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Quantum Jacobi forms and balanced unimodal sequences ☆

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A B S T R A C T

The notion of a quantum Jacobi form was defined in 2016 by Bringmann and the second author in [1], marrying Zagier's notion of a quantum modular form [12] with that of a Jacobi form. Only one example of such a function has been given to-date (see [1]). Here, we prove that two combinatorial rank generating functions for certain balanced unimodal sequences, studied previously by Kim, Lim and Lovejoy [8], are also natural examples of quantum Jacobi forms. These two combinatorial functions are also duals to partial theta functions studied by Ramanujan. Additionally, we prove that these two functions have the stronger property that they exhibit mock Jacobi transformations in $\mathbb{C} \times \mathbb{H}$ as well as quantum Jacobi transformations in $\mathbb{Q} \times \mathbb{Q}$. As corollaries to these results, we use quantum Jacobi properties to establish new, simpler expressions for these functions as simple Laurent polynomials when evaluated at pairs of rational numbers.

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1. Introduction and statement of results

A sequence of integers $\{a_j\}_{j=1}^{s} (s \in \mathbb{N})$ is called a *strongly unimodal sequence of size* n if there exists a positive integer k such that

$$0 < a_1 < a_2 < \cdots < a_k > a_{k+1} > \cdots > a_s > 0,$$

and $a_1 + \cdots + a_s = n$. The *rank* of a strongly unimodal sequence is equal to $2k - s - 1$.[1] *Odd-balanced unimodal sequences* allow odd parts to repeat on either side of the peak a_k, but they must be identical on either side, and the peak must be even. Let $\nu(m, n)$ be the number of such sequences with rank m and size $2n$.[2] The two-variable rank generating function for odd-balanced unimodal sequences satisfies [8]

$$q\mathcal{V}(w; q) := \sum_{n \geq 0} \frac{(-wq; q)_n (-w^{-1}q; q)_n q^{n+1}}{(q; q^2)_{n+1}} = \sum_{\substack{n \geq 1 \\ m \in \mathbb{Z}}} \nu(m, n) w^m q^n. \tag{1.1}$$

Here and throughout, the q-Pochhammer symbol is defined by $(w; q)_n := \prod_{j=0}^{n-1}(1 - wq^j)$ for $n \in \mathbb{N}_0 \cup \{\infty\}$.

In [1], the notion of a *quantum Jacobi form* is introduced by Bringmann and the second author, marrying Zagier's notion of a quantum modular form [12] with that of a Jacobi form (see §2 for a precise definition and more details). In [1] it is also proved that the two-variable combinatorial generating function for ranks of strongly unimodal sequences is an example of such a form – this is the first and only example of such a function in the literature to-date. Here we prove that two additional combinatorial functions are quantum Jacobi forms, one of which is a normalized version of the function in (1.1). Precisely, we define

$$\mathcal{V}^+(z; \tau) := 2\cos(\pi z)\mathcal{V}(w; q)q^{\frac{7}{8}}, \tag{1.2}$$

on $\mathbb{C} \times \mathbb{H}$ with $w := e(z), q := e(\tau)$, where here and throughout $e(\alpha) := e^{2\pi i \alpha}$. In Theorem 1.1, we prove that $\mathcal{V}^+$ is a quantum Jacobi form with respect to the congruence subgroup $\Gamma_0(4) \subseteq \mathrm{SL}_2(\mathbb{Z})$. In fact, we prove that $\mathcal{V}^+$ has the stronger property that it exhibits mock Jacobi transformations in $\mathbb{C} \times \mathbb{H}$ as well as quantum Jacobi transformations in $\mathbb{Q} \times \mathbb{Q}$. By exploiting these quantum Jacobi properties, we also obtain a new simple expression for $\mathcal{V}^+$ as a Laurent polynomial when evaluated at pairs of rational numbers (see Theorem 1.3).

In order to state our results related to $\mathcal{V}^+$ precisely, we first define the "errors of modularity"

[1] Here, we use the definition of rank as in [8]. Other sources such as [2] define rank to be $-(2k - s - 1) = s - 2k + 1$.

[2] Note that $\nu(m, n) = v(m, n - 1)$, where v is as defined in [8].

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$$H_1(z;\tau) := -\frac{1}{2}\sqrt{\frac{i}{\tau}}e^{\frac{\pi i z^2}{\tau}}h\left(\frac{z}{\tau};\frac{-1-4\tau}{\tau}\right) - \frac{1}{2}h(z;\tau), \tag{1.3}$$

$$H_2(z;\tau) := \frac{\eta(2\tau)\vartheta\left(z+\frac{1}{2};\tau\right)}{2\eta^2(\tau)}\left(-i\sqrt{\frac{i}{2\tau}}e^{\frac{2\pi i z^2}{\tau}}h\left(\frac{z}{\tau};\frac{-1-4\tau}{2\tau}\right) - h(2z;2\tau)\right), \tag{1.4}$$

where for $z \in \mathbb{C}, \tau \in \mathbb{H}$, the Mordell integral h is given by

$$h(z;\tau) := \int\limits_{\mathbb{R}} \frac{e^{\pi i \tau t^2 - 2\pi z t}}{\cosh(\pi t)}dt, \tag{1.5}$$

and η, ϑ are the familiar holomorphic modular and Jacobi forms defined in (2.3).

Theorem 1.1 below establishes the quantum Jacobi (and mock Jacobi) transformation properties of the two-variable combinatorial generating function $\mathcal{V}^+$. The set $\mathcal{Q}_{\mathcal{V}} \subseteq \mathbb{Q}\times\mathbb{Q}$ is defined explicitly in §3.

Theorem 1.1. *The following transformation properties hold.*

(i) *For $(z,\tau) \in (\mathbb{C}\times\mathbb{H}) \cup \mathcal{Q}_{\mathcal{V}}$, we have that*

$$\mathcal{V}^+(z;\tau) - e^{\frac{\pi i}{4}}\mathcal{V}^+(z;\tau+1) = 0, \tag{1.6}$$

$$\mathcal{V}^+(z;\tau) - \mathcal{V}^+(-z;\tau) = 0, \tag{1.7}$$

$$\mathcal{V}^+(z;\tau) + (4\tau+1)^{-\frac{1}{2}}e^{\frac{4\pi i z^2}{4\tau+1}}\mathcal{V}^+\left(\frac{z}{4\tau+1};\frac{\tau}{4\tau+1}\right) = H_1(z;\tau) + H_2(z;\tau), \quad \left(\tau \neq -\tfrac{1}{4}\right), \tag{1.8}$$

$$\mathcal{V}^+(z;\tau) + \mathcal{V}^+(z+1;\tau) = 0, \tag{1.9}$$

$$\mathcal{V}^+(z;\tau) + e^{-2\pi i z - \pi i \tau}\mathcal{V}^+(z+\tau;\tau) = -e^{-2\pi i z - \frac{\pi i \tau}{2}}\frac{\vartheta\left(z+\frac{1}{2};\tau\right)\eta(2\tau)}{\eta^2(\tau)} - e^{-\pi i z - \frac{\pi i \tau}{4}}. \tag{1.10}$$

(ii) *In particular, for $(z,\tau) \in \mathcal{Q}_{\mathcal{V}}$, we have that*

$$\mathcal{V}^+(z;\tau) + (4\tau+1)^{-\frac{1}{2}}e^{\frac{4\pi i z^2}{4\tau+1}}\mathcal{V}^+\left(\frac{z}{4\tau+1};\frac{\tau}{4\tau+1}\right) = H_1(z;\tau), \quad \left(\tau \neq -\tfrac{1}{4}\right), \tag{1.11}$$

$$\mathcal{V}^+(z;\tau) + e^{-2\pi i z - \pi i \tau}\mathcal{V}^+(z+\tau;\tau) = -e^{-\pi i z - \frac{\pi i \tau}{4}}. \tag{1.12}$$

The function H_1 on the right-hand side of (1.11) extends to a C^∞ function on $(\mathbb{R}\setminus(\mathbb{Z}+\{\frac{1}{2},\pm\frac{1}{8},\pm\frac{3}{8}\}))\times(\mathbb{R}\setminus\{-\frac{1}{4}\})$, and the function on the right-hand-side of (1.12) extends to a C^∞ function on $\mathbb{R}\times\mathbb{R}$.

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Remarks.

(1) Theorem 1.1 shows that the function $\mathcal{V}^+$ is a quantum Jacobi form of weight $\frac{1}{2}$ and index $-\frac{1}{2}$ with respect to $\Gamma_0(4)$, which is generated by $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right)$. Direct calculation shows that $\mathcal{Q}_{\mathcal{V}}$ is invariant under the action of $\Gamma_0(4)$.
(2) In (1.8) and (1.11), we must exclude pairs (z,τ) with $\tau = -\frac{1}{4}$ in order to avoid singularities. For any $\gamma \in \Gamma_0(4)$, we similarly must exclude pairs (z,τ) with $\tau = \gamma^{-1}(i\infty)$ in the analogues to (1.8) and (1.11).

We also study the quantum Jacobi properties of a second combinatorial q-hypergeometric series, namely, the function

$$\mathcal{W}(w;q) := \sum_{n\geq 0} \frac{(wq;q^2)_n(w^{-1}q;q^2)_n q^{2n}}{(-q;q)_{2n+1}}.$$

It is explained in [8] that a normalized version of $\mathcal{W}$ also has combinatorial meaning as a two-variable partition rank generating function related to partitions without repeated odd parts. Parallel to Theorem 1.1, in Theorem 1.2 below, we establish the quantum Jacobi (and mock Jacobi) properties of the normalized function

$$\mathcal{W}^+(z;\tau) := 2q^{\frac{3}{8}}\cos(\pi z)\mathcal{W}(w;q^{\frac{1}{2}}). \tag{1.13}$$

The "error" functions G_1 and G_2 in (1.16) and (1.19) are defined in (4.15) and (4.16) respectively; like the functions H_1 and H_2 in (1.3) and (1.4), they are defined in terms of the Mordell integral h defined in (1.5). The set $\mathcal{Q}_W \subseteq \mathbb{Q}\times\mathbb{Q}$ is defined in §3.

Theorem 1.2. *The following transformation properties hold.*

(i) *For $(z,\tau) \in (\mathbb{C}\times\mathbb{H}) \cup \mathcal{Q}_{\mathcal{W}}$, we have that*

$$\mathcal{W}^+(z;\tau) - i\mathcal{W}^+(z;\tau+1) = 0, \tag{1.14}$$

$$\mathcal{W}^+(z;\tau) - \mathcal{W}^+(-z;\tau) = 0, \tag{1.15}$$

$$\mathcal{W}^+(z;\tau) + ie^{\frac{2\pi i z^2}{2\tau+1}}(2\tau+1)^{-\frac{1}{2}}\mathcal{W}^+\left(\frac{z}{2\tau+1};\frac{\tau}{2\tau+1}\right) = G_1(z;\tau) + G_2(z;\tau), \tag{1.16}$$

$$\mathcal{W}^+(z;\tau) + \mathcal{W}^+(z+1;\tau) = 0, \tag{1.17}$$

$$\mathcal{W}^+(z;\tau) + e^{-2\pi i z - \pi i \tau}\mathcal{W}^+(z+\tau;\tau)$$
$$= 2e^{-\pi i z}e^{\frac{-\pi i\tau}{4}} + i(e^{-2\pi i z} - e^{\pi i \tau})e^{\frac{-7\pi i \tau}{8}}\ \frac{\eta(\frac{\tau}{2})\vartheta(z+\frac{\tau}{2};\tau)}{\eta^2(\tau)}. \tag{1.18}$$

(ii) *In particular, for* $(z, \tau) \in \mathcal{Q}_{\mathcal{W}}$, *we have that*

$$\mathcal{W}^+(z;\tau) + ie^{\frac{2\pi i z^2}{2\tau+1}}(2\tau+1)^{-\frac{1}{2}}\mathcal{W}^+\left(\frac{z}{2\tau+1};\frac{\tau}{2\tau+1}\right) = G_1(z;\tau), \tag{1.19}$$

$$\mathcal{W}^+(z;\tau) + e^{-2\pi i z - \pi i \tau}\mathcal{W}^+(z+\tau;\tau) = 2w^{-\frac{1}{2}}q^{-\frac{1}{8}}. \tag{1.20}$$

The function G_1 *on the right-hand-side of* (1.19) *extends to a* C^∞ *function on* $(\mathbb{R} \setminus (\mathbb{Z} + \{\frac{1}{2}, \pm\frac{1}{4}\})) \times (\mathbb{R} \setminus \{-\frac{1}{2}\})$, *and the function on the right-hand-side of* (1.20) *extends to a* C^∞ *function on* $\mathbb{R} \times \mathbb{R}$.

Remark. Theorem 1.2 implies that the function $\mathcal{W}^+(z;\tau)$ is a quantum Jacobi form of weight $1/2$ and index $-1/2$ with respect to the group $\Gamma_0(2)$, which is generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$. Direct calculation shows that $\mathcal{Q}_{\mathcal{W}}$ is invariant under the action of $\Gamma_0(2)$. See also the second remark after Theorem 1.1; an analogous statement applies here.

Using quantum properties established in Theorem 1.1 and Theorem 1.2, we show in Theorem 1.3 that $\mathcal{V}^+$ and $\mathcal{W}^+$ can be expressed as simple Laurent polynomials when evaluated at pairs of rationals in $\mathcal{Q}_{\mathcal{V}}$ and $\mathcal{Q}_{\mathcal{W}}$, respectively. We point out that these evaluations are nontrivial, in that they do not simply follow from the q-hypergeometric definitions of the functions $\mathcal{V}$ and $\mathcal{W}$. Evaluations of the type exhibited in Theorem 1.3 have been of recent interest, as they pertain to radial limits of mock theta functions [1,3,6,7,13].

Theorem 1.3. *The following identities hold.*

(i) *For* $(\frac{a}{b}, \frac{h}{k}) \in \mathcal{Q}_{\mathcal{V}}$, *we have that*

$$\mathcal{V}^+(\tfrac{a}{b}, \tfrac{h}{k}) = -\frac{1}{2}\zeta_{2b}^{-a}\zeta_{8k}^{7h}\sum_{j=0}^{k-1}(-1)^j\zeta_{2k}^{-hj(j+1)}\zeta_b^{-aj}.$$

(ii) *For* $(\frac{a}{b}, \frac{h}{k}) \in \mathcal{Q}_{\mathcal{W}}$, *we have that*

$$\mathcal{W}^+\left(\tfrac{a}{b};\tfrac{h}{k}\right) = \zeta_{2b}^{a}\zeta_{8k}^{-h}\sum_{j=0}^{k-1}(-1)^j\zeta_{2k}^{-hj(j+1)}\zeta_b^{aj}.$$

The remainder of this paper is organized as follows. In §2, we recall the definition of a quantum Jacobi form and discuss properties of some required (mock) Jacobi and modular forms. In §3, we introduce some auxiliary lemmas that are instrumental to our work. We present the proofs of our main theorems in §4.

2. Preliminaries

2.1. Quantum modular forms

The definition of a quantum modular form was originally introduced by Zagier in [12] and has since been slightly modified to allow for half-integral weights, subgroups of $\mathrm{SL}_2(\mathbb{Z})$, etc. (See, for example, [1,4].)

Definition 2.1. A weight $k \in \frac{1}{2}\mathbb{Z}$ quantum modular form is a complex-valued function f on $\mathbb{Q}$, such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the functions $h_\gamma : \mathbb{Q} \setminus \gamma^{-1}(i\infty) \to \mathbb{C}$ defined by

$$h_\gamma(x) := f(x) - \varepsilon^{-1}(\gamma)(cx+d)^{-k} f\left(\frac{ax+b}{cx+d}\right)$$

satisfy a "suitable" property of continuity or analyticity in a subset of $\mathbb{R}$.

Remarks.

(1) The complex numbers $\varepsilon(\gamma)$, which satisfy $|\varepsilon(\gamma)| = 1$, are such as those appearing in the theory of half-integral weight modular forms.
(2) We may modify Definition 2.1 appropriately to allow transformations on finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$. We may also restrict the domains of the functions h_γ to be suitable subsets of $\mathbb{Q}$.

Since the time of origin of this definition in 2010, the theory of quantum modular forms has been widely explored (see for example [4], and references therein). Some recent and relevant examples of quantum modular forms arise from the combinatorial generating functions $\mathcal{V}$ and $\mathcal{W}$. More precisely, Kim, Lim and Lovejoy showed in [8] that the three functions

$$q^{-7}\mathcal{V}(1;q^{-8}) = -\frac{1}{2}\sum_{n\geq 0}\chi(n)q^{n^2}, \qquad q^{-3}\mathcal{W}(1,q^{-4}) = \sum_{n\geq 0}\chi(n)q^{n^2}, \tag{2.1}$$

$$q^{-7}\mathcal{V}(-1;q^{-8}) = -\frac{1}{2}\sum_{n\geq 0}(2n+1)q^{(2n+1)^2}, \tag{2.2}$$

where $\chi(n)$ is a Dirichlet character (mod 4), are quantum modular forms. In particular, the functions in (2.1) are of weight $1/2$, and the function in (2.2) is of weight $3/2$. Note that the parameters w above in $\mathcal{V}(w;q)$ and $\mathcal{W}(w;q)$ are fixed to be ± 1, and these functions are viewed in [8] as one-variable quantum modular forms in $\tau \in \mathbb{Q}$, where $q = e^{2\pi i\tau}$. Our work described in §1 and discussed further in the following sections establishes the general two-variable quantum Jacobi (and mock Jacobi) properties of the functions $\mathcal{V}(w;q)$ and $\mathcal{W}(w;q)$ in the variables $(w,q)=(e^{2\pi iz},e^{2\pi i\tau})$, where $(z,\tau)\in(\mathbb{Q}\times\mathbb{Q})\cup(\mathbb{C}\times\mathbb{H})$.

2.2. Quantum Jacobi forms

The definition of a quantum Jacobi form was originally introduced by Bringmann and the second author in [1], marrying Zagier's definition of a quantum modular form [12] with that of a Jacobi form, the theory of which was largely developed by Eichler and Zagier [5]. That is, a quantum Jacobi form is a two-variable analogue of a quantum modular form, similar to how a (holomorphic) Jacobi form is a two-variable analogue of a (holomorphic) modular form. We recall the definition of a quantum Jacobi form from [1] here.

Definition 2.2. A weight $k\in\frac{1}{2}\mathbb{Z}$ and index $m\in\frac{1}{2}\mathbb{Z}$ quantum Jacobi form is a complex-valued function on $\mathbb{Q}\times\mathbb{Q}$ such that for all $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$ and $(\lambda,\mu)\in\mathbb{Z}\times\mathbb{Z}$, the functions $h_\gamma:\mathbb{Q}\times(\mathbb{Q}\setminus\gamma^{-1}(i\infty))\to\mathbb{C}$ and $g_{(\lambda,\mu)}:\mathbb{Q}\times\mathbb{Q}\to\mathbb{C}$ defined by

$$h_\gamma(z;\tau):=\phi(z;\tau)-\varepsilon_1^{-1}(\gamma)(c\tau+d)^{-k}e^{\frac{-2\pi imcz^2}{c\tau+d}}\phi\left(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right),$$

$$g_{(\lambda,\mu)}(z;\tau):=\phi(z;\tau)-\varepsilon_2^{-1}((\lambda,\mu))e^{2\pi im(\lambda^2\tau+2\lambda z)}\phi(z+\lambda\tau+\mu;\tau),$$

satisfy a "suitable" property of continuity or analyticity in a subset of $\mathbb{R}\times\mathbb{R}$.

Remarks.

(1) The complex numbers $\varepsilon_1(\gamma)$ and $\varepsilon_2((\lambda,\mu))$ satisfy $|\varepsilon_1(\gamma)|=|\varepsilon_2((\lambda,\mu))|=1$; in particular, the $\varepsilon_1(\gamma)$ are such as those appearing in the theory of half-integral weight forms.
(2) We may modify the definition to allow modular transformations on subgroups of $\mathrm{SL}_2(\mathbb{Z})$. We may also restrict the domains of the functions h_γ and $g_{(\lambda,\mu)}$ to be suitable subsets of $\mathbb{Q}\times\mathbb{Q}$.

As mentioned in §1, the first example of a quantum Jacobi form is provided in [1]. In particular, Bringmann and the second author proved that the function on $\mathbb{C}\times\mathbb{H}$ defined by

$$Y^+(z;\tau):=-2i\sin(\pi z)q^{-\frac{1}{24}}\mathcal{U}(z;\tau)$$

is a quantum Jacobi form of weight $\frac{1}{2}$ and index $-\frac{3}{2}$, where

$$\mathcal{U}(w;q):=\sum_{n=0}^{\infty}(wq;q)_n(w^{-1}q;q)_nq^{n+1}.$$

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Like $\mathcal{V}$ and $\mathcal{W}$, the function $\mathcal{U}$ also has combinatorial meaning: it is the rank generating function for strongly unimodal sequences.

2.3. "Modular" and "elliptic" forms

Here we recall some transformation properties of various "modular" and "elliptic" objects, where we use these terms loosely to encompass various types of forms (nonholomorphic, Jacobi, mock, etc.). We begin with the Mordell integral h defined in §1. The following elliptic properties were given by Zwegers in [14].

Lemma 2.3. *The following properties hold:*

(i) h *is an even function of* z,
(ii) $h(z;\tau) + h(z+1;\tau) = \dfrac{2}{\sqrt{-i\tau}} e^{\frac{\pi i (z+1/2)^2}{\tau}}$,
(iii) $h(z;\tau) + e^{-2\pi i z - \pi i \tau} h(z+\tau;\tau) = 2e^{-\pi i \left(z + \frac{\tau}{4}\right)}$.

Zwegers also shows in [14] how to express h, under certain specializations of variables, in terms of the unary theta functions $g_{a,b}$, defined for $a, b \in \mathbb{R}$ and $\tau \in \mathbb{H}$ by

$$g_{a,b}(\tau) := \sum_{\nu \in a+\mathbb{Z}} \nu e^{\pi i \nu^2 \tau + 2\pi i \nu b}.$$

The functions $g_{a,b}$ are modular forms of weight $\frac{3}{2}$. In particular, we have the following transformation properties [11,14].

Lemma 2.4. *With hypotheses as above, the functions* $g_{a,b}$ *satisfy:*

(i) $g_{a+1,b}(\tau) = g_{a,b}(\tau)$,
(ii) $g_{a,b+1}(\tau) = e^{2\pi i a} g_{a,b}(\tau)$,
(iii) $g_{a,b}(\tau+1) = e^{-\pi i a(a+1)} g_{a,a+b+\frac{1}{2}}(\tau)$,
(iv) $g_{a,b}\left(-\frac{1}{\tau}\right) = i e^{2\pi i ab} (-i\tau)^{\frac{3}{2}} g_{b,-a}(\tau)$,
(v) $g_{-a,-b}(\tau) = -g_{a,b}(\tau)$.

From [14], the Mordell integrals h can be expressed as period integrals of $g_{a,b}$ when $z = a\tau - b$, and $a, b \in (-\frac{1}{2}, \frac{1}{2})$.

Lemma 2.5. *For* $a, b \in (-\frac{1}{2}, \frac{1}{2})$,

$$\int_0^{i\infty} \frac{g_{a+\frac{1}{2},b+\frac{1}{2}}(z)}{\sqrt{-i(z+\tau)}} dz = -e^{-\pi i a^2 \tau + 2\pi i a\left(b+\frac{1}{2}\right)} h(a\tau - b;\tau).$$

Next we define some related functions whose (mock) modular or Jacobi transformation properties will be used in §4 to prove Theorem 1.1. We begin with the well-known modular and Jacobi forms η and ϑ (respectively), defined for $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$ by

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n), \qquad \vartheta(z;\tau) := \sum_{n \in \mathbb{Z}+\frac{1}{2}} e^{\pi i n^2 \tau + 2\pi i n (z+\frac{1}{2})}. \tag{2.3}$$

The functions η and ϑ are well-known to satisfy the following properties. (See, for example, [10].)

Lemma 2.6. *For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$,*

$$\eta(\gamma\tau) = \chi(\gamma)(c\tau+d)^{\frac{1}{2}}\eta(\tau),$$

where for $c > 0$,

$$\chi(\gamma) = \begin{cases} \frac{1}{\sqrt{i}} \left(\frac{d}{c}\right) i^{(1-c)/2} e^{\pi i (bd(1-c^2)+c(a+d))/12} & \text{if } c \text{ is odd}, \\ \frac{1}{\sqrt{i}} \left(\frac{c}{d}\right) e^{\pi i d/4} e^{\pi i (ac(1-d^2)+d(b-c))/12} & \text{if } d \text{ is odd}. \end{cases} \tag{2.4}$$

Lemma 2.7. *For $\lambda, \mu \in \mathbb{Z}, \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, and $(z,\tau) \in \mathbb{C} \times \mathbb{H}$,*

(i) $\vartheta(z+\lambda\tau+\mu;\tau) = (-1)^{\lambda+\mu} q^{-\frac{\lambda^2}{2}} e^{-2\pi i \lambda z} \vartheta(z;\tau)$,

(ii) $\vartheta\left(\frac{z}{c\tau+d};\gamma\tau\right) = \chi^3(\gamma)(c\tau+d)^{\frac{1}{2}} e^{\frac{\pi i c z^2}{c\tau+d}} \vartheta(z;\tau)$,

(iii) $\vartheta(z;\tau) = -iq^{\frac{1}{8}} w^{-\frac{1}{2}} \prod_{n=1}^{\infty} (1-q^n)(1-wq^{n-1})(1-w^{-1}q^n)$.

Next we define Zwegers' mock Jacobi form μ (see [14]), defined for $\tau \in \mathbb{H}$ and $u, v \in \mathbb{C}$ $(u, v \notin \mathbb{Z}\tau + \mathbb{Z})$ by

$$\mu(u,v;\tau) := \frac{e^{\pi i u}}{\vartheta(v;\tau)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{\pi i (n^2+n)\tau + 2\pi i n v}}{1 - e^{2\pi i n \tau + 2\pi i u}}. \tag{2.5}$$

The transformation properties of μ are studied via its completion $\widehat{\mu}$ into a nonholomorphic Jacobi form. Precisely, from [14], we have that

$$\widehat{\mu}(u,v;\tau) := \mu(u,v;\tau) + \frac{i}{2} R(u-v;\tau), \tag{2.6}$$

where the nonholomorphic function R is defined by

$$R(u;\tau) := \sum_{\nu \in \frac{1}{2}+\mathbb{Z}} \left\{ \mathrm{sgn}(\nu) - E\left((\nu+\alpha)\sqrt{2y}\right) \right\} (-1)^{\nu-\frac{1}{2}} e^{-\pi i \nu^2 \tau - 2\pi i \nu u}, \tag{2.7}$$

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with $y := \mathrm{Im}(\tau)$, $\alpha := \frac{\mathrm{Im}(u)}{\mathrm{Im}(\tau)}$ and

$$E(z) := 2\int_0^z e^{-\pi u^2} du.$$

From [14, Proposition 1.9, Proposition 1.10, Theorem 1.11], we have the following transformation properties of R and $\widehat{\mu}$.

Lemma 2.8. *With hypotheses as above, R satisfies the following transformation properties:*

(i) $R(u;\tau+1) = e^{-\frac{\pi i}{4}} R(u;\tau)$,
(ii) $\dfrac{1}{\sqrt{-i\tau}} e^{\frac{\pi i u^2}{\tau}} R\left(\frac{u}{\tau}; -\frac{1}{\tau}\right) + R(u;\tau) = h(u;\tau)$,
(iii) $R(u;\tau) = R(-u;\tau)$.

Lemma 2.9. *With hypotheses as above, for $k, l, m, n \in \mathbb{Z}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the function $\widehat{\mu}$ satisfies the following (nonholomorphic) Jacobi transformation properties:*

(i) $\widehat{\mu}(u + k\tau + l, v + m\tau + n; \tau) = (-1)^{k+l+m+n} e^{\pi i(k-m)^2\tau + 2\pi i(k-m)(u-v)} \widehat{\mu}(u, v; \tau)$,
(ii) $\widehat{\mu}\left(\dfrac{u}{c\tau+d}, \dfrac{v}{c\tau+d}; \dfrac{a\tau+b}{c\tau+d}\right) = \chi^{-3}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) (c\tau+d)^{1/2} e^{-\frac{\pi i c(u-v)^2}{c\tau+d}} \widehat{\mu}(u, v; \tau)$.

3. Auxiliary lemmas

In this section, we establish a number of auxiliary lemmas which will be used in §4 towards the proofs of our main results stated in §1. In order to establish the quantum Jacobi transformation properties of $\mathcal{V}^+$ and $\mathcal{W}^+$ in the next section, we first make precise infinite subsets $\mathcal{Q}_{\mathcal{V}}$ and $\mathcal{Q}_{\mathcal{W}}$ of $\mathbb{Q}\times\mathbb{Q}$ on which $\mathcal{V}$ (hence $\mathcal{V}^+$) and $\mathcal{W}$ (hence $\mathcal{W}^+$) converge, respectively. Namely, we define

$$\mathcal{Q}_{\mathcal{V}} := \left\{ \left(\tfrac{a}{b}, \tfrac{h}{k}\right) \in \mathbb{Q}\times\mathbb{Q} \,|\, a, h \in \mathbb{Z}, b \in \mathbb{N}, k \in 2\mathbb{N}, \gcd(a,b) = \gcd(h,k) = 1, b \mid k \right\},$$
$$\mathcal{Q}_{\mathcal{W}} := \left\{ \left(\tfrac{a}{b}, \tfrac{h}{k}\right) \in \mathbb{Q}\times\mathbb{Q} \mid a \in \mathbb{Z}, b, k \in \mathbb{N}, h \in 2\mathbb{Z}, \gcd(a,b) = \gcd(h,k) = 1, b \mid k \right\}.$$

Lemma 3.1. *The following identities hold.*

(i) *If $(z,\tau) = \left(\frac{a}{b}, \frac{h}{k}\right) \in \mathcal{Q}_{\mathcal{V}}$, then there is a constant $C_{a,b,h,k}$ such that*

$$\mathcal{V}(e^{2\pi i z}; e^{2\pi i\tau}) = \mathcal{V}(\zeta_b^a; \zeta_k^h) = \sum_{n=0}^{C_{a,b,h,k}} \frac{(-\zeta_b^a\zeta_k^h; \zeta_k^h)_n (-\zeta_b^{-a}\zeta_k^h; \zeta_k^h)_n \zeta_k^{hn}}{\left(\zeta_k^h; \zeta_k^{2h}\right)_{n+1}}. \tag{3.1}$$

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(ii) *If* $(z,\tau)=\left(\frac{a}{b},\frac{h}{k}\right)\in\mathcal{Q}_{\mathcal{W}}$*, then there is a constant* $D_{a,b,h,k}$ *such that*

$$\mathcal{W}(e^{2\pi iz};e^{\pi i\tau})=\mathcal{W}(\zeta_b^a;\zeta_{2k}^h)=\sum_{n=0}^{D_{a,b,h,k}}\frac{(\zeta_b^a\zeta_{2k}^h;\zeta_k^h)_n(\zeta_b^{-a}\zeta_{2k}^h;\zeta_k^h)_n\zeta_k^{hn}}{(-\zeta_{2k}^h;\zeta_{2k}^h)_{2n+1}}.$$

Proof. The proof of (ii) is very similar to the proof of (i) and relatively straightforward, so for brevity we omit it, and explain only (i) here. By definition of $\mathcal{V}$, we have that

$$\mathcal{V}(\zeta_b^a;\zeta_k^h)=\sum_{n=0}^{\infty}\frac{(-\zeta_b^a\zeta_k^h;\zeta_k^h)_n(-\zeta_b^{-a}\zeta_k^h;\zeta_k^h)_n\zeta_k^{hn}}{\left(\zeta_k^h;\zeta_k^{2h}\right)_{n+1}}. \tag{3.2}$$

If k is even, the denominators of the summands in (3.2) are nonzero for every term in the series. On the other hand, suppose that

$$\pm\frac{a}{b}+j\frac{h}{k}=m+\frac{1}{2}\in\mathbb{Z}+\frac{1}{2} \tag{3.3}$$

for some $j\in\mathbb{N}$ and $m\in\mathbb{Z}$. Then the numerators of the summands in (3.2) vanish for sufficiently large n. Thus, it remains to be seen that such a j and m exist. Since $(\frac{a}{b},\frac{h}{k})\in\mathcal{Q}_{\mathcal{V}}$, there is some integer h' such that $hh'\equiv -1 \pmod{k}$. Moreover, since $b|k$, we may write $bb'=k$ for some integer b'. We define j to be a positive integer which is congruent to $\pm ab'h'-kh'/2 \pmod{k}$. Note that $kh'/2\in\mathbb{Z}$, since k is even. This congruence is equivalent to the congruence $jh\equiv\mp ab'+k/2 \pmod{k}$, which is equivalent to the existence of some integer m such that $jh=\mp ab'+k/2+mk$, which after some rewriting is equivalent to (3.3). □

In [9], Mortenson derived expressions directly related to $\mathcal{V}$ and $\mathcal{W}$ in terms of theta functions j and Appell–Lerch series m defined as follows:

$$j(x;q):=(x;q)_\infty(x^{-1}q;q)_\infty(q;q)_\infty,$$
$$m(\rho,q,x):=\frac{1}{j(x;q)}\sum_{n\in\mathbb{Z}}\frac{(-1)^nq^{\frac{n(n-1)}{2}}x^n}{1-\rho xq^{n-1}}.$$

The functions j and m are nearly identical to the functions ϑ and μ defined in (2.3) and (2.5) after suitable changes of variables. The next lemma follows immediately from the definitions of j,m and μ, and from Lemma 2.7 iii).

Lemma 3.2. *The following identities hold:*

(i) $j(w;q)=iw^{\frac{1}{2}}q^{-\frac{1}{8}}\vartheta(z;\tau)$,
(ii) $m\left(w,q,-1\right)=iw^{-\frac{1}{2}}q^{\frac{1}{8}}\mu\left(z+\frac{1}{2},\frac{1}{2};\tau\right)$.

Using Lemma 3.2 and equations (4.23) and (4.32) in [9], we establish the following expressions for $\mathcal{V}^+$ and $\mathcal{W}^+$ in terms of the Jacobi ϑ-function, and Zwegers' mock Jacobi form μ.

Lemma 3.3. *We have that*

$$\mathcal{V}^+(z;\tau) = \mathcal{V}_1^+(z;\tau) + \mathcal{V}_2^+(z;\tau) + \mathcal{V}_3^+(z;\tau), \tag{3.4}$$

and

$$\mathcal{W}^+(z;\tau) = \mathcal{W}_1^+(z;\tau) + \mathcal{W}_2^+(z;\tau) + \mathcal{W}_3^+(z;\tau), \tag{3.5}$$

where

$$\begin{aligned}
\mathcal{V}_1^+(z;\tau) &:= -i\mu\left(z+\tfrac{1}{2},\tfrac{1}{2};\tau\right),\\
\mathcal{V}_2^+(z;\tau) &:= -i\frac{\eta(2\tau)\vartheta\left(z+\frac{1}{2};\tau\right)}{\eta^2(\tau)}\mu\left(2z+\tfrac{1}{2},\tfrac{1}{2};2\tau\right),\\
\mathcal{V}_3^+(z;\tau) &:= -iq^{\frac{1}{4}}w\frac{\eta^3(4\tau)\vartheta(z;\tau)\vartheta(2z+\tau;2\tau)}{\eta^3(2\tau)\vartheta(4z;4\tau)},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{W}_1^+(z;\tau) &:= 2i\mu\left(z+\tfrac{1}{2},\tfrac{1}{2};\tau\right),\\
\mathcal{W}_2^+(z;\tau) &:= q^{\frac{1}{8}}w^{\frac{1}{2}}\frac{\eta(\frac{\tau}{2})\vartheta(z+\frac{\tau}{2};\tau)}{\eta^2(\tau)}\mu\left(z+\tfrac{1}{2},\tfrac{1}{2};\tfrac{\tau}{2}\right),\\
\mathcal{W}_3^+(z;\tau) &:= \frac{i}{2}q^{\frac{1}{8}}w^{\frac{1}{2}}\frac{\eta^5(\frac{\tau}{2})\vartheta(z+\frac{\tau}{2};\tau)}{\eta^4(\tau)\vartheta\left(z+\frac{1}{2};\frac{\tau}{2}\right)}.
\end{aligned}$$

The next result (Lemma 3.4) will be used repeatedly in the next section to simplify various expressions and equations.

Lemma 3.4. *The following are true.*

(i) *If* $(z,\tau)\in\mathcal{Q}_{\mathcal{V}}$, *then* $\dfrac{\vartheta\left(z+\frac{1}{2};\tau\right)}{\eta(\tau)}=0.$

(ii) *If* $(z,\tau)\in\mathcal{Q}_{\mathcal{W}}$, *then* $\dfrac{\vartheta\left(z+\frac{\tau}{2};\tau\right)}{\eta(\tau)}=0.$

Proof. To prove (i), by Lemma 2.7, we have that

$$\frac{\vartheta\left(z+\frac{1}{2};\tau\right)}{\eta(\tau)} = -iq^{\frac{1}{12}}(-w)^{-\frac{1}{2}}\prod_{n=1}^{\infty}\left(1+wq^{n-1}\right)\left(1+w^{-1}q^n\right). \tag{3.6}$$

By an argument similar to the argument given in the proof of Lemma 3.1, we find that the function in (3.6) vanishes for $(z, \tau) \in \mathcal{Q}_{\mathcal{V}}$. The proof of (ii) follows similarly using Lemma 2.7 and the definition of $\mathcal{Q}_{\mathcal{W}}$. $\square$

4. Proof of main theorems

4.1. Proof of Theorem 1.1

We first point out that equations (1.6), (1.7), and (1.9) follow directly from the definition of $\mathcal{V}^+$ in (1.2). Equations (1.11) and (1.12) follow directly from (1.8) and (1.10), respectively, using Lemma 3.4, and that $\eta(2\tau)/\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1+q^n)$ also vanishes for $\tau = h/k$ arising from $\mathcal{Q}_{\mathcal{V}}$. Thus, we are left to prove (1.8), (1.10), and the analytic properties claimed in Theorem 1.1.

We begin with (1.10). Using the functional equation that appears immediately below equation (4.32) in [9], as well as Lemma 3.2, and the definition of $\mathcal{V}^+$ in (1.2), we find

$$\mathcal{V}^+\left(z+\tau;\tau\right) + w^{\frac{1}{2}}q^{\frac{3}{8}} + wq^{\frac{1}{2}}\mathcal{V}^+\left(z;\tau\right) = i\frac{\vartheta\left(z+\frac{1}{2};\tau\right)}{\vartheta\left(\tau;2\tau\right)} = -q^{\frac{1}{4}}\frac{\eta(2\tau)}{\eta^2(\tau)}\vartheta\left(z+\tfrac{1}{2};\tau\right).$$

Equation (1.10) follows after rearrangement of terms.

To prove (1.8), we begin with the expressions given for $\mathcal{V}^+$ in Lemma 3.3. We define the following completed functions:

$$\begin{aligned}
\widehat{\mathcal{V}}_1^+(z;\tau) &:= -i\widehat{\mu}\left(z+\tfrac{1}{2},\tfrac{1}{2};\tau\right),\\
\widehat{\mathcal{V}}_2^+(z;\tau) &:= -i\widehat{\mu}\left(2z+\tfrac{1}{2},\tfrac{1}{2};2\tau\right)\frac{\eta(2\tau)}{\eta^2(\tau)}\vartheta\left(z+\tfrac{1}{2};\tau\right),\\
\widehat{\mathcal{V}}_3^+(z;\tau) &:= \mathcal{V}_3^+(z;\tau).
\end{aligned}$$

Note that $\widehat{\mathcal{V}}_3^+ = \mathcal{V}_3^+$ because, as we shall show, $\mathcal{V}_3^+$ is a Jacobi form (see (4.5)). Using [14, Theorem 1.11 (2)], Lemma 2.6, and Lemma 2.7, after some calculation and simplification, we obtain the following Jacobi transformation properties, which hold for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, and $j \in \{1,2,3\}$:

$$\widehat{\mathcal{V}}_j^+\left(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right) = \psi_j(c\tau+d)^{\frac{1}{2}}e^{\frac{-\pi icz^2}{c\tau+d}}\widehat{\mathcal{V}}_j^+(z;\tau), \tag{4.1}$$

where

$$\begin{aligned}
\psi_1 = \psi_1\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) &:= \chi^{-3}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right),\\
\psi_2 = \psi_2\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) &:= \chi^{-2}\begin{pmatrix} a & 2b \\ \frac{c}{2} & d \end{pmatrix}\chi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(-1)^{\frac{d-1}{2}}e^{\frac{\pi icd}{4}},\\
\psi_3 = \psi_3\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) &:= \chi^{3}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(-1)^{\frac{a-1}{2}+b}e\left(\tfrac{ab}{4}\right).
\end{aligned}$$

A lengthy but straightforward calculation using (2.4) reveals that ψ_1, ψ_2, and ψ_3 are equal for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, so that for $j \in \{1,2,3\}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, we have

$$\widehat{\mathcal{V}}_j^+\left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = \psi(c\tau+d)^{\frac{1}{2}} e^{\frac{-\pi icz^2}{c\tau+d}} \widehat{\mathcal{V}}_j^+(z;\tau), \tag{4.2}$$

where $\psi = \psi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \psi_1 = \psi_2 = \psi_3$.

Now, by definition, we have that

$$\widehat{\mathcal{V}}_1^+(z;\tau) = \mathcal{V}_1^+(z;\tau) + \frac{1}{2} R(z;\tau),$$
$$\widehat{\mathcal{V}}_2^+(z;\tau) = \mathcal{V}_2^+(z;\tau) + \frac{1}{2} \frac{\vartheta(z+\frac{1}{2};\tau)\eta(2\tau)}{\eta^2(\tau)} R(2z;2\tau).$$

Thus, we obtain from (4.1) and the discussion following, that for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$,

$$\begin{aligned} &\mathcal{V}_1^+\left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) - \psi(c\tau+d)^{\frac{1}{2}} e^{\frac{-\pi icz^2}{c\tau+d}} \mathcal{V}_1^+(z;\tau) \\ &\qquad = -\frac{1}{2} R\left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) + \frac{1}{2}\psi(c\tau+d)^{\frac{1}{2}} e^{\frac{-\pi icz^2}{c\tau+d}} R(z;\tau), \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\mathcal{V}_2^+\left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) - \psi(c\tau+d)^{\frac{1}{2}} e^{\frac{-\pi icz^2}{c\tau+d}} \mathcal{V}_2^+(z;\tau) \\ &\qquad = \frac{-\vartheta(\frac{z}{c\tau+d}+\frac{1}{2}; \frac{a\tau+b}{c\tau+d})\eta(\frac{2a\tau+2b}{c\tau+d})}{2\eta^2(\frac{a\tau+b}{c\tau+d})} R\left(\frac{2z}{c\tau+d}; \frac{2a\tau+2b}{c\tau+d}\right) \\ &\qquad\quad + \psi(c\tau+d)^{\frac{1}{2}} e^{\frac{-\pi icz^2}{c\tau+d}} \frac{\vartheta(z+\frac{1}{2};\tau)\eta(2\tau)}{2\eta^2(\tau)} R(2z;2\tau), \end{aligned} \tag{4.4}$$

$$\mathcal{V}_3^+\left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) - \psi(c\tau+d)^{\frac{1}{2}} e^{\frac{-\pi icz^2}{c\tau+d}} \mathcal{V}_3^+(z;\tau) = 0. \tag{4.5}$$

If we denote the right-hand sides of (4.3) and (4.4) by $F_1(z;\tau)$ and $F_2(z;\tau)$ respectively, then we have shown (for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$) that

$$\mathcal{V}^+\left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) - \psi(c\tau+d)^{\frac{1}{2}} e^{\frac{-\pi icz^2}{c\tau+d}} \mathcal{V}^+(z;\tau) = F_1(z;\tau) + F_2(z;\tau). \tag{4.6}$$

Next we express $\left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right)$ as $S^{-1}T^{-4}S$ (where $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$) and apply Lemma 2.8 repeatedly. After some lengthy but straightforward calculations, we obtain

$$\begin{aligned} R\left(\frac{z}{4\tau+1}; \frac{\tau}{4\tau+1}\right) &= \sqrt{\frac{i}{\tau}}(4\tau+1)^{\frac{1}{2}} e^{\frac{\pi iz^2}{(4\tau+1)\tau}} \\ &\quad \times \left(h\left(\frac{z}{\tau}; \frac{-1-4\tau}{\tau}\right) + \sqrt{-i\tau} e^{\frac{-\pi iz^2}{\tau}}\left(h(z;\tau) - R(z;\tau)\right)\right), \end{aligned} \tag{4.7}$$

and

$$R\left(\frac{2z}{4\tau+1};\frac{2\tau}{4\tau+1}\right)=\sqrt{\frac{i}{2\tau}}(4\tau+1)^{\frac{1}{2}}e^{\frac{2\pi iz^2}{(4\tau+1)\tau}}$$
$$\times\left(h\left(\frac{z}{\tau};\frac{-1-4\tau}{2\tau}\right)-\sqrt{2i\tau}e^{\frac{-2\pi iz^2}{\tau}}\Big(h(2z;2\tau)-R(2z;2\tau)\Big)\right). \quad (4.8)$$

Substituting (4.7) and (4.8) into (4.6) with $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)=\left(\begin{smallmatrix} 1 & 0\\ 4 & 1\end{smallmatrix}\right)$, applying Lemma 2.6, Lemma 2.7, and rearranging, we obtain (1.8).

It remains to show that the "errors" on the right-hand sides of (1.12) and (1.11) extend to C^∞ functions on $\mathbb{R}\times\mathbb{R}$ and $(\mathbb{R}\setminus(\mathbb{Z}+\{\frac{1}{2},\pm\frac{1}{8},\pm\frac{3}{8}\}))\times(\mathbb{R}\setminus\{-\frac{1}{4}\})$, respectively. The former is clear. For the latter, we will first prove H_1 is C^∞ on $(-\frac{1}{8},\frac{1}{8})\times(\mathbb{R}\setminus\{-\frac{1}{4}\})$, and then explain at the conclusion of the proof why it suffices to prove analyticity on this restricted interval in z. In short, this is due to the fact that by Lemma 2.3, translating $h(z;\tau)$ by integers or integer multiples of τ in the elliptic variable preserves analyticity ($\tau\neq 0$). We save further discussion on this point until the conclusion of the proof, and begin by applying Lemma 2.4 and Lemma 2.5, with $a=0$ and $b=-z$, with $z\in(-\frac{1}{2},\frac{1}{2})$, to write the Mordell integral $h(z;\tau)$ as the period integral

$$h(z;\tau)=-\int_0^{i\infty}\frac{g_{\frac{1}{2},-z+\frac{1}{2}}(\rho)}{\sqrt{-i(\tau+\rho)}}d\rho. \quad (4.9)$$

Similarly, with the substitutions $a=-z$ and $b=4z$, with $z\in(-\frac{1}{8},\frac{1}{8})$, and $\tau\mapsto\frac{-1-4\tau}{\tau}$, we obtain

$$h\left(\frac{z}{\tau};\frac{-1-4\tau}{\tau}\right)=-e^{-\pi iz^2\left(\frac{1+4\tau}{\tau}\right)+2\pi iz\left(4z+\frac{1}{2}\right)}\int_0^{i\infty}\frac{g_{-z+\frac{1}{2},4z+\frac{1}{2}}(\rho)}{\sqrt{-i\left(\rho+\frac{-1-4\tau}{\tau}\right)}}d\rho. \quad (4.10)$$

Making the change of variable $\rho\mapsto 4-\frac{1}{\rho}$, and applying Lemma 2.4 and Lemma 2.5, after some calculation and simplification we find that (4.10) equals

$$i\sqrt{i\tau}e^{\frac{-\pi iz^2}{\tau}}\int_{\frac{1}{4}}^{0}\frac{g_{\frac{1}{2},-z+\frac{1}{2}}(\rho)}{\sqrt{-i(\tau+\rho)}}d\rho. \quad (4.11)$$

Finally, substituting (4.9) and (4.11) into (1.3), we find that for $z\in(-\frac{1}{8},\frac{1}{8})$,

$$H_1(z;\tau)=\frac{1}{2}\int_{\frac{1}{4}}^{i\infty}\frac{g_{\frac{1}{2},-z+\frac{1}{2}}(\rho)d\rho}{\sqrt{-i(\tau+\rho)}}. \quad (4.12)$$

The proof that the function in (4.12) is C^∞ follows by an almost identical argument as given by Bringmann and the second author in [1]; note that the same integrand $g_{\frac{1}{2},-z+\frac{1}{2}}(\rho)/\sqrt{-i(\tau+\rho)}$ appears in [1, Proof of Theorem 1.1, p. 375]. We integrate on the vertical path from $\frac{1}{4}$ to $\frac{1}{4}+i\infty$, and on the horizontal path from $\frac{1}{4}+i\infty$ to $i\infty$. It is not difficult to show the integral vanishes on the latter path, so we are left to analyze

$$\int_0^\infty \frac{g_{\frac{1}{2},-z+\frac{1}{2}}(\frac{1}{4}+it)dt}{\sqrt{-i(\tau+\frac{1}{4}+it)}}.$$

For $\ell \in \mathbb{N}_0$, we employ the following bound pertaining to the theta function in the integrand, which was established in [1]:

$$\frac{\partial^\ell}{\partial z^\ell} g_{\frac{1}{2},-z+\frac{1}{2}}(\tfrac{1}{4}+it) \ll e^{-\frac{\pi}{4}t}.$$

The argument that $H_1(z,\tau)$ is C^∞ on $(-\frac{1}{8},\frac{1}{8})\times(\mathbb{R}\setminus\{-\frac{1}{4}\})$ now follows as in [1] by the Leibniz Rule.

For general $z \in (\mathbb{R}\setminus(\mathbb{Z}+\{\frac{1}{2},\pm\frac{1}{8},\pm\frac{3}{8}\}))$, we use Lemma 2.3 to translate the Mordell integral h in the elliptic variable by integers or integer multiples of τ up to addition of analytic functions ($\tau \neq 0$). (See also [1, Proof of Theorem 1.1, p. 375].) More precisely, for $z \in (\mathbb{R}\setminus(\mathbb{Z}+\{\frac{1}{2},\pm\frac{1}{8},\pm\frac{3}{8}\}))$, there is some $m \in \mathbb{Z}$ such that $z+m \in (\frac{j}{8},\frac{j+1}{8}) \subseteq (-\frac{1}{2},\frac{1}{2})$, where $j \in \{-4,\pm3,\pm2,\pm1,0\}$. Repeatedly applying Lemma 2.3 (ii) relates $h(z;\tau)$ and $h(z+m;\tau)$ up to the addition of an analytic function, and we may proceed as with (4.9), using $a=0$ and $b=-(z+m)$, to re-write $h(z+m;\tau)$ as a period integral. There also exists an $n \in \mathbb{Z}$ (which depends on m and j) so that $4(z+m)+n \in (-\frac{1}{2},\frac{1}{2})$. Let $\widetilde{\tau} := \frac{-1-4\tau}{\tau}$. Using Lemma 2.3 (ii) and (iii) repeatedly again relates $h\left(\frac{z}{\tau};\widetilde{\tau}\right)$ and $h\left(\frac{z+m}{\tau}-n;\widetilde{\tau}\right) = h\left(\frac{z}{\tau}-m\widetilde{\tau}-4m-n;\widetilde{\tau}\right)$ up to the addition of analytic functions, and we may proceed as with (4.10), using $a=-(z+m)$ and $b=4(z+m)+n$, to re-write $h\left(\frac{z+m}{\tau}-n;\widetilde{\tau}\right)$ as a period integral. The rest of the argument then follows as above.

4.2. Proof of Theorem 1.2

The proof of Theorem 1.2 is very similar to the proof of Theorem 1.1 given above. We provide a detailed sketch. Equations (1.14), (1.15), and (1.17) follow directly from the definition of $\mathcal{W}^+$ in (1.13). Properties in (1.19) and (1.20) follow directly from (1.16) and (1.18), respectively, using Lemma 3.4, and simplifying the quotient $\eta(\frac{\tau}{2})/\eta(\tau)$. Equation (1.18) follows after a short calculation using [9, (4.24)], the definition of $\mathcal{W}^+$ in (1.13), and Lemma 3.2. We are thus left to prove (1.16) and the analytic properties claimed in Theorem 1.2.

To prove (1.16), we begin with the expressions given for $\mathcal{W}^+$ in Lemma 3.3, and define the completed functions

$$\widehat{\mathcal{W}}_1^+(z;\tau) := 2i\widehat{\mu}\left(z+\tfrac{1}{2},\tfrac{1}{2};\tau\right),$$
$$\widehat{\mathcal{W}}_2^+(z;\tau) := q^{\frac{1}{8}}w^{\frac{1}{2}}\frac{\eta(\frac{\tau}{2})\vartheta(z+\frac{\tau}{2};\tau)}{\eta^2(\tau)}\widehat{\mu}\left(z+\tfrac{1}{2},\tfrac{1}{2};\tfrac{\tau}{2}\right),$$
$$\widehat{\mathcal{W}}_3^+(z;\tau) := \frac{i}{2}q^{\frac{1}{8}}w^{\frac{1}{2}}\frac{\eta^5(\frac{\tau}{2})\vartheta(z+\frac{\tau}{2};\tau)}{\eta^4(\tau)\vartheta\left(z+\frac{1}{2};\frac{\tau}{2}\right)}.$$

Using Lemma 2.6, Lemma 2.7, and Lemma 2.9, we find after some straightforward calculations and simplifications for $j \in \{1,2,3\}$ that

$$\widehat{\mathcal{W}}_j^+(z;\tau) + i(2\tau+1)^{-\frac{1}{2}}e^{\frac{2\pi iz^2}{2\tau+1}}\widehat{\mathcal{W}}_j^+\left(\frac{z}{2\tau+1};\frac{\tau}{2\tau+1}\right) = 0. \tag{4.13}$$

Using (4.13), the decomposition given for $\mathcal{W}^+$ in Lemma 3.3, and the definitions of the functions $\widehat{\mathcal{W}}_j^+$, we find that

$$\mathcal{W}^+(z;\tau) + i(2\tau+1)^{-\frac{1}{2}}e^{\frac{2\pi iz^2}{2\tau+1}}\mathcal{W}^+\left(\frac{z}{2\tau+1};\frac{\tau}{2\tau+1}\right) = \mathcal{E}_1(z;\tau)+\mathcal{E}_2(z;\tau), \tag{4.14}$$

with"error" functions $\mathcal{E}_1$ and $\mathcal{E}_2$ defined using the Mordell integral h in (1.5) and the nonholomorphic function R in (2.7) (similar to the functions F_1 and F_2 from (4.3) and (4.4)). We express $\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$ as $S^{-1}T^{-2}S$ and apply Lemma 2.8 repeatedly. After some lengthy but straightforward calculations as in the proof of Theorem 1.1, we find that $\mathcal{E}_1(z;\tau)+\mathcal{E}_2(z;\tau) = G_1(z;\tau)+G_2(z;\tau)$, where

$$G_1(z;\tau) := h(z;\tau) + \tau^{-\frac{1}{2}}\zeta_8^3 e^{\frac{\pi iz^2}{\tau}}h\left(\frac{-z}{\tau};\frac{-1-2\tau}{\tau}\right), \tag{4.15}$$

$$G_2(z;\tau) := \frac{-iq^{\frac{1}{8}}w^{\frac{1}{2}}\eta(\frac{\tau}{2})\vartheta(z+\frac{\tau}{2};\tau)}{2\eta^2(\tau)}\left(h(z;\tfrac{\tau}{2}) + \tau^{-\frac{1}{2}}\zeta_8\sqrt{2}\,e^{\frac{2\pi iz^2}{\tau}}h\left(\frac{-2z}{\tau};\frac{-2-4\tau}{\tau}\right)\right). \tag{4.16}$$

This proves (1.16).

The claim that (1.20) extends to a C^∞ function on $\mathbb{R}\times\mathbb{R}$ is clear. To prove that G_1 extends to a C^∞ function on the claimed subset of $\mathbb{R}\times\mathbb{R}$, we proceed as in the proof of Theorem 1.1. After some calculations, using Lemma 2.4 and Lemma 2.5, we re-write G_1 for $z \in (-\frac{1}{4},\frac{1}{4})$ as the period integral

$$-\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{1}{2},-z+\frac{1}{2}}(\rho)}{\sqrt{-i(\tau+\rho)}}d\rho.$$

That this function is C^∞ on $(-\frac{1}{4},\frac{1}{4})\times(\mathbb{R}\setminus\{-\frac{1}{2}\})$ follows as in the proof of Theorem 1.1. We extend to $z \in \mathbb{R}\setminus(\mathbb{Z}+\{\frac{1}{2},\pm\frac{1}{4}\})$ as argued in the proof of Theorem 1.1 using Lemma 2.3.

4.3. Proof of Theorem 1.3

We proceed in a similar manner as Bringmann and the second author in [1, Proof of Theorem 1.2].

We begin with the proof of (i) in Theorem 1.3. Suppose a function $f(z;\tau)$ transforms in the elliptic variable as

$$f(z+\tau;\tau) = -wq^{\frac{1}{2}}f(z;\tau) + wq^{\frac{1}{2}}r(z;\tau) \tag{4.17}$$

for some function $r(z;\tau)$. By induction on $m \in \mathbb{N}_0$, it is not difficult to show that for all $m \in \mathbb{N}_0$

$$f(z+m\tau;\tau) = (-1)^m w^m q^{\frac{1}{2}m^2} f(z;\tau) + \mathcal{G}_{m,r}(z;\tau),$$

where

$$\mathcal{G}_{m,r}(z;\tau) := (-1)^{m+1}\sum_{j=0}^{m-1}(-1)^j r(z+j\tau;\tau)w^{m-j}q^{\frac{1}{2}(m^2-j^2)}.$$

It is clear from (1.12) that for $(z,\tau) \in \mathcal{Q}_{\mathcal{V}}$ that the function $\mathcal{V}^+(z;\tau)$ satisfies (4.17) with

$$r(z;\tau) := -w^{-\frac{1}{2}}q^{\frac{7}{8}}.$$

If we set $m = k$, we thus have

$$\mathcal{V}^+(z+h;\tau)\Big|_{(z,\tau)=\left(\frac{a}{b},\frac{h}{k}\right)\in\mathcal{Q}_{\mathcal{V}}} = \left((-1)^k w^k q^{\frac{k^2}{2}}\mathcal{V}^+(z;\tau) + \mathcal{G}_{k,r}(z;\tau)\right)\Big|_{(z,\tau)=\left(\frac{a}{b},\frac{h}{k}\right)\in\mathcal{Q}_{\mathcal{V}}}.$$

We also deduce from (1.9) that for any $h \in \mathbb{Z}$,

$$\mathcal{V}^+(z+h;\tau) = (-1)^h\mathcal{V}^+(z;\tau).$$

Thus, we obtain

$$\left((-1)^h - (-1)^k w^k q^{\frac{k^2}{2}}\right)\mathcal{V}^+(z;\tau)\Big|_{(z,\tau)=\left(\frac{a}{b},\frac{h}{k}\right)\in\mathcal{Q}_{\mathcal{V}}} = \mathcal{G}_{k,r}(z;\tau)\Big|_{(z,\tau)=\left(\frac{a}{b},\frac{h}{k}\right)\in\mathcal{Q}_{\mathcal{V}}}.$$

The factor in front of $\mathcal{V}^+$ in this equation evaluates to -2, using the fact that k is even and h is odd. After some simplification, part (i) of Theorem 1.3 now follows.

The proof of part (ii) of Theorem 1.3 is very similar to the above proof of part (i), so for brevity's sake, we provide a detailed sketch of proof. We first deduce that for any $m \in \mathbb{N}_0$ and $h \in \mathbb{Z}$, that

$$\mathcal{W}^+(z+m\tau;\tau) = (-1)^m w^m q^{\frac{m^2}{2}} \mathcal{W}^+(z;\tau) + 2\sum_{j=1}^{m} (-1)^{j+1} w^{j-\frac{1}{2}} q^{\frac{(2j-1)(4m-2j+1)}{8}}, \tag{4.18}$$

$$\mathcal{W}^+(z+h;\tau) = (-1)^h \mathcal{W}^+(z;\tau). \tag{4.19}$$

Substituting $m = k$ and $(z, \tau) = (\frac{a}{b}, \frac{h}{k}) \in \mathcal{Q}_{\mathcal{W}}$ in (4.18), using that $b|k$, h is even, and k is odd, we obtain

$$\mathcal{W}^+(\tfrac{a}{b}+h;\tfrac{h}{k}) = -\mathcal{W}^+(\tfrac{a}{b};\tfrac{h}{k}) + 2\sum_{j=1}^{k} (-1)^{j+1} \zeta_{2b}^{a(2j-1)} \zeta_{8k}^{h(2j-1)(4k-2j+1)}. \tag{4.20}$$

On the other hand, from (4.19) with these same substitutions for z and τ, we obtain

$$\mathcal{W}^+(\tfrac{a}{b}+h;\tfrac{h}{k}) = \mathcal{W}^+(\tfrac{a}{b};\tfrac{h}{k}). \tag{4.21}$$

After substituting (4.21) into (4.20), rearranging terms, simplifying exponents that appear, and changing the index of summation, we find the expression given in part (ii) of Theorem 1.3.

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