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p-Adic gamma function and the trace of Frobenius of elliptic curves

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A B S T R A C T

In [12], McCarthy defined a function ${}_nG_n[\cdots]$ using the Teichmüller character of finite fields and quotients of the p-adic gamma function, and expressed the trace of Frobenius of elliptic curves in terms of special values of ${}_2G_2[\cdots]$. We establish two different expressions for the traces of Frobenius of elliptic curves in terms of the function ${}_2G_2[\cdots]$. As a result, we obtain two transformation formulas of the function ${}_2G_2[\cdots]$ with different parameters.

1. Introduction and statement of results

Let $q = p^r$ be a power of an odd prime, and let $\mathbb{F}_q$ be the finite field of q elements. Let $\mathbb{Z}_p$ denote the ring of p-adic integers. Let $\Gamma_p(.)$ denote the Morita's p-adic gamma function, and let ω denote the Teichmüller character of $\mathbb{F}_q$. We denote by $\overline{\omega}$ the inverse of ω. For $x \in \mathbb{Q}$ we let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x and $\langle x \rangle$ denote

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the fractional part of x, i.e. $x - \lfloor x \rfloor$. Also, we denote by $\mathbb{Z}^+$ and $\mathbb{Z}_{\geqslant 0}$ the set of positive integers and non negative integers, respectively. In [12], McCarthy defined a function ${}_nG_n[\cdots]$ as given below which can best be described as an analogue of hypergeometric series in the p-adic setting.

Definition 1.1. (See [12, Definition 5.1].) Let $q = p^r$, for p an odd prime and $r \in \mathbb{Z}^+$, and let $t \in \mathbb{F}_q$. For $n \in \mathbb{Z}^+$ and $1 \leqslant i \leqslant n$, let $a_i, b_i \in \mathbb{Q} \cap \mathbb{Z}_p$. Then the function ${}_nG_n[\cdots]$ is defined by

$$\begin{aligned} &{}_nG_n\left[\begin{matrix} a_1, & a_2, & \dots, & a_n \\ b_1, & b_2, & \dots, & b_n \end{matrix} \;\middle|\; t\right]_q \\ &:= \frac{-1}{q-1}\sum_{j=0}^{q-2}(-1)^{jn}\overline{\omega}^j(t)\prod_{i=1}^{n}\prod_{k=0}^{r-1}(-p)^{-\lfloor\langle a_ip^k\rangle-\frac{jp^k}{q-1}\rfloor-\lfloor\langle -b_ip^k\rangle+\frac{jp^k}{q-1}\rfloor} \\ &\quad\times \frac{\Gamma_p(\langle(a_i-\frac{j}{q-1})p^k\rangle)}{\Gamma_p(\langle a_ip^k\rangle)}\frac{\Gamma_p(\langle(-b_i+\frac{j}{q-1})p^k\rangle)}{\Gamma_p(\langle -b_ip^k\rangle)}. \end{aligned}$$

This function has many interesting properties. For further details, see [12]. For an earlier version of this G-function, see [11]. In [5], Greene introduced the notion of hypergeometric functions over finite fields. Since then, many interesting connections between hypergeometric functions over finite field and algebraic curves have been found. But these results are restricted to primes satisfying certain congruence conditions. For example, see [1,2,4,9,10]. Let $E/\mathbb{F}_q$ be an elliptic curve given in the Weierstrass form. Then the trace of Frobenius $a_q(E)$ of E is given by

$$a_q(E) := q + 1 - \#E(\mathbb{F}_q), \tag{1}$$

where $\#E(\mathbb{F}_q)$ denotes the number of $\mathbb{F}_q$-points on E including the point at infinity. Let $j(E)$ denote the j-invariant of the elliptic curve E. Let ϕ be the quadratic character of $\mathbb{F}_q^\times$ extended to all of $\mathbb{F}_q$ by setting $\phi(0) := 0$. Using the function ${}_2G_2[\cdots]$, McCarthy expressed the trace of Frobenius of elliptic curves defined over $\mathbb{F}_p$ without any congruence condition on the prime. The statement of his result is given below.

Theorem 1.2. *(See [12, Theorem 1.2].) Let $p > 3$ be a prime. Consider an elliptic curve $E_s/\mathbb{F}_p$ of the form E_s: $y^2 = x^3 + ax + b$ with $j(E_s) \neq 0, 1728$. Then*

$$a_p(E_s) = \phi(b) \cdot p \cdot {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{3}, & \frac{2}{3} \end{matrix} \;\middle|\; -\frac{27b^2}{4a^3}\right]_p. \tag{2}$$

In [2], the first author and Kalita gave two formulas for the trace of Frobenius of the elliptic curve $E_{a,b}$: $y^2 = x^3 + ax + b$ defined over $\mathbb{F}_q$ under the conditions $q \equiv 1 \pmod 6$ and $q \equiv 1 \pmod 4$, respectively. In this paper, we prove the following two expressions for the trace of Frobenius of the elliptic curve $E_{a,b}/\mathbb{F}_q$ in terms of special values of the function ${}_2G_2[\cdots]$ without any congruence conditions on q.

Theorem 1.3. *Let $q=p^r$, $p>3$ be a prime. Consider an elliptic curve $E_{a,b}/\mathbb{F}_q$ of the form $E_{a,b}$: $y^2=x^3+ax+b$ with $j(E_{a,b})\neq 0$. If $(-a/3)$ is a quadratic residue in $\mathbb{F}_q$, then*

$$a_q(E_{a,b})=\phi\big(k^3+ak+b\big)\cdot q\cdot {}_2G_2\left[\begin{matrix}\frac{1}{2},\ \frac{1}{2}\\ \frac{1}{3},\ \frac{2}{3}\end{matrix}\ \Big|\ -\frac{k^3+ak+b}{4k^3}\right]_q,$$

where $3k^2+a=0$.

Theorem 1.4. *Let $q=p^r$, $p>3$ be a prime. Consider an elliptic curve $E_{a,b}/\mathbb{F}_q$ of the form $E_{a,b}$: $y^2=x^3+ax+b$ with $j(E_{a,b})\neq 1728$. If $x^3+ax+b=0$ has a non zero solution in $\mathbb{F}_q$, then*

$$a_q(E_{a,b})=\phi\big(-3h^2-a\big)\cdot q\cdot {}_2G_2\left[\begin{matrix}\frac{1}{2},\ \frac{1}{2}\\ \frac{1}{4},\ \frac{3}{4}\end{matrix}\ \Big|\ \frac{4(3h^2+a)}{9h^2}\right]_q,$$

where $h^3+ah+b=0$.

McCarthy proved Theorem 1.2 over $\mathbb{F}_p$. Along the proof of Theorem 1.3 and Theorem 1.4 (which are proved for $\mathbb{F}_q$), we have verified that Theorem 1.2 is also true for $\mathbb{F}_q$. Hence, we have the following corollary which gives nice transformation formulas between special values of the function ${}_2G_2[\cdots]$ with different parameters. Apart from the transformations which can be implied from the analogous hypergeometric functions over finite fields, no proper transformations for ${}_nG_n[\cdots]$ have been shown to date.

Corollary 1.5. *Let $q=p^r$, $p>3$ be a prime. Let $a,b\in\mathbb{F}_q^\times$ and $-\frac{27b^2}{4a^3}\neq 1$. Then*

$$\begin{aligned}
&{}_2G_2\left[\begin{matrix}\frac{1}{4},\ \frac{3}{4}\\ \frac{1}{3},\ \frac{2}{3}\end{matrix}\ \Big|\ -\frac{27b^2}{4a^3}\right]_q\\
&=\begin{cases}\phi(b(k^3+ak+b))\cdot {}_2G_2\left[\begin{matrix}\frac{1}{2},\ \frac{1}{2}\\ \frac{1}{3},\ \frac{2}{3}\end{matrix}\ \Big|\ -\frac{k^3+ak+b}{4k^3}\right]_q & \text{if } a=-3k^2;\\ \phi\big(-b\big(3h^2+a\big)\big)\cdot {}_2G_2\left[\begin{matrix}\frac{1}{2},\ \frac{1}{2}\\ \frac{1}{4},\ \frac{3}{4}\end{matrix}\ \Big|\ \frac{4(3h^2+a)}{9h^2}\right]_q & \text{if } h^3+ah+b=0.\end{cases}
\end{aligned}$$

2. Preliminaries

Let $\widehat{\mathbb{F}_q^\times}$ denote the group of multiplicative characters of $\mathbb{F}_q^\times$. We extend the domain of each $\chi \in \mathbb{F}_q^\times$ to $\mathbb{F}_q$ by setting $\chi(0) := 0$ including the trivial character ε. The *orthogonality relations* for multiplicative characters are listed in the following lemma.

Lemma 2.1. *(See [7, Chapter 8].) We have*

$$\sum_{x\in\mathbb{F}_q} \chi(x) = \begin{cases} q-1 & \text{if } \chi = \varepsilon; \\ 0 & \text{if } \chi \neq \varepsilon. \end{cases} \tag{1}$$

$$\sum_{\chi\in\widehat{\mathbb{F}_q^\times}} \chi(x) = \begin{cases} q-1 & \text{if } x = 1; \\ 0 & \text{if } x \neq 1. \end{cases} \tag{2}$$

Let $\mathbb{Z}_p$ denote the ring of p-adic integers, $\mathbb{Q}_p$ the field of p-adic numbers, $\overline{\mathbb{Q}_p}$ the algebraic closure of $\mathbb{Q}_p$, and $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}_p}$. Let $\mathbb{Z}_q$ be the ring of integers in the unique unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. Recall that $\mathbb{Z}_q^\times$ contains all $(q-1)$-th root of unity. Therefore, we can consider multiplicative characters of $\mathbb{F}_q^\times$ to be maps $\chi : \mathbb{F}_q^\times \to \mathbb{Z}_q^\times$.

We now introduce some properties of Gauss sums. For further details, see [3]. Let ζ_p be a fixed primitive root of unity in $\overline{\mathbb{Q}_p}$. Then the additive character $\theta : \mathbb{F}_q \to \mathbb{Q}_p(\zeta_p)$ is defined by

$$\theta(\alpha) = \zeta_p^{\operatorname{tr}(\alpha)},$$

where $\operatorname{tr} : \mathbb{F}_q \to \mathbb{F}_p$ is the trace map given by

$$\operatorname{tr}(\alpha) = \alpha + \alpha^p + \alpha^{p^2} + \cdots + \alpha^{p^{r-1}}.$$

For $\chi \in \widehat{\mathbb{F}_q^\times}$, the *Gauss sum* is defined by

$$G(\chi) := \sum_{x\in\mathbb{F}_q} \chi(x)\theta(x).$$

We let T denote a fixed generator of $\widehat{\mathbb{F}_q^\times}$. The Gauss sum $G(T^m)$ is denoted by G_m. The following lemma provides a formula for the multiplicative inverse of a Gauss sum.

Lemma 2.2. *(See [5, Eq. 1.12].) If $k \in \mathbb{Z}$ and $T^k \neq \varepsilon$, then*

$$G_k G_{-k} = qT^k(-1).$$

Using orthogonality, we can write θ in terms of Gauss sums as given in the following lemma.

Lemma 2.3. *(See [4, Lemma 2.2].) For all $\alpha \in \mathbb{F}_q^\times$,*

$$\theta(\alpha) = \frac{1}{q-1} \sum_{m=0}^{q-2} G_{-m} T^m(\alpha).$$

Theorem 2.4 *(Davenport–Hasse relation). (See [8].) Let m be a positive integer and let $q = p^r$ be a prime power such that $q \equiv 1 \pmod{m}$. For multiplicative characters $\chi, \psi \in \widehat{\mathbb{F}_q^\times}$, we have*

$$\prod_{\chi^m=1} G(\chi\psi) = -G\big(\psi^m\big)\psi\big(m^{-m}\big) \prod_{\chi^m=1} G(\chi). \tag{3}$$

We now recall the definition of p-adic gamma function. For $n \in \mathbb{Z}^+$, the p-adic gamma function $\Gamma_p(n)$ is defined as

$$\Gamma_p(n) := (-1)^n \prod_{0<j<n,\ p\nmid j} j$$

and one extends it to all $x \in \mathbb{Z}_p$ by setting $\Gamma_p(0) := 1$ and

$$\Gamma_p(x) := \lim_{n \to x} \Gamma_p(n)$$

for $x \neq 0$, where n runs through any sequence of positive integers p-adically approaching x. This limit exists, is independent of how n approaches x, and determines a continuous function on $\mathbb{Z}_p$ with values in $\mathbb{Z}_p^\times$.

We now state a product formula for the p-adic gamma function from [6, Theorem 3.1]. Let $\omega : \mathbb{F}_q^\times \to \mathbb{Z}_q^\times$ be the Teichmüller character. For $a \in \mathbb{F}_q^\times$, the value $\omega(a)$ is just the $(q-1)$-th root of unity in $\mathbb{Z}_q$ such that $\omega(a) \equiv a \pmod{p}$. We denote by $\overline{\omega}$ the inverse of ω. If $m \in \mathbb{Z}^+$, $p \nmid m$ and $x \in \mathbb{Q}$ satisfies $0 \leqslant x \leqslant 1$ and $(q-1)x \in \mathbb{Z}$, then

$$\prod_{i=0}^{r-1} \prod_{h=0}^{m-1} \Gamma_p\left(\left\langle \left(\frac{x+h}{m}\right) p^i \right\rangle\right) = \omega\big(m^{(1-x)(1-q)}\big) \prod_{i=0}^{r-1} \Gamma_p\big(\langle xp^i \rangle\big) \prod_{h=1}^{m-1} \Gamma_p\left(\left\langle \frac{hp^i}{m} \right\rangle\right). \tag{4}$$

We note that the argument of ω, namely, $m^{(1-x)(1-q)} \in \mathbb{F}_p^\times$ and $\omega|_{\mathbb{F}_p^\times}$ is the Teichmüller character on $\mathbb{F}_p^\times$ with values in $\mathbb{Z}_p^\times$. Also,

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{x_0}, \tag{5}$$

where $x_0 \in \{1, 2, \ldots, p\}$ satisfies $x_0 \equiv x \pmod{p}$.

The Gross–Koblitz formula allows us to relate the Gauss sums and the p-adic gamma function. Let $\pi \in \mathbb{C}_p$ be the fixed root of $x^{p-1} + p = 0$ which satisfies $\pi \equiv \zeta_p - 1 \pmod{(\zeta_p - 1)^2}$. Then we have the following result.

Theorem 2.5. *(See Gross, Koblitz [6].) For $a \in \mathbb{Z}$ and $q = p^r$,*

$$G(\overline{\omega}^a) = -\pi^{(p-1)\sum_{i=0}^{r-1}\langle\frac{ap^i}{q-1}\rangle} \prod_{i=0}^{r-1} \Gamma_p\left(\left\langle \frac{ap^i}{q-1} \right\rangle\right).$$

3. Proof of the results

We first prove a lemma which we will use to prove the main results. This lemma is a generalization of Lemma 4.1 in [12] and the proof proceeds along similar lines.

Lemma 3.1. *Let p be a prime and $q = p^r$. For $0 \leqslant j \leqslant q-2$ and $t \in \mathbb{Z}^+$ with $p \nmid t$, we have*

$$\omega(t^{tj}) \prod_{i=0}^{r-1} \Gamma_p\left(\left\langle \frac{tp^i j}{q-1} \right\rangle\right) \prod_{h=1}^{t-1} \Gamma_p\left(\left\langle \frac{hp^i}{t} \right\rangle\right) = \prod_{i=0}^{r-1}\prod_{h=0}^{t-1} \Gamma_p\left(\left\langle \frac{p^i h}{t} + \frac{p^i j}{q-1} \right\rangle\right) \tag{6}$$

and

$$\omega(t^{-tj}) \prod_{i=0}^{r-1} \Gamma_p\left(\left\langle \frac{-tp^i j}{q-1} \right\rangle\right) \prod_{h=1}^{t-1} \Gamma_p\left(\left\langle \frac{hp^i}{t} \right\rangle\right)$$
$$= \prod_{i=0}^{r-1}\prod_{h=0}^{t-1} \Gamma_p\left(\left\langle \frac{p^i(1+h)}{t} - \frac{p^i j}{q-1} \right\rangle\right). \tag{7}$$

Proof. Fix $0 \leqslant j \leqslant q-2$, and let $k \in \mathbb{Z}_{\geqslant 0}$ be defined such that

$$k\left(\frac{q-1}{t}\right) \leqslant j < (k+1)\left(\frac{q-1}{t}\right). \tag{8}$$

Putting $m = t$ and $x = \frac{tj}{q-1} - k$ in (4), we obtain

$$\prod_{i=0}^{r-1}\prod_{h=0}^{t-1} \Gamma_p\left(\left\langle \left(\frac{j}{q-1} + \frac{h-k}{t}\right)p^i \right\rangle\right)$$
$$= \omega\left(t^{(1-\frac{tj}{q-1}+k)(1-q)}\right) \prod_{i=0}^{r-1} \Gamma_p\left(\left\langle \left(\frac{tj}{q-1} - k\right)p^i \right\rangle\right) \prod_{h=1}^{t-1} \Gamma_p\left(\left\langle \frac{hp^i}{t} \right\rangle\right). \tag{9}$$

We observe that $0 \leqslant k < t$. Therefore, we have

$$\prod_{i=0}^{r-1}\prod_{h=0}^{t-1} \Gamma_p\left(\left\langle \left(\frac{h-k}{t} + \frac{j}{q-1}\right)p^i \right\rangle\right)$$
$$= \prod_{i=0}^{r-1}\prod_{h=0}^{k-1} \Gamma_p\left(\left\langle \left(\frac{t+h-k}{t} + \frac{j}{q-1}\right)p^i \right\rangle\right) \prod_{h=k}^{t-1} \Gamma_p\left(\left\langle \left(\frac{h-k}{t} + \frac{j}{q-1}\right)p^i \right\rangle\right)$$

$$= \prod_{i=0}^{r-1} \prod_{h=t-k}^{t-1} \Gamma_p\left(\left\langle\left(\frac{h}{t}+\frac{j}{q-1}\right)p^i\right\rangle\right) \prod_{h=0}^{t-k-1} \Gamma_p\left(\left\langle\left(\frac{h}{t}+\frac{j}{q-1}\right)p^i\right\rangle\right)$$

$$= \prod_{i=0}^{r-1} \prod_{h=0}^{t-1} \Gamma_p\left(\left\langle\left(\frac{h}{t}+\frac{j}{q-1}\right)p^i\right\rangle\right). \tag{10}$$

Again by our choice of k, for any nonnegative integer i we have

$$\left\langle \frac{p^i tj}{q-1} \right\rangle = \left\langle \left(\frac{tj}{q-1} - k\right) p^i \right\rangle.$$

This gives us

$$\Gamma_p\left(\left\langle\left(\frac{p^i tj}{q-1}\right)\right\rangle\right) = \Gamma_p\left(\left\langle\left(\frac{tj}{q-1} - k\right)p^i\right\rangle\right). \tag{11}$$

Now substituting (10) and (11) into (9) we obtain (6).

We prove (7) following [12, Lemma 4.1] and using similar arguments as given in the proof of (6). □

Lemma 3.2. *For $1 \leqslant l \leqslant q-2$ and $0 \leqslant i \leqslant r-1$, we have*

$$\left\lfloor -\frac{lp^i}{q-1} \right\rfloor - \left\lfloor -\frac{2lp^i}{q-1} \right\rfloor - \left\lfloor -\frac{2lp^i}{q-1} \right\rfloor - \left\lfloor \frac{3lp^i}{q-1} \right\rfloor - 1$$
$$= -2\left\lfloor \left\langle \frac{p^i}{2} \right\rangle - \frac{lp^i}{q-1} \right\rfloor - \left\lfloor \left\langle -\frac{p^i}{3} \right\rangle + \frac{lp^i}{q-1} \right\rfloor - \left\lfloor \left\langle -\frac{2p^i}{3} \right\rangle + \frac{lp^i}{q-1} \right\rfloor. \tag{12}$$

Proof. Since $\lfloor \frac{6lp^i}{q-1} \rfloor$ can be written as $6u+v$, for some $u, v \in \mathbb{Z}$ such that $0 \leqslant v \leqslant 5$, (12) can be verified by considering the cases $v = 0, 1, \ldots, 5$. For the case $v = 0$ we have $\lfloor \frac{6lp^i}{q-1} \rfloor = 6u$, and then it is easy to check that both the sides of (12) are equal to zero. Similarly, for other values of v one can verify the result. □

Lemma 3.3. *For $0 \leqslant l \leqslant q-2$ and $0 \leqslant i \leqslant r-1$, we have*

$$\left\lfloor \frac{2lp^i}{q-1} \right\rfloor + 2\left\lfloor \frac{-lp^i}{q-1} \right\rfloor - 2\left\lfloor \frac{-2lp^i}{q-1} \right\rfloor - \left\lfloor \frac{4lp^i}{q-1} \right\rfloor$$
$$= -2\left\lfloor \left\langle \frac{p^i}{2} \right\rangle - \frac{lp^i}{q-1} \right\rfloor - \left\lfloor \left\langle -\frac{p^i}{4} \right\rangle + \frac{lp^i}{q-1} \right\rfloor - \left\lfloor \left\langle -\frac{3p^i}{4} \right\rangle + \frac{lp^i}{q-1} \right\rfloor. \tag{13}$$

Proof. Since $\lfloor \frac{4lp^i}{q-1} \rfloor$ can be written as $4u+v$, for some $u, v \in \mathbb{Z}$ such that $0 \leqslant v \leqslant 3$, (13) can be verified by considering the cases $v = 0, 1, 2, 3$. For the case $v = 0$ we have $\lfloor \frac{4lp^i}{q-1} \rfloor = 4u$, and then it is easy to verify that both the sides of (13) are equal to zero. Similarly, for other values of v we can verify the result. □

Now, we are going to prove Theorem 1.3. The proof will follow as a consequence of the next theorem. We consider an elliptic curve E_1 over $\mathbb{F}_q$ in the form

$$E_1:\quad y^2 = x^3 + cx^2 + d,$$

where $c \neq 0$. We express the trace of Frobenius endomorphism on the curve E_1 as a special value of the function ${}_2G_2[\cdots]$ in the following way.

Theorem 3.4. *Let $q = p^r$, $p > 3$ be a prime. The trace of Frobenius on E_1 is given by*

$$a_q(E_1) = q \cdot \phi(d) \cdot {}_2G_2\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ \frac{1}{3}, & \frac{2}{3} \end{matrix} \,\Big|\, -\frac{27d}{4c^3}\right]_q.$$

Proof. We have $\#E_1(\mathbb{F}_q) - 1 = \#\{(x,y) \in \mathbb{F}_q \times \mathbb{F}_q\colon\ y^2 = x^3 + cx^2 + d\}$.

Let $P(x,y) = x^3 + cx^2 + d - y^2$. Using the identity

$$\sum_{z \in \mathbb{F}_q} \theta\bigl(zP(x,y)\bigr) = \begin{cases} q, & \text{if } P(x,y) = 0; \\ 0, & \text{if } P(x,y) \neq 0, \end{cases} \tag{14}$$

we obtain

$$\begin{aligned} q.\bigl(\#E_1(\mathbb{F}_q) - 1\bigr) &= \sum_{x,y,z \in \mathbb{F}_q} \theta\bigl(zP(x,y)\bigr) \\ &= q^2 + \sum_{z \in \mathbb{F}_q^\times} \theta(zd) + \sum_{y,z \in \mathbb{F}_q^\times} \theta(zd)\theta\bigl(-zy^2\bigr) + \sum_{x,z \in \mathbb{F}_q^\times} \theta(zd)\theta\bigl(zx^3\bigr)\theta\bigl(zcx^2\bigr) \\ &\quad + \sum_{x,y,z \in \mathbb{F}_q^\times} \theta(zd)\theta\bigl(zx^3\bigr)\theta\bigl(zcx^2\bigr)\theta\bigl(-zy^2\bigr) \\ &= q^2 + A + B + C + D. \end{aligned} \tag{15}$$

From the proof of [2, Theorem 3.1], we have $A = -1$, $B = 1 + qT^{\frac{q-1}{2}}(d)$ and $D = -C + D_{\frac{q-1}{2}}$, where

$$\begin{aligned} D_{\frac{q-1}{2}} &= \frac{1}{(q-1)^3} \sum_{l,m,n=0}^{q-2} G_{-l}G_{-m}G_{-n}G_{\frac{q-1}{2}} T^l(d)T^n(c)T^{\frac{q-1}{2}}(-1) \\ &\quad \times \sum_{x \in \mathbb{F}_q^\times} T^{3m+2n}(x) \sum_{z \in \mathbb{F}_q^\times} T^{l+m+n+\frac{q-1}{2}}(z), \end{aligned}$$

which is non zero only if $m = -\frac{2}{3}n$ and $n = -3l - \frac{3(q-1)}{2}$. Since $G_{3l+\frac{3(q-1)}{2}} = G_{3l+\frac{q-1}{2}}$ and $G_{-2l-(q-1)} = G_{-2l}$, we have

$$D_{\frac{q-1}{2}} = \frac{1}{q-1}\sum_{l=0}^{q-2} G_{-l}G_{-2l}G_{3l+\frac{q-1}{2}}G_{\frac{q-1}{2}}T^{l}(d)T^{-3l+\frac{q-1}{2}}(c)T^{\frac{q-1}{2}}(-1). \tag{16}$$

Replacing l by $l - \frac{q-1}{2}$ we obtain

$$\begin{aligned} D_{\frac{q-1}{2}} &= \frac{1}{q-1}\sum_{l=0}^{q-2} G_{-l+\frac{q-1}{2}}G_{-2l}G_{3l}G_{\frac{q-1}{2}}T^{l-\frac{q-1}{2}}(d)T^{-3l}(c)T^{\frac{q-1}{2}}(-1) \\ &= \frac{\phi(-d)}{q-1}\sum_{l=0}^{q-2} G_{-l+\frac{q-1}{2}}G_{-2l}G_{3l}G_{\frac{q-1}{2}}T^{l}(d)T^{-3l}(c). \end{aligned} \tag{17}$$

Using Davenport–Hasse relation (Theorem 2.4) for $m = 2$, $\psi = T^{-l}$, we deduce that

$$G_{-l+\frac{q-1}{2}} = \frac{G_{\frac{q-1}{2}}G_{-2l}T^{l}(4)}{G_{-l}}. \tag{18}$$

Substituting (18) into (17) and using Lemma 2.2 we find that

$$D_{\frac{q-1}{2}} = \frac{q\phi(d)}{q-1}\sum_{l=0}^{q-2} \frac{G_{-2l}G_{-2l}G_{3l}}{G_{-l}}T^{l}\left(\frac{4d}{c^3}\right).$$

Putting the values of A, B, C and D in (15) we obtain

$$q\cdot\left(\#E_1(\mathbb{F}_q) - 1\right) = q^2 + q\phi(d) + D_{\frac{q-1}{2}},$$

which yields

$$a_q(E_1) = -\phi(d) - \frac{\phi(d)}{q-1}\sum_{l=0}^{q-2} \frac{G_{-2l}G_{-2l}G_{3l}}{G_{-l}}T^{l}\left(\frac{4d}{c^3}\right). \tag{19}$$

We take T to be the inverse of the Teichmüller character, that is, $T = \overline{\omega}$ and use the Gross–Koblitz formula (Theorem 2.5) to convert the above expression to an expressing involving the p-adic gamma function. This gives

$$\begin{aligned} a_q(E_1) = -\phi(d) - \frac{\phi(d)}{q-1}\sum_{l=0}^{q-2}(-p)^{\sum_{i=0}^{r-1}\{\langle\frac{-2lp^i}{q-1}\rangle + \langle\frac{-2lp^i}{q-1}\rangle + \langle\frac{3lp^i}{q-1}\rangle - \langle\frac{-lp^i}{q-1}\rangle\}} \\ \times \prod_{i=0}^{r-1}\frac{\Gamma_p(\langle\frac{-2lp^i}{q-1}\rangle)\Gamma_p(\langle\frac{-2lp^i}{q-1}\rangle)\Gamma_p(\langle\frac{3lp^i}{q-1}\rangle)}{\Gamma_p(\langle\frac{-lp^i}{q-1}\rangle)}\overline{\omega}^{l}\left(\frac{4d}{c^3}\right). \end{aligned}$$

If we put $s=\sum_{i=0}^{r-1}\{\langle\frac{-2lp^i}{q-1}\rangle+\langle\frac{-2lp^i}{q-1}\rangle+\langle\frac{3lp^i}{q-1}\rangle-\langle\frac{-lp^i}{q-1}\rangle\}$, then the above equation becomes

$$a_q(E_1)=-\phi(d)-\frac{\phi(d)}{q-1}\sum_{l=0}^{q-2}(-p)^s\overline{\omega}^l\left(\frac{4d}{c^3}\right)\\ \times\prod_{i=0}^{r-1}\frac{\Gamma_p(\langle\frac{-2lp^i}{q-1}\rangle)\Gamma_p(\langle\frac{-2lp^i}{q-1}\rangle)\Gamma_p(\langle\frac{3lp^i}{q-1}\rangle)}{\Gamma_p(\langle\frac{-lp^i}{q-1}\rangle)}. \tag{20}$$

Next we use Lemma 3.1 and simplify (20) to obtain

$$a_q(E_1)=-\phi(d)-\frac{\phi(d)}{q-1}\sum_{l=0}^{q-2}(-p)^s\overline{\omega}^l\left(\frac{27d}{4c^3}\right)\\ \times\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\left(1-\frac{l}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(\frac{l}{q-1}\right)p^i\right\rangle\right)\\ \times\frac{\Gamma_p(\langle(\frac{1}{2}-\frac{l}{q-1})p^i\rangle)\Gamma_p(\langle(\frac{1}{2}-\frac{l}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{p^i}{2}\rangle)\Gamma_p(\langle\frac{p^i}{2}\rangle)}\\ \times\frac{\Gamma_p(\langle(\frac{1}{3}+\frac{l}{q-1})p^i\rangle)\Gamma_p(\langle(\frac{2}{3}+\frac{l}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{p^i}{3}\rangle)\Gamma_p(\langle\frac{2p^i}{3}\rangle)}. \tag{21}$$

Calculating s we deduce that

$$s=\sum_{i=0}^{r-1}\left\{\left\lfloor\frac{-lp^i}{q-1}\right\rfloor-\left\lfloor\frac{-2lp^i}{q-1}\right\rfloor-\left\lfloor\frac{-2lp^i}{q-1}\right\rfloor-\left\lfloor\frac{3lp^i}{q-1}\right\rfloor\right\}. \tag{22}$$

By (5), for $0<l\leqslant q-2$, we have

$$\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\left(1-\frac{l}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(\frac{l}{q-1}\right)p^i\right\rangle\right)=(-1)^r\overline{\omega}^l(-1). \tag{23}$$

Therefore,

$$a_q(E_1)=-\frac{q\phi(d)}{q-1}-\frac{q\phi(d)}{q-1}\sum_{l=1}^{q-2}(-p)^{s-r}\overline{\omega}^l\left(-\frac{27d}{4c^3}\right)\\ \times\prod_{i=0}^{r-1}\frac{\Gamma_p(\langle(\frac{1}{2}-\frac{l}{q-1})p^i\rangle)\Gamma_p(\langle(\frac{1}{2}-\frac{l}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{p^i}{2}\rangle)\Gamma_p(\langle\frac{p^i}{2}\rangle)}\\ \times\frac{\Gamma_p(\langle(\frac{1}{3}+\frac{l}{q-1})p^i\rangle)\Gamma_p(\langle(\frac{2}{3}+\frac{l}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{p^i}{3}\rangle)\Gamma_p(\langle\frac{2p^i}{3}\rangle)}.$$

Now using the following relation for $0\leqslant l\leqslant q-2$

$$\begin{aligned}
&\prod_{i=0}^{r-1} \Gamma_p\left(\left\langle\left(\frac{1}{3}+\frac{l}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(\frac{2}{3}+\frac{l}{q-1}\right)p^i\right\rangle\right) \\
&\quad = \prod_{i=0}^{r-1} \Gamma_p\left(\left\langle\left(-\frac{1}{3}+\frac{l}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(-\frac{2}{3}+\frac{l}{q-1}\right)p^i\right\rangle\right)
\end{aligned} \tag{24}$$

and Lemma 3.2, we deduce that

$$\begin{aligned}
a_q(E_1) &= -\frac{q\phi(d)}{q-1}\sum_{l=0}^{q-2}\prod_{i=0}^{r-1}(-p)^{-\lfloor\langle\frac{p^i}{2}\rangle-\frac{lp^i}{q-1}\rfloor-\lfloor\langle\frac{p^i}{2}\rangle-\frac{lp^i}{q-1}\rfloor-\lfloor\langle-\frac{p^i}{3}\rangle+\frac{lp^i}{q-1}\rfloor-\lfloor\langle-\frac{2p^i}{3}\rangle+\frac{lp^i}{q-1}\rfloor} \\
&\quad\times \frac{\Gamma_p(\langle(\frac{1}{2}-\frac{l}{q-1})p^i\rangle)\Gamma_p(\langle(\frac{1}{2}-\frac{l}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{p^i}{2}\rangle)\Gamma_p(\langle\frac{p^i}{2}\rangle)} \\
&\quad\times \frac{\Gamma_p(\langle(-\frac{1}{3}+\frac{l}{q-1})p^i\rangle)\Gamma_p(\langle(-\frac{2}{3}+\frac{l}{q-1})p^i\rangle)}{\Gamma_p(\langle-\frac{p^i}{3}\rangle)\Gamma_p(\langle-\frac{2p^i}{3}\rangle)}\overline{\omega}^l\left(-\frac{27d}{4c^3}\right) \\
&= q\cdot\phi(d)\cdot {}_2G_2\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2} \\ \frac{1}{3}, & \frac{2}{3}\end{matrix}\;\middle|\; -\frac{27d}{4c^3}\right]_q.
\end{aligned}$$

This completes the proof of the theorem. □

Proof of Theorem 1.3. We have $j(E_{a,b}) \neq 0$. Hence $a \neq 0$. Since $(-a/3)$ is a quadratic residue in $\mathbb{F}_q$, we find $k \in \mathbb{F}_q^\times$ such that $3k^2 + a = 0$. A change of variables $(x, y) \mapsto (x + k, h)$ takes the elliptic curve $E_{a,b}$: $y^2 = x^3 + ax + b$ to

$$E'\colon \quad y^2 = x^3 + 3kx^2 + \left(k^3 + ak + b\right). \tag{25}$$

Clearly $a_q(E_{a,b}) = a_q(E')$. Using Theorem 3.4 for the elliptic curve E', we complete the proof. □

Now, we are going to prove Theorem 1.4. The proof will follow as a consequence of the next theorem. We consider an elliptic curve E_2 over $\mathbb{F}_q$ in the form

$$E_2\colon \quad y^2 = x^3 + fx^2 + gx,$$

where $f \neq 0$. We express the trace of Frobenius endomorphism on the curve E_2 as a special value of the function ${}_2G_2[\cdots]$ in the following way.

Theorem 3.5. *Let $q = p^r$, $p > 3$ be a prime. The trace of Frobenius on E_2 is given by*

$$a_q(E_2) = q\cdot\phi(-g)\cdot {}_2G_2\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2} \\ \frac{1}{4}, & \frac{3}{4}\end{matrix}\;\middle|\; \frac{4g}{f^2}\right]_q.$$

Proof. We recall that $\#E_2(\mathbb{F}_q) - 1 = \#\{(x,y) \in \mathbb{F}_q \times \mathbb{F}_q\colon\ y^2 = x^3 + fx^2 + gx\}$.

Let $P(x,y) = x^3 + fx^2 + gx - y^2$. Using the identity

$$\sum_{z\in\mathbb{F}_q} \theta\bigl(zP(x,y)\bigr) = \begin{cases} q, & \text{if } P(x,y)=0;\\ 0, & \text{if } P(x,y)\neq 0,\end{cases} \tag{26}$$

we obtain

$$\begin{aligned} q\cdot\bigl(\#E_2(\mathbb{F}_q)-1\bigr) &= \sum_{x,y,z\in\mathbb{F}_q} \theta\bigl(zP(x,y)\bigr)\\ &= q^2 + \sum_{z\in\mathbb{F}_q^\times}\theta(0) + \sum_{y,z\in\mathbb{F}_q^\times}\theta\bigl(-zy^2\bigr) + \sum_{x,z\in\mathbb{F}_q^\times}\theta\bigl(zx^3\bigr)\theta\bigl(zfx^2\bigr)\theta(zgx)\\ &\quad+ \sum_{x,y,z\in\mathbb{F}_q^\times}\theta\bigl(zx^3\bigr)\theta\bigl(zfx^2\bigr)\theta(zgx)\theta\bigl(-zy^2\bigr)\\ &= q^2 + (q-1) + A + B + C. \end{aligned} \tag{27}$$

From [2, Theorem 3.2] we have $A = -(q-1)$ and $C = -B + C_{\frac{q-1}{2}}$, where

$$\begin{aligned} C_{\frac{q-1}{2}} &= \frac{G_{\frac{q-1}{2}}}{(q-1)^3}\sum_{l,m,n=0}^{q-2} G_{-l}G_{-m}G_{-n}T^m(f)T^n(g)T^{\frac{q-1}{2}}(-1)\\ &\quad\times\sum_{x\in\mathbb{F}_q^\times}T^{3l+2m+n}(x)\sum_{z\in\mathbb{F}_q^\times}T^{l+m+n+\frac{q-1}{2}}(z), \end{aligned}$$

which is non zero only if $n = l$ and $m = -2l + \frac{q-1}{2}$. This gives

$$C_{\frac{q-1}{2}} = \frac{G_{\frac{q-1}{2}}\phi(-1)}{q-1}\sum_{l=0}^{q-2} G_{-l}G_{2l+\frac{q-1}{2}}G_{-l}T^l\left(\frac{g}{f^2}\right).$$

Substituting the values of A, B and C in (27) we obtain

$$q\cdot\bigl(\#E_2(\mathbb{F}_q)-1\bigr) = q^2 + \frac{G_{\frac{q-1}{2}}\phi(-1)}{q-1}\sum_{l=0}^{q-2} G_{-l}G_{2l+\frac{q-1}{2}}G_{-l}T^l\left(\frac{g}{f^2}\right). \tag{28}$$

Replacing l by $l - \frac{q-1}{2}$ we deduce that

$$\begin{aligned} q\cdot\bigl(\#E_2(\mathbb{F}_q)-1\bigr) &= q^2 + \frac{G_{\frac{q-1}{2}}\phi(-1)}{q-1}\sum_{l=0}^{q-2} G_{-l+\frac{q-1}{2}}G_{2l+\frac{q-1}{2}}G_{-l+\frac{q-1}{2}}T^{l-\frac{q-1}{2}}\left(\frac{g}{f^2}\right)\\ &= q^2 + \frac{G_{\frac{q-1}{2}}\phi(-g)}{q-1}\sum_{l=0}^{q-2} G_{-l+\frac{q-1}{2}}G_{2l+\frac{q-1}{2}}G_{-l+\frac{q-1}{2}}T^{l}\left(\frac{g}{f^2}\right). \end{aligned} \tag{29}$$

Using Davenport–Hasse relation (Theorem 2.4) for $m = 2$, $\psi = T^{-l}$ and $\psi = T^{2l}$ successively, we have

$$G_{-l+\frac{q-1}{2}} = \frac{G_{\frac{q-1}{2}} G_{-2l} T^{l}(4)}{G_{-l}}$$

and

$$G_{2l+\frac{q-1}{2}} = \frac{G_{4l} G_{\frac{q-1}{2}} T^{-l}(16)}{G_{2l}}.$$

Putting these values in (29) and using Lemma 2.2, we obtain

$$q \cdot \big(\#E_2(\mathbb{F}_q) - 1\big) = q^2 + \frac{q^2\phi(-g)}{q-1} \sum_{l=0}^{q-2} \frac{G_{-2l}G_{-2l}G_{4l}}{G_{-l}G_{-l}G_{2l}} T^{l}\left(\frac{g}{f^2}\right).$$

We now put $T = \overline{\omega}$. Then (1) and Gross–Koblitz formula (Theorem 2.5) yield

$$\begin{aligned}
a_q(E_2) &= -\frac{q\phi(-g)}{q-1} \sum_{l=0}^{q-2} (-p)^{\sum_{i=0}^{r-1}\{2\langle -\frac{2lp^i}{q-1}\rangle + \langle \frac{4lp^i}{q-1}\rangle - 2\langle -\frac{lp^i}{q-1}\rangle - \langle \frac{2lp^i}{q-1}\rangle\}} \\
&\quad \times \prod_{i=0}^{r-1} \frac{\Gamma_p(\langle \frac{-2lp^i}{q-1}\rangle)\Gamma_p(\langle \frac{-2lp^i}{q-1}\rangle)\Gamma_p(\langle \frac{4lp^i}{q-1}\rangle)}{\Gamma_p(\langle \frac{-lp^i}{q-1}\rangle)\Gamma_p(\langle \frac{-lp^i}{q-1}\rangle)\Gamma_p(\langle \frac{2lp^i}{q-1}\rangle)} \overline{\omega}^{l}\left(\frac{g}{f^2}\right) \\
&= -\frac{q\phi(-g)}{q-1} \sum_{l=0}^{q-2} (-p)^{s} \overline{\omega}^{l}\left(\frac{g}{f^2}\right) \\
&\quad \times \prod_{i=0}^{r-1} \frac{\Gamma_p(\langle \frac{-2lp^i}{q-1}\rangle)\Gamma_p(\langle \frac{-2lp^i}{q-1}\rangle)\Gamma_p(\langle \frac{4lp^i}{q-1}\rangle)}{\Gamma_p(\langle \frac{-lp^i}{q-1}\rangle)\Gamma_p(\langle \frac{-lp^i}{q-1}\rangle)\Gamma_p(\langle \frac{2lp^i}{q-1}\rangle)},
\end{aligned} \tag{30}$$

where $s = \sum_{i=0}^{r-1}\{2\langle \frac{-2lp^i}{q-1}\rangle + \langle \frac{4lp^i}{q-1}\rangle - 2\langle \frac{-lp^i}{q-1}\rangle - \langle \frac{2lp^i}{q-1}\rangle\}$.

Next we use Lemma 3.1 and after simplification we obtain

$$\begin{aligned}
a_q(E_2) &= -\frac{q\phi(-g)}{q-1} \sum_{l=0}^{q-2} (-p)^{s} \overline{\omega}^{l}\left(\frac{4g}{f^2}\right) \\
&\quad \times \prod_{i=0}^{r-1} \frac{\Gamma_p(\langle (\frac{1}{2} - \frac{l}{q-1})p^i\rangle)\Gamma_p(\langle (\frac{1}{2} - \frac{l}{q-1})p^i\rangle)}{\Gamma_p(\langle \frac{p^i}{2}\rangle)\Gamma_p(\langle \frac{p^i}{2}\rangle)} \\
&\quad \frac{\Gamma_p(\langle (\frac{1}{4} + \frac{l}{q-1})p^i\rangle)\Gamma_p(\langle (\frac{3}{4} + \frac{l}{q-1})p^i\rangle)}{\Gamma_p(\langle \frac{p^i}{4}\rangle)\Gamma_p(\langle \frac{3p^i}{4}\rangle)}.
\end{aligned} \tag{31}$$

We simplify the expression for s and find that

$$s=\sum_{i=0}^{r-1}\left\{\left\lfloor\frac{2lp^i}{q-1}\right\rfloor+2\left\lfloor\frac{-lp^i}{q-1}\right\rfloor-2\left\lfloor\frac{-2lp^i}{q-1}\right\rfloor-\left\lfloor\frac{4lp^i}{q-1}\right\rfloor\right\}. \tag{32}$$

The following relation for $0\leqslant l\leqslant q-2$

$$\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\left(\frac{1}{4}+\frac{l}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(\frac{3}{4}+\frac{l}{q-1}\right)p^i\right\rangle\right)$$
$$=\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\left(-\frac{1}{4}+\frac{l}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(-\frac{3}{4}+\frac{l}{q-1}\right)p^i\right\rangle\right)$$

and Lemma 3.3 yield

$$\begin{aligned}a_q(E_2)&=-\frac{q\phi(-g)}{q-1}\sum_{l=0}^{q-2}\prod_{i=0}^{r-1}(-p)^{-\lfloor\langle\frac{p^i}{2}\rangle-\frac{lp^i}{q-1}\rfloor-\lfloor\langle\frac{p^i}{2}\rangle-\frac{lp^i}{q-1}\rfloor-\lfloor\langle-\frac{p^i}{4}\rangle+\frac{lp^i}{q-1}\rfloor-\lfloor\langle-\frac{3p^i}{4}\rangle+\frac{lp^i}{q-1}\rfloor}\\&\quad\times\frac{\Gamma_p(\langle(\frac{1}{2}-\frac{l}{q-1})p^i\rangle)\Gamma_p(\langle(\frac{1}{2}-\frac{l}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{p^i}{2}\rangle)\Gamma_p(\langle\frac{p^i}{2}\rangle)}\\&\quad\times\frac{\Gamma_p(\langle(-\frac{1}{4}+\frac{l}{q-1})p^i\rangle)\Gamma_p(\langle(-\frac{3}{4}+\frac{l}{q-1})p^i\rangle)}{\Gamma_p(\langle-\frac{p^i}{4}\rangle)\Gamma_p(\langle-\frac{3p^i}{4}\rangle)}\overline{\omega}^l\left(\frac{4g}{f^2}\right)\\&=q\cdot\phi(-g)\cdot{}_2G_2\left[\begin{matrix}\frac{1}{2},&\frac{1}{2}\\\frac{1}{4},&\frac{3}{4}\end{matrix}\;\middle|\;\frac{4g}{f^2}\right]_q.\end{aligned}$$

This completes the proof of the theorem. □

Proof of Theorem 1.4. Here $j(E_{a,b})\neq 1728$ and hence $b\neq 0$. Let $h\in\mathbb{F}_q^\times$ be such that $h^3+ah+b=0$. A change of variables $(x,y)\mapsto(x+h,y)$ takes the elliptic curve $E_{a,b}$: $y^2=x^3+ax+b$ to

$$E'':\quad y^2=x^3+3hx^2+\left(3h^2+a\right)x. \tag{33}$$

Clearly $a_q(E_{a,b})=a_q(E'')$ and $3h\neq 0$. Using Theorem 3.5 for the elliptic curve E'', we complete the proof. □

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