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ON THE NON-NEGATIVITY OF THE SPT-CRANK FOR PARTITIONS WITHOUT REPEATED ODD PARTS

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Abstract In this paper, we establish the non-negativity of the spt-crank for partitions without repeated odd parts which was first conjectured by Garvan and Jennings-Shaffer. As corollaries, we prove inequalities between the positive rank and crank moments of such partitions.
Keywords: Partitions without repeated odd parts, Spt-crank, Rank moments
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1. Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers that whose sum is n. In the study of integer partitions, people are often interested in weighted counts of partitions (see [3, 4, 24, 22], for example). Recently, Andrews [6] introduced the smallest parts function $spt(n)$ as the weighted counting of partitions of n with respect to the number of occurrences of the smallest part, i.e., $spt(n)$ is the total number of appearances of smallest parts in all the partitions of n. Andrews proved

$$\mathrm{spt}(5n+4) \equiv 0 \pmod 5, \tag{1.1}$$
$$\mathrm{spt}(7n+5) \equiv 0 \pmod 7, \tag{1.2}$$
$$\mathrm{spt}(13n+6) \equiv 0 \pmod{13}.$$

To give combinatorial interpretations of the above congruences, Andrews, Garvan and Liang introduce the spt-crank of an S-partition. Let $\mathcal{P}$ denote the set of partitions and $\mathcal{D}$ denote the set of partitions into distinct parts. For $\pi \in \mathcal{P}$, we let $s(\pi)$ denote the smallest part of π with $s(\emptyset) = \infty$ for the empty partition, and $\sharp(\pi)$ be the number of parts in π. Define

$$S := \{(\pi_1, \pi_2, \pi_3) \in \mathcal{D} \times \mathcal{P} \times \mathcal{P} : \pi_1 \neq \emptyset \text{ and } s(\pi_1) \leq \min s(\pi_2), s(\pi_3)\}.$$

For $\overrightarrow{\pi} = (\pi_1, \pi_2, \pi_3) \in S$, we define the weight $\omega_1(\overrightarrow{\pi}) := (-1)^{\sharp(\pi_1)-1}$, the crank $(\overrightarrow{\pi}) := \sharp(\pi_2) - \sharp(\pi_3)$, and $\mid \overrightarrow{\pi} \mid := \mid \pi_1 \mid + \mid \pi_2 \mid + \mid \pi_3 \mid$ where $\mid \pi_j \mid$ is the sum of the parts of π_j. We say $\overrightarrow{\pi}$ is a vector partition of n if $\mid \overrightarrow{\pi} \mid = n$. For an integer $n \geq 1$ and any integer m, let $N_S(m, n)$ denote the number of vector partitions of n in S with crank m counted according to the weight ω_1, that is,

$$N_S(m,n) := \sum_{\substack{\overrightarrow{\pi} \in S, \mid \overrightarrow{\pi} \mid = n \\ \mathrm{crank}(\overrightarrow{\pi}) = m}} \omega_1(\overrightarrow{\pi}).$$

Then Andrews, Garvan and Liang [10] show that, for integers $n \geq 0$,

$$N_S(k, 5, 5n+4) = \frac{spt(5n+4)}{5}, \text{ for } 0 \leq k \leq 4, \tag{1.3}$$

and

$$N_S(k,7,7n+5)=\frac{spt(7n+5)}{7},\ \text{for } 0\le k\le 6, \tag{1.4}$$

where $N_S(m,t,n):=\sum_{k\equiv m \pmod t} N_S(k,n)$. Clearly, equation (1.3) (resp. (1.4)) implies (1.1) (resp. (1.2)). In the same paper, Andrews, Garvan and Liang also show that

$$N_S(m,n)\ge 0,$$

for all integers m and $n\ge 1$. We remark that Chen, Ji and Zang [21] found a definition of spt-crank for the ordinary partitions. More studies on spt-function and spt-crank of ordinary partitions can be found in [1, 11, 18, 26].

In this paper, we study the spt-crank for the partitions without repeated odd parts. Recall that the spt-function for such partitions were first introduced by Ahlgren, Bringmann and Lovejoy [2]. Let M2spt(n) denote the total number of smallest parts in all partitions of n without repeated odd parts and the smallest part is even. For an odd prime p, let $\left(\frac{\cdot}{p}\right)$ denote the Legendre symbol [28, p. 85]. Then the authors of [2] proved that for any prime $l\ge 3$, any integer $m\ge 1$, and $n\ge 0$ such that $\left(\frac{-n}{l}\right)=1$, we have

$$\mathrm{M2spt}\left(\frac{l^{2m}n+1}{8}\right)\equiv 0 \pmod{l^m}.$$

In [27], Garvan and Jennings-Shaffer provided more congruences for M2spt(n). For example, they proved that, for integers $n\ge 0$,

$$\mathrm{M2spt}(3n+1)\equiv 0 \pmod 3, \tag{1.5}$$

$$\mathrm{M2spt}(5n+1)\equiv 0 \pmod 5, \tag{1.6}$$

$$\mathrm{M2spt}(5n+3)\equiv 0 \pmod 5. \tag{1.7}$$

To obtain the above congruences, they introduced the spt-crank for partitions without repeated odd parts as follows. Following the notations used in the definition of the spt-crank of the S-partition, we let $\overline{V}:=\mathcal{D}\times\mathcal{P}\times\mathcal{P}\times\mathcal{D}$. For $\overrightarrow{\pi}=(\pi_1,\pi_2,\pi_3,\pi_4)\in\overline{V}$, we define the weight $\omega(\overrightarrow{\pi}):=(-1)^{\sharp(\pi_1)-1}$, the crank$(\overrightarrow{\pi}):=\sharp(\pi_2)-\sharp(\pi_3)$, and the norm $|\overrightarrow{\pi}|:=|\pi_1|+|\pi_2|+|\pi_3|+|\pi_4|$. Then $\overline{S}$ denotes the subset of $\overline{V}$ defined by

$$\overline{S}:=\left\{(\pi_1,\pi_2,\pi_3,\pi_4)\in\overline{V}:1\le s(\pi_1)<\infty, s(\pi_1)\le s(\pi_2), s(\pi_1)<s(\pi_4)\right\}.$$

For $\pi\in\mathcal{P}$, we also define $n_o(\pi)$ and $n_e(\pi)$ to be the number of odd and even part of π, respectively. Let $S2$ denote the subset of $\overline{S}$ given by

$$S2:=\left\{(\pi_1,\pi_2,\pi_3,\pi_4)\in\overline{S}:n_o(\pi_1)=n_o(\pi_2)=n_o(\pi_3)=n_e(\pi_4)=0\right\}.$$

Then, for an integer $n\ge 1$ and any integer m, we denote $N_{S2}(m,n)$ to be the number of vector partitions of n in $S2$ with crank m counted according to the weight ω, that is

$$N_{S2}(m,n):=\sum_{\substack{\overrightarrow{\pi}\in S2,\ |\overrightarrow{\pi}|=n\\ \mathrm{crank}(\overrightarrow{\pi})=m}}\omega(\overrightarrow{\pi}).$$

By [27, Eq. (2.3)], we have the following generating function for $N_{S2}(m,n)$:

$$S2(z,q) := \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N_{S2}(m,n) z^m q^n = \sum_{n=1}^{\infty} \frac{q^{2n}(q^{2n+2}, -q^{2n+1}; q^2)_\infty}{(zq^{2n}, q^{2n}/z; q^2)_\infty}. \tag{1.8}$$

In equation (1.8) and for the rest of this article, we use the notations

$$(x_1, x_2, \ldots, x_k; q)_m := \prod_{n=0}^{m-1} (1 - x_1 q^n)(1 - x_2 q^n) \cdots (1 - x_k q^n),$$

for $m \in \mathbb{N} \cup \{\infty\}$ and we require $|q| < 1$ for absolute convergence.

Recall the q-binomial coefficient

$$\begin{bmatrix} L \\ K \end{bmatrix}_q := \begin{cases} 0, & \text{if } K < 0 \text{ or } K > L, \\ \dfrac{(q;q)_L}{(q;q)_K (q;q)_{L-K}}, & \text{otherwise.} \end{cases}$$

For $L \geq K \geq 0$, it is well known that $\begin{bmatrix} L \\ K \end{bmatrix}_q$ is a polynomial with non-negative integral coefficients. See [5, p. 33].

For integers $n \geq 1$ and $0 \leq m \leq t-1$, let $N_{S2}(m,t,n) := \sum_{k \equiv m \pmod t} N_{S2}(k,n)$. Then Garvan and Jennings-Shaffer showed in [27] that

$$N_{S2}(k,3,3n+1) = \frac{\mathrm{M2spt}(3n+1)}{3}, \text{ for } 0 \leq k \leq 2,$$
$$N_{S2}(k,5,5n+1) = \frac{\mathrm{M2spt}(5n+1)}{5}, \text{ for } 0 \leq k \leq 4,$$

and

$$N_{S2}(k,5,5n+3) = \frac{\mathrm{M2spt}(5n+3)}{5}, \text{ for } 0 \leq k \leq 4,$$

which imply the congruences (1.5), (1.6) and (1.7), respectively. At the end of [27], Garvan and Jennings-Shaffer made the following conjecture:

Conjecture 1.1. *For $n \geq 1$ and $m \in \mathbb{Z}$, we have*

$$N_{S2}(m,n) \geq 0.$$

The first result of this paper is the following theorem:

Theorem 1.2. *Conjecture 1.1 is true. In particular, for $m \in \mathbb{Z}$ and $n = |\, 2m \,| + 2$ or $n \geq |\, 2m \,| + 4$, we have*

$$N_{S2}(m,n) > 0. \tag{1.9}$$

Remark. Equation (1.8) implies $N_{S2}(m,n) = N_{S2}(-m,n)$ for $m \in \mathbb{Z}$ and $n \geq 1$. Thus, we only consider $N_{S2}(m,n)$ with $m \geq 0$ in the proof of Theorem 1.2.

In [27], Garvan and Jennings-Shaffer also defined another three spt-cranks in terms of $\overline{S}$-partitions. These objects are proved to be closely related to the spt-functions of overpartitions (see [27, Section 3]). In particular, Garvan and Jennings-Shaffer established the non-negativity for such spt-cranks. Armed with this result, the author [34] proved inequalities between some rank and crank moments for overpartitions. In this paper, as an application of Theorem 1.2, we prove inequalities between rank and crank moments of partitions without repeated odd parts with the same method adopted in [34]. We recall that Berkovich and Garvan [12] introduced what they called the M_2- rank of such partitions. The M_2-rank of a partition λ without repeated odd parts is defined by

$$M_2\text{-rank}(\lambda) := \left\lceil \frac{l(\lambda)}{2} \right\rceil - \nu(\lambda),$$

where $l(\lambda)$ is the largest part of λ, $\nu(\lambda)$ is the number of parts of λ and $\lceil . \rceil$ is the ceiling function. In [27], Garvan and Jennings-Shaffer defined a residual crank for partitions without repeated odd parts. Recall that the crank of an ordinary partition is the largest part if there are no ones and is the number of parts larger than the number of ones minus the number of ones otherwise. Then, for a partition π of n with distinct odd parts, we define the residual crank as the crank of the partition $\frac{\pi_e}{2}$ obtained by takeing the subpartition π_e, of the even parts of π, and halving each part of π_e. For an integer $n \geq 1$ and any integer m, let $N2(m,n)$ (resp. $M2(m,n)$) denote the number of partitions of n without repeated odd parts whose M_2-rank (resp. residual crank) is m. Define the positive rank and crank moments by

$$N2_k^+(n) := \sum_{m=1}^{\infty} m^k N2(m,n), \qquad M2_k^+(n) := \sum_{m=1}^{\infty} m^k M2(m,n),$$

and their generating functions by

$$N2_k(q) := \sum_{n=1}^{\infty} N2_k^+(n) q^n, \qquad C2_k(q) := \sum_{n=1}^{\infty} M2_k^+(n) q^n. \tag{1.10}$$

Then Jennings-Shaffer constructed various quasimodular forms using $N2_{2k}(q)$ and $M2_{2k}(q)$ (see [30, Theorem 1.1]). In [29], Jennings-Shaffer used the theory of Bailey pairs to prove that

$$M2_{2k}^+(n) > N2_{2k}^+(n), \tag{1.11}$$

for all $k \geq 1$ and $n > 4$. For more inequalities between rank and crank moments of partitions, see [8, 13, 17, 19, 25, 31, 32]. Studies on asymptotic formulas for these combinatorial objects can be found [14, 15, 16, 33, 35].

The second result of this paper is a generalization of (1.11).

Theorem 1.3. *For positive integers k, n, we have*

$$M2_k^+(n) \geq N2_k^+(n).$$

In particular, we have $M2_k^+(n) > N2_k^+(n)$ when $k \geq 1, n \geq 4$.

The paper is organized as follows. In Section 2 and Section 3, we prove Theorem 1.2 and Theorem 1.3, respectively. At the end of this paper, we give a conjecture on the monotonicity of $N_{S2}(m,n)$ in a short concluding section.

2. Proof of Theorem 1.2

The proof of Theorem 1.2 is motivated by the works in [10, Section 5].

For $l \geq 0$, let $B_l(q) := \sum_{n=1}^{\infty} N_{S2}(l,n) q^n$. Then we have

$$B_0(q) + \sum_{l=1}^{\infty} B_l(q)(z^l + z^{-l}) = S2(z,q).$$

Thus, to prove $N_{S2}(l,n)\geq 0$, it suffices to show that $B_l(q)$ has non-negative coefficients for all $l\geq 0$.

First, we recall the q-binomial theorem [5, p. 17, (2.2.1)] and its finite form [5, p. 36, (3.3.6)]:

$$\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}x^k=\frac{(ax;q)_\infty}{(x;q)_\infty}, \tag{2.1}$$

and

$$\sum_{k=0}^{n-1}(-1)^kq^{k(k-1)/2}x^k\begin{bmatrix}n-1\\k\end{bmatrix}_q=(x;q)_{n-1}. \tag{2.2}$$

Next, using (1.8), we rewrite $S2(z,q)$ as follows:

$$\begin{aligned}S2(z,q)&=\sum_{n=1}^{\infty}\frac{q^{2n}(-q^{2n+1};q^2)_\infty}{(zq^{2n};q^2)_\infty}\sum_{m=0}^{\infty}\frac{(zq^2;q^2)_m(q^{2n}/z)^m}{(q^2;q^2)_m} \qquad \text{(by (2.1))}\\&=\sum_{n=1}^{\infty}q^{2n}(-q^{2n+1};q^2)_\infty(zq^2;q^2)_{n-1}\sum_{m=0}^{\infty}\frac{(q^{2n}/z)^m}{(q^2;q^2)_m(zq^{2m+2};q^2)_\infty}\\&=\sum_{n=1}^{\infty}q^{2n}(-q^{2n+1};q^2)_\infty\sum_{h=0}^{n-1}\begin{bmatrix}n-1\\h\end{bmatrix}_{q^2}(-1)^hq^{h(h+1)}z^h\\&\quad\times\sum_{m,i\geq 0}\frac{(q^{2n}/z)^m(zq^{2m+2})^i}{(q^2;q^2)_m(q^2;q^2)_i},\end{aligned} \tag{2.3}$$

where the last equality follows from (2.2) and (2.1). Rewriting the summations on the right hand side of (2.3), we obtain

$$\begin{aligned}S2(z,q)&=\sum_{m=0}^{\infty}\frac{z^{-m}}{(q^2;q^2)_m}\sum_{i=0}^{\infty}\frac{q^{(2m+2)i}z^i}{(q^2;q^2)_i}\sum_{h=0}^{\infty}z^h\\&\quad\times\sum_{n=h+1}^{\infty}q^{2n}(-q^{2n+1};q^2)_\infty\begin{bmatrix}n-1\\h\end{bmatrix}_{q^2}(-1)^hq^{h(h+1)+2nm}\\&=\sum_{m=0}^{\infty}\frac{z^{-m}}{(q^2;q^2)_m}\sum_{j=0}^{\infty}z^j\\&\quad\times\sum_{h=0}^{j}\sum_{n=h+1}^{\infty}q^{2n}(-q^{2n+1};q^2)_\infty\begin{bmatrix}n-1\\h\end{bmatrix}_{q^2}(-1)^hq^{h(h+1)+2nm}\frac{q^{(2m+2)(j-h)}}{(q^2;q^2)_{j-h}}.\end{aligned}$$

Extracting the coefficient in z^l for $l=j-m\geq 0$, this implies that we have

$$\begin{aligned}B_l(q)&=\sum_{j=l}^{\infty}\sum_{h=0}^{j}\sum_{n=h+1}^{\infty}\frac{q^{2n}(-q^{2n+1};q^2)_\infty\left[\begin{smallmatrix}n-1\\h\end{smallmatrix}\right]_{q^2}(-1)^hq^{h(h+1)+2n(j-l)+(2(j-l)+2)(j-h)}}{(q^2;q^2)_{j-h}(q^2;q^2)_{j-l}}\\&=\sum_{j=0}^{\infty}\sum_{h=0}^{j+l}\sum_{n=h+1}^{\infty}(-q^{2n+1};q^2)_\infty\begin{bmatrix}n-1\\h\end{bmatrix}_{q^2}\\&\quad\times\frac{(-1)^hq^{h(h+1)+2n(j+1)+2(j+1)(j+l-h)}}{(q^2;q^2)_{j+l-h}(q^2;q^2)_j}.\end{aligned} \tag{2.4}$$

Letting $n\to\infty$ and replacing q,x by $q^2,-q^{2n+1}$, respectively, in (2.2), we obtain

$$(-q^{2n+1};q^2)_\infty=\sum_{k=0}^{\infty}\frac{q^{k^2+2kn}}{(q^2;q^2)_k}.$$

Substituting the above equation into (2.4), we find that

$$\begin{aligned}B_l(q)=&\sum_{k=0}^{\infty}\frac{q^{k^2}}{(q^2;q^2)_k}\sum_{j=0}^{\infty}\sum_{h=0}^{j+l}\frac{(-1)^hq^{h(h+1)+2(j+1)(j+l-h)}}{(q^2;q^2)_{j+l-h}(q^2;q^2)_j}\\&\times\sum_{n=h+1}^{\infty}\begin{bmatrix}n-1\\h\end{bmatrix}_{q^2}q^{2n(j+1+k)}.\end{aligned}\tag{2.5}$$

Replacing q by q^2 and setting $a=c=0,b=q^{2(h+1)},t=q^{2(k+1+j)}$ in [5, p.19, Corollary 2.3] which can be obtained from the q-binomial theorem (see (2.1)), we find that

$$\sum_{n=0}^{\infty}\begin{bmatrix}n+h\\h\end{bmatrix}_{q^2}q^{2n(k+1+j)}=\frac{(q^2;q^2)_{k+j}}{(q^2;q^2)_h}\sum_{n=0}^{\infty}\begin{bmatrix}n+j+k\\n\end{bmatrix}_{q^2}q^{2n(h+1)}.$$

This gives that

$$\sum_{n=h+1}^{\infty}\begin{bmatrix}n-1\\h\end{bmatrix}_{q^2}q^{2n(k+1+j)}=\frac{q^{2(h+1)(k+j+1)}(q^2;q^2)_{k+j}}{(q^2;q^2)_h}\sum_{n=0}^{\infty}\begin{bmatrix}n+j+k\\n\end{bmatrix}_{q^2}q^{2n(h+1)},$$

which together with (2.5) implies that

$$\begin{aligned}B_l(q)=&\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{k^2+2(j+1+k)+2n+2(j+1)(j+l)}(q^2;q^2)_{j+k}}{(q^2;q^2)_k(q^2;q^2)_j}\begin{bmatrix}n+j+k\\n\end{bmatrix}_{q^2}\\&\times\sum_{h=0}^{j+l}\frac{(-1)^hq^{h(h+1)+2h(n+k)}}{(q^2;q^2)_{j+l-h}(q^2;q^2)_h}.\end{aligned}\tag{2.6}$$

Next, we note that

$$\begin{aligned}&\sum_{h=0}^{j+l}\frac{(-1)^hq^{h(h+1)+2h(n+k)}}{(q^2;q^2)_{j+l-h}(q^2;q^2)_h}\\&=\frac{1}{(q^2;q^2)_{j+l}}\sum_{h=0}^{j+l}\frac{(-1)^hq^{h(h+1)+2h(n+k)}(q^{2(j+l+1-h)};q^2)_h}{(q^2;q^2)_h}\\&=\frac{1}{(q^2;q^2)_{j+l}}\sum_{h=0}^{j+l}\frac{q^{2h(l+j+1+n+k)}(q^{-2(j+l)};q^2)_h}{(q^2;q^2)_h}\\&=\frac{(q^{2(n+k+1)};q^2)_\infty}{(q^2;q^2)_{j+l}(q^{2(n+k+1+j+l)};q^2)_\infty}\qquad\text{(by (2.1))}\\&=\frac{(q^2;q^2)_{n+k+j+l}}{(q^2;q^2)_{j+l}(q^2;q^2)_{n+k}}.\end{aligned}\tag{2.7}$$

Substituting (2.7) into (2.6), we obtain

$$\begin{aligned} B_l(q) &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \frac{q^{k^2+2(j+1+k)+2n+2(j+1)(j+l)}(q^2;q^2)_{j+k}}{(q^2;q^2)_k(q^2;q^2)_j} \begin{bmatrix} n+j+k \\ n \end{bmatrix}_{q^2} \\ &\quad \times \frac{(q^2;q^2)_{n+k+j+l}}{(q^2;q^2)_{j+l}(q^2;q^2)_{n+k}} \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty} q^{k^2+2(j+1+k)+2n+2(j+1)(j+l)} \\ &\quad \times \begin{bmatrix} j+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n+j+k \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n+j+k+l \\ j+l \end{bmatrix}_{q^2}. \end{aligned} \tag{2.8}$$

Since the polynomial $\begin{bmatrix} L \\ K \end{bmatrix}_q$ has non-negative coefficients, equation (2.8) implies that $B_l(q)$ has non-negative coefficients. This proves the first statement in Theorem 1.2.

To prove (1.9), it suffices to show that, for $l \geq 0$, the power series $B_l(q)$ has positive coefficients for q^n when $n = 2l+2$ or $n \geq 2l+4$.

We rewrite (2.8) as follows:

$$B_l(q) = \sum_{n=0}^{\infty} q^{2n+2l+2} \begin{bmatrix} n+l \\ l \end{bmatrix}_{q^2} + \sum_{n=0}^{\infty} q^{2n+2l+5} \begin{bmatrix} n+1 \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n+1+l \\ l \end{bmatrix}_{q^2} + B_l'(q), \tag{2.9}$$

where the first (resp. second) sum of the above equation is obtained by setting the summation indexes $k = 0, j = 0$ (resp. $k = 1, j = 0$) in (2.8) and the power series $B_l'(q)$ has non-negative coefficients. By the definition of the q-binomial coefficient, it is easy to see that the constant term of the polynomial $\begin{bmatrix} L \\ K \end{bmatrix}_q$ is 1. This together with (2.9) implies that

$$B_l(q) = \sum_{n=0}^{\infty} q^{2n+2l+2} + \sum_{n=0}^{\infty} q^{2n+2l+5} + B_l''(q), \tag{2.10}$$

where the power series $B_l''(q)$ has non-negative coefficients. Clearly, equation (2.10) implies that the power series $B_l(q)$ has positive coefficients for q^n when $n = 2l+2$ or $n \geq 2l+4$. This completes the proof of Theorem 1.2.

3. Inequalities between rank and crank moments

In this section, we prove Theorem 1.3. To do this, we need to establish some lemmas. First, we prove the following result on the generating function for $N_{S2}(m,n)$.

Lemma 3.1. *For all non-negative integers m, we have*

$$\sum_{n=1}^{\infty} N_{S2}(m,n)q^n = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n+1)+2mn}(1-q^{n^2})}{1-q^{2n}}. \tag{3.1}$$

Proof. From the proof of [27, Theorem 2.2], we find that

$$\begin{aligned} S2(z,q) &= \frac{(-q;q^2)_\infty}{(1-z)(1-z^{-1})(q^2;q^2)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(2n+1)}(1+q^{2n})}{(1-zq^{2n})(1-z^{-1}q^{2n})} \right) \\ &\quad - \frac{(-q,q^2;q^2)_\infty}{(1-z)(1-z^{-1})(zq^2,q^2/z;q^2)_\infty}. \end{aligned} \tag{3.2}$$

In section 2 of [29], Jennings-Shaffer showed that

$$\frac{(-q,q^2;q^2)_\infty}{(zq^2,q^2/z;q^2)_\infty}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^nq^{n(n+1)}(1+q^{2n})}{(1-zq^{2n})(1-z^{-1}q^{2n})}\right),$$

which together with (3.2) gives

$$S2(z,q)=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^nq^{n(n+1)}(1+q^{2n})(q^{n^2}-1)}{(1-zq^{2n})(1-z^{-1}q^{2n})}. \tag{3.3}$$

It is easy to verify that

$$\frac{1}{(1-zq^{2n})(1-z^{-1}q^{2n})}=\frac{1}{1-q^{4n}}+\sum_{m=1}^{\infty}\frac{(z^m+z^{-m})q^{2nm}}{1-q^{4n}}.$$

Substituting the above equation into (3.3), we obtain

$$\begin{aligned}S2(z,q)=&\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(n+1)}(1-q^{n^2})}{1-q^{2n}}+\\&+\sum_{m=1}^{\infty}(z^m+z^{-m})\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(n+1+2m)}(1-q^{n^2})}{1-q^{2n}},\end{aligned}$$

which implies (3.1) □

From the content of Section **5** of [7] (see also [31, Lemma 4.2]), we can deduce the following lemma on the Euler polynomials.

Lemma 3.2. *Let $A_1(t)=1$. Then, for $k\geq 1$, we have*

$$\left(z\frac{\partial}{\partial z}\right)^{k-1}\frac{z}{(1-zq^{2n})^2}=\frac{zA_k(zq^{2n})}{(1-zq^{2n})^{k+1}}$$

where the Euler polynomial

$$A_k(t):=A_{k,0}+A_{k,1}t+\cdots+A_{k,k}t^{k-1}$$

is a polynomial of degree $k-1$ whose coefficients $A_{k,m}$ satisfy the recursive relation

$$\begin{aligned}&A_{k,m}=(m+1)A_{k-1,m}+(k-m)A_{k-1,m-1}\qquad (1\leq m\leq k-1);\\&A_{k,0}=1\qquad (k\geq 1);\qquad A_{k,m}=0\qquad (m\geq k).\end{aligned}$$

Remark: Using the recursion above, one can easily find that the coefficients of $A_k(t)$ are all positive. See [23] for more details on Euler polynomials.

Armed with Lemma 3.2, we find the following generating function for the positive rank and crank moments.

Lemma 3.3. *For all positive integers k, we have*

$$C2_k(q)=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(n+1)}A_k(q^{2n})}{(1-q^{2n})^k}, \tag{3.4}$$

and

$$R2_k(q)=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(2n+1)}A_k(q^{2n})}{(1-q^{2n})^k}. \tag{3.5}$$

Proof. We prove (3.4) first. Jennings-Shaffer [29] proved that

$$\begin{aligned} C2(z,q) :&= \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M2(m,n) z^m q^n \\ &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left[1+\sum_{n=1}^{\infty}(-1)^n q^{n(n+1)}\left(\frac{1-z}{1-zq^{2n}}+\frac{1-z^{-1}}{1-z^{-1}q^{2n}}\right)\right]. \end{aligned} \tag{3.6}$$

Extracting the coefficients in z^l for $l>0$ in (3.6) and recalling the definition of $C2_k(q)$ in (1.10), we find that

$$C2_k(q) = \lim_{z\to 1}\left(z\frac{\partial}{\partial z}\right)^k \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(1-z)(-1)^n q^{n(n+1)}}{1-zq^{2n}}. \tag{3.7}$$

Applying Lemma 3.2, we find that

$$\left(z\frac{\partial}{\partial z}\right)^k \frac{1-z}{1-zq^{2n}} = \frac{z(q^{2n}-1)A_k(zq^{2n})}{(1-zq^{2n})^{k+1}}.$$

Substituting the above equation into (3.7), we obtain

$$C2_k(q) = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(n+1)}A_k(q^{2n})}{(1-q^{2n})^k}.$$

This completes the proof of (3.4).

Using the following analog of (3.6) (see [29, p. 296]):

$$\begin{aligned} R2(z,q) :&= \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N2(m,n) z^m q^n \\ &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left[1+\sum_{n=1}^{\infty}(-1)^n q^{n(2n+1)}\left(\frac{1-z}{1-zq^{2n}}+\frac{1-z^{-1}}{1-z^{-1}q^{2n}}\right)\right], \end{aligned}$$

we can proceed as above to prove (3.5) with Lemma 3.2. □

Now we are in a position to prove Theorem 1.3.

Proof of Theorem 1.3. To prove $M2_k^+(n) \geq N2_k^+(n)$, it suffices to show that $C2_k(q) - R2_k(q)$ has non-negative coefficients. By Lemma 3.3, we find that

$$C2_k(q) - R2_k(q) = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(n+1)}(1-q^{n^2})A_k(q^{2n})}{(1-q^{2n})^k} \tag{3.8}$$

For $k \geq 1$, we write

$$\frac{A_k(q^{2n})}{(1-q^{2n})^{k-1}} =: \sum_{m=0}^{\infty} b_k(m) q^{2nm},$$

where the coefficients $b_k(m)$ are all non-negative, in particular, we have $b_k(0) = 1$ since the constant term of the Euler polynomial is 1. Substituting the above equation into (3.8), we obtain

$$\begin{aligned}C2_k(q) - R2_k(q) &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(n+1)}(1-q^{n^2})}{1-q^{2n}}\sum_{m=0}^{\infty}b_k(m)q^{2nm}\\ &= \sum_{m=0}^{\infty}b_k(m)\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(n+1)+2mn}(1-q^{n^2})}{1-q^{2n}}\\ &= \sum_{m=0}^{\infty}b_k(m)\sum_{n=1}^{\infty}N_{S2}(m,n)q^n, \qquad (3.9)\end{aligned}$$

where the last equality follows from Lemma 3.1. Thus, Theorem 1.2 implies that $C2_k(q) - R2_k(q)$ has non-negative coefficients. This proves the first statement of Theorem 1.3.

By (3.9), we have

$$C2_k(q) - R2_k(q) = \sum_{n=1}^{\infty}N_{S2}(0,n)q^n + \sum_{m=1}^{\infty}b_k(m)\sum_{n=1}^{\infty}N_{S2}(m,n)q^n. \qquad (3.10)$$

Theorem 1.2 implies that $N_{S2}(0,n) > 0$ when $n \geq 4$. Thus, equation (3.10) together with the non-negativity of $N_{S2}(m,n)$ and $b_k(m)$ gives that the power series $C2_k(q)-R2_k(q)$ has positive coefficients for q^n when $k \geq 1, n \geq 4$. This means that, for $k \geq 1, n \geq 4$, we have $M2_k^+(n) > N2_k^+(n)$. Hence the proof of Theorem 1.3 is complete. □

4. Concluding Remark

Andrews, Dyson and Rhoades [9] conjectured that

$$N_S(m,n) \geq N_S(m+1,n), \qquad (4.1)$$

for all $m \geq 0$ and $n \geq 0$. Chen, Ji and Zang [20] found a combinatorial proof of (4.1). Computer evidence suggests that an analog of (4.1) might hold. Namely, we made the following conjecture:

Conjecture 4.1. *We have*

$$N_{S2}(m,n) \geq N_{S2}(m+1,n),$$

for all $m \geq 0$ and $n \geq 0$.

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