d\mid k\\ d>1}} \mu(d)\delta_{d,r} + O(\log x). \tag{15}$$

Moreover, note that

$$\sum_{k\leqslant x}\sum_{\substack{d\mid k\\ d>1}} \mu(d)\delta_{d,r} = \sum_{1<d\leqslant x} \mu(d)\delta_{d,r}\left[\frac{x}{d}\right] = \sum_{1<d\leqslant x} \mu(d)\delta_{d,r}\left(\frac{x}{d} + O(1)\right)$$
$$= x\sum_{d=2}^{\infty}\frac{\mu(d)\delta_{d,r}}{d} + O(\log x), \tag{16}$$

where we used the fact that $0 < \delta_{d,r} < \frac{1}{d}$. Combining (14)–(16), the desired average value over k is determined to be

$$\frac{6}{\pi^2} - \frac{1}{r+1} - \sum_{d=2}^{\infty}\frac{\mu(d)\delta_{d,r}}{d} = \frac{6}{\pi^2} - \frac{1}{r+1} + \frac{\delta_{2,r}}{2} + \frac{\delta_{3,r}}{3} + \frac{\delta_{5,r}}{5} - \sum_{d=6}^{\infty}\frac{\mu(d)\delta_{d,r}}{d}$$
$$> \frac{6}{\pi^2} - \frac{1}{r+1} - \sum_{d=6}^{\infty}\frac{\mu(d)\delta_{d,r}}{d}. \tag{17}$$

Observing that

$$\left|\sum_{d=6}^{\infty} \frac{\mu(d)\delta_{d,r}}{d}\right| \leqslant \sum_{d=6}^{\infty} \frac{1}{d^2} \leqslant \int_{5}^{\infty} \frac{1}{t^2}\, dt = \frac{1}{5}, \tag{18}$$

we deduce from (17) and (18) that

$$\frac{6}{\pi^2} - \frac{1}{r+1} - \sum_{d=2}^{\infty} \frac{\mu(d)\delta_{d,r}}{d} \geqslant \frac{6}{\pi^2} - \frac{8}{15} > 0$$

for $r \geqslant 2$. But when $r = 1$, the average value over k is $\frac{3}{\pi^2}$. Thus the average value over k is positive for all $r \geqslant 1$. Finally when $k \geqslant 2$ is fixed, using (13) and the fact that for any fixed $d > 1$,

$$\lim_{r\to\infty} \delta_{d,r} = \lim_{r\to\infty} \left(\frac{1}{r+1} - \frac{S_r(d)}{d^{r+1}}\right) = 0,$$

we see that

$$\lim_{r\to\infty} \left(\frac{1}{k^{r+1}} \sum_{j=1}^{k} j^r c_k(j)\right) = \frac{\varphi(k)}{k} > 0 \tag{19}$$

which happens to be true also when $k = 1$. Therefore, if r is large enough only in terms of k, then the normalization is positive by (19). This completes the proof.

3. Proof of Theorem 2

First observe that since M is infinite, we have from (19) that

$$\lim_{\substack{r\in M\\ r\to\infty}} \left(\frac{1}{k^{r+1}} \sum_{j=1}^{k} j^r c_k(j)\right) = \frac{\varphi(k)}{k}. \tag{20}$$

Hence as a result of (20), to show that any number in $[0,1]$ is a limit point of U, we need to show that the numbers $\frac{\varphi(k)}{k}$ are dense in $[0,1]$ as k ranges in (P). To this end, let us define an arithmetic function $\phi : \mathbb{N} \to \mathbb{Q}$ as follows. If $n \in (P)$, then we take $\phi(n) = \frac{n}{\varphi(n)}$. This makes ϕ a multiplicative function on (P). Then extend ϕ as a multiplicative function on $\mathbb{N}$ by defining $\phi(p^m) = 1$ for any prime $p \notin P$ and $m \geqslant 1$. Note that $f_\phi(n) = \log\phi(n)$ is an additive function defined on $\mathbb{N}$. Thus, if $n \in (P)$, then $f_\phi(n) = \log(\frac{n}{\varphi(n)})$. Moreover, observe that for any prime $p \in P$ and $m \geqslant 1$, we have

$$\left|f_\phi(p^m)\right| = \left|\log\left(1 - \frac{1}{p}\right)\right| = -\log\left(1 - \frac{1}{p}\right) \leqslant \frac{2}{p} \tag{21}$$

and

$$f_\phi(p) = \frac{1}{p} + O\left(\frac{1}{p^2}\right). \tag{22}$$

Next we employ the main result of [8]. Given an additive function $f : \mathbb{N} \to \mathbb{R}$, assume that the following conditions are satisfied for some $\delta > 0$ and $\lambda > 0$:

$$\sum_{\substack{f(p)>0 \\ p \text{ prime}}} f(p) = \infty, \tag{23}$$

$$\left|f\left(p^m\right)\right| \leqslant \frac{C(f)}{p^\delta} \tag{24}$$

for any prime p and $m \geqslant 1$ with $C(f) > 0$ depending only on f. There exists $t_0(f) > 0$ depending only on f such that for any $0 < t \leqslant t_0(f)$, there is a prime p satisfying

$$t - t^{1+\lambda} \leqslant f(p) \leqslant t. \tag{25}$$

Then there is a $\beta > 0$ (depending on δ, λ) such that for all appropriate $\alpha \in \mathbb{R}$

$$\left|f(n) - \alpha\right| < \frac{1}{n^\beta}$$

holds for infinitely many n. Since $f_\phi(p^m) = 0$ when $p \notin P$ and P has positive lower density by our assumption, it follows from (22) that

$$\sum_{\substack{f_\phi(p)>0 \\ p \text{ prime}}} f_\phi(p) = \sum_{p \in P} \left(\frac{1}{p} + O\left(\frac{1}{p^2}\right)\right) = \infty$$

and (23) is satisfied. Using (21), we see that (24) holds with $C(f_\phi) = 2$ and $\delta = 1$. In order to verify (25), we consider

$$t - t^{1+\lambda} \leqslant \frac{1}{p} + O\left(\frac{1}{p^2}\right) \leqslant t \tag{26}$$

for $p \in P$. It is easy to see that (26) is equivalent to

$$x \leqslant p + O(1) \leqslant x + x^{1-\lambda} \tag{27}$$

for all large enough x. Since for all large enough x, there exists $p \in P$ with $x \leqslant p \leqslant x + x^\theta$ by our assumption, we may choose $\lambda < 1 - \theta$ to see that (27) holds as well. It follows that the values of f_ϕ are dense in $[0, \infty)$ and therefore that the values $\frac{\varphi(k)}{k}$ are dense in $[0, 1]$ as k ranges in (P). This completes the proof.

4. Proof of Corollary 1

Using the evaluation of Bernoulli polynomials at $x = 1$, it is easy to see that

$$\frac{1}{2} - \frac{1}{r+1} = \frac{1}{(r+1)} \sum_{m=1}^{[\frac{r}{2}]} \binom{r+1}{2m} B_{2m} \tag{28}$$

for any $r \geqslant 2$. Assume that $r \geqslant 2$ is fixed and $\{k_n\}$ is an increasing sequence of integers satisfying

$$\lim_{n\to\infty} \frac{J_2(k_n)}{k_n^2} = 1. \tag{29}$$

Clearly, (29) gives

$$\lim_{n\to\infty} \frac{J_{2m}(k_n)}{k_n^{2m}} = 1 \tag{30}$$

for any $m \geqslant 1$. Using (1), one gets

$$a_r(k_n) = \frac{\varphi(k_n)}{2k_n} + \frac{1}{(r+1)} \sum_{m=1}^{[\frac{r}{2}]} \binom{r+1}{2m} B_{2m} \frac{J_{2m}(k_n)}{k_n^{2m}}. \tag{31}$$

Combining (28), (30) and (31), we deduce that

$$\liminf_{n\to\infty} a_r(k_n) \geqslant \frac{1}{2} - \frac{1}{r+1}.$$

To prove the other claim our main idea is to sieve by the first N primes $2, 3, \dots, p_N$, where N is a parameter. Note that when $r = 1$, $a_1(k)$ is positive for any $k \geqslant 1$. Therefore, we may assume that $r \geqslant 2$. Given $\epsilon > 0$, one can find N large enough such that

$$1 - \epsilon < \prod_{p > p_N} \left(1 - \frac{1}{p^2}\right) < 1$$

and consequently that

$$1 - \epsilon < \prod_{p > p_N} \left(1 - \frac{1}{p^{2m}}\right) < 1$$

for any $m \geqslant 1$. It follows that with this choice of N, if k is not divisible by the first N primes, then

$$1 - \epsilon < \frac{J_{2m}(k)}{k^{2m}} = \prod_{p|k} \left(1 - \frac{1}{p^{2m}}\right) \leqslant 1$$

holds for any $m \geqslant 1$. Using now (28) it is clear that

$$\frac{1}{(r+1)}\sum_{m=1}^{[\frac{r}{2}]}\binom{r+1}{2m}B_{2m}\frac{J_{2m}(k)}{k^{2m}}$$

can be made as close as we wish to $\frac{1}{2}-\frac{1}{r+1}>0$ when $r \geqslant 2$. To do this, N would depend only on r. Since the set of k's not divisible by the first N primes has positive density depending only on r, one obtains from (1) that $a_r(k)$ is guaranteed to be positive for these values of k.

5. Proof of Corollary 2

Since $a_r(1) = 1$ for $r \geqslant 1$, the inequalities hold when $k = 1$. Therefore, we may assume $k \geqslant 2$. We only prove the inequality for $r = 6$. The other inequalities can be obtained similarly. Indeed taking $r = 6$ in (1), we obtain

$$a_6(k) = \frac{\varphi(k)}{2k} + \frac{J_2(k)}{2k^2} - \frac{J_4(k)}{6k^4} + \frac{J_6(k)}{42k^6} > \frac{J_2(k)}{2k^2} - \frac{J_4(k)}{6k^4} + \frac{J_6(k)}{42k^6}. \tag{32}$$

Using the inequalities

$$\frac{J_2(k)}{k^2} > \frac{1}{\zeta(2)} = \frac{6}{\pi^2}, \qquad \frac{J_4(k)}{k^4} < 1, \qquad \frac{J_6(k)}{k^6} > \frac{1}{\zeta(6)} = \frac{945}{\pi^6}$$

for $k \geqslant 2$, we obtain from (32) the desired inequality for $a_6(k)$.

6. Proof of Theorem 3

First note that when $k = 1$, the required maximum is infinite and the inequality trivially holds. We may then assume that $k \geqslant 2$. In this case one has

$$\sum_{j=1}^{k} c_k(j) = 0 \tag{33}$$

and the required maximum is finite. Taking $r = 3$ in (1), we obtain that

$$\sum_{j=1}^{k} j^3 c_k(j) = \left(\frac{\varphi(k)}{2k} + \frac{J_2(k)}{4k^2}\right)k^4 \tag{34}$$

for $k \geqslant 2$. Applying Abel's summation to the left side of (34) and using (33), one sees that

$$\sum_{j=1}^{k} j^3 c_k(j) = -\int_1^k 3t^2\left(\sum_{1\leqslant j\leqslant t} c_k(j)\right)dt. \tag{35}$$

It follows from (35) that

$$\left|\sum_{j=1}^{k} j^3 c_k(j)\right| \leqslant \left(\max_N \left|\sum_{j=1}^{N} c_k(j)\right|\right) k^3. \tag{36}$$

The desired estimate follows from (34) and (36).

7. Proof of Corollary 3

Again when $k = 1$, the inequality holds by (28). Hence we may assume that $k \geqslant 2$. From (1), we have

$$\sum_{j=1}^{k} j^r c_k(j) = \left(\frac{\varphi(k)}{2k} + \frac{1}{(r+1)} \sum_{m=1}^{[\frac{r}{2}]} \binom{r+1}{2m} B_{2m} \frac{J_{2m}(k)}{k^{2m}}\right) k^{r+1}. \tag{37}$$

Applying Abel's summation to the left side of (37), one obtains

$$\left|\sum_{j=1}^{k} j^r c_k(j)\right| \leqslant \left(\max_N \left|\sum_{j=1}^{N} c_k(j)\right|\right) k^r \leqslant \left(\sum_{j=1}^{k} \left|c_k(j)\right|\right) k^r. \tag{38}$$

Let

$$k = \prod_{v=1}^{s} p_v^{a_v}$$

be the prime factorization of k into distinct primes. Then note that $c_k(j) = 0$ unless $\frac{k}{(k,j)}$ is a square-free number. If we let $d = (k, j)$, then

$$d = \prod_{v=1}^{s} p_v^{b_v}$$

with $a_v - 1 \leqslant b_v \leqslant a_v$ for each v and there are exactly $2^{\omega(k)}$ values of d (which are divisors of k) such that $c_k(j) \neq 0$. Using the estimate

$$\left|c_k(j)\right| \leqslant \frac{\varphi(k)}{\varphi(\frac{k}{(k,j)})},$$

we have

$$\sum_{j=1}^{k} \left|c_k(j)\right| = \sum_{\substack{1 \leqslant j \leqslant k \\ c_k(j) \neq 0}} \left|c_k(j)\right| \leqslant \sum_{\substack{1 \leqslant j \leqslant k \\ c_k(j) \neq 0}} \frac{\varphi(k)}{\varphi(\frac{k}{(k,j)})}. \tag{39}$$

The conditions $1 \leqslant j \leqslant k$ and $d = (k, j)$ are equivalent to $1 \leqslant \frac{j}{d} \leqslant \frac{k}{d}$ and $(\frac{k}{d}, \frac{j}{d}) = 1$. Therefore, the number of such integers is $\varphi(\frac{k}{d})$ and (39) gives that

$$\sum_{j=1}^{k} \left|c_k(j)\right| \leqslant \varphi(k)2^{\omega(k)}. \tag{40}$$

The desired inequality follows from (37), (38) and (40).

8. Proof of Theorem 4

We may assume that b_n's are periodic with period k. Note that if $(s, k) = 1$, then multiplication by s permutes residue classes (or reduced residue classes) modulo k. Thus we have

$$\begin{aligned}\left|\sum_n b_n\right|^q \sum_{j=1}^{k} w(j)\left|c_k(j)\right|^q &= \sum_{j=1}^{k} w(j)\left|\left(\sum_n b_n\right)\left(\sum_s e^{\frac{2\pi isj}{k}}\right)\right|^q \\ &= \sum_{j=1}^{k} w(j)\left|\sum_n \sum_s b_{sn} e^{\frac{2\pi isj}{k}}\right|^q. \end{aligned} \tag{41}$$

Using Hölder's inequality when $q > 1$ and triangle inequality when $q = 1$, one obtains

$$\left|\sum_n \sum_s b_{sn} e^{\frac{2\pi isj}{k}}\right| \leqslant \varphi(k)^{\frac{1}{q'}} \left(\sum_n \left|\sum_s b_{sn} e^{\frac{2\pi isj}{k}}\right|^q\right)^{\frac{1}{q}}, \tag{42}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. Combining (41), (42) and using the fact that w is nonnegative, the left side of (41) is

$$\leqslant \varphi(k)^{\frac{q}{q'}} \sum_{j=1}^{k} w(j) \sum_n \left|\sum_s b_{sn} e^{\frac{2\pi isj}{k}}\right|^q = \varphi(k)^{\frac{q}{q'}} \sum_n \sum_{j=1}^{k} w(j) \left|\sum_s b_{sn} e^{\frac{2\pi isn\bar{n}j}{k}}\right|^q, \tag{43}$$

where $\bar{n}$ is the inverse of n modulo k. For each fixed $1 \leqslant n \leqslant k$ with $(n, k) = 1$, the numbers $\bar{n}j$ run over a complete residue system modulo k and the numbers sn run over a reduced residue system modulo k. Noting that the number of such n is $\varphi(k)$ and $1 + \frac{q}{q'} = q$, we see that the right side of (43) equals

$$\varphi(k)^q \sum_{j=1}^{k} w(j) \left|\sum_n b_n e^{\frac{2\pi inj}{k}}\right|^q. \tag{44}$$

The desired inequality now follows from (41), (43) and (44). Indeed taking $w(j) = j^r$ for some integer $r \geqslant 2$ and $q = 1$, one has

$$\sum_{j=1}^{k} j^r \left| \sum_n b_n e^{\frac{2\pi i n j}{k}} \right| \geqslant \frac{|\sum_n b_n|}{\varphi(k)} \sum_{j=1}^{k} j^r \left|c_k(j)\right| \geqslant \frac{|\sum_n b_n|}{\varphi(k)} \left| \sum_{j=1}^{k} j^r c_k(j) \right|. \tag{45}$$

Therefore, the desired inequality follows from (37) when $k \geqslant 2$ and from (28) when $k = 1$. If $w(j) = j$ and $q = 1$, then we have

$$\sum_{j=1}^{k} j \left| \sum_n b_n e^{\frac{2\pi i n j}{k}} \right| \geqslant \frac{|\sum_n b_n|}{\varphi(k)} \sum_{j=1}^{k} j \left|c_k(j)\right|. \tag{46}$$

Using evaluation of Ramanujan sum and the fact that $Q_1(d) = \frac{d\varphi(d)}{2}$ when $d \geqslant 2$, one obtains

$$\begin{aligned}\sum_{j=1}^{k} j\left|c_k(j)\right| &= k\varphi(k) \sum_{d|k} \frac{\mu^2(d)}{d\varphi(d)} Q_1(d) = k\varphi(k)\left(1 + \frac{1}{2}\sum_{\substack{d|k \\ d>1}} \mu^2(d)\right) \\ &= \frac{1}{2}\left(2^{\omega(k)} + 1\right)k\varphi(k).\end{aligned} \tag{47}$$

Combining (46) and (47) gives

$$\sum_{j=1}^{k} j \left| \sum_n b_n e^{\frac{2\pi i n j}{k}} \right| \geqslant \frac{1}{2} \left| \sum_n b_n \right| \left(2^{\omega(k)} + 1\right) k.$$

When $r \in \{2, 3, 4, 5, 6\}$, using (45), one has

$$\sum_{j=1}^{k} j^r \left| \sum_n b_n e^{\frac{2\pi i n j}{k}} \right| \geqslant \left(\frac{|\sum_n b_n|}{\varphi(k)} \right) \left|a_r(k)\right| k^{r+1}$$

and

$$\sum_{j=1}^{k} j^r \left| \sum_n b_n e^{\frac{2\pi i n j}{k}} \right| > c_r \left(\frac{|\sum_n b_n|}{\varphi(k)} \right) k^{r+1}$$

is a consequence of Corollary 2. To show the next inequality, one can use an idea of the proof of Corollary 1. Indeed if N is large enough and k is not divisible by the first N primes $2, 3, \ldots, p_N$, then for any given $r \geqslant 2$

$$\frac{1}{(r+1)} \sum_{m=1}^{[\frac{r}{2}]} \binom{r+1}{2m} B_{2m} \frac{J_{2m}(k)}{k^{2m}}$$

can be made arbitrarily close to $\frac{1}{2} - \frac{1}{r+1}$. Since $\frac{1}{2} - \frac{1}{r+1} \geqslant \frac{1}{6}$ for $r \geqslant 2$, we may safely assume that

$$\frac{1}{(r+1)} \sum_{m=1}^{[\frac{r}{2}]} \binom{r+1}{2m} B_{2m} \frac{J_{2m}(k)}{k^{2m}} > \frac{1}{7} \tag{48}$$

for any k not divisible by the first N primes and N is large enough only in terms of r. It is well-known that record low values of $\varphi(k)$ occur when k is a product of the first R consecutive primes, where R is a parameter. Moreover,

$$\varphi(k) \asymp \frac{k}{\log\log k} \tag{49}$$

for these specific values of k. When k is not divisible by the first N primes, the record low values of $\varphi(k)$ occur when k is a product of the first R consecutive primes coming after p_N, namely $p_{N+1}, p_{N+2}, \ldots, p_{N+R}$. Since

$$\prod_{p \leqslant p_N} \left(1 - \frac{1}{p}\right)$$

is a constant depending only on r, it follows from (49) that

$$\varphi(k) \asymp_r \frac{k}{\log\log k} \tag{50}$$

when k is a product of the first R consecutive primes coming after p_N. Therefore, combining (37), (45), (48) and (50), we deduce that there exists a positive constant M_r depending only on r and infinitely many k not divisible by the first N primes and satisfying (50) such that

$$\begin{aligned}\sum_{j=1}^{k} j^r \left|\sum_n b_n e^{\frac{2\pi i n j}{k}}\right| &\geqslant \left|\sum_n b_n\right| \left|\frac{1}{2} + \left(\frac{1}{(r+1)} \sum_{m=1}^{[\frac{r}{2}]} \binom{r+1}{2m} B_{2m} \frac{J_{2m}(k)}{k^{2m}}\right) \frac{k}{\varphi(k)}\right| k^r \\ &> M_r \left|\sum_n b_n\right| k^r \log\log k.\end{aligned}$$

To complete the proof, note that by the Cauchy–Schwarz inequality,

$$\sum_{j=1}^{k} j^r \left|\sum_n b_n e^{\frac{2\pi i n j}{k}}\right| \leqslant \left(\sum_{j=1}^{k} j^{2r}\right)^{\frac{1}{2}} \left(\sum_{j=1}^{k} \left|\sum_n b_n e^{\frac{2\pi i n j}{k}}\right|^2\right)^{\frac{1}{2}} \tag{51}$$

holds. Clearly, we have

$$\left(\sum_{j=1}^{k} j^{2r}\right)^{\frac{1}{2}} \ll_r k^{r+\frac{1}{2}}, \tag{52}$$

where the implied constant depends only on r. Using the observation

$$\sum_{j=1}^{k}\left|\sum_{n} b_n e^{\frac{2\pi i n j}{k}}\right|^2 = \sum_{m,n} b_n\overline{b_m}\sum_{j=1}^{k} e^{\frac{2\pi i(n-m)j}{k}} = k\sum_{n}|b_n|^2 \tag{53}$$

and gathering (51), (52) and (53), one may obtain the desired upper bound.

9. Proof of Theorem 5

First using the definition of Ramanujan sum, we may write

$$\sum_{j=1}^{k} j^r c_k(j) = k^r\varphi(k) + \sum_{j=1}^{k-1} j^r c_k(j) = k^r\varphi(k) + \sum_{\substack{1\leqslant m\leqslant k\\(m,k)=1}}\sum_{j=1}^{k-1} j^r e^{\frac{2\pi i j m}{k}}. \tag{54}$$

It is shown in [2] that

$$\sum_{j=1}^{k-1} j^r e^{\frac{2\pi i j m}{k}} = \sum_{j=1}^{r}\binom{r}{j}k^j \lim_{w\to e^{\frac{2\pi i m}{k}}}\frac{d^{r-j}}{dw^{r-j}}\left(\frac{1}{e^w-1}\right) \tag{55}$$

for $k \geqslant 2$ and $r \geqslant 1$. Again from [2], recall that when $j \leqslant r-1$,

$$\frac{d^{r-j}}{dw^{r-j}}\left(\frac{1}{e^w-1}\right) = \sum_{s=1}^{r-j+1}\frac{A_{r-j,s}}{(e^w-1)^s} \tag{56}$$

holds, where $A_{q,j}$'s are integers defined uniquely by the conditions given in the statement of the theorem. Combining (54), (55) and (56), we see that

$$\begin{aligned}\sum_{j=1}^{k} j^r c_k(j) = k^r\varphi(k) &+ \sum_{j=1}^{r-1}\sum_{s=1}^{r-j+1}\binom{r}{j}A_{r-j,s}k^j\sum_{\substack{1\leqslant m\leqslant k\\(m,k)=1}}\frac{1}{(e^{\frac{2\pi i m}{k}}-1)^s}\\ &+ k^r\sum_{\substack{1\leqslant m\leqslant k\\(m,k)=1}}\frac{1}{(e^{\frac{2\pi i m}{k}}-1)}.\end{aligned} \tag{57}$$

Since the left hand side of (57) is a real number, it is enough to keep only the real part of the right hand side of (57). Thus we get

$$\sum_{j=1}^{k} j^r c_k(j) = \frac{k^r\varphi(k)}{2} + \sum_{j=1}^{r-1}\sum_{s=1}^{r-j+1}\binom{r}{j}A_{r-j,s}k^j\mathfrak{R}\left(\sum_{\substack{1\leqslant m\leqslant k\\(m,k)=1}}\frac{1}{(e^{\frac{2\pi i m}{k}}-1)^s}\right). \tag{58}$$

Clearly, we have

$$\begin{aligned}\Re\Bigg(\sum_{\substack{1\leqslant m\leqslant k\\(m,k)=1}}\frac{1}{(e^{\frac{2\pi im}{k}}-1)^s}\Bigg)&=\Re\Bigg(\sum_{\substack{1\leqslant m\leqslant k\\(m,k)=1}}\frac{e^{-\frac{\pi ims}{k}}}{(2i\sin(\frac{\pi m}{k}))^s}\Bigg)\\&=\sum_{\substack{1\leqslant m\leqslant k\\(m,k)=1}}\frac{g(k,m,s)}{(2\sin(\frac{\pi m}{k}))^s},\end{aligned}\tag{59}$$

where $g(k,m,s)$ is defined modulo 4 as in the statement of the theorem. Changing the order of j and s in (58) imposes the conditions $1\leqslant s\leqslant r$ and $1\leqslant j\leqslant \min(r-1,r-s+1)$. Combining (1), (58) and (59), we complete the proof of Theorem 5.

References

[1] E. Alkan, On the mean square average of special values of L-functions, J. Number Theory 131 (2011) 1470–1485.
[2] E. Alkan, Values of Dirichlet L-functions, Gauss sums and trigonometric sums, Ramanujan J. 26 (2011) 375–398.
[3] E. Alkan, Distribution of averages of Ramanujan sums, Ramanujan J. 29 (2012) 385–408.
[4] E. Alkan, Ramanujan sums and the Burgess zeta function, Int. J. Number Theory 8 (2012) 2069–2092.
[5] E. Alkan, On linear combinations of special values of L-functions, Manuscripta Math. 139 (2012) 473–494.
[6] E. Alkan, Averages of values of L-series, Proc. Amer. Math. Soc. 141 (2013) 1161–1175.
[7] E. Alkan, H. Göral, On sums over the Möbius function and discrepancy of fractions, J. Number Theory 133 (2013) 2217–2239.
[8] E. Alkan, K. Ford, A. Zaharescu, Diophantine approximation with arithmetic functions. I, Trans. Amer. Math. Soc. 361 (2009) 2263–2275.
[9] G. Bachman, On an optimality property of Ramanujan sums, Proc. Amer. Math. Soc. 125 (1997) 1001–1003.
[10] B.C. Berndt, A. Zaharescu, Finite trigonometric sums and class numbers, Math. Ann. 330 (2004) 551–575.
[11] R. Carmichael, Expansions of arithmetical functions in infinite series, Proc. Lond. Math. Soc. 34 (1932) 1–26.
[12] O.-Y. Chan, Weighted trigonometric sums over a half period, Adv. in Appl. Math. 38 (2007) 482–504.
[13] H. Davenport, Multiplicative Number Theory, third edition, Grad. Texts in Math., Springer, New York, 2000.
[14] B. Green, T. Tao, The Möbius function is strongly orthogonal to nilsequences, Ann. of Math. 175 (2012) 541–566.
[15] A.M. Güloğlu, Sums with convolutions of Dirichlet characters to cube-free modulus, Proc. Amer. Math. Soc. 139 (2011) 3195–3202.
[16] J. Knopfmacher, Abstract Analytic Number Theory, second edition, Dover, New York, 1990.
[17] A. Ledoan, A. Zaharescu, Real moments of the restrictive factor, Proc. Indian Acad. Sci. Math. Sci. 119 (2009) 559–566.
[18] H.L. Montgomery, R.C. Vaughan, Exponential sums with multiplicative coefficients, Invent. Math. 43 (1977) 69–82.
[19] H.L. Montgomery, R.C. Vaughan, Multiplicative Number Theory I. Classical Theory, Cambridge Stud. Adv. Math., vol. 97, Cambridge University Press, Cambridge, 2007.
[20] R.E.A.C. Paley, A theorem on characters, J. Lond. Math. Soc. 7 (1932) 28–32.
[21] S. Ramanujan, On certain trigonometrical sums and their applications in the theory of numbers, Trans. Cambridge Philos. Soc. 22 (1918) 259–276.
[22] W. Schwarz, Ramanujan expansions of arithmetical functions, in: Ramanujan Revisited, Proc. Centenary Conference, Urbana, 1987, Academic Press, Boston, 1987, pp. 187–214.

[23] W. Schwarz, J. Spilker, Arithmetical Functions, London Math. Soc. Lecture Note Ser., vol. 184, Cambridge University Press, Cambridge, 1994.
[24] J. Singh, Defining power sums of n and $\varphi(n)$ integers, Int. J. Number Theory 5 (2009) 41–53.
[25] T. Tao, A remark on partial sums involving the Möbius function, Bull. Aust. Math. Soc. 81 (2010) 343–349.