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# Quantitative versions of the joint distributions of Hecke eigenvalues

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ABSTRACT

In 2009, Omar and Mazhouda proved that as $k \to \infty$, $\{\lambda_f(p^2) : f \in H_k\}$ and $\{\lambda_f(p^3) : f \in H_k\}$ are equidistributed with respect to some measures respectively, where $H_k$ is the set of all the normalized primitive holomorphic cusp forms of weight $k$ for $SL_2(\mathbb{Z})$. In this paper, we obtain a quantitative version of Omar and Mazhouda's result. Moreover, we find out that $\{\lambda_f(p^4) : f \in H_k\}$ and $\{\lambda_f(p^r) - \lambda_f(p^{r-2}) : f \in H_k$ and $r \geq 2\}$ follow some nice distribution laws respectively as $k \to \infty$ and get quantitative versions of these distributions. In the context of Maass cusp forms, we establish analogous results.



## 1. Introduction

Let $H_k$ be the set of normalized primitive holomorphic cusp forms of even integral weight $k$ for the modular group $\Gamma = SL_2(\mathbb{Z})$. Given any $f \in H_k$ and prime $p$, the

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distribution of the normalized Hecke eigenvalues $\lambda_f(p)$ is an interesting and difficult problem. The generalized Ramanujan conjecture for primitive holomorphic cusp forms implies that $|\lambda_f(p)| \leq 2$ for any $f \in H_k$ and prime $p$. This conjecture was proved by Deligne [4] in 1974.

Inspired by the Sato–Tate conjecture, Serre studied the asymptotic distribution of the Hecke eigenvalues $\lambda_f(p)$, as $f$ is fixed and the primes $p$ vary. In the 1960's, Serre conjectured that for any $f \in H_k$, as $x \to \infty$, $\lambda_f(p)$, $p \leq x$, are equidistributed in $[-2, 2]$ with respect to the Sato–Tate measure

$$d\mu_\infty(x) = \begin{cases} \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}\,dx & \text{if } x \in [-2,2], \\ 0 & \text{otherwise,} \end{cases}$$

which is also called the Sato–Tate conjecture. This conjecture was proved by Barnet-Lamb, Geraghty, Harris and Taylor [1] in 2011.

For a fixed prime $p$, as $k \to \infty$, the values of $\lambda_f(p)$, $f \in H_k$, also follow some nice distribution laws. Conrey, Duke and Farmer [3] and Serre [11] figured out that they are equidistributed with respect to the $p$-adic measure

$$d\mu_p(x) = \begin{cases} \frac{p+1}{2\pi}\frac{\sqrt{4-x^2}}{(p^{1/2}+p^{-1/2})^2-x^2}\,dx & \text{if } x \in [-2,2], \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Moreover, Murty and Sinha [8] proved that the rate of convergence to the above distribution is $O\left(\frac{\log p}{\log k}\right)$. Lau and Wang [7] later generalized Murty and Sinha's result to a joint distribution.

On the other hand, it is well-known that for any $m, n \geq 1$

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right)$$

and so

$$\lambda_f(p^n) = X_n\left(\frac{\lambda_f(p)}{2}\right),$$

where $X_n$ is the $n$th Chebychev polynomial of the second kind. One may naturally consider the distribution of $\{\lambda_f(p^n) : f \in H_k\}$ for some fixed prime $p$ and as $k$ goes to infinity. In this direction, Omar and Mazhouda [9] proved that $\{\lambda_f(p^2) : f \in H_k\}$ and $\{\lambda_f(p^3) : f \in H_k\}$ are equidistributed with respect to the measures

$$d\mu_{p,2}(x) = \begin{cases} m_{p,2}(x)\,dx & \text{if } x \in [-1,3], \\ 0 & \text{otherwise,} \end{cases}$$

and

$$d\mu_{p,3}(x) = \begin{cases} m_{p,3}(x)\,dx & \text{if } x \in [-4,4], \\ 0 & \text{otherwise,} \end{cases}$$

respectively. Here

$$m_{p,2}(x) = \frac{p+1}{2\pi}\frac{1}{(p^{1/2}+p^{-1/2})^2-(x+1)}\sqrt{\frac{3-x}{1+x}}$$

and $m_{p,3}$ is given by [9, formula (3.1)] with

$$\phi_1(x) = \frac{2\sqrt{6}}{3}\cos\frac{2\pi + \arccos\frac{3\sqrt{6}}{8}x}{3},\ \phi_2(x) = \frac{2\sqrt{6}}{3}\cos\frac{4\pi + \arccos\frac{3\sqrt{6}}{8}x}{3},$$
$$\phi_3(x) = \frac{2\sqrt{6}}{3}\cos\frac{\arccos\frac{3\sqrt{6}}{8}x}{3}$$

for $x \in \left[-\frac{4\sqrt{6}}{9}, \frac{4\sqrt{6}}{9}\right]$. In fact, when $n < 5$, we can easily solve the following types of inequalities

$$a \le X_n\left(\frac{\lambda_f(p)}{2}\right) \le b \tag{2}$$

for some real numbers $a, b \in \mathbb{R}$. Then applying Lau and Wang's result, we obtain a quantitative version of Omar and Mazhouda's theorems. Furthermore, we also get a quantitative version of the joint distribution of $\{\lambda_f(p^4) : f \in H_k\}$.

**Theorem 1.** *Let $r = 2, 3, 4$. There exists a small constant $\delta > 0$ such that for all sufficiently large $k$,*

$$\frac{1}{|H_k|}\#\{f \in H_k :\ (\lambda_f(p_1^r), \dots, \lambda_f(p_N^r)) \in I_r\}$$
$$= \int_{I_r}\prod_{n=1}^{N} d\mu_{p_n,r} + O\left(\frac{r^N \log(p_1p_2\cdots p_N)}{\log k}\right)$$

*holds uniformly for any integer $N \ge 1$ and distinct primes $p_1, p_2, \dots, p_N$ satisfying*

$$r^N \log(p_1p_2\cdots p_N) \le \delta \log k,$$

*and uniformly for*

$$I_2 = \prod_{n=1}^{N}[a_{2,n}, b_{2,n}] \subset [-1,3]^N,\ I_3 = \prod_{n=1}^{N}[a_{3,n}, b_{3,n}] \subset [-4,4]^N,$$

*and*

$$I_4 = \prod_{n=1}^{N} [a_{4,n}, b_{4,n}] \subset [-5/4, 5]^N.$$

*Here, the intervals* $[a_{2,n}, b_{2,n}]$, $[a_{3,n}, b_{3,n}]$ *and* $[a_{4,n}, b_{4,n}]$ *are any subintervals of* $[-1, 3]$, $[-4, 4]$ *and* $[-5/4, 5]$, *respectively. Furthermore,*

$$d\mu_{p,4}(x) = \begin{cases} \frac{1}{2\sqrt{x+\frac{5}{4}}} \left( m_{p,2}\left(\frac{1}{2} + \sqrt{x+\frac{5}{4}}\right) + m_{p,2}\left(\frac{1}{2} - \sqrt{x+\frac{5}{4}}\right)\right) dx & \textit{if } x \in [-5/4, 5]; \\ 0 & \textit{otherwise.} \end{cases}$$

When $n \geq 5$, the $n$th Chebychev polynomial is complicated and it is not easy to solve the inequality (2). We cannot figure out the precise distributions of $\{\lambda_f(p^n) : f \in H_k\}$ for $n \geq 5$. However, we find some interesting relationships between $\lambda_f(p^n)$ and $\lambda_f(p^{n-2})$ for $n \geq 2$.

**Theorem 2.** *There exists a small constant* $\delta > 0$ *such that for any integer* $r \geq 2$ *and all sufficiently large* $k$,

$$\frac{1}{|H_k|} \# \left\{ f \in H_k : \ (\lambda_f(p_1^r) - \lambda_f(p_1^{r-2}), \ldots, \lambda_f(p_N^r) - \lambda_f(p_N^{r-2}) \in I \right\}$$
$$= \int_I \prod_{n=1}^{N} d\mu^*_{p_n,r}(x) + O\left(\frac{rN \log(p_1 p_2 \cdots p_N)}{\log k}\right)$$

*holds uniformly for any integer* $N \geq 1$ *and distinct primes* $p_1, p_2, \ldots, p_N$ *satisfying*

$$rN \log(p_1 p_2 \cdots p_N) \leq \delta \log k,$$

*and uniformly for*

$$I = \prod_{n=1}^{N} [a_n, b_n] \subset [-2, 2]^N.$$

*Here,* $d\mu^*_{p,r}(x)$ *is defined by*

$$d\mu^*_{p,r}(x) = \begin{cases} \frac{1}{\pi\sqrt{4-x^2}} \frac{p^r + p^{1-r} - (1+p)(x^2-2)}{(p^{r/2} + p^{-r/2})^2 - x^2} \, dx & \textit{if } x \in [-2, 2] \textit{ and } r \geq 3 \textit{ is odd;} \\ \frac{1}{\pi\sqrt{4-x^2}} \frac{p^{r/2} + p^{1-r/2} - (1+p)x}{(p^{r/4} + p^{-r/4})^2 - 2(1+x)} \, dx & \textit{if } x \in [-2, 2] \textit{ and } r \geq 2 \textit{ is even;} \\ 0 & \textit{if } |x| > 2. \end{cases}$$

In the context of Maass forms, we have similar results. Let $\mathcal{C}$ denote the space consisting of all the Maass cusp forms for $\Gamma = SL(2, \mathbb{Z})$. Let $\{u_j : j \geq 0\}$ be a complete orthonormal basis for $\mathcal{C}$ consisting of common eigenfunctions of the Hecke operators $T_n$, $n = 1, 2, \dots$ and the Laplacian $\Delta = -y^2\big((\partial/\partial x)^2 + (\partial/\partial y)^2\big)$ with

$$T_n u_j = \lambda_j(n) u_j \quad \text{and} \quad \Delta u_j = (1/4 + t_j^2) u_j,$$

where $u_0$ is a constant function. It is well-known that $0 < t_1 \leq t_2 \leq \cdots$ and (Weyl's law, see [5])

$$r(T) = \#\{j : 0 < t_j \leq T\} = \frac{1}{12}T^2 + O(T \log T).$$

Moreover, we have $\lambda_j(n) \in \mathbb{R}$ and the generalized Ramanujan conjecture for Maass forms asserts that

$$|\lambda_j(p)| \leq 2$$

for any prime $p$. Unfortunately, this conjecture is still open and the best result

$$|\lambda_j(p)| \leq p^{7/64} + p^{-7/64} \qquad \text{for any prime } p \tag{3}$$

was obtained by Kim and Sarnak [6].

Let

$$x_j = \{(\lambda_j(2), \lambda_j(3), \lambda_j(5), \dots)\}.$$

Then $x_j \in \prod_p [-p^{7/64} - p^{-7/64}, p^{7/64} + p^{-7/64}]$. In [10], Sarnak pointed out that $\{x_j\}$, $j = 1, 2, 3, \dots$ is equidistributed with respect to $\prod_p d\mu_p(x)$. Inspired by Murty and Sinha [8], Lau and Wang [7] obtained a quantitative version of Sarnak's result.

It is well-known that for any $m, n \geq 1$

$$\lambda_j(m)\lambda_j(n) = \sum_{d|(m,n)} \lambda_j\left(\frac{mn}{d^2}\right) \tag{4}$$

and

$$\lambda_j(p^n) = X_n\left(\frac{\lambda_j(p)}{2}\right),$$

where $X_n$ is the $n$th Chebychev polynomial of the second kind. We consider the analogues of Theorem 1 and Theorem 2 and obtain the following two theorems.

**Theorem 3.** *Let $r = 2, 3, 4$. There exists a small constant $\delta > 0$ such that for all sufficiently large $T$,*

$$\frac{1}{r(T)}\#\{0 < t_j \le T:\ (\lambda_j(p_1^r), \ldots, \lambda_j(p_N^r)) \in I_r\}$$
$$= \int_{I_r} \prod_{n=1}^{N} d\mu_{p_n}^{m}(x) + O\left(\frac{r^N \log(p_1 p_2 \cdots p_N)}{\log T}\right)$$

*holds uniformly for any integer $N \ge 1$ and distinct primes $p_1, p_2, \ldots, p_N$ satisfying*

$$r^N \log(p_1 p_2 \cdots p_N) \le \delta \log T,$$

*and uniformly for*

$$I_2 = \prod_{n=1}^{N} [a_{2,n}, b_{2,n}] \subset [-1, 3]^N,\ \ I_3 = \prod_{n=1}^{N} [a_{3,n}, b_{3,n}] \subset [-4, 4]^N,$$

*and*

$$I_4 = \prod_{n=1}^{N} [a_{4,n}, b_{4,n}] \subset [-5/4, 5]^N.$$

**Theorem 4.** *There exists a small constant $\delta > 0$ such that for any integer $r \ge 2$ and all sufficiently large $T$,*

$$\frac{1}{r(T)}\#\left\{0 < t_j \le T:\ (\lambda_j(p_1^r) - \lambda_j(p_1^{r-2}), \ldots, \lambda_j(p_N^r) - \lambda_j(p_N^{r-2})) \in I\right\}$$
$$= \int_{I} \prod_{n=1}^{N} d\mu_{p_n,r}^{*}(x) + O\left(\frac{rN \log(p_1 p_2 \cdots p_N)}{\log T}\right)$$

*holds uniformly for any integer $N \ge 1$ and distinct primes $p_1, p_2, \ldots, p_N$ satisfying*

$$rN \log(p_1 p_2 \cdots p_N) \le \delta \log T,$$

*and uniformly for*

$$I = \prod_{n=1}^{N} [a_n, b_n] \subset [-2, 2]^N.$$

We shall omit the proofs of Theorem 1 and Theorem 2 because they are very similar to the proofs of Theorem 3 and Theorem 4 respectively.

## 2. Proof of Theorem 3

By (4), we have

$$\lambda_j(p^2) = \lambda_j^2(p) - 1.$$

Then $\lambda_j(p_n^2) \in [a_{2,n}, b_{2,n}] \subset [-1, 3]$ is equivalent to

$$\lambda_j(p_n) \in U_{2,n} = \left[\sqrt{a_{2,n}+1}, \sqrt{b_{2,n}+1}\right] \cup \left[-\sqrt{b_{2,n}+1}, -\sqrt{a_{2,n}+1}\right].$$

For $n = 1, 2, 3, \ldots, N$, put

$$\left[\sqrt{a_{2,n}+1}, \sqrt{b_{2,n}+1}\right] = 1 \times \left[\sqrt{a_{2,n}+1}, \sqrt{b_{2,n}+1}\right],$$

$$\left[-\sqrt{b_{2,n}+1}, -\sqrt{a_{2,n}+1}\right] = -1 \times \left[\sqrt{a_{2,n}+1}, \sqrt{b_{2,n}+1}\right]$$

and

$$U_2 = \bigcup_{n=1}^{N} \bigcup_{\epsilon_n=\pm 1} \prod_{n=1}^{N} \epsilon_n \times \left[\sqrt{a_{2,n}+1}, \sqrt{b_{2,n}+1}\right].$$

Then $(\lambda_j(p_1^2), \ldots, \lambda_j(p_N^2)) \in I_2$ is equivalent to $(\lambda_j(p_1), \ldots, \lambda_j(p_N)) \in U_2$. Hence, by [7, Theorem 1], we have

$$\begin{aligned}
&\frac{1}{r(T)} \# \left\{0 < t_j \le T :\ (\lambda_j(p_1^2), \ldots, \lambda_j(p_N^2)) \in I_2\right\} \\
&= \frac{1}{r(T)} \# \left\{0 < t_j \le T :\ (\lambda_j(p_1), \ldots, \lambda_j(p_N)) \in U_2\right\} \\
&= \int\limits_{U_2} \prod_{n=1}^{N} d\mu_{p_n}(x) + O\left(\frac{2^N \log(p_1 p_2 \cdots p_N)}{\log T}\right).
\end{aligned}$$

Making a change of variable $x^2 - 1 \to t$, we obtain

$$\begin{aligned}
&\frac{1}{r(T)} \# \left\{0 < t_j \le T :\ (\lambda_j(p_1^2), \ldots, \lambda_j(p_N^2)) \in I_2\right\} \\
&= \int\limits_{I_2} \prod_{n=1}^{N} d\mu_{p_n,2}(x) + O\left(\frac{2^N \log(p_1 p_2 \cdots p_N)}{\log k}\right).
\end{aligned}$$

Noting that

$$\lambda_j(p^3) = \lambda_j^3(p) - 2\lambda_j(p) \qquad \text{and} \qquad \lambda_j(p^4) = \lambda_j^2(p^2) - \lambda_j(p^2) - 1,$$

we can prove Theorem 3 for $r = 3, 4$ similarly. $\square$

## 3. Preparations for Theorem 4

To begin with, we cite some notation and results from [2,7,12] which will be used to prove Theorem 4. Let $\varphi_{u,v}: \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be the normalized characteristic functions defined as

$$\varphi_{u,v}(x) = \begin{cases} 1 & \text{if } u < x - n < v \text{ for some } n \in \mathbb{Z}, \\ \dfrac{1}{2} & \text{if } u - x \in \mathbb{Z} \text{ or if } v - x \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

where $u < v < u + 1$. For our purpose, we take $0 \le u < v \le 1/2$ and define

$$\widetilde{\varphi}_{u,v}(x) = \varphi_{u,v}(x) + \varphi_{-v,-u}(x) \in [0,1]$$

for any $x \in \mathbb{R}$, since the two intervals $(u,v)$ and $(-v,-u)$ do not overlap in $\mathbb{R}/\mathbb{Z}$. Furthermore, Barton, Montgomery and Vaaler [2, §2] obtained

$$\varphi_{u,v}(x) = (v - u) + \psi(u - x) + \psi(x - v)$$

with $\psi(x) = x - [x] - 1/2$ for $x \notin \mathbb{Z}$ and $\psi(x) = 0$ for $x \in \mathbb{Z}$.

Let $M$ be a positive integer to be determined later. Define

$$k_M(x) = \sum_{|\ell| \le M} \left(1 - \frac{|\ell|}{M+1}\right) e(\ell x) = \frac{1}{M+1} \left(\frac{\sin \pi(M+1)x}{\sin \pi x}\right)^2 \tag{5}$$

and

$$j_M(x) = \sum_{|\ell| \le M} \widehat{J}\left(\frac{\ell}{M+1}\right) e(\ell x),$$

where

$$\widehat{J}(t) = \begin{cases} \pi t(1 - |t|) \cot \pi t + |t|, & \text{if } 0 < |t| < 1; \\ 1, & \text{if } t = 0. \end{cases} \tag{6}$$

Furthermore, define

$$\widetilde{\beta}_{u,v}(x) = (2M+2)^{-1} \big(k_M(x-u) + k_M(x-v) + k_M(x+u) + k_M(x+v)\big)$$

and

$$\widetilde{\alpha}_{u,v}(x) = \varphi_{u,v} * j_M(x) + \varphi_{-v,-u} * j_M(x), \tag{7}$$

where $f * g$ is the convolution given by

$$f * g(x) = \int_0^1 f(y)g(x-y)dy.$$

By direct calculation, we rewrite $\widetilde{\alpha}_{u,v}$ and $\widetilde{\beta}_{u,v}$ in cosine series (see [7, (2.6)])

$$\widetilde{\alpha}_{u,v}(x) = \widehat{\alpha}_{u,v}(0) + \sum_{1\le|\ell|\le M} \widehat{\alpha}_{u,v}(\ell)\cos(2\pi\ell x) \tag{8}$$

and

$$\widetilde{\beta}_{u,v}(x) = (2M+2)^{-1} \sum_{|\ell|\le M} \widehat{\beta}_{u,v}(\ell)\cos(2\pi\ell x), \tag{9}$$

where

$$\begin{aligned}\widehat{\alpha}_{u,v}(\ell) &= \begin{cases} (\pi\ell i)^{-1}\widehat{J}\left(\frac{\ell}{M+1}\right)(e(-\ell u) - e(-\ell v)) & \text{if } \ell \ne 0; \\ 2(v-u) & \text{if } \ell = 0; \end{cases} \\ \widehat{\beta}_{u,v}(\ell) &= \begin{cases} 2\left(1 - \frac{\ell}{M+1}\right)(e(-\ell u) - e(-\ell v)) & \text{if } \ell \ne 0; \\ 4 & \text{if } \ell = 0. \end{cases}\end{aligned} \tag{10}$$

Moreover, for $x \in \mathbb{R}$, it is known that (see [2, (2.6)] or [7, (2.4)])

$$|\widetilde{\varphi}_{u,v}(x) - \widetilde{\alpha}_{u,v}(x)| \le \widetilde{\beta}_{u,v}(x) \tag{11}$$

and (see [7, (2.8)] and the line below it)

$$0 \le \widetilde{\alpha}_{u,v}(x) \le 1 \qquad \text{and} \qquad 0 \le \widetilde{\beta}_{u,v}(x) \le 2. \tag{12}$$

Define

$$\Phi_{\underline{u},\underline{v}}(\underline{x}) = \prod_{n=1}^{N} \widetilde{\varphi}_{u_n,v_n}(x_n), \tag{13}$$

where $\underline{u} = (u_1, u_2, \ldots, u_N)$ and $\underline{v} = (v_1, v_2, \ldots, v_N)$ with $0 \le u_n < v_n \le 1/2$ and $u_n(M+1), v_n(M+1) \in \mathbb{Z}$ for all $n = 1, 2, \ldots, N$. For simplicity, we put

$$\begin{aligned}&\widetilde{\varphi}_n(x) = \widetilde{\varphi}_{u_n,v_n}(x),\ \ \widetilde{\alpha}_n(x) = \widetilde{\alpha}_{u_n,v_n}(x),\ \ \widetilde{\beta}_n(x) = \widetilde{\beta}_{u_n,v_n}(x), \\ &\widehat{\alpha}_n(\ell) = \widehat{\alpha}_{u_n,v_n}(\ell),\ \ \ \widehat{\beta}_n(\ell) = \widehat{\beta}_{u_n,v_n}(\ell).\end{aligned}$$

The following result is one of our main tools.

**Lemma 3.1** *([7, Proposition 1]). Let $\Phi_{\underline{u},\underline{v}} : (\mathbb{R}/\mathbb{Z})^N \to \mathbb{R}$ be defined as in (13) where $0 \le u_n < v_n \le 1/2$ and $u_n(M+1), v_n(M+1) \in \mathbb{Z}$. Then for $\underline{x} = (x_1, \dots, x_N)$,*

$$|\Phi_{\underline{u},\underline{v}}(\underline{x}) - \widetilde{\boldsymbol{\alpha}}(\underline{x})| \le B(\underline{x}),$$

*where*

$$\widetilde{\boldsymbol{\alpha}}(\underline{x}) = \prod_{n=1}^{N} \widetilde{\alpha}_n(x_n) = \sum_{\underline{\ell} \in ([-M,M]\cap\mathbb{Z})^N} \widehat{\boldsymbol{\alpha}}(\underline{\ell}) \,\mathbf{cos}(2\pi \underline{\ell} \diamond \underline{x}),$$

$$B(\underline{x}) = \sum_{n=1}^{N} \widetilde{\beta}_n(x_n) = \frac{1}{2(M+1)} \sum_{n=1}^{N} \sum_{|m|\le M} \widehat{\beta}_n(m) \cos(2\pi m x_n).$$

*Here,*

$$\widehat{\boldsymbol{\alpha}}(\underline{\ell}) = \prod_{n=1}^{N} \widehat{\alpha}_n(\ell_n), \quad \mathbf{cos}(2\pi \underline{\ell} \diamond \underline{x}) = \prod_{n=1}^{N} \cos(2\pi \ell_n x_n) \tag{14}$$

*and $\underline{\ell} \diamond \underline{x} := (\ell_1 x_1, \dots, \ell_N x_N)$ denotes the Hadamard product; the values of $\widehat{\alpha}_n(\ell)$ and $\widehat{\beta}_n(m)$ are defined in (10). Moreover,*

$$|\widehat{\alpha}_n(\ell)| \le 2\widehat{k}_M(\ell), \quad |\widehat{\beta}_n(m)| \le 4\widehat{k}_M(m) \tag{15}$$

*where $\widehat{k}_M(\ell) = (1 - |\ell|/(M+1))$.*

Another main tool is the following lemma.

**Lemma 3.2** *([7, Lemma 3.4]). Let $S$ be a finite set of distinct primes. Then we have for $m_p \ge 1$ $(p \in S)$,*

$$\begin{aligned} &\sum_{1\le j\le r(T)} \prod_{p\in S} \left(a_p \lambda_j(p^{m_p}) + b_p \lambda_j(p^{m_p-2})\right) \\ &= \frac{T^2}{12} \prod_{p\in S} \delta_{2|m_p} \left(\frac{a_p}{p^{m/2}} + \frac{b_p}{p^{m/2-1}}\right) + O\left(2^{\#(S)} T^{2-\kappa} \prod_{p\in S} \left(p^{m_p \eta} \max(|a_p|, |b_p|)\right)\right) \end{aligned}$$

*where $\delta_{2|h} = 1$ if $2|h$ and $0$ otherwise, $a_p, b_p$ are constants depending only on $p$, $\#(S)$ denotes the cardinality of $S$, $0 < \kappa < \kappa_0 = \frac{11}{155}$ and $\eta > \eta_0 = \frac{43}{620}$ are any absolute constants. When $m_p = 1$, $\lambda_j(p^{m_p-2})$ denotes $0$.*

*Further, for $M \ge 1$, we have*

$$\sum_{1\le j\le r(T)} \prod_{p\in S} \lambda_j(p^M)^2 \ll r(T) \prod_{p\in S} \frac{\sigma(p^M)}{p^M} + T^{2-\kappa} \Big(\prod_{p\in S} p\Big)^{2M\eta},$$

*where $\sigma(m) = \sum_{d|m} d$.*

The Hecke eigenvalue $\lambda_j(p)$ can be expressed in the form $\lambda_j(p) = \alpha_{j,p} + \beta_{j,p}$ with $\alpha_{j,p}, \beta_{j,p} \in \mathbb{C}$ and $\alpha_{j,p}\beta_{j,p} = 1$. By (3), $|\alpha_{j,p}| \le p^\theta$ with $\theta = 7/64$. Hence, there exists a unique $\theta_j(p) \in [0,\pi] \cup i(0, \theta\log p] \cup \pi + i(0, \theta\log p]$ such that $\alpha_{j,p} = e^{i\theta_j(p)}$ and

$$\lambda_j(p) = 2\cos\theta_j(p).$$

It is well-known that

$$\lambda_j(p^r) = X_r(\cos\theta_j(p)) = \frac{\sin(r+1)\theta_j(p)}{\sin\theta_j(p)}. \tag{16}$$

Then we easily obtain that for $r \ge 2$,

$$\lambda_j(p^r) - \lambda_j(p^{r-2}) = 2\cos r\theta_j(p) \in [-2,2]$$

and $\lambda_j(p^r) - \lambda_j(p^{r-2}) \in (a,b) \subset [-2,2]$ is equivalent to

$$r\theta_j(p)/(2\pi) \in (k_{r,j,p} + b^*, k_{r,j,p} + a^*) \cup (k_{r,j,p} - a^*, k_{r,j,p} - b^*)$$

for some integer $k_{r,j,p}$ depending on $r, p$ and $j$, where

$$a^* = \frac{\arccos(a/2)}{2\pi} \qquad \text{and} \qquad b^* = \frac{\arccos(b/2)}{2\pi}.$$

Hence $\lambda_j(p^r) - \lambda_j(p^{r-2}) \in (a,b) \subset [-2,2]$ if and only if $\widetilde{\varphi}_{b^*,a^*}(r\theta_j(p)/(2\pi)) = 1$. Therefore, for any $(a,b) \subset [-2,2]$ and $r \ge 2$

$$\sum_{\substack{1\le j\le r(T)\\ \lambda_j(p^r)-\lambda_j(p^{r-2})\in(a,b)}} 1 = \sum_{\substack{1\le j\le r(T)\\ \theta_j(p)\in[0,\pi]}} \widetilde{\varphi}_{b^*,a^*}\Big(\frac{r}{2\pi}\theta_j(p)\Big). \tag{17}$$

Let $p_1, \ldots, p_N$ be distinct primes. To prove Theorem 4, we shall consider

$$\Phi_{\underline{u},\underline{v}}\Big(\frac{m}{2\pi}\underline{\theta}_j(\underline{p})\Big),$$

where $\underline{\theta}_j(\underline{p}) = (\theta_j(p_1), \ldots, \theta_j(p_N))$ and $\theta_j(p_n)$ is defined as above. In light of Lemma 3.1, we are led to prove the following two lemmas.

**Lemma 3.3.** *For $r\underline{\ell} = (r\ell_1, \ldots, r\ell_N)$ where $|\ell_n| \le M$, we have*

$$\Bigg|\sum_{1\le j\le r(T)} \widetilde{\boldsymbol{\alpha}}\Big(\frac{r}{2\pi}\underline{\theta}_j(\underline{p})\Big) - r(T)\sum_{\underline{\ell}\in([-M,M]\cap\mathbb{Z})^N} \widehat{\boldsymbol{\alpha}}(\underline{\ell})\mathbf{c}_{\underline{p}}(r\underline{\ell})\Bigg|$$
$$\ll (2M+2)^N T^{2-\kappa}(p_1\cdots p_N)^{2rM\eta}, \tag{18}$$

*where $\mathbf{c}_{\underline{p}}(r\underline{\ell}) = \prod_{n=1}^N c_{p_n}(r\ell_n)$ with $c_p(0) = 1$, and*

$$c_p(r\ell) = \begin{cases} \frac{1}{2}p^{-|\ell|r/2}(1-p) & \textit{if } r\ell \textit{ is even} \\ 0 & \textit{if } r\ell \textit{ is odd.} \end{cases}$$

*Here* $0 < \kappa < \kappa_0 = \frac{11}{155}$ *and* $\eta > \eta_0 = \frac{43}{620}$ *are any absolute constants and the implied constant in (18) depends only on* $\eta$.

**Proof.** By (15), we only have to prove

$$\left| \sum_{1\le j\le r(T)} \mathbf{cos}(r\underline{\ell} \diamond \underline{\theta}_j(\underline{p})) - r(T)\mathbf{c}_{\underline{p}}(r\underline{\ell}) \right| \ll T^{2-\kappa}(p_1 \cdots p_N)^{rM\eta}. \tag{19}$$

Using $2\cos(\ell\theta) = X_{|\ell|}(2\cos\theta) - X_{|\ell|-2}(2\cos\theta)$ for $|\ell| \ge 2$, we have

$$\cos(\ell\theta_j(p)) = \begin{cases} \frac{1}{2}\big(\lambda_j(p^{|\ell|}) - \lambda_j(p^{|\ell|-2})\big), & |\ell| \ge 2, \\ \frac{1}{2}\lambda_j(p), & |\ell| = 1, \\ 1, & \ell = 0. \end{cases}$$

Therefore, if we denote $Q(X,Y) = \frac{1}{2}(X-Y)$, then

$$\mathbf{cos}(r\underline{\ell} \diamond \underline{\theta}_j(\underline{p})) = \prod_{n:\,|\ell_n|\ge 1} Q(\lambda_j(p_n^{r|\ell_n|}), \lambda_j(p_n^{r|\ell_n|-2})).$$

Applying Lemma 3.2 with $a_{p_n} = 1/2$, $b_{p_n} = -1/2$ and $m_{p_n} = r|\ell_n|$,

$$\begin{aligned} &\sum_{1\le j\le r(T)} \prod_{n:\,|\ell_n|\ge 1} Q(\lambda_j(p_n^{r|\ell_n|}), \lambda_j(p_n^{r|\ell_n|-2})) \\ &= \frac{T^2}{12} \prod_{n:|\ell_n|\ge 1} \delta_{2|r\ell_n}\left(\frac{1}{2}\frac{1}{p^{r|\ell_n|/2}} - \frac{1}{2}\frac{1}{p^{r|\ell_n|/2-1}}\right) + O\Bigg(T^{2-\kappa} \prod_{n:|\ell_n|\ge 1} p_n^{r|\ell_n|\eta}\Bigg) \\ &= \frac{T^2}{12}\mathbf{c}_{\underline{p}}(r\underline{\ell}) + O\big(T^{2-\kappa}(p_1\cdots p_N)^{rM\eta}\big). \end{aligned}$$

Here we have used the assumption $|\ell_n| \le M$ in the last step. The proof is complete. $\square$

Define

$$F_{p,r}(y) = \sum_{\ell=-\infty}^{\infty} c_p(r\ell)e(\ell y), \tag{20}$$

where $c_p(r\ell)$ is defined as in Lemma 3.3. By the identity

$$\sum_{m=0}^{\infty} t^m \cos mx = \frac{1 - t\cos x}{|1 - te^{ix}|^2} \ (t, x \in \mathbb{R}, |t| < 1),$$

we get

$$F_{p,r}(y) = \begin{cases} \frac{p^r + p^{1-r} - (1+p)\cos 4\pi y}{(p^{r/2}+p^{-r/2})^2 - 4\cos^2(2\pi y)} & \text{if } r \text{ is odd;} \\ \frac{p^{r/2}+p^{1-r/2}-(1+p)\cos 2\pi y}{(p^{r/4}+p^{-r/4})^2 - 4\cos^2(\pi y)} & \text{if } r \text{ is even.} \end{cases} \tag{21}$$

Note that $F_{p,r}(y) \geq 0$ and $\int_0^1 F_{p,r}(y)\,dy = 1$, yielding a probability density function on the space $\mathbb{R}/\mathbb{Z}$.

**Lemma 3.4.** *Let $\widehat{\boldsymbol{\alpha}}(\underline{\ell})$ and $\mathbf{c}_{\underline{p}}(r\underline{\ell})$ be defined as in Lemma 3.1 and Lemma 3.3, respectively. Then we have*

$$\sum_{\underline{\ell}\in([-M,M]\cap\mathbb{Z})^N} \widehat{\boldsymbol{\alpha}}(\underline{\ell})\mathbf{c}_{\underline{p}}(r\underline{\ell}) = \prod_{n=1}^{N} \int_{u_n}^{v_n} 2F_{p_n,r}(y)\,dy + O\Big(\frac{N}{M}\Big).$$

**Proof.** By the definition of $\widehat{\boldsymbol{\alpha}}(\underline{\ell})$, $\mathbf{c}_{\underline{p}}(r\underline{\ell})$, we have

$$\sum_{\underline{\ell}\in([-M,M]\cap\mathbb{Z})^N} \widehat{\boldsymbol{\alpha}}(\underline{\ell})\mathbf{c}_{\underline{p}}(r\underline{\ell}) = \prod_{n=1}^{N} \sum_{|\ell_n|\leq M} \widehat{\alpha}_n(\ell_n)c_{p_n}(r\ell_n) = \prod_{n=1}^{N} \Sigma_{p_n,r},$$

where

$$\Sigma_{p_n,r} = \sum_{|\ell_n|\leq M} \widehat{\alpha}_n(\ell_n)c_{p_n}(r\ell_n).$$

By (8) and (20), we obtain

$$\Sigma_{p_n,r} = \int_0^1 \widetilde{\alpha}_{u_n,v_n}(y)F_{p_n,r}(-y)dy = \widetilde{\alpha}_{u_n,v_n} * F_{p_n,r}(0).$$

By the nonnegativity of $F_{p_n,r}(y)$ (see (21)) and (12), we have

$$0 \leq \Sigma_{p_n,r} = \int_0^1 \widetilde{\alpha}_{u_n,v_n}(y)F_{p_n,r}(-y)dy \leq \int_0^1 F_{p_n,r}(-y)dy = 1. \tag{22}$$

Moreover, we have

$$|\Sigma_{p_n,r} - \widetilde{\varphi}_{u_n,v_n} * F_{p_n,r}(0)| = \left|\int_0^1 (\widetilde{\alpha}_{u_n,v_n}(y) - \widetilde{\varphi}_{u_n,v_n}(y))F_{p_n,r}(-y)dy\right|$$

$$\leq \int_0^1 |\widetilde{\alpha}_{u_n,v_n}(y) - \widetilde{\varphi}_{u_n,v_n}(y)| F_{p_n,r}(-y) dy$$
$$\leq \widetilde{\beta}_{u_n,v_n} * F_{p_n,r}(0). \tag{23}$$

Here we have used (11) in the last step. By the nonnegativity of $\widetilde{\beta}_{u_n,v_n}$ (see (12)) and (9), we have

$$\widetilde{\beta}_{u_n,v_n} * F_{p_n,r}(0) \leq \max_{y\in[0,1]} |F_{p_n,r}(y)| \int_0^1 \widetilde{\beta}_{u_n,v_n}(y)\, dy \ll \frac{1}{M+1}. \tag{24}$$

Recall that $0 \leq u < v \leq 1/2$. It is easy to see

$$\widetilde{\varphi}_{u,v} * F_{p,r}(0) = \int_u^v 2F_{p,r}(y)\, dy \in [0,1]. \tag{25}$$

Combining (25) with (23) and (24), we get

$$\Sigma_{p_n,r} = \int_{u_n}^{v_n} 2F_{p_n,r}(y) + O\Big(\frac{1}{M}\Big). \tag{26}$$

Suppose for $S \geq 1$, we have

$$\prod_{n=1}^{S} \Sigma_{p_n,r} = \prod_{n=1}^{S} \int_{u_n}^{v_n} 2F_{p_n,r}(y)\, dy + O\Big(\frac{S}{M}\Big).$$

By (22), (25) and (26), we have

$$\begin{aligned}\Sigma_{p_{S+1},r} \times \prod_{n=1}^{S} \Sigma_{p_n,r} &= \Sigma_{p_{S+1},r} \times \prod_{n=1}^{S} \int_{u_n}^{v_n} 2F_{p_n,r}(y)\, dy + O\Big(\frac{S}{M}\Big) \\ &= \prod_{n=1}^{S+1} \int_{u_n}^{v_n} 2F_{p_n,r}(y)\, dy + O\Big(\frac{1}{M}\Big) + O\Big(\frac{S}{M}\Big) \\ &= \prod_{n=1}^{S+1} \int_{u_n}^{v_n} 2F_{p_n,r}(y)\, dy + O\Big(\frac{S+1}{M}\Big).\end{aligned}$$

Therefore, by induction,

$$\sum_{\underline{\ell}\in([-M,M]\cap\mathbb{Z})^N} \widehat{\boldsymbol{\alpha}}(\underline{\ell})\mathbf{c}_{\underline{p}}(r\underline{\ell}) = \prod_{n=1}^{N} \Sigma_{p_n,r} = \prod_{n=1}^{N} \int_{u_n}^{v_n} 2F_{p_n,r}(y)\, dy + O\Big(\frac{N}{M}\Big). \quad \square$$

## 4. Proof of Theorem 4

For $N\log(p_1\cdots p_N)\le\delta\log T$, we have $N\ll\sqrt{\log T}$ as $p_i\ge 2$. Define

$$\Theta_T(p)=\{1\le j\le r(T):\ \theta_j(p)\in[0,\pi]\}.$$

We have the following result.

**Lemma 4.1** *([7, Lemma 4.3]). Let $p$ be a prime. Then for all sufficiently large $T$,*

$$\frac{1}{r(T)}(r(T)-|\Theta_T(p)|)\ll\left(\frac{\log p}{\log T}\right)^2$$

*where $|\Theta_T(p)|$ denotes the cardinality of $\Theta_T(p)$ and the implied constant is absolute.*

Define

$$\Theta=\bigcap_{n=1}^{N}\Theta_T(p_n)$$

and

$$\Theta'=\{j:\ 1\le j\le r(T)\}\setminus\Theta=\bigcup_{n=1}^{N}(\{j:\ 1\le j\le r(T)\}\setminus\Theta_T(p_n)).$$

Here we suppress symbols for the dependence on $T$ and $p_1,\ldots,p_N$, as no ambiguity will arise. By Lemma 4.1, we obtain

$$|\Theta'|\ll\frac{r(T)}{(\log T)^2}\sum_{n=1}^{N}(\log p_n)^2\ll r(T)\left(\frac{\log(p_1\cdots p_N)}{\log T}\right)^2. \tag{27}$$

With the notation as in Section 3, we prove the following lemma.

**Lemma 4.2.** *Let $\prod_{n=1}^N[u_n,v_n]\subset[0,1]^N$ satisfy the conditions in Lemma 3.1. Then we have*

$$\begin{aligned}&\frac{1}{r(T)}\sum_{j\in\Theta}\Phi_{\underline{u},\underline{v}}\Big(\frac{r}{2\pi}\underline{\theta}_j(\underline{p})\Big)\\&=\prod_{n=1}^{N}\int_{u_n}^{v_n}2F_{p_n,r}(y)\,dy+O\left(\frac{rN\log(p_1\cdots p_N)}{\log T}\right).\end{aligned} \tag{28}$$

**Proof.** By Lemma 3.3 and Lemma 3.4, we obtain

$$\left| \sum_{1\le j\le T} \widetilde{\boldsymbol{\alpha}}\big(\frac{r}{2\pi}\underline{\theta}_j(\underline{p})\big) - r(T)\prod_{n=1}^{N}\int_{u_n}^{v_n} 2F_{p_n,r}(y)\,dy \right|$$
$$\ll (2M+2)^N T^{2-\kappa}(p_1\cdots p_N)^{2rM\eta} + r(T)\frac{N}{M}. \tag{29}$$

Note that the complete sum in (29) shows that it suffices to estimate

$$\sum_{j\in\Theta'} \widetilde{\boldsymbol{\alpha}}\Big(\frac{r}{2\pi}\underline{\theta}_j(\underline{p})\Big).$$

If $\lambda_j(p_n)\in[-2,2]$, then $\theta_j(p_n)$ is real and by (12)

$$|\widetilde{\alpha}_n(\frac{r}{2\pi}\theta_j(p_n))| \le 1.$$

If $|\lambda_j(p_n)|>2$, then $\theta_j(p_n)=i\vartheta_j(p_n)$ or $\pi+i\vartheta_j(p_n)$ for some real $\vartheta_j(p_n)$. By (8) and (15), we have

$$\begin{aligned}|\widetilde{\alpha}_{u,v}(r\theta_j(p_n)/(2\pi))| &\le \sum_{|\ell|\le M} |\widehat{\alpha}_n(\ell)|\cosh(\ell r\vartheta_j(p_n)) \\ &\le 2\sum_{|\ell|\le M} \widehat{k}_M(\ell)\cosh(\ell r\vartheta_j(p_n)) \\ &\le 2\sum_{|\ell|\le M} \widehat{k}_M(\ell)\cosh(2\ell r\vartheta_j(p_n))\end{aligned}$$

as $\cosh(\phi)\le\cosh(2\phi)$ for real $\phi$. Thus by (5) the last line gives

$$\begin{aligned}|\widetilde{\alpha}_{u,v}(r\theta_j(p_n)/(2\pi))| &\le 2k_M\big(\frac{r\theta_j(p_n)}{\pi}\big) \\ &= \frac{2}{M+1}\left(\frac{\sin(M+1)r\theta_j(p_n)}{\sin r\theta_j(p_n)}\right)^2.\end{aligned} \tag{30}$$

It is easy to see that (noting that $r\ge 2$)

$$\begin{aligned}(\sin r\theta_j(p_n))^2 &= (e^{2r\vartheta_j(p_n)}+e^{-2r\vartheta_j(p_n)}-2)/4 \\ &> (e^{2\vartheta_j(p_n)}+e^{-2\vartheta_j(p_n)}-2)/4 \\ &= (\sin\theta_j(p_n))^2.\end{aligned}$$

Combining the above formula with (30), we obtain

$$
\begin{aligned}
|\widetilde{\alpha}_{u,v}(r\theta_j(p_n)/(2\pi))| &\le \frac{2}{M+1}\left(\frac{\sin(M+1)r\theta_j(p_n)}{\sin\theta_j(p_n)}\right)^2 \\
&= \frac{2}{M+1}\left(X_{(M+1)r-1}(\cos(\theta_j(p_n)))\right)^2 \\
&= \frac{2}{M+1}\lambda_j(p_n^{rM+r-1})^2
\end{aligned}
$$

by (16). Therefore, for both cases, we have

$$
\widetilde{\boldsymbol{\alpha}}\Big(\frac{r}{2\pi}\underline{\theta}_j(\underline{p})\Big) \ll \prod_{n=1}^{N}\Big(1+\frac{2\lambda_j(p_n^{rM+r-1})^2}{M+1}\Big).
$$

By the second part of Lemma 3.2 and the fact $\sigma(p^M)/p^M \le 2$, we have

$$
\begin{aligned}
&\sum_{j\in\Theta'} \widetilde{\boldsymbol{\alpha}}\big(\frac{r}{2\pi}\underline{\theta}_j(\underline{p})\big) \\
&\le \sum_{j\in\Theta'}\prod_{n=1}^{N}\Big(1+\frac{2}{M+1}\lambda_j(p_n^{rM+r-1})^2\Big) \\
&\ll |\Theta'| + T^2\sum_{h=1}^{N}\binom{N}{h}\big(\frac{4}{M+1}\big)^h + 2^N T^{2-\kappa}(p_1\cdots p_N)^{2(rM+r-1)\eta} \\
&\ll |\Theta'| + r(T)\frac{N}{M} \qquad (31)
\end{aligned}
$$

if

$$
M \le \frac{\kappa}{4r\eta}\frac{\log T}{\log(p_1\cdots p_N)}. \qquad (32)
$$

Here we have used the fact that the sum over $h$ is $\ll (1+\frac{4}{M+1})^N - 1 \ll N/M$.

By Lemma 3.1, we also need to estimate

$$
\sum_{1\le j\le T} B\Big(\frac{r}{2\pi}\underline{\theta}_j(\underline{p})\Big).
$$

By (19), we have

$$
\begin{aligned}
\sum_{1\le j\le T} B\Big(\frac{r}{2\pi}\underline{\theta}_j(\underline{p})\Big) &= \frac{r(T)}{2M+2}\sum_{n=1}^{N}\sum_{|\ell|\le M}\widehat{\beta}_n(\ell)c_{p_n}(r\ell) + O\Big(T^{2-\kappa}\sum_{n=1}^{N}p_n^{2rM\eta}\Big) \\
&\ll r(T)\frac{N}{M} \qquad (33)
\end{aligned}
$$

provided that (32) holds.

Combining (33), (29), (31) and (27) with Lemma 3.1, we conclude that

$$\left|\sum_{j\in\Theta}\Phi_{\underline{u},\underline{v}}\Big(\frac{r}{2\pi}\underline{\theta}_j(\underline{p})\Big)-r(T)\prod_{n=1}^{N}\int_{u_n}^{v_n}2F_{p_n,r}(y)\,dy\right|\ll r(T)\frac{rN\log(p_1\cdots p_N)}{\log T}$$

by taking

$$M=\left[\frac{\kappa}{4r\eta}\frac{\log T}{\log(p_1\cdots p_N)}\right]. \quad \square \tag{34}$$

Now we are ready to complete the proof. Let $T$ be sufficiently large, and write $I=\prod_{n=1}^{N}[a_n,b_n]$ with $[a_n,b_n]\subset(-2,2)$ where $n=1,\dots,N$. Then $2\cos r\theta\in[a_n,b_n]$ is equivalent to

$$r\theta/(2\pi)\in(k_{r,\theta}+b_n^*,k_{r,\theta}+a_n^*)\cup(k_{r,\theta}-a_n^*,k_{r,\theta}-b_n^*)$$

for some integer $k_{r,\theta}$ depending on $r$ and $\theta$, where

$$a_n^*=\frac{\arccos(a_n/2)}{2\pi}\qquad\text{and}\qquad b_n^*=\frac{\arccos(b_n/2)}{2\pi}.$$

Hence $2\cos r\theta\in[a_n,b_n]$ if and only if $\widetilde{\varphi}_{b_n^*,a_n^*}(r\theta/(2\pi))=1$. Therefore,

$$\chi_{[a_n,b_n]}(2\cos r\theta)=\widetilde{\varphi}_{b_n^*,a_n^*}(r\theta/(2\pi)),$$

where $\chi_{[a,b]}$ denotes the characteristic function over $[a,b]$.

Next, we choose $[u_n,v_n]\subset[b_n^*,a_n^*]\subset[u_n',v_n']$ $(\subset[0,1/2])$ such that $\mathrm{u}(M+1)$, $\mathrm{v}(M+1)\in\mathbb{Z}$ for $(\mathrm{u},\mathrm{v})=(u_n,v_n)$ and $(u_n',v_n')$, the complement has a small measure

$$\big|[u_n',v_n']\setminus[u_n,v_n]\big|\ll 1/M$$

where $M$ takes the value as in (34), and also,

$$\widetilde{\varphi}_{u_n,v_n}(r\theta/(2\pi))\le\chi_{[a_n,b_n]}(2\cos r\theta)\le\widetilde{\varphi}_{u_n',v_n'}(r\theta/(2\pi)).$$

We denote

$$\underline{u}=(u_1,\dots,u_N),\qquad \underline{v}=(v_1,\dots,v_N),$$

and write $\underline{u}'$, $\underline{v}'$ similarly. Applying Lemma 4.2 to $\Phi_{\underline{u},\underline{v}}$ and $\Phi_{\underline{u}',\underline{v}'}$, we obtain lower and upper bounds of the form in the right-side of (28) for

$$\frac{1}{r(T)}\#\{1\le j\le r(T):(\lambda_j(p_1^r)-\lambda_j(p_1^{r-2}),\dots,\lambda_j(p_N^r)-\lambda_j(p_N^{r-2}))\in I\}.$$

Write $\underline{\mathrm{u}} = (\mathrm{u}_1, \ldots, \mathrm{u}_N)$ and $\underline{\mathrm{v}} = (\mathrm{v}_1, \ldots, \mathrm{v}_N)$. It remains to show

$$\prod_{n=1}^{N} \int_{\mathrm{u}_n}^{\mathrm{v}_n} 2F_{p_n,r}(y) = \int_I \prod_{n=1}^{N} d\mu^*_{p_n,r}(x) + O\Big(\frac{N}{M}\Big) \tag{35}$$

for $(\underline{\mathrm{u}}, \underline{\mathrm{v}}) = (\underline{u}, \underline{v})$ and $(\underline{u}', \underline{v}')$. For odd $r$, we make a change of variable $x = 2\cos 2\pi y$. Then we get that (with the subscript $n$ suppressed)

$$\begin{aligned}\int_{\mathrm{u}}^{\mathrm{v}} 2F_{p,r}(y)\,dy &= 2\int_{\mathrm{u}}^{\mathrm{v}} \frac{p^r + p^{1-r} - (1+p)\cos 4\pi y}{(p^{r/2} + p^{-r/2})^2 - 4\cos^2(2\pi y)}\,dy \\ &= \frac{1}{\pi} \int_{2\cos 2\pi \mathrm{v}}^{2\cos 2\pi \mathrm{u}} \frac{p^r + p^{1-r} - (1+p)(x^2-2)}{(p^{r/2} + p^{-r/2})^2 - x^2} \frac{1}{\sqrt{4-x^2}}\,dx.\end{aligned}$$

As $[2\cos 2\pi v, 2\cos 2\pi u] \subset [a,b] \subset [2\cos 2\pi v', 2\cos 2\pi u']$, we get

$$\int_{\mathrm{u}}^{\mathrm{v}} 2F_{p,r}(y)\,dy = \int_a^b d\mu^*_{p,r}(x) + O(1/M)$$

and (35) follows. By similar arguments, we get (35) for even $r$. Finally we relax the condition $[a_n, b_n] \subset (-2,2)$ to $[-2,2]$ with (35) and the proof is complete.

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