Journal of Number Theory 171 (2017) 213–226

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

No general Riemann–Hurwitz formula for relative p-class groups

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ARTICLE INFO

Article history:
Received 1 February 2016
Received in revised form 7 July 2016
Accepted 14 July 2016
Available online 6 September 2016
Communicated by D. Goss

Keywords:
Ideal class groups
Kida's formula
Iwasawa theory
Class field theory
p-ramification
Ambiguous classes

ABSTRACT

We disprove, by means of numerical examples and theoretical arguments, illustrated with $p = 3$, the existence of a Riemann–Hurwitz formula for the p-ranks of relative class groups in a p-ramified p-extension K/k of number fields of CM-type containing μ_p in contradiction with a result published in 1996. In the cyclic case of degree p, under some assumptions on the p-class group of k and the decomposition of the p-places, we prove some results on the structure of the p-class group of K and justify that some theoretical structures do not exist in this particular situation. In this context, an analogue of Kida's formula is valid for the p-ranks if and only if the p-class group of K is reduced to the group of ambiguous classes, which is not always the case, as shown by a numerical table for $p = 3$.

1. Generalities

In [Ki, 1980], Y. Kida proved an analogue of the Riemann–Hurwitz formula for the minus part of the Iwasawa λ-invariant (under the nullity of the μ-invariant) of the

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http://dx.doi.org/10.1016/j.jnt.2016.07.016
0022-314X/

cyclotomic $\mathbb{Z}_p$-extension k_∞ of an algebraic number field k of CM-type containing the group μ_p of pth roots of unity:

$$\lambda^-(K) - 1 = [K : k] \cdot (\lambda^-(k) - 1) + \sum_{v^+} \big(e_{v^+}(K_\infty^+/k_\infty^+) - 1\big), \tag{1}$$

where K/k is a finite p-extension of CM-fields (the maximal totally real subfields of k and K are denoted k^+ and K^+, respectively; to simplify we have supposed $K \cap k_\infty = k$); then v^+ ranges over all non-p-places of K_∞^+, split in K_∞/K_∞^+, e_{v^+} being the corresponding ramification index. See also [Iw, § 9] giving again this formula.

When K/k is p-ramified (i.e., unramified outside p) the formula reduces to:

$$\lambda^-(K) - 1 = [K : k] \cdot (\lambda^-(k) - 1). \tag{2}$$

Many generalizations of Kida's formula were given as for instance in [Sch]. In [FOO] it is shown (for the primes 2, 3, and 5) the existence of $\mathbb{Z}_p$-extensions with prescribed Iwasawa λ-invariant by using Kida's formula.

A general formula was established at the origin by Iwasawa [Iw, 1981, Theorem 6] for the whole λ-invariant; for instance, when K/k is cyclic of degree p and the μ-invariant is 0, one gets:

$$\lambda(K) - 1 = p \cdot (\lambda(k) - 1) + (p - 1) \cdot (\chi(G, E_{K_\infty}) + 1) + \sum_{w} \big(e_w(K_\infty/k_\infty) - 1\big),$$

where w ranges over all non-p-places of K_∞, where $p^{\chi(G,E_{K_\infty})}$ is the Herbrand quotient $\dfrac{\mathrm{H}^2(G, E_{K_\infty})}{\mathrm{H}^1(G, E_{K_\infty})}$ of the group E_{K_∞} of units of K_∞ and $G = \mathrm{Gal}(K_\infty/k_\infty)$ (see [Sch, Theorem 1.5 and Theorem 1.9] for some generalizations). For generalized class groups with ramification and decomposition, see [JaMa,JaMi].

Of course, in the CM-fields case for relative class groups, the units of infinite order "disappear" as is well known, and one obtains Kida's formula using only obvious parameters from K/k (still assuming the nullity of the μ-invariant).

Definitions 1.1. (i) For any number field F, let $\mathcal{C}\ell_F$ be its p-class group and let $\mathcal{C}\ell_F^\pm$ be its two usual components when F is a CM-field, so that $\mathcal{C}\ell_F = \mathcal{C}\ell_F^+ \oplus \mathcal{C}\ell_F^-$ (to simplify, we shall suppose $p \neq 2$).
(ii) We denote by H_F the p-Hilbert class field of F.
(iii) The p-rank of a $\mathbb{Z}_p$-module M is the $\mathbb{F}_p$-dimension of the $\mathbb{F}_p$-space M/M^p.

An interesting question is to ask if a Kida's formula can be valid, in a p-extension K/k of CM-fields, for the p-ranks (r_K^- and r_k^-) of the relative class groups $\mathcal{C}\ell_K^-$ and $\mathcal{C}\ell_k^-$. If so, such a formula should be (in the simplest case where K/k is p-ramified and contains μ_p) analogous to formula (2), that is to say:

$$r_K^- - 1 = [K : k] \cdot (r_k^- - 1). \tag{3}$$

Indeed, in a work using Iwasawa theory and published by K. Wingberg [W, 1996], it is proposed, in a very general framework, an analog of Kida's formula for the p-ranks of relative S-class groups in some S-ramified p-extensions containing μ_p, for a set of finite places S containing the p-places [W, Corollary 2.2 (i)] and applied to the following data (taking for S the set of p-places):

k is a CM-field containing μ_p such that no prime of k^+ above p splits in k, and K^+ is a totally real p-ramified p-extension of k^+; then $K := K^+(\mu_p) = K^+k$.

This leads, for K/k, exactly to formula (3) that we intend to disprove (see [W, Corollary 2.2 (ii) & Remark 2.3]).

We can be astonished by a result which is not really "arithmetical" since many of our class group investigations show that such a "regularity" only happens at infinity (Iwasawa theory). The analytic proof of Kida's formula given by W. Sinnott [Sin, 1984], using p-adic L-functions, is probably the most appropriate to see the transition from one aspect to the other giving the λ-invariants as "ultimate p-ranks" of class groups in the cyclotomic $\mathbb{Z}_p$-extensions.

Indeed, finding a relation between the p-ranks of p-class groups (e.g., in K/k cyclic of degree p with Galois group G), depends on non-obvious structures of finite $\mathbb{Z}_p[G]$-modules M $(= \mathcal{C}\!\ell_K)$ provided with an arithmetical norm $\mathrm{N}_{K/k}$ and a transfer map $i_{K/k}$ (with $i_{K/k} \circ \mathrm{N}_{K/k} = \nu_{K/k} := 1 + \sigma + \cdots + \sigma^{p-1}$ where σ is a generator of G), the filtration of the $M_i := \big\{h \in M,\ h^{(1-\sigma)^i} = 1\big\}$ playing an important non-algebraic role because all the orders $\#(M_{i+1}/M_i)$ depend on *arithmetical local normic computations* by means of formulas, given in [Gr3, 1973] and systematized in [Gr2, 1994] or [Gr4, 2016], similar to that of the case $i = 0$ of Chevalley's ambiguous class number formula (see § 4.1).

To be more convincing, we shall give numerical computations and we shall see that it is not difficult to conjecture that there are infinitely many counterexamples to the formula (3) which may be true in some cases.

2. A numerical counterexample

Using PARI (from [P]), we give in § 4.2 a program which can be used by the reader to compute easily, for the case $p = 3$, the 3-ranks of class groups in some 3-ramified cubic cyclic extensions K/k over a biquadratic field k containing a primitive 3rd root of unity ζ, and such that its 3-class group is (for instance) of order 3.

2.1. Definitions and assumptions

Consider the following diagram:

$$\begin{tikzcd}
K^{+} \arrow[rr, dash, "s"] \arrow[d, dash] & & K = k(\sqrt[3]{\alpha}) \arrow[d, dash, "{[K:k]=3}"] \\
k^{+} = \mathbb{Q}(\sqrt{3\cdot d}) \arrow[rr, dash] \arrow[dd, dash] & & k = k^{+}(\zeta) \arrow[dd, dash] \arrow[dl, dash, "\mathrm{t}"'] \\
& k^{-} = \mathbb{Q}(\sqrt{-d}) \arrow[dl, dash] & \\
\mathbb{Q} \arrow[rr, dash] & & \mathbb{Q}(\zeta) = \mathbb{Q}(\sqrt{-3})
\end{tikzcd}$$

Recall, in this particular context about the 3-class group of k, the hypothesis of [W, Corollary 2.2 (ii)]; we shall suppose these conditions satisfied in all the sequel.

Hypothesis 2.1. (i) $d > 0$, d squarefree, $d \not\equiv 0 \pmod 3$,
(ii) $p = 3$ does not split in $k/\mathbb{Q}$ (hence $d \equiv 1 \pmod 3$),
(iii) K^+/k^+ is a 3-ramified cubic cyclic extension and $K = K^+(\zeta)$,
(iv) K^+ is not contained in the cyclotomic $\mathbb{Z}_3$-extension of k^+,[1]
(v) $\mathcal{C}\ell_{k^+} = 1$ & $\mathcal{C}\ell_{k^-} \simeq \mathbb{Z}/3\mathbb{Z}$ (equivalent to $\mathcal{C}\ell_k \simeq \mathbb{Z}/3\mathbb{Z}$).

From (v), the ambiguous class number formula shall give $\mathcal{C}\ell_{K^+} = \mathcal{C}\ell_K^+ = 1$, so that $\mathcal{C}\ell_K^- = \mathcal{C}\ell_K$ (Lemma 2.3); this is convenient for numerical calculations because this avoids the computation of $\mathcal{C}\ell_K^+$ since PARI gives the whole class group. Moreover, in spite of appearances, this context is rich enough for our purpose.

2.2. Recalls on Kummer and class field theories

Starting from a 3-ramified cubic cyclic extension K^+/k^+, the associated Kummer extension K/k is $k(\sqrt[3]{\beta})/k$, for $\beta \in k^\times \setminus k^{\times 3}$, $(\beta) = \mathfrak{b}^3$, up to a power of $\mathfrak{p} = (\sqrt{-3})$, for an ideal $\mathfrak{b}$ of k, and $\beta^{s+1} \in k^{\times 3}$, where $s \in \mathrm{Gal}(K/K^+)$ is the complex conjugation (usual decomposition criterion of a Kummer extension over a subfield) which gives $(\beta) = \mathfrak{b}^3$ where we can choose $\mathfrak{b}$ prime to p. More precisely:

Lemma 2.2. *Under the above conditions, a system of solutions β giving independent extensions $k(\sqrt[3]{\beta})/k$ is for instance $\{\alpha, \alpha\cdot\zeta, \alpha\cdot\zeta^2, \zeta\}$, the number $\alpha \in k^-$ being obtained from the non-trivial 3-class of k^-.*

Proof. The Kummer extension K/k is 3-ramified if and only if $\beta \in \langle 3, \zeta, \varepsilon, \alpha\rangle_{\mathbb{Z}} \cdot k^{\times 3}$, where ε is the fundamental unit of k^+. But the cubic subextensions of $k(\sqrt[3]{3}, \sqrt[3]{\varepsilon})$ are

[1] Conditions (iii) and (iv) imply that the "minimal" base field $k \supset \mu_3$ is a suitable biquadratic field.

not decomposed over k^+ and $k(\sqrt[3]{\zeta})$ is decomposed over k^+ but is not allowed for our purpose because of the condition (iv) (the 3-rank of the Galois group of the maximal Abelian 3-ramified 3-extension of k^+ is $r^- + 1 = 2$ because 3 does not split in k; see [Gr1, Proposition III.4.2.2] for the general statement).

So, $\beta \in \langle \zeta, \alpha \rangle_{\mathbb{Z}} \cdot k^{\times 3}$ with α in k^- such that $(\alpha) = \mathfrak{a}^3$ ($\mathfrak{a}$ must be a non-principal ideal since E_{k^-} is trivial); this defines a canonical 3-ramified cubic cyclic extension K^+ of k^+ and the numbers $\alpha{\cdot}\zeta$, $\alpha{\cdot}\zeta^2$ define the two other 3-ramified cubic cyclic extensions K_1^+, K_2^+ of k^+, distinct from the first step of k_∞^+ defined by ζ (which explains the "dimension 2"). $\square$

In all our numerical examples, we shall only use the canonical "pseudo-unit" α since the Galois group of $K^+/\mathbb{Q}$ is the dihedral group D_6 while the Galois group of the Galois closure of $K_1^+/\mathbb{Q}$ or of $K_2^+/\mathbb{Q}$ is of order 18 (group $F_{18}(6) = [3^2]2 = 3 \text{ wr } 2$, from [KM]). For instance, with $k = \mathbb{Q}(\sqrt{-211}, \sqrt{-3})$, $\alpha = \frac{17+\sqrt{-211}}{2}$ (example of the § 2.3 below), a polynomial defining $K^+/\mathbb{Q}$ is $x^3 - 15\,x + 17$ (the discriminant of $K^+/\mathbb{Q}$ is $3^3 \cdot 211$) while a polynomial defining the Galois closure of $K_1^+/\mathbb{Q}$ is given by $x^6 - 30\,x^4 + 17\,x^3 + 225\,x^2 - 255\,x - 86$ (the discriminant of $K_1^+/\mathbb{Q}$ is $3^9 \cdot 211^3$).

Lemma 2.3. *We have* $\#\mathcal{C}\ell_{K^+} = 1$*, hence* $\mathcal{C}\ell_K = \mathcal{C}\ell_K^-$ *since* $\mathcal{C}\ell_K^+ \simeq \mathcal{C}\ell_{K^+} = 1$*,* $\#\mathcal{C}\ell_K^G = \#\mathcal{C}\ell_k = 3$*. We have* $\#\mathcal{C}\ell_{H_k} = 1$*.*

Proof. Using Chevalley's formula in K^+/k^+ (see e.g. [Gr1, Lemma II.6.1.2]) with a trivial 3-class group for k^+, the formula reduces to

$$\#\mathcal{C}\ell_{K^+}^G = \frac{3}{3 \cdot (E_{k^+} : E_{k^+} \cap \mathrm{N}_{K^+/k^+} K^{+\times})} = 1,$$

since the product of ramification indices is equal to 3 ($\mathcal{C}\ell_{k^+} = 1$ implies that K^+/k^+ is totally ramified at the single prime of k^+ above 3). The same formula in K/k is

$$\#\mathcal{C}\ell_K^G = \frac{\#\mathcal{C}\ell_k \cdot 3}{3 \cdot (E_k : E_k \cap \mathrm{N}_{K/k} K^\times)}$$

with $E_k = \langle \varepsilon, \zeta \rangle$, where ε is the fundamental unit of k^+; but, as for K^+/k^+, there is by assumption a single prime ideal of k ramified in K/k, thus, using the product formula, the Hasse norm theorem shows that all these units are local norms everywhere hence global norms. So $\#\mathcal{C}\ell_K^G = 3$ since $\#\mathcal{C}\ell_k = \#\mathcal{C}\ell_{k^-} = 3$.

With $G' := \mathrm{Gal}(H_k/k) \simeq \mathbb{Z}/3\mathbb{Z}$, we get $\#\mathcal{C}\ell_{H_k}^{G'} = \dfrac{\#\mathcal{C}\ell_k}{3 \cdot (E_k : E_k \cap \mathrm{N}_{H_k/k} H_k^\times)} = 1$. $\square$

2.3. Numerical data for a first counterexample

We have a first counterexample with $d = 211 \equiv 1 \pmod 3$ and $\alpha = \frac{17+\sqrt{-211}}{2}$ where $(\alpha) = \mathfrak{p}_5^3$ for a non-principal prime ideal dividing 5 in k^-. The class number of k^+

is 1 and that of k^- is 3, which is coherent with the fact that the fundamental unit $\varepsilon = 440772247 + 17519124\sqrt{3{\cdot}211}$ of k^+ is 3-primary ($\varepsilon \equiv 1 + 3\sqrt{3{\cdot}211} \pmod 9$), which implies that H_{k^-}/k^- is given via $k(\sqrt[3]{\varepsilon})/k$, decomposed over k^-.

So all the five conditions (i) to (v) of Hypothesis 2.1 are fulfilled.

The PARI program gives in "*component* $(H, 5)$" the class number and the structure of the whole class group of K; the program needs an irreducible polynomial defining K; it is given by "$P =$ *polcompositum* $(x^2 + x + 1, Q)$" where $Q = x^6 - 17\,x^3 + 5^3$ is the irreducible polynomial of $\sqrt[3]{\alpha}$ over $\mathbb{Q}$, from the general formula:

$$Q = x^6 - \mathrm{Tr}_{k^-/\mathbb{Q}}(\alpha)\, x^3 + \mathrm{N}_{k^-/\mathbb{Q}}(\alpha);$$

one obtains $P = x^{12} - 6\,x^{11} + 21\,x^{10} - 84\,x^9 + 243\,x^8 - 432\,x^7 + 1037\,x^6 - 1896\,x^5 - 204\,x^4 - 966\,x^3 + 5949\,x^2 + 4905\,x + 11881$. The program gives

$$\mathcal{C}\!\ell_K = \mathcal{C}\!\ell_K^- \simeq \mathbb{Z}/9\,\mathbb{Z} \bigoplus \mathbb{Z}/3\,\mathbb{Z}, \quad \text{i.e., } r_K = 2,$$

for a whole class number equal to 27. This yields, from Lemma 2.3, the 3-rank $r_K^- = 2$ for $\mathcal{C}\!\ell_K^-$ when the 3-rank r_k^- of $\mathcal{C}\!\ell_k^-$ is 1, which is incompatible with formula (3):

$$r_K^- - 1 = 3 \cdot (r_k^- - 1).$$

We do not know if the problem in [W] comes from the "group theory and cohomology" part or from the "number fields" part. But this "Riemann–Hurwitz formula" is valid if and only if $\mathcal{C}\!\ell_K^- = (\mathcal{C}\!\ell_K^-)^G \simeq \mathbb{Z}/3\,\mathbb{Z}$ (no exceptional 3-classes). Such a case is also very frequent (see § 4.3). From [DFKS], we get $\lambda^-(k) = 2$ (for $p = 3$) whence $\lambda^-(K) = 4$ from (2), which illustrates the difference of nature between λ^- and r^-.

3. Some structural results

Denote by M a finite $\mathbb{Z}_p[\Gamma]$-module, where we assume that Γ is an Abelian Galois group of the form $G \times g$, where $G = \mathrm{Gal}(K/k) =: \langle\sigma\rangle$ is cyclic of order p and $g \simeq \mathrm{Gal}(k/k_0)$ (of order prime to p), where k_0 is a suitable subfield of k (so that $K = k\,K_0$ with $K_0 := K^g$). The existence of g allows us to take isotypic components of M (as the $\pm$-components when the fields are of CM-type). In our example, $p = 3$, $g = \langle s\rangle$, $k_0 = k^+$ and $K_0 = K^+$.

For our purpose we shall have $M = \mathcal{C}\!\ell_K^- = \mathcal{C}\!\ell_K$ since $\mathcal{C}\!\ell_K^+ = 1$; to simplify, we shall use the notations $\mathcal{C}\!\ell_k, \mathcal{C}\!\ell_K$ instead of the minus parts. By class field theory, K/k being totally ramified at the unique prime ideal $\mathfrak{p} = (\sqrt{-3}) \mid 3$ of k, the arithmetical norm $\mathrm{N}_{K/k} : \mathcal{C}\!\ell_K \longrightarrow \mathcal{C}\!\ell_k$ is surjective. Another important fact for the structure of $\mathcal{C}\!\ell_K$ in our particular context, is that the class of order 3 of k capitulates in K because the equality $(\alpha) = \mathfrak{a}^3$ becomes $(\sqrt[3]{\alpha}) = (\mathfrak{a})_K$ in K (the transfer map $i_{K/k} : \mathcal{C}\!\ell_k \longrightarrow \mathcal{C}\!\ell_K$ is not injective). This has the following tricky consequence for $\nu := \nu_{K/k} = i_{K/k} \circ \mathrm{N}_{K/k}$:

$$\mathrm{N}_{K/k}(\mathcal{C}\!\ell_K) = \mathcal{C}\!\ell_k \;\;\&\;\; (\mathcal{C}\!\ell_K)^\nu = 1.$$

3.1. Structure of $\mathbb{Z}_p[G]$-modules M such that $M^\nu = 1$ and $\# M^G = p$

Return to the cyclic case G of order p (for any prime p) with a $\mathbb{Z}_p[G]$-module M of finite p-power order, and suppose $M^\nu = 1$ for the algebraic norm $\nu = 1 + \sigma + \cdots + \sigma^{p-1}$, and $\# M^G = p$. So M is a module over

$$\mathbb{Z}_p[G]/(\nu) \simeq \mathbb{Z}_p[X]/(1 + X + \cdots + X^{p-1}) \simeq \mathbb{Z}_p[\zeta],$$

where ζ is a primitive pth root of unity. We have, canonically $M \simeq \bigoplus_{j=1}^{m} \mathbb{Z}_p[\zeta]/(1-\zeta)^{n_j}$, $1 \leq n_1 \leq n_2 \leq \cdots \leq n_m,\ m \geq 0$, and the exact sequence $1 \to M^G \longrightarrow M \xrightarrow{1-\sigma} M^{1-\sigma} \to 1$ which becomes in the $\mathbb{Z}_p[\zeta]$-structure:

$$\begin{aligned} 1 \to \bigoplus_{j=1}^{m} (1-\zeta)^{n_j - 1}\, \mathbb{Z}_p[\zeta]/(1-\zeta)^{n_j} \longrightarrow \\ \bigoplus_{j=1}^{m} \mathbb{Z}_p[\zeta]/(1-\zeta)^{n_j} \xrightarrow{1-\zeta} \bigoplus_{j=1}^{m} (1-\zeta)\, \mathbb{Z}_p[\zeta]/(1-\zeta)^{n_j} \to 1. \end{aligned} \tag{4}$$

Since $\# M^G = p$, we get $m = 1$ and $M \simeq \mathbb{Z}_p[\zeta]/(1-\zeta)^n,\ n \geq 1$ (M is $\mathbb{Z}_p[\zeta]$-monogenic).

Lemma 3.1. *Put $n = a\,(p-1) + b$, $a \geq 0$ and $0 \leq b \leq p-2$. Then*

$$M \simeq \mathbb{Z}_p[\zeta]/(1-\zeta)^n \simeq (\mathbb{Z}/p^{a+1}\mathbb{Z})^b \bigoplus (\mathbb{Z}/p^a\mathbb{Z})^{p-1-b}.$$

Proof. We have $\mathbb{Z}_p[\zeta]/(1-\zeta)^n \simeq \mathbb{Z}_p[\zeta]/p^a\,(1-\zeta)^b$. So, to have the Abelian group structure, it is sufficient to compute the p^r-ranks for all $r \geq 1$ (i.e., the dimensions over $\mathbb{F}_p$ of $M^{p^{r-1}}/M^{p^r}$), which is immediate since this p^r-rank is $p-1$ for $r \leq a$, b for $r = a+1$ and 0 for $r > a+1$. $\square$

This implies that the p-rank of M is $R = p-1$ if $a \geq 1$ and $R = b$ if $a = 0$ (i.e., $b = n \leq p-2$). So the parameters a and b will be important in a theoretical and numerical point of view.

3.2. The structure $\mathcal{C}\ell_K^- \simeq \mathbb{Z}/3^a\mathbb{Z} \bigoplus \mathbb{Z}/3^a\mathbb{Z}$ does not exist

We shall prove that in our particular example of the case $p = 3$, we always have $b = 1$ in Lemma 3.1 (i.e., $M := \mathcal{C}\ell_K^- \simeq \mathbb{Z}_3[\zeta]/3^a\,(1-\zeta),\ \ a \geq 0$).

Theorem 3.2. *Under the Hypothesis 2.1, there exists $a \geq 0$ such that*

$$\mathcal{C}\ell_K = \mathcal{C}\ell_K^- \simeq \mathbb{Z}_3[\zeta]/3^a\,(1-\zeta) \simeq \mathbb{Z}/3^{a+1}\mathbb{Z} \bigoplus \mathbb{Z}/3^a\mathbb{Z}.$$

The case $a = 0$ is equivalent to $\mathcal{C}\ell_K = \mathcal{C}\ell_K^G$.

Proof. Suppose that $M \simeq \mathbb{Z}_3[\zeta]/(3^a) = \mathbb{Z}_3[\zeta]/(1-\zeta)^{2a} \simeq \mathbb{Z}/3^a\mathbb{Z} \bigoplus \mathbb{Z}/3^a\mathbb{Z}$, $a \geq 1$. Consider the following diagram; since $\mathrm{Gal}(H_K/KH_k) \simeq {}_{\mathrm{N}}M$ and since $M^{1-\sigma} \subseteq {}_{\mathrm{N}}M$ is of index $\#M^G = 3$ in M, we get $\mathrm{Gal}(H_K/KH_k) \simeq M^{1-\sigma}$:

$$
\begin{tikzcd}
K \arrow[r, dash, "3"] \arrow[rr, dash, bend left, "M=\langle h\rangle"] \arrow[d, dash, "G=\langle\sigma\rangle"'] & KH_k \arrow[r, dash] \arrow[r, dash, bend left, "M^{1-\sigma}"] \arrow[d, dash] & H_K \\
k \arrow[r, dash] & H_k &
\end{tikzcd}
$$

Let I_K be the group of ideals of K (by abuse of notation, some writings may be sometimes in $I_K \otimes \mathbb{Z}_p$). Let $h = c\ell(\mathfrak{A})$, $\mathfrak{A} \in I_K$, be a 3-class generating M as G-module. Then we have (as Abelian group) $M = \langle h\rangle_{\mathbb{Z}} \bigoplus \langle h^{\omega}\rangle_{\mathbb{Z}}$, where $\omega = \sigma(1-\sigma)$ in $\mathbb{Z}_3[G]$ (or $\omega = \zeta\,(1-\zeta)$ in $\mathbb{Z}_3[\zeta]$ so that $\omega^2 = -3$ in $\mathbb{Z}_3[\zeta]$). Let $h_1 = c\ell(\mathfrak{A}_1)$, $\mathfrak{A}_1 \in I_K$, be a class generating M^G. We know from (4) that we can suppose $h_1 = h^{\omega^{2a-1}} = h^{(-3)^{a-1}\omega}$.

(i) Action of s and t on M (see the first schema in § 2). Let s be the complex conjugation on K and let t be an extension to K of the generator of $\mathrm{Gal}(k/k^-)$ (t is non-unique modulo $\mathrm{Gal}(K/k)$, but t is of order 2 since $\mathrm{Gal}(K/k^-) = D_6$; moreover, we shall see that the calculations do not depend on this choice).[2] Thus s and t operate by conjugation on $\mathrm{Gal}(H_K/K)$ since $H_K/\mathbb{Q}$ is Galois (if $\tau_h \in \mathrm{Gal}(H_K/K)$ is the Artin symbol of $h \in M$, then $\tau_{h^{\varphi}} = \varphi \cdot \tau_h \cdot \varphi^{-1}$ for any $\varphi \in \mathrm{Gal}(K/\mathbb{Q})$). We have

$$t \cdot \sigma \cdot t^{-1} = \sigma^{-1} \;\;\&\;\; h^s = h^{-1} \text{ for all } h \in M.$$

The key fact is the action of t on M. Since h generates M and $\mathrm{N}(\mathcal{C}\ell_K) = \mathcal{C}\ell_k$, we have $\mathrm{N}(h) = h_0$, the class of order 3 of k^- seen in k; of course $h_0^t = h_0$, thus $(\mathrm{N}(h))^t = \mathrm{N}(h)$. Since $t \cdot \nu_{K/k} = \nu_{K/k} \cdot t$ and $\nu_{K/k} = i_{K/k} \circ \mathrm{N}_{K/k}$, we get easily that $\mathrm{N}(h^t) = \mathrm{N}(h)$, and it follows that $h^{t-1} \in {}_{\mathrm{N}}M = M^{1-\sigma}$. So we can put

$$h^t = h^{1+\Omega}, \quad \Omega \in \omega \cdot A,$$

where A and ω mean $\mathbb{Z}_3[G]$ or $\mathbb{Z}_3[\zeta]$ and $\sigma\,(1-\sigma)$ or $\zeta(1-\zeta) = \sqrt{-3}$, respectively. We must have $h^{t^2} = h$, whence $h = (h^{1+\Omega})^t := h^{t\cdot(1+\Omega)}$ (law of left modules). We have to compute, in general, $h^{t\cdot(a+b\sigma+c\sigma^2)}$, but $t \cdot (a + b\sigma + c\sigma^2) = (a + b\sigma^2 + c\sigma) \cdot t$, which can be translated into $h^{t\cdot(1+\Omega)} = h^{(1+\overline{\Omega})\cdot t}$, where $\overline{\Omega}$ is the "conjugate" of Ω in an obvious meaning. So we finally obtain $h = h^{(1+\overline{\Omega})\cdot t} = (h^{1+\Omega})^{1+\overline{\Omega}} = h^{(1+\Omega)(1+\overline{\Omega})}$, which implies $(1 + \Omega)(1 + \overline{\Omega}) \equiv 1 \pmod{\omega^n}$, where $\#M = 3^n$; in the local ring $\mathbb{Z}_3[\zeta]$ this defines a principal unit $u \equiv 1 \pmod 3$, but we do not need its precise value.

[2] One can find similar Galois operations in [L] in a dihedral context and some generalizations.

(ii) Computation of $(h^{\omega^k})^t = h^{t\cdot\omega^k}$ for any $k \geq 0$, $k = 2\,\ell$ or $2\,\ell + 1$.
– If $h^{\omega^k} = h^{(-3)^\ell}$, then $(h^{(-3)^\ell})^t = h^{(-3)^\ell\cdot(1+\Omega)} = h^{\omega^k\cdot(1+\Omega)}$;
– if $h^{\omega^k} = h^{(-3)^\ell\,\omega}$, then $(h^{(-3)^\ell\,\omega})^t = h^{(-3)^\ell\,\overline{\omega}\,(1+\Omega)} = h^{-\omega^k\cdot(1+\Omega)}$ since $\overline{\omega} = -\sqrt{-3}$.

(iii) Direct computation of h_1^t. We have $h_1 = c\ell(\mathfrak{A}_1)$ such that $\mathfrak{A}_1^{1-\sigma} = (x)$, $x \in K^\times$, with $\mathrm{N}(x) = \eta \in E_k$; since we are in relative class groups, the only solution is $\eta = \zeta$ (indeed, if $\eta \in E_{k^+}$, this implies $\mathfrak{A}_1 \in I_K^G$, hence $h_1 = 1$ (absurd)). Then the relation $\mathrm{N}(x) = \zeta$ yields $(\mathrm{N}(x))^t = \mathrm{N}(x^t) = \zeta^{-1}$, hence $\mathrm{N}(x^t) = \mathrm{N}(x^{-1})$ whence

$$x^t = x^{-1}\cdot y^{1-\sigma},\quad y \in K^\times,$$

so

$$(\mathfrak{A}_1^{1-\sigma})^t = \mathfrak{A}_1^{t\cdot(1-\sigma)} = \mathfrak{A}_1^{-(1-\sigma)}\cdot(y)^{1-\sigma},$$

but

$$\mathfrak{A}_1^{t\cdot(1-\sigma)} = \mathfrak{A}_1^{(1-\sigma^2)\cdot t} = (\mathfrak{A}_1^t)^{1-\sigma^2} = \mathfrak{A}_1^{-(1-\sigma)}\cdot(y)^{1-\sigma};$$

then “suppressing” $1-\sigma$ and writing $\sim$ for equivalence modulo principal ideals:

$$(\mathfrak{A}_1^t)^{1+\sigma} \sim \mathfrak{A}_1^{-1}\cdot(\mathfrak{a}_0)_K\cdot\mathfrak{P}^e \sim \mathfrak{A}_1^{-1},\quad \mathfrak{a}_0 \in I_k,$$

because the class of any $\mathfrak{a}_0$ capitulates in K and the ramified prime ideal $\mathfrak{P} \mid 3$ in K is invariant but principal (see Remark (i) below). The element $1+\sigma$ is invertible with inverse $\frac{1}{2}\,(1-\sigma+\sigma^2)$ giving $\mathfrak{A}_1^t \sim \mathfrak{A}_1^{-\frac{1}{2}\,(1-\sigma+\sigma^2)} \sim \mathfrak{A}_1$, i.e., $h_1^t = h_1$, since we have $-\frac{1}{2}\,(1-\sigma+\sigma^2) \equiv 1 \pmod{(3, 1-\sigma)}$. But the case $h_1 = h^{(-3)^{a-1}\omega}$ would give (from the second case of (ii))

$$h_1^t = h^{-(-3)^{a-1}\omega\cdot(1+\Omega)} = h_1^{-(1+\Omega)} = h_1^{-1}$$

since $h_1^\Omega = 1$ (absurd). □

Remarks 3.3. (i) Since 3 is non-split in $k/\mathbb{Q}$, the prime ideal $\mathfrak{P}\,|\,3$ in K is 3-principal: indeed, $\mathfrak{P}^{1+s} = \mathfrak{P}^2$ gives the extension of $\mathfrak{P}^+\,|\,3$ in K^+ which is 3-principal (Lemma 2.3); so $c\ell_K(\mathfrak{P}) = 1$ in $\mathcal{C}\ell_K$. By class field theory, $\mathfrak{P}$ splits completely in H_K/K.

(ii) The parameter a can probably take any value; we have for instance obtained the following examples:

For $d = 12058$, $\#\mathcal{C}\ell_{k^+} = 2$, $\mathcal{C}\ell_{k^-} \simeq \mathbb{Z}/3\mathbb{Z}$, $\alpha = 989 + 26\sqrt{-d}$ of norm 209^3, $\mathcal{C}\ell_K^- \simeq \mathbb{Z}/3^4\mathbb{Z} \bigoplus \mathbb{Z}/3^3\mathbb{Z}$. A polynomial defining K is:

$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 4006\,x^9 + 17892\,x^8 - 35730\,x^7 + 22212821\,x^6 - 66531354\,x^5 - 113482743\,x^4 - 35777798264\,x^3 + 54059937672\,x^2 + 54106942656\,x + 83308554531904$.

For $d = 86942$, $\#\mathcal{C}\ell_{k^+} = 4$, $\mathcal{C}\ell_{k^-} \simeq \mathbb{Z}/3\mathbb{Z}$, $\alpha = 557 + 3\sqrt{-d}$ of norm 103^3, $\mathcal{C}\ell_K^- \simeq \mathbb{Z}/3^5\mathbb{Z} \bigoplus \mathbb{Z}/3^4\mathbb{Z}$. A polynomial defining K is:

$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 2278\,x^9 + 10116\,x^8 - 20178\,x^7 + 3449985\,x^6 - 10289502\,x^5 - 8954865\,x^4 - 2399550304\,x^3 + 3642928674\,x^2 + 3641624304\,x + 1191621124996.$

For $d = 954163$, $\#\mathcal{C}\ell_{k^+} = 2$, $\mathcal{C}\ell_{k^-} \simeq \mathbb{Z}/3\mathbb{Z}$, $\alpha = \frac{1}{2}\,(691 + \sqrt{-d})$ of norm 71^3, $\mathcal{C}\ell_K^- \simeq \mathbb{Z}/3^6\mathbb{Z} \bigoplus \mathbb{Z}/3^5\mathbb{Z}$. A polynomial defining K is:

$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 1432\,x^9 + 6309\,x^8 - 12564\,x^7 + 1207955\,x^6 - 3586254\,x^5 - 2509908\,x^4 - 483644468\,x^3 + 739433307\,x^2 + 738375807\,x + 127606842841.$

4. Numerical results and heuristic aspects

4.1. A filtration for the computation of the invariants a and b

We briefly recall some theoretical computations showing that, *in an heuristic point of view*, there is a large degree of possibilities for the values of the p-ranks in a more general situation of a p-extension K/k with arbitrary $\mathcal{C}\ell_k$ (the fields k and K are not assumed of CM-type). To simplify the approach, we consider $M = \mathcal{C}\ell_K$ in a cyclic extension K/k of degree p with Galois group $G = \langle\sigma\rangle$ and we suppose K/k totally ramified at a single place $\mathfrak{q}$ (not necessarily above p). Then we assume that $M^\nu = 1$ & $\#M^G = p$, which is equivalent to $\#\mathcal{C}\ell_k = p$ and $j_{K/k}(\mathcal{C}\ell_k) = 1$.

Put $M_i := \{h \in M,\ h^{(1-\sigma)^i} = 1\}$, $i \geq 0$, and let n be the least integer i such that $M_i = M$. We have $M_i \subset M_{i+1}$ for $i = 0, \ldots, n-1$. From the exact sequence:

$$1 \longrightarrow M_1 = M^G \longrightarrow M_{i+1} \xrightarrow{1-\sigma} M_{i+1}^{1-\sigma} \subseteq M_i \longrightarrow 1,$$

with $\#M_1 = p$ by assumption, we obtain $M_{i+1}^{1-\sigma} = M_i$ and $\#(M_{i+1}/M_i) = p$, for $i = 0, \ldots, n-1$, which gives immediately $\#M = p^n$ and the structure theorem $M \simeq (\mathbb{Z}/p^{a+1}\mathbb{Z})^b \bigoplus (\mathbb{Z}/p^a\mathbb{Z})^{p-1-b}$, $n = a\,(p-1) + b$, $a \geq 0$ and $0 \leq b \leq p-2$.

We use the general formula [Gr2, Corollaire 2.8]:

$$\#\big(M/M_i\big)^G = \#\big(M_{i+1}/M_i\big) = \frac{\#\mathcal{C}\ell_k \cdot \prod e_v(K/k)}{[K:k] \cdot \#\mathrm{N}_{K/k}(M_i) \cdot (\Lambda_i : \Lambda_i \cap \mathrm{N}_{K/k}(K^\times))},$$

where $\mathrm{N}_{K/k}(M_i) := c\ell_k(\mathrm{N}_{K/k}(I_i))$ for a suitable ideal group I_i such that $c\ell_K(I_i) = M_i$, and where $\Lambda_i = \{x \in k^\times,\ (x) \in \mathrm{N}_{K/k}(I_i)\}$.

In our particular case there is a single prime ideal $\mathfrak{q}$ ramified in K/k, and the elements of Λ_i, being norms of ideals, are everywhere local norms except perhaps at $\mathfrak{q}$; so $(\Lambda_i : \Lambda_i \cap \mathrm{N}_{K/k}(K^\times)) = 1$ by the product formula and the Hasse norm theorem; this yields $\#\big(M_{i+1}/M_i\big) = \dfrac{p}{\#\mathrm{N}_{K/k}(M_i)}$ which is trivial as soon as $\mathrm{N}_{K/k}(M_i) = \mathcal{C}\ell_k$.

So, in this case where the p-rank of $\mathrm{N}_{K/k}(M) = \mathcal{C}\ell_k \simeq \mathbb{Z}/p\mathbb{Z}$ is $r_k = 1$ and that of $M = \mathcal{C}\ell_K$ is r_K, we have the Riemann–Hurwitz formula $r_K - 1 = p \cdot (r_k - 1)$ if and only if $b = n = 1$, which is equivalent to $M = M^G$. Otherwise, r_K can take a priori any value in $[1, p-1]$ since it depends only on local data with computable probabilities.

For structural results, when $M^\nu \neq 1$, see [Gr3, Chapter IV, § 2, Proposition 4.3] valid for any $p \geq 2$, or [Gr4, Theorem 4.3]. With our biquadratic case and $p = 3$, we obtain interesting structures for which a theoretical study should be improved. From the general PARI program of § 4.2 we have obtained the following numerical examples with $\#\mathcal{C}\ell_{k^+} = 1$ (hence $\#\mathcal{C}\ell_{K^+} = 1$) but $\mathcal{C}\ell_{k^-}$ cyclic of 3-power order larger than 3:

(i) For $d = 1759$, $\mathcal{C}\ell_{k^-} \simeq \mathbb{Z}/27\mathbb{Z}$, $\alpha = 37 + 20\sqrt{-d}$ of norm 89^3,
$\mathcal{C}\ell_K^- \simeq \mathbb{Z}/27\mathbb{Z}$ (i.e., $\mathcal{C}\ell_K^- = (\mathcal{C}\ell_K^-)^G$).
(ii) For $d = 2047$, $\mathcal{C}\ell_{k^-} \simeq \mathbb{Z}/9\mathbb{Z}$, $\alpha = 332 + 11\sqrt{-d}$ of norm 71^3,
$\mathcal{C}\ell_K^- \simeq \mathbb{Z}/9\mathbb{Z} \bigoplus \mathbb{Z}/3\mathbb{Z} \bigoplus \mathbb{Z}/3\mathbb{Z}$.
(iii) For $d = 1579$, $\mathcal{C}\ell_{k^-} \simeq \mathbb{Z}/9\mathbb{Z}$, $\alpha = \frac{1}{2}(115 + 3\sqrt{-d})$ of norm 19^3,
$\mathcal{C}\ell_K^- \simeq \mathbb{Z}/27\mathbb{Z} \bigoplus \mathbb{Z}/9\mathbb{Z} \bigoplus \mathbb{Z}/3\mathbb{Z}$.

In conclusion, the structure of $\mathcal{C}\ell_K^-$ strongly depends on the order of $\mathcal{C}\ell_k^-$ and not only of its p-rank (here the 3-rank is 1 but we have chosen orders larger than 3).

4.2. PARI program

We keep the same assumptions ((i) to (v)) of Hypothesis 2.1 about $k = \mathbb{Q}(\sqrt{-d}, \zeta)$ and the 3-class groups of $k^+ = \mathbb{Q}(\sqrt{3{\cdot}d})$ and $k^- = \mathbb{Q}(\sqrt{-d})$ (especially $\mathcal{C}\ell_{k^+} = 1$, $\mathcal{C}\ell_{k^-} \simeq \mathbb{Z}/3\mathbb{Z}$), the non-splitting of 3 in $k/\mathbb{Q}$, and the Kummer construction of the 3-ramified cubic cyclic extension K/k decomposed over k^+. We give explicit numerical computations of $\mathcal{C}\ell_K$, for various biquadratic fields k.

The following PARI program gives in "*component* $(H, 5)$" the class number and the structure of the whole class group $\mathcal{C}\ell_K$ of K in the form:

$$\textit{class group} \ : \ [\#\mathcal{C}\ell_K, [c_1, \ldots, c_\lambda]]$$

such that $\mathcal{C}\ell_K \simeq \oplus_{i=1}^{\lambda} \mathbb{Z}/c_i\mathbb{Z}$.

For simplicity we compute an $\alpha \in k^- = \mathbb{Q}(\sqrt{-d})$ being an integer without non-trivial rational divisor; so (α) is the cube of an ideal if and only if $\mathrm{N}_{k^-/\mathbb{Q}}(\alpha) \in \mathbb{Q}^{\times 3}$. Then the irreducible polynomial defining K is given by $P = \textit{polcompositum}(x^2 + x + 1, Q)$ where $Q = x^6 - 2\,u\,x^3 + u^2 + d\,v^2$ where $\alpha = u + v\sqrt{-d}$ (u, v integers or half-integers). If by accident, $\alpha \in k^{\times 3}$, PARI gives a list of three polynomials of degree 4. But the program uses the least odd $A \in \mathbb{N}$ such A^3 is the norm of an integer α as above; so if $\alpha = \beta^3$, then $A = \mathrm{N}_{k^-/\mathbb{Q}}(\beta)$ which may be very rare (or impossible) because of the inequality $A > \frac{d}{4}$.

```
{d=1; while(d<10^6, d=d+3; if(core(d)==d, D=3*d; if(Mod(D,4)!=1, D=4*D);
h=qfbclassno(D); if(Mod(h,3)!=0, K=bnfinit(x^2+d); hm=bnrinit(K,1);
hm=component(component(hm,5),1); if(Mod(hm,3)==0&Mod(hm,9)!=0, A=1; Test=0;
while(Test==0, A=A+2; Z=bnfisintnorm(K,A^3);
if(component(matsize(Z),2)>1, Y=component(Z,2); u=component(Y,1); v=component(Y,2);
if(gcd(u,v)<=1, Test=1; T=2*u; N=u^2+d*v^2; Q=x^6-T*x^3+N;
P=polcompositum(x^2+x+1,Q); R=component(P,1);
H=bnrinit(bnfinit(R,1),1); F=component(H,5); G=component(F,1);
if(Mod(G,3)==0&Mod(G,9)!=0, print(""); print("d=",d); print("u=",u); print("v=",v);
print("hm=",hm,"h=",h); print(P); print("classgroup:",F)))))))))}
```

The condition "*Mod (G, 3)==0 & Mod (G, 9)!=0* " must be adapted to the relevant needed structure $(3^{a+1}, 3^a)$ (i.e., write "*Mod (G,* 3^{2a+1}*)==0 & Mod (G,* 3^{2a+2}*)!=0* "); here we test the case $a = 0$. We give below an extract of the examples we have obtained from another program giving others u, v (but the same K) with prescribed $a \geq 0$; for a more complete table up to 10^5, please use the link:
https://www.dropbox.com/s/6rlwrt34ami4383/TABLE-Riemann-Hurwitz.pdf?dl=0

4.3. Case $\mathcal{C}\ell_K \simeq \mathbb{Z}/3\mathbb{Z}$ $(a = 0)$, equivalent to $\mathcal{C}\ell_K = \mathcal{C}\ell_K^G \simeq \mathbb{Z}/3\mathbb{Z}$

$d = 31, u = 1/2, v = 1/2, \#\mathcal{C}\ell_{k^-} = 3,\ \#\mathcal{C}\ell_{k^+} = 1$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 52\,x^9 + 99\,x^8 - 144\,x^7 + 179\,x^6 - 186\,x^5 - 33\,x^4 + 268\,x^3 - 87\,x^2 - 24\,x + 64$
$classgroup : [3, [3]]$

$d = 61, u = 8, v = 1, \#\mathcal{C}\ell_{k^-} = 6,\ \#\mathcal{C}\ell_{k^+} = 2$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 82\,x^9 + 234\,x^8 - 414\,x^7 + 983\,x^6 - 1788\,x^5 - 393\,x^4 - 506\,x^3 + 5394\,x^2 + 4620\,x + 12100$
$classgroup : [12, [6, 2]]$

...

$d = 913, u = 321, v = 4, \#\mathcal{C}\ell_{k^-} = 12,\ \#\mathcal{C}\ell_{k^+} = 8$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 1334\,x^9 + 5868\,x^8 - 11682\,x^7 + 661085\,x^6 - 1948290\,x^5 + 702561\,x^4 - 149227072\,x^3 + 227288688\,x^2 + 224655360\,x + 13690872064$
$classgroup : [768, [24, 4, 4, 2]]$

$d = 970, u = 563, v = 20, \#\mathcal{C}\ell_{k^-} = 12,\ \#\mathcal{C}\ell_{k^+} = 4$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 2302\,x^9 + 10224\,x^8 - 20394\,x^7 + 2701601\,x^6 - 8043702\,x^5 - 2977323\,x^4 - 1568242964\,x^3 + 2378397756\,x^2 + 2373361968\,x + 495396376336$
$classgroup : [600, [30, 10, 2]]$

4.4. Case $\mathcal{C}\ell_K \simeq \mathbb{Z}/9\mathbb{Z} \bigoplus \mathbb{Z}/3\mathbb{Z}$ $(a = 1)$

$d = 211, u = 17/2, v = 1/2, \#\mathcal{C}\ell_{k^-} = 3,\ \#\mathcal{C}\ell_{k^+} = 1$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 84\,x^9 + 243\,x^8 - 432\,x^7 + 1037\,x^6 - 1896\,x^5 - 204\,x^4 - 966\,x^3 + 5949\,x^2 + 4905\,x + 11881$
$classgroup : [27, [9, 3]]$

$d = 214, u = 89, v = 6, \#\mathcal{C}\ell_{k^-} = 6,\ \#\mathcal{C}\ell_{k^+} = 2$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 406\,x^9 + 1692\,x^8 - 3330\,x^7 + 66813\,x^6 - 190530\,x^5 - 45783\,x^4 - 5155600\,x^3 + 8296296\,x^2 + 8156544\,x + 238640704$
$classgroup : [54, [18, 3]]$

...

$d = 4531, u = 403/2, v = 5/2, \#\mathcal{C}\ell_{k^-} = 12,\ \#\mathcal{C}\ell_{k^+} = 2$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 856\,x^9 + 3717\,x^8 - 7380\,x^7 + 308855\,x^6 - 904506\,x^5 - 62898\,x^4 - 53921936\,x^3 + 83258895\,x^2 + 82428357\,x + 4694853361$
$classgroup : [1728, [36, 12, 4]]$

$d = 4639, u = 361, v = 2, \#\mathcal{C}\ell_{k^-} = 51,\ \#\mathcal{C}\ell_{k^+} = 1$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 1494\,x^9 + 6588\,x^8 - 13122\,x^7 + 834341\,x^6 - 2463738\,x^5 + 888141\,x^4 - 212657184\,x^3 + 323349156\,x^2 + 320016960\,x + 21950200336$
$classgroup : [7344, [612, 12]]$

4.5. Case $\mathcal{C}\ell_K \simeq \mathbb{Z}/27\mathbb{Z} \bigoplus \mathbb{Z}/9\mathbb{Z}$ $(a = 2)$

$d = 1141, u = 449, v = 8, \#\mathcal{C}\ell_{k^-} = 24,\ \#\mathcal{C}\ell_{k^+} = 4$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 1846\,x^9 + 8172\,x^8 - 16290\,x^7 + 1374653\,x^6 - 4075170\,x^5 + 711057\,x^4 - 487867520\,x^3 + 740542656\,x^2 + 735780864\,x + 74927017984$
$classgroup : [7776, [216, 18, 2]]$

$d = 1174, u = 21, v = 5, \#\mathcal{C}\ell_{k^-} = 30,\ \#\mathcal{C}\ell_{k^+} = 2$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 134\,x^9 + 468\,x^8 - 882\,x^7 + 62369\,x^6 - 184542\,x^5 - 436569\,x^4 - 1322320\,x^3 + 3316650\,x^2 + 3570000\,x + 885062500$
$classgroup : [2430, [270, 9]]$

...

$d = 4087, u = 357, v = 8, \#\mathcal{C}\ell_{k^-} = 30,\ \#\mathcal{C}\ell_{k^+} = 2$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 1478\,x^9 + 6516\,x^8 - 12978\,x^7 + 1302965\,x^6 - 3870042\,x^5 - 2782815\,x^4 - 543509224\,x^3 + 830485104\,x^2 + 829417344\,x + 150779996416$
$classgroup : [19440, [270, 18, 2, 2]]$

$d = 4567, u = 195/2, v = 1/2, \#\mathcal{C}\ell_{k^-} = 33,\ \#\mathcal{C}\ell_{k^+} = 7$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 440\,x^9 + 1845\,x^8 - 3636\,x^7 + 63557\,x^6 - 179844\,x^5 + 66765\,x^4 - 3988930\,x^3 + 6294021\,x^2 + 6052866\,x + 109286116$
$classgroup : [18711, [2079, 9]]$

4.6. Case $\mathcal{C}\ell_K \simeq \mathbb{Z}/81\mathbb{Z} \bigoplus \mathbb{Z}/27\mathbb{Z}$ $(a = 3)$

$d = 12058, u = 989, v = 26, \#\mathcal{C}\ell_{k^-} = 42,\ \#\mathcal{C}\ell_{k^+} = 2$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 4006\,x^9 + 17892\,x^8 - 35730\,x^7 + 22212821\,x^6 - 66531354\,x^5 - 113482743\,x^4 - 35777798264\,x^3 + 54059937672\,x^2 + 54106942656\,x + 83308554531904$
$classgroup : [30618, [1134, 27]]$

$d = 15607, u = 534, v = 1, \#\mathcal{C}\ell_{k^-} = 39,\ \#\mathcal{C}\ell_{k^+} = 1$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 2186\,x^9 + 9702\,x^8 - 19350\,x^7 + 1764719\,x^6 - 5236188\,x^5 + 2322777\,x^4 - 638361238\,x^3 + 965957748\,x^2 + 958427808\,x + 89817692416$
$classgroup : [28431, [1053, 27]]$

...

$d = 45517, u = 845, v = 6, \#\mathcal{C}\ell_{k^-} = 120 \#\mathcal{C}\ell_{k^+} = 4$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 3430\,x^9 + 15300\,x^8 - 30546\,x^7 + 7597005\,x^6 - 22699458\,x^5 - 18168075\,x^4 - 7877764840\,x^3 + 11909686236\,x^2 + 11905200672\,x + 5526956498704$
$classgroup : [174960, [3240, 54]]$

$d = 47194, u = 293, v = 2, \#\mathcal{C}\ell_{k^-} = 120 \#\mathcal{C}\ell_{k^+} = 4$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 1222\,x^9 + 5364\,x^8 - 10674\,x^7 + 905093\,x^6 - 2683338\,x^5 - 2064183\,x^4 - 313267016\,x^3 + 480721224\,x^2 + 480118080\,x + 75097921600$
$classgroup : [699840, [3240, 54, 2, 2]]$

Acknowledgments

I thank Christian Maire for telling me about difficulties with the techniques developed in [W], then Thong Nguyen Quang Do and Jean-François Jaulent for a similar thinking about it. My sincere thanks to the referee for valuable suggestions and indications of some inaccuracies, especially about recalls on Kida's formulas.

References

[P] K. Belabas, et al., Pari/gp, Version 2.5.3, Laboratoire A2X, Université de Bordeaux I, http://sagemath.org/.
[DFKS] D.S. Dummit, D. Ford, H. Kisilevsky, J.W. Sands, Computation of Iwasawa Lambda invariants for imaginary quadratic fields, J. Number Theory 37 (1) (1991) 100–121, http://www.sciencedirect.com/science/article/pii/S0022314X05800277.
[FOO] S. Fujii, Y. Ohgi, M. Ozaki, Construction of $\mathbb{Z}_p$-extensions with prescribed Iwasawa λ-invariants, J. Number Theory 118 (2) (2006) 200–207, http://www.sciencedirect.com/science/article/pii/S0022314X05002040.
[Gr4] G. Gras, Invariant generalized ideal classes – Structure theorems for p-class groups in p-extensions. A survey, submitted to a Journal, 2016.
[Gr3] G. Gras, Sur les ℓ-classes d'idéaux dans les extensions cycliques relatives de degré premier ℓ, Ann. Inst. Fourier 23 (3) (1973) 1–48, http://www.numdam.org/item?id=AIF_1973___23_3_1_0.
[Gr2] G. Gras, Classes généralisées invariantes, J. Math. Soc. Japan 46 (3) (1994) 467–476, http://projecteuclid.org/euclid.jmsj/1227104692.
[Gr1] G. Gras, Class Field Theory: from Theory to Practice, Springer Monogr. Math., Springer-Verlag, 2003; second corrected printing 2005, http://dx.doi.org/10.1007/978-3-662-11323-3.

[Iw] K. Iwasawa, Riemann–Hurwitz formula and p-adic Galois representations for number fields, Tohoku Math. J. 33 (2) (1981) 263–288, https://www.jstage.jst.go.jp/article/tmj1949/33/2/33_2_263/_pdf.
[JaMa] J.-F. Jaulent, C. Maire, Sur les invariants d'Iwasawa des tours cyclotomiques, Canad. Math. Bull. 46 (2003) 178–190, https://cms.math.ca/openaccess/cmb/v46/jaulent8160.pdf.
[JaMi] J.-F. Jaulent, A. Michel, Classes des corps surcirculaires et des corps de fonctions, Progr. Math. 102 (1992) 141–161, http://www.math.u-bordeaux1.fr/~jjaulent/Articles/ClCpsSurc.pdf.
[Ki] Y. Kida, ℓ-extensions of CM-fields and cyclotomic invariants, J. Number Theory 12 (4) (1980) 519–528, http://www.sciencedirect.com/science/article/pii/0022314X80900426.
[KM] J. Klüners, G. Malle, A Database for Number Fields, http://galoisdb.math.upb.de/groups?sort=size&deg=6.
[L] F. Lemmermeyer, Class groups of dihedral extensions, Math. Nachr. 278 (6) (2005) 679–691, http://onlinelibrary.wiley.com/doi/10.1002/mana.200310263/abstract.
[Sch] J. Schettler, Generalizations of Iwasawa's "Riemann–Hurwitz" formula for cyclic p-extensions of number fields, arXiv:1211.4140v3, 2013, http://arxiv.org/pdf/1211.4140v3.pdf. From: The change in lambda invariants for cyclic p-extensions of $\mathbb{Z}_p$-fields, PhD thesis, The University of Arizona, 2012, http://arizona.openrepository.com/arizona/handle/10150/217113.
[Sin] W. Sinnott, On p-adic L-functions and the Riemann–Hurwitz genus formula, Compos. Math. 53 (1) (1984) 3–17, http://archive.numdam.org/ARCHIVE/CM/CM_1984___53_1/CM_1984___53_1_3_0/CM_1984___53_1_3_0.pdf.
[W] K. Wingberg, A Riemann–Hurwitz formula for the p-rank of ideal class groups of CM-fields, J. Number Theory 56 (2) (1996) 319–328, http://www.sciencedirect.com/science/article/pii/S0022314X96900219.