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Sums of coefficients of L -functions and applications



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ABSTRACT

In this paper, we establish a general summation formula for the coefficients of a class of L -functions, without assuming the generalized Ramanujan conjecture. As an application, we consider integral power moments of Fourier coefficients of Hecke–Maass cusp forms.

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1. Introduction

It is an important problem to estimate various sums for coefficients of L -functions in number theory. In this paper we shall consider a general summation formula for the coefficients of a class of L -functions satisfying the following conditions. See e.g. Iwaniec and Kowalski [3, Pg. 94]:

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- **Dirichlet Series.** Let $\mathcal{A} = \{a_n\}$ be a sequence of complex numbers not all the terms of which are zero. Then our general L -function $L(s, \mathcal{A})$ will be given by

$$L(s, \mathcal{A}) = \sum_{n \geq 1} \frac{a_n}{n^s},$$

where we assume that the series are absolutely convergent for $\Re(s) > 1$.

- **Analytic Continuation.** There is some $m = m(\mathcal{A})$ such that $L(s, \mathcal{A})$ can be continued analytically over all of \mathbb{C} except possibly for a pole of order m at $s = 1$.
- **Growth.** There exists some $\delta > 0$ such that $L(s, \mathcal{A}) \ll \exp \exp(\epsilon |t|)$, for all $\epsilon > 0$, as $|t| \rightarrow \infty$, uniformly in $-\delta < \sigma < 1 + \delta$.
- **Functional Equation.** There exists another sequence of complex matrices \mathcal{A}^* satisfying the above 3 axioms, and there exist $Q_{\mathcal{A}} \in \mathbb{R}$, $\beta_i \in \mathbb{C}$ with $\Re(\beta_i) > -1$, $1 \leq i \leq d$, and $\omega \in \mathbb{C}$ with $|\omega| = 1$ such that

$$\Lambda(s, \mathcal{A}) := Q_{\mathcal{A}}^{s/2} \prod_{i=1}^d \Gamma\left(\frac{s + \beta_i}{2}\right) L(s, \mathcal{A}),$$

and

$$\Lambda(s, \mathcal{A}^*) := Q_{\mathcal{A}}^{s/2} \prod_{i=1}^d \Gamma\left(\frac{s + \bar{\beta}_i}{2}\right) L(s, \mathcal{A}^*),$$

satisfy the functional equation

$$\Lambda(1 - s, \mathcal{A}) = \omega \Lambda(s, \mathcal{A}^*).$$

The generalized Ramanujan conjecture states that

$$|a_n| \ll_{\varepsilon} n^{\varepsilon} \tag{1.1}$$

hold for any $\varepsilon > 0$, which is often an obstacle to investigate the oscillating behavior of coefficients of L -functions. Under the generalized Ramanujan conjecture, Friedlander and Iwaniec [2] proved a fairly general summation formula. As a corollary, they proved that for any $\varepsilon > 0$,

$$\sum_{n \leq x} a_n = R(x) + O\left(Q_{\mathcal{A}}^{\frac{1}{d+1}} x^{\frac{d-1}{d+1} + \varepsilon}\right), \tag{1.2}$$

where $R(x) = \text{Res}_{s=1} L(s, \mathcal{A}) x^s / s$, and the implied constant depends only on ε and the parameters $\beta_1, \beta_2, \dots, \beta_d$.

It should be remarked that Landau's classical method gives that

$$\sum_{n \leq x} a_n = R(x) + O\left(x^{\frac{d-1}{d+1}+\varepsilon}\right), \quad (1.3)$$

if $a_n \geq 0$, otherwise

$$\sum_{n \leq X} a_n = R(X) + O\left(x^{\frac{(d-1)(1-\eta)}{2}+\varepsilon}\right) + O\left(\sum_{X < n < X+X^\eta} |a_n|\right) \quad (1.4)$$

for every $0 < \eta < 1$, without assuming the generalized Ramanujan conjecture. See e.g. Chandrasekharan and Narasimhan [1, Theorem 4.1]. One should note when a_n twisted with some smooth test functions, there often exist very good bounds, which have many applications.

Moreover, if $L(s, \mathcal{A})$ is not *primitive*, which means that $L(s, \mathcal{A})$ can be decomposed into a product of L -functions of lower degrees. Suppose that

$$L(s, \mathcal{A}) = L(s, \mathcal{A}_1)L(s, \mathcal{A}_2), \quad (1.5)$$

where $L(s, \mathcal{A}_1), L(s, \mathcal{A}_2)$ are L -functions of degree d_1, d_2 respectively, satisfying $d_1 + d_2 = d \geq 3$. Friedlander and Iwaniec [2] mentioned that the mean square estimate (see Lemma 2.2) yields

$$\sum_{n \leq x} a_n = R(x) + O\left(Q_{\mathcal{A}}^{\frac{1}{2}} x^{\theta(d_1, d_2)+\varepsilon}\right), \quad (1.6)$$

under the generalized Ramanujan conjecture. Here

$$\theta(0, d) = \frac{d}{d+1}, \quad \theta(1, d-1) = \frac{d-1}{d+1}, \quad \theta(d_1, d_2) = \frac{d-2}{d} \quad \text{if } d_1, d_2 \geq 2.$$

See also Lau and Lü [5] in the context of holomorphic cusp forms.

Our aim in this paper is to estimate the sum of the coefficients a_n of a class of imprimitive L -functions, without assuming the generalized Ramanujan conjecture (1.1). We are able to prove the following result.

Theorem 1.1. *Let $L(s, \mathcal{A})$ is not primitive as in (1.5) with $d_1, d_2 \geq 2$. Assume that the coefficients $a_{i,n}$ of L -functions $L(s, \mathcal{A}_i)$ such that $\sum_{n \leq x} |a_{i,n}|^2 \ll x^{1+\varepsilon}$ with $i = 1, 2$. Then for any $\varepsilon > 0$, we have*

$$\begin{aligned} \sum_{n \leq X} a_n &= R(X) + O\left(X^{\eta+\varepsilon}\right) + O\left(Q_{\mathcal{A}}^{\frac{1}{4}} X^{\frac{1}{2}+(\frac{d}{4}-1)(1-\eta)+\varepsilon}\right) \\ &\quad + O\left(\sum_{X < n < X+X^\eta} |a_n|\right) \end{aligned} \quad (1.7)$$

for every $0 < \eta < 1$, where $R(x) = \text{Res}_{s=1} L(s, \mathcal{A}) x^s / s$, and the implied constant depends only on ε and the parameters $\beta_1, \beta_2, \dots, \beta_d$.

If, in addition, $a_n \geq 0$, then we have

$$\sum_{n \leq X} a_n = R(X) + O\left(Q_{\mathcal{A}}^{\frac{1}{d}} X^{1-\frac{2}{d}+\varepsilon}\right), \quad (1.8)$$

where the implied constant also depends only on ε and the parameters $\beta_1, \beta_2, \dots, \beta_d$.

Remark 1.2. The assumption is a natural condition which plays an important role in the proof of Lemma 2.2. However, it is not too restrictive and holds for many applications. For example, the automorphic L -functions, for which the condition holds by Rankin–Selberg theory.

Remark 1.3. Similar to (1.4), the equation (1.7) produces a formula for $\sum_{n \leq X} a_n$ in terms of a sum of a_n over a short interval. It can be removed by bounding individual coefficient a_n . However, there exists more effective approaches to estimate the sum in short intervals for some special cases. If $a_n \geq 0$, our result (1.8) is not only superior to (1.3), but also gives the explicit dependence of $Q_{\mathcal{A}}$, and we enhance the aspect of $Q_{\mathcal{A}}$ and remove the generalized Ramanujan conjecture when comparing to the result (1.6) of Friedlander–Iwaniec in the case $d_1, d_2 \geq 2$. For the general a_n , (1.7) is better than (1.4) if the parameter η is chosen with $\eta \leq \frac{d}{d+2}$.

Corollary 1.4. Let $m \geq 2$. Suppose that $L(s, \mathcal{A})$ has a more general decomposition

$$L(s, \mathcal{A}) = U(s) \prod_{i=1}^m L(s, \mathcal{A}_i), \quad (1.9)$$

and the degree of $L(s, \mathcal{A}_i)$ is d_i , $i = 1, 2, \dots, m$. Here $d_1, d_2 \geq 2$ and $d_i \geq 0$ for $3 \leq i \leq m$. Assume that the coefficients $a_{i,n}$ of L -functions $L(s, \mathcal{A}_i)$ such that $\sum_{n \leq x} |a_{i,n}|^2 \ll x^{1+\varepsilon}$ with $i = 1, 2$ and $U(s)$ admits a Dirichlet series convergent absolutely in $\Re s > \sigma_0$, where $1/2 < \sigma_0 < 1$. Let $d = d_1 + d_2 + \dots + d_m$ denote the degree of $L(s, \mathcal{A})$. Then we also have

$$\begin{aligned} \sum_{n \leq X} a_n &= R(X) + O(X^{\eta+\varepsilon}) + O\left(Q_{\mathcal{A}}^{\frac{1-\sigma_0}{2}} X^{\sigma_0+(\frac{d}{2}(1-\sigma_0)-1)(1-\eta)+\varepsilon}\right) \\ &\quad + O\left(\sum_{X < n < X+X^\eta} |a_n|\right). \end{aligned} \quad (1.10)$$

If $a_n \geq 0$, then we have

$$\sum_{n \leq X} a_n = R(X) + O\left(Q_{\mathcal{A}}^{\frac{1}{d}} X^{1-\frac{2}{d}+\varepsilon}\right) + O\left(Q_{\mathcal{A}}^{\frac{1-\sigma_0}{2}} X^{\sigma_0+\varepsilon}\right). \quad (1.11)$$

Remark 1.5. As an application, we consider integral power moments of Fourier coefficients of Hecke–Maass cusp forms. Since the assumptions in [Corollary 1.4](#) are unconditional in this context, we unconditionally improve previous results. See [Theorem 4.1](#) in [Section 4](#).

2. Convexity bounds and mean values for $L(s, \mathcal{A})$

We need the individual and averaged convexity bounds for $L(\mathcal{A}, s)$ in the critical strip.

Lemma 2.1. *Let $L(s, \mathcal{A})$ have a pole of order l at $s = 1$. Then for any $\varepsilon > 0$, we have*

$$\left(\frac{s-1}{s+1}\right)^l L(\sigma + it, \mathcal{A}) \ll (Q_{\mathcal{A}}(|t| + 1)^d)^{\max\{\frac{1-\sigma}{2}, 0\} + \varepsilon} \quad (2.1)$$

uniformly in $-\varepsilon \leq \sigma \leq 1 + \varepsilon$ and $t \in \mathbb{R}$.

Proof. It follows from [\[3, Eq. \(5.20\)\]](#). \square

Lemma 2.2. *Assume that the coefficients a_n of L -functions $L(s, \mathcal{A})$ such that $\sum_{n \leq x} |a_n|^2 \ll x^{1+\varepsilon}$. For any $\varepsilon > 0$, we have that for $\frac{1}{2} < \sigma < 1$,*

$$\int_1^T |L(\sigma + it, \mathcal{A})|^2 dt \ll_{\varepsilon} (Q_{\mathcal{A}}(T + 1)^d)^{1-\sigma+\varepsilon}, \quad (2.2)$$

where d is the degree of $L(s, \mathcal{A})$.

Proof. See [Perelli \[8\]](#). \square

3. Proof of [Theorem 1.1](#) and [Corollary 1.4](#)

Let $1 \leq Y \leq X/2$. We introduce a smooth compactly supported function $w(x)$ satisfying: $w(x) = 1$ for $x \in [2Y, X]$, $w(x) = 0$ for $n < Y$ and $n > X + Y$, and $w^{(j)}(x) \ll_j Y^{-j}$ for all $j \geq 0$.

Then we have

$$\sum_{n \leq X} a_n = \sum_n a_n w(n) + O\left(\sum_{n < 2Y} |a_n|\right) + O\left(\sum_{X < n < X+Y} |a_n|\right). \quad (3.1)$$

Moreover, by Mellin's inverse transform we have

$$\sum_n a_n w(n) = \frac{1}{2\pi i} \int_{(2)} \tilde{w}(s) L(s, \mathcal{A}) ds,$$

where the Mellin transform $\tilde{w}(s)$ is

$$\tilde{w}(s) = \int_0^\infty w(x)x^s \frac{dx}{x}.$$

By Cauchy's residue Theorem, we obtain that

$$\sum_n a_n w(n) = \operatorname{Res}_{s=1} \tilde{w}(s)L(s, \mathcal{A}) + \frac{1}{2\pi i} \int_{(1/2)} \tilde{w}(s)L(s, \mathcal{A})ds. \quad (3.2)$$

To go further, we notice that by partial integration,

$$\tilde{w}(s) = \frac{1}{s(s+1)\cdots(s+i-1)} \int_0^\infty w^{(i)}(x)x^{s+i-1}dx \ll \frac{Y}{X^{1-\sigma}} \cdot \left(\frac{X}{|s|Y}\right)^i \quad (3.3)$$

for any $i \geq 1$.

Then obviously the contribution from the integration over $|s| \geq T = X^{1+\varepsilon}/Y$ on the right hand side of (3.2) is negligibly small (i.e. $O(X^{-N})$ for any $N > 0$) if we choose a sufficiently large i .

Hence, we have

$$\begin{aligned} & \frac{1}{2\pi i} \int_{(1/2)} \tilde{w}(s)L(s, \mathcal{A})ds \\ & \ll \int_{1/2-iT}^{1/2+iT} |\tilde{w}(s)L(s, \mathcal{A})|dt + X^{-N} \\ & \ll X^{1/2} \int_{1/2-iT}^{1/2+iT} |L(s, \mathcal{A})| \cdot |s|^{-1}dt + X^{-N}. \end{aligned} \quad (3.4)$$

Here we have used (3.3) with $i = 1$. By (2.1) and standard arguments, we have

$$\begin{aligned} & \frac{1}{2\pi i} \int_{(1/2)} \tilde{w}(s)L(s, \mathcal{A})ds \\ & \ll Q_{\mathcal{A}}^{\frac{1}{4}} X^{\frac{1}{2}+\varepsilon} + X^{\frac{1}{2}} \max_{1 \leq T_1 \leq T} \frac{1}{T_1} \int_{\frac{T_1}{2}}^{T_1} \left| L\left(\frac{1}{2} + it, \mathcal{A}\right) \right| dt. \end{aligned} \quad (3.5)$$

By the Cauchy inequality and Lemma 2.2, we have

$$\begin{aligned}
& \frac{1}{2\pi i} \int_{(1/2)} \tilde{w}(s) L(s, \mathcal{A}) ds \\
& \ll X^{\frac{1}{2}} \max_{1 \leq T_1 \leq T} \frac{1}{T_1} \left(\int_{\frac{T_1}{2}}^{T_1} \left| L\left(\frac{1}{2} + it, \mathcal{A}_1\right) \right|^2 dt \right)^{\frac{1}{2}} \left(\int_{\frac{T_1}{2}}^{T_1} \left| L\left(\frac{1}{2} + it, \mathcal{A}_2\right) \right|^2 dt \right)^{\frac{1}{2}} \quad (3.6) \\
& \quad + Q_{\mathcal{A}}^{\frac{1}{4}} X^{\frac{1}{2} + \varepsilon} \\
& \ll Q_{\mathcal{A}}^{\frac{1}{4}} T^{\frac{d}{4} - 1} X^{\frac{1}{2} + \varepsilon}.
\end{aligned}$$

On inserting (3.6) into (3.2), we derive

$$\sum_n a_n w(n) = R(X) + O\left(Q_{\mathcal{A}}^{\frac{1}{4}} T^{\frac{d}{4} - 1} X^{\frac{1}{2} + \varepsilon}\right), \quad (3.7)$$

where $R(x) = \text{res}_{s=1} L(s, \mathcal{A}) x^s / s$. Let $Y = X^\eta$ with any $0 < \eta < 1$. We obtain from (3.1) that

$$\sum_{n \leq X} a_n = R(X) + O(X^{\eta + \varepsilon}) + O\left(Q_{\mathcal{A}}^{\frac{1}{4}} X^{\frac{1}{2} + (\frac{d}{4} - 1)(1 - \eta) + \varepsilon}\right) + O\left(\sum_{X < n < X + X^\eta} |a_n|\right). \quad (3.8)$$

For case $a_n \geq 0$, we choose two smooth compactly supported functions $w_\pm(x)$: $w_+(x) = 1$ for $x \in [2Y, X]$, $w_+(x) = 0$ for $n < Y$ and $n > X + Y$; $w_-(x) = 1$ for $x \in [2Y, X - Y]$, $w_-(x) = 0$ for $x \geq X$ and $x \leq Y$; and $w_\pm^{(j)}(x) \ll Y^{-j}$ for all $j \geq 0$. Then we have

$$\sum_n a_n w_-(n) + O(X^{\eta + \varepsilon}) \leq \sum_{n \leq X} a_n \leq \sum_n a_n w_+(n) + O(X^{\eta + \varepsilon}). \quad (3.9)$$

Following the above arguments, we could get

$$\sum_n a_n w_\pm(n) = R(X) + O(X^{\eta + \varepsilon}) + O\left(Q_{\mathcal{A}}^{\frac{1}{4}} X^{\frac{1}{2} + (\frac{d}{4} - 1)(1 - \eta) + \varepsilon}\right). \quad (3.10)$$

On choosing $\eta = 1 - \frac{2}{d} + \frac{\log Q_{\mathcal{A}}}{d \log X}$, we have

$$\sum_{n \leq X} a_n = R(X) + O\left(Q_{\mathcal{A}}^{\frac{1}{4}} X^{1 - \frac{2}{d} + \varepsilon}\right). \quad (3.11)$$

This completes the proof of Theorem 1.1.

As for the proof of Corollary 1.4, instead of shifting the integration along $\Re s = 2$ to the parallel line $\Re s = 1/2$, we move it to $\Re s = \sigma_0 + \epsilon$. Then the contribution of $U(s)$ can

be absorbed into the O -constant, since it is absolutely convergent in $\Re s > \sigma_0$. It suffices to estimate

$$X^{\sigma_0+\varepsilon} \max_{1 \leq T_1 \leq T} \frac{1}{T_1} \int_{\frac{T_1}{2}}^{T_1} |L(\sigma_0 + \varepsilon + it, \mathcal{A})| dt.$$

We first take $L(s, \mathcal{A}_i)$, $i = 3, 4, \dots, m$ out of the integral, and then apply [Lemmas 2.1 and 2.2](#) with $\sigma = \sigma_0 + \varepsilon$ to get

$$\begin{aligned} & X^{\sigma_0+\varepsilon} \max_{1 \leq T_1 \leq T} \frac{1}{T_1} \int_{\frac{T_1}{2}}^{T_1} |L(\sigma_0 + \varepsilon + it, \mathcal{A})| dt \\ & \ll X^{\sigma_0+\varepsilon} \max_{1 \leq T_1 \leq T} \frac{1}{T_1} (Q_{\mathcal{A}_3} Q_{\mathcal{A}_4} \cdots Q_{\mathcal{A}_m})^{\frac{1-\sigma_0}{2}} T_1^{(d-d_1-d_2)\frac{1-\sigma_0}{2}} \times \\ & \quad \times \int_{\frac{T_1}{2}}^{T_1} |L(\sigma_0 + \varepsilon + it, \mathcal{A}_1) L(\sigma_0 + \varepsilon + it, \mathcal{A}_2)| dt \\ & \ll X^{\sigma_0+\varepsilon} \max_{1 \leq T_1 \leq T} \frac{1}{T_1} (Q_{\mathcal{A}_3} Q_{\mathcal{A}_4} \cdots Q_{\mathcal{A}_m})^{\frac{1-\sigma_0}{2}} T_1^{(d-d_1-d_2)\frac{1-\sigma_0}{2}} \times \\ & \quad \times \left(\int_{\frac{T_1}{2}}^{T_1} |L(\sigma_0 + \varepsilon + it, \mathcal{A}_1)|^2 dt \right)^{\frac{1}{2}} \left(\int_{\frac{T_1}{2}}^{T_1} |L(\sigma_0 + \varepsilon + it, \mathcal{A}_2)|^2 dt \right)^{\frac{1}{2}} \\ & \ll Q_{\mathcal{A}}^{\frac{1-\sigma_0}{2}} T^{\frac{d(1-\sigma_0)}{2}-1} X^{\sigma_0+\varepsilon}. \end{aligned} \tag{3.12}$$

This essentially determines the result of [Corollary 1.4](#).

4. Applications to Fourier coefficients of Maass forms

We consider Maass forms for the full modular group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, which are eigenfunctions of all the Hecke operators T_n . Let S_r be the set of normalized primitive Maass cusp form of eigenvalue $\lambda = 1/4 + r^2$, then $f(z) \in S_r$ has the Fourier expansion

$$f(z) = \sum_{n \neq 0} \lambda_f(n) \sqrt{y} K_{ir}(2\pi |n| y) e(nx), \tag{4.1}$$

where the coefficients $\lambda_f(n) \in \mathbb{R}$ are eigenvalues of T_n and K_{ir} is the K -Bessel function. The Ramanujan conjecture asserts that

$$|\lambda_f(n)| \leq d(n), \tag{4.2}$$

where $d(n)$ is the number of positive divisors of n . The result of Kim and Sarnak [4] (see also [9, Theorem 1.2]) implies

$$|\lambda_f(n)| \leq n^{\frac{7}{64}} d(n). \quad (4.3)$$

In order to generalize Rankin and Selberg's famous result

$$\sum_{n \leq x} \lambda_f(n)^2 = cx + O(x^{\frac{3}{5}}), \quad (4.4)$$

Lau and Lü [5] considered the ℓ th power moment

$$S_\ell(f; x) := \sum_{n \leq x} \lambda_f(n)^\ell.$$

They obtained that for $\ell = 4, 6, 8$

$$S_\ell(f; x) = xP_\ell(\log x) + O_{f,\varepsilon}(x^{\theta_\ell+\varepsilon}) \quad (\ell = 4, 6, 8), \quad (4.5)$$

where $P_4(t), P_6(t), P_8(t)$ are polynomials of degree 1, 4, 13 respectively and

$$\theta_4 = \frac{15}{17} = 0.8823\dots, \quad \theta_6 = \frac{63}{65} = 0.9692\dots, \quad \theta_8 = \frac{255}{257} = 0.9922\dots \quad (4.6)$$

Up to now, we do not find nontrivial results for $S_\ell(f; x)$ ($\ell = 3, 5, 7$).

By applying Corollary 1.4, we shall refine the exponents θ_ℓ for $\ell = 4, 6, 8$ and give nontrivial results for $S_\ell(f; x)$ ($\ell = 3, 5, 7$).

Theorem 4.1. *Under the previous notation, we have*

$$\begin{aligned} S_\ell(f; x) &= xP_\ell(\log x) + O_{f,\varepsilon}(x^{\theta_\ell+\varepsilon}), \\ \theta_3 &= \frac{455}{594} = 0.76599\dots, \quad \theta_5 = \frac{15}{16} = 0.9375, \quad \theta_7 = \frac{63}{64} = 0.984375, \\ \theta_4 &= \frac{7}{8} = 0.875, \quad \theta_6 = \frac{31}{32} = 0.96875, \quad \theta_8 = \frac{127}{128} = 0.99218\dots, \end{aligned}$$

where $P_4(t), P_6(t), P_8(t)$ are polynomials of degree 1, 4, 13 respectively and $P_\ell(t) \equiv 0$ for $\ell = 3, 5, 7$.

Remark 4.2. The results in Theorem 4.1 can be improved further, if we apply subconvexity bounds for $\zeta(s), L(s, f)$ and $L(s, \text{sym}^2 f)$. For example,

$$\zeta(\sigma + it) \ll_\varepsilon (|t| + 1)^{\max\{(1/3)(1-\sigma), 0\} + \varepsilon}, \quad (4.7)$$

$$L(\sigma + it, f) \ll_{f,\varepsilon} (|t| + 1)^{\max\{(2/3)(1-\sigma), 0\} + \varepsilon}, \quad (4.8)$$

and

$$L(\sigma + it, \text{sym}^2 f) \ll_{f, \varepsilon} (|t| + 1)^{\max\{(11/8)(1-\sigma), 0\} + \varepsilon} \quad (4.9)$$

uniformly for $\frac{1}{2} \leq \sigma \leq 2$. These results are due to Weyl, Meurman [7] and Li [6] respectively.

4.1. Some estimates related to $\lambda_f(n)$ and decomposition of Dirichlet series

Define the Hecke L -function associated to $f(z)$ by

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\beta_p}{p^s}\right)^{-1}, \quad (4.10)$$

where the local parameters α_p and β_p satisfy

$$\alpha_p + \beta_p = \lambda_f(p) \quad \text{and} \quad \alpha_p \beta_p = 1. \quad (4.11)$$

The result towards the Ramanujan conjecture by Kim and Sarnak [4] states that

$$|\alpha_p|, |\beta_p| \leq p^{7/64}. \quad (4.12)$$

We define the m th symmetric power L -function by the degree $m+1$ Euler product

$$L(s, \text{sym}^m f) = \prod_p \prod_{0 \leq j \leq m} \left(1 - \frac{\alpha_p^{m-j} \beta_p^j}{p^s}\right)^{-1}, \quad (4.13)$$

and the Rankin–Selberg convolution L -function of $\text{sym}^m f$ and $\text{sym}^n f$ by the degree $m+n+2$ Euler product

$$L(s, \text{sym}^m f \times \text{sym}^n f) = \prod_p \prod_{0 \leq j \leq m} \prod_{0 \leq i \leq n} \left(1 - \frac{\alpha_p^{m-2j} \alpha_p^{n-2i}}{p^s}\right)^{-1}. \quad (4.14)$$

It is easy to see that

$$\begin{cases} L(s, \text{sym}^0 f) = \zeta(s), \\ L(s, \text{sym}^1 f) = L(s, f), \\ L(s, \text{sym}^m f \times \text{sym}^0 f) = L(s, \text{sym}^m f). \end{cases}$$

Lemma 4.3. *Let us define*

$$F_\ell(s) := \sum_{n \geq 1} \lambda_f(n)^\ell n^{-s}. \quad (4.15)$$

Then $F_\ell(s)$ can be decomposed into

$$F_\ell(s) = G_\ell(s)H_\ell(s) \quad (4.16)$$

for $3 \leq \ell \leq 8$, where

$$G_3(s) = L(s, f)^2 L(s, \text{sym}^3 f),$$

$$G_4(s) = \zeta(s)^2 L(s, \text{sym}^2 f)^3 L(s, \text{sym}^4 f),$$

$$G_5(s) = L(s, f)^5 L(s, \text{sym}^3 f)^3 L(s, \text{sym}^4 f \times f),$$

$$G_6(s) = \zeta(s)^5 L(s, \text{sym}^2 f)^8 L(s, \text{sym}^4 f)^4 L(s, \text{sym}^4 f \times \text{sym}^2 f),$$

$$G_7(s) = L(s, f)^{13} L(s, \text{sym}^3 f)^8 L(s, \text{sym}^4 f \times f)^5 L(s, \text{sym}^4 f \times \text{sym}^3 f),$$

$$G_8(s) = \zeta(s)^{13} L(s, \text{sym}^2 f)^{21} L(s, \text{sym}^4 f)^{13} L(s, \text{sym}^4 f \times \text{sym}^2 f)^6 L(s, \text{sym}^4 f \times \text{sym}^4 f),$$

and the functions $H_\ell(s) := \prod_p L_{\ell,p}(p^{-s})$. $L_{\ell,p}$ are polynomials of finite degree. Furthermore, $H_\ell(s)$ admit Dirichlet series convergent absolutely in $\Re s \geq 1/2 + \varepsilon$ except $H_7(s)$ and $H_8(s)$ in which cases converge absolutely in $\Re s \geq 39/64 + \varepsilon$ and $\Re s \geq 23/32 + \varepsilon$ respectively, and $H_\ell(s) \neq 0$ for $\Re s = 1$.

Proof. From the multiplicative property of $\lambda_f(n)^\ell$, we have the Euler product identity

$$F_\ell(s) = \prod_p \left(\sum_{v=0}^{\infty} \lambda_f(p^v)^\ell p^{-vs} \right).$$

Now we shall take $T = p^{-s}$ in the following. Then by the same arguments as those in [5, Lemma 7.1], we can get the decomposition of $F_\ell(s)$ above and know that all coefficients of $G_\ell(s)$ are nonnegative if ℓ is even. To see the property of $H_\ell(s)$, a straightforward calculation shows that (to simplify the notation we put $a = \lambda_f(p)$)

$$\begin{aligned} L_{3,p}(T) &= 1 - (3a^2 - 3)T^2 + 2a^3T^3 - (3a^2 - 3)T^4 + T^6, \\ L_{4,p}(T) &= 1 - (6a^4 - 12a^2 + 7)T^2 + \cdots - T^{14}, \\ L_{5,p}(T) &= 1 - (10a^6 - 30a^4 + 35a^2 - 15)T^2 + \cdots + T^{30}, \\ L_{6,p}(T) &= 1 - (15a^8 - 60a^6 + 105a^4 - 90a^2 + 31)T^2 + \cdots - T^{62}, \\ L_{7,p}(T) &= 1 - (21a^{10} - 108a^8 + 245a^6 - 315a^4 + 217a^2 - 63)T^2 \\ &\quad + \cdots + T^{126}, \\ L_{8,p}(T) &= 1 - (28a^{12} - 168a^{10} + 490a^8 - 840a^6 + 868a^4 - 504a^2 + 127)T^2 \\ &\quad + \cdots - T^{254}. \end{aligned} \quad (4.17)$$

All the coefficients of $L_{\ell,p}(T)$ can be calculated explicitly. Some are omitted here because of their complexities. From the absolute convergence of $L(s, \text{sym}^m f \times \text{sym}^m f)$ in $\sigma = \Re s > 1 + \varepsilon$ for $m \leq 4$, that is $L(\sigma + it, \text{sym}^m f \times \text{sym}^m f) \ll 1$, we know

$$\prod_p \left(1 + \frac{|\alpha_p|^{2m} + |\beta_p|^{2m}}{p^\sigma} \right) \ll 1, \quad \text{for } m \leq 4. \quad (4.18)$$

The reason is that if the Ramanujan conjecture holds at place p , that is $|\alpha_p| = |\beta_p| = 1$ then (4.18) holds obviously, otherwise α_p and β_p are real, then (4.18) also holds by positivity the Euler factor at p in (4.14). Combining (4.11), (4.12), the expression (4.17) of $L_{\ell,p}(T)$ and (4.18), we deduce that $H_7(s)$ and $H_8(s)$ converge absolutely in the region $\Re s \geq 39/64 + \varepsilon$ and $\Re s \geq 23/32 + \varepsilon$ respectively, the others in $\Re s \geq 1/2 + \varepsilon$. This completes the proof of Lemma 4.3. \square

Lemma 4.4. *We have the following inequality*

$$|\lambda_f(n)|^8 \ll n^\varepsilon \sum_{d|n} |\lambda_f(d^4)|^2. \quad (4.19)$$

Proof. It is known that

$$\lambda_f(p^k) = \sum_{0 \leq m \leq k} \alpha_p^m \beta_p^{k-m} = \sum_{0 \leq m \leq k} \alpha_p^{k-2m}. \quad (4.20)$$

If the Ramanujan bound holds at prime p , that is $|\alpha_p| = |\beta_p| = 1$. Then we have

$$|\lambda_f(p^k)| \leq k + 1. \quad (4.21)$$

If the Ramanujan bound does not hold at prime p , we learn that α_p and β_p are real. By the multiplicative property of $\lambda_f(n)$, we get

$$\lambda_f(p^k)^2 = \sum_{d|p^k} \lambda_f \left(\frac{p^{2k}}{d^2} \right) = \lambda_f(p^{2k}) + \lambda_f(p^{2k-2}) + \cdots + 1. \quad (4.22)$$

Since each term α_p^{k-2m} in (4.20) for $0 \leq m \leq k$ has the same sign, then

$$\lambda_f(p^{2k}) > \lambda_f(p^{2k-2}) > \cdots > \lambda_f(p^2) > 1.$$

Thus, we have

$$\lambda_f(p^k)^4 \leq (k+1)^2 \lambda_f(p^{2k})^2 \leq (k+1)^4 \lambda_f(p^{4k}). \quad (4.23)$$

Suppose now that the integer n has the standard decomposition

$$n = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s} \quad (4.24)$$

such that Ramanujan conjecture holds at p_1, \dots, p_i (after possible reordering). Then we derive

$$\begin{aligned}
\lambda_f(n)^4 &= \lambda_f(p_1^{a_1})^4 \cdots \lambda_f(p_i^{a_i})^4 \lambda_f(p_{i+1}^{a_{i+1}})^4 \cdots \lambda_f(p_s^{a_s})^4 \\
&\leq (a_1 + 1)^4 \cdots (a_s + 1)^4 \lambda_f(p_{i+1}^{4k}) \cdots \lambda_f(p_s^{4k}) \\
&\leq (a_1 + 1)^4 \cdots (a_s + 1)^4 \lambda_f(p_{i+1}^{4k} \cdots p_s^{4k}) \\
&\leq d(n)^4 \sum_{d|n} |\lambda_f(d^4)|.
\end{aligned} \tag{4.25}$$

Therefore, [Lemma 4.4](#) immediately follows from the Cauchy–Schwarz inequality. \square

Lemma 4.5. *For any $\varepsilon > 0$, we have*

$$\sum_{n \leq x} \sum_{d|n} |\lambda_f(d^4)|^2 = b_1 x \log x + b_0 x + O\left(x^{\frac{25}{27} + \varepsilon}\right), \tag{4.26}$$

where b_0 and b_1 are constants depended on f .

Proof. Similar to the proof of [Lemma 4.3](#), we can derive

$$\sum_{n \geq 1} \sum_{d|n} |\lambda_f(d^4)|^2 n^{-s} = \zeta(s) L(s, \text{sym}^4 f \times \text{sym}^4 f) U(s), \tag{4.27}$$

where $U(s)$ is absolutely convergent in $\Re s \geq 23/32 + \varepsilon$. Thus this lemma follows from [\(1.3\)](#) (see e.g. Chandrasekharan and Narasimhan [\[1\]](#)). \square

4.2. Proof of [Theorem 4.1](#)

For $3 \leq \ell \leq 8$, obviously $F_\ell(s)$ satisfies the conditions in [Corollary 1.4](#). For example, if $\ell = 4, 6, 8$, we can take

$$L(s, \mathcal{A}_1) = L(s, \text{sym}^2 f), \quad L(s, \mathcal{A}_2) = L(s, \text{sym}^4 f).$$

Hence the condition that the Rankin–Selberg L -functions $L(s, \mathcal{A}_1 \otimes \overline{\mathcal{A}_1})$ and $L(s, \mathcal{A}_2 \otimes \overline{\mathcal{A}_2})$ are absolutely convergent for $\Re(s) > 1$ follows from the automorphy of the k th symmetric power lift $\text{sym}^k f$ ($1 \leq k \leq 4$) and the Rankin–Selberg theory on automorphic cuspidal representations.

After applying [Corollary 1.4](#) to the Dirichlet series $F_\ell(s)$, we get

$$\sum_{n \leq x} \lambda_f(n)^\ell = x P_\ell(\log x) + O_{f, \varepsilon}(x^{\theta_\ell + \varepsilon}) \quad (\ell = 4, 6, 8), \tag{4.28}$$

where $P_4(t), P_6(t), P_8(t)$ are polynomials of degree 1, 4, 13 respectively and

$$\theta_4 = \frac{7}{8} = 0.875, \quad \theta_6 = \frac{31}{32} = 0.9687 \dots, \quad \theta_8 = \frac{127}{128} = 0.9921 \dots \tag{4.29}$$

For $\ell = 3, 5, 7$, we have

$$\begin{aligned}
 & \sum_{n \leq x} \lambda_f(n)^\ell \\
 & \ll x^{\eta+\varepsilon} + x^{\sigma_0 + (2^{\ell-1}(1-\sigma_0)-1)(1-\eta)+\varepsilon} + \sum_{x < n < x+x^\eta} |\lambda_f(n)|^\ell \\
 & \ll x^{\eta+\varepsilon} + x^{\sigma_0 + (2^{\ell-1}(1-\sigma_0)-1)(1-\eta)+\varepsilon} \\
 & \quad + \left(\sum_{x < n < x+x^\eta} |\lambda_f(n)|^2 \right)^{\frac{8-\ell}{6}} \left(\sum_{x < n < x+x^\eta} |\lambda_f(n)|^8 \right)^{\frac{\ell-2}{6}}.
 \end{aligned} \tag{4.30}$$

By (4.4), Lemmas 4.4 and 4.5, we have

$$\begin{aligned}
 & \sum_{n \leq x} \lambda_f(n)^\ell \\
 & \ll x^{\eta+\varepsilon} + x^{\sigma_0 + (2^{\ell-1}(1-\sigma_0)-1)(1-\eta)+\varepsilon} + x^{\frac{(8-\ell)\eta}{6}+\varepsilon} \left(\sum_{x < n < x+x^\eta} \sum_{d|n} |\lambda_f(d^4)|^2 \right)^{\frac{\ell-2}{6}} \\
 & \ll x^{\eta+\varepsilon} + x^{\sigma_0 + (2^{\ell-1}(1-\sigma_0)-1)(1-\eta)+\varepsilon} + x^{\frac{(8-\ell)\eta}{6} + \frac{25}{27} \cdot \frac{\ell-2}{6} + \varepsilon},
 \end{aligned}$$

provided that $\eta \geq 3/5$.

If $\ell = 3$, on taking $\eta = \frac{218}{297}$, we have (note that $3/5 \leq \eta \leq 25/27$)

$$\sum_{n \leq x} \lambda_f(n)^3 \ll x^{\eta+\varepsilon} + x^{\frac{1}{2}+1-\eta+\varepsilon} + x^{\frac{5\eta}{6} + \frac{25}{27} \cdot \frac{1}{6} + \varepsilon} \ll x^{\frac{455}{594} + \varepsilon}.$$

If $\ell = 5$, on taking $\eta = \frac{15}{16}$, we have (note that $\eta \geq 25/27$)

$$\sum_{n \leq x} \lambda_f(n)^5 \ll x^{\eta+\varepsilon} + x^{\frac{1}{2}+7-7\eta+\varepsilon} \ll x^{\frac{15}{16} + \varepsilon}.$$

If $\ell = 7$, on taking $\eta = \frac{63}{64}$, we have (note that $\sigma_0 = 39/64$ and $\eta \geq 25/27$)

$$\sum_{n \leq x} \lambda_f(n)^7 \ll x^{\eta+\varepsilon} + x^{\frac{39}{64}+24-24\eta+\varepsilon} \ll x^{\frac{63}{64} + \varepsilon}.$$

This completes the proof of Theorem 4.1.

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