Journal of Number Theory 203 (2019) 12–31

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

General Section

Level aspect subconvexity for twisted L-functions

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ARTICLE INFO

Article history:
Received 9 October 2017
Received in revised form 18 March 2019
Accepted 31 March 2019
Available online 16 April 2019
Communicated by F. Pellarin

MSC:
11F67
11F11
11L07

Keywords:
Automorphic forms
Rankin-Selberg convolution
Subconvexity
Large sieve inequality

ABSTRACT

Let M be a square-free integer and P be a prime such that $(M,P)=1$. We prove a new level aspect hybrid subconvexity bound for $L(1/2, f\otimes\chi)$ where f is a primitive (either holomorphic or Maass) cusp form of level P and χ a primitive Dirichlet character modulo M satisfying $P\sim M^\eta$ with $0<\eta<3/2-3\vartheta$, where ϑ is the current known approximation towards the Ramanujan-Petersson conjecture. Particularly we obtain a stronger subconvexity for $\max\{6\vartheta, 1/2\}<\eta<(3-6\vartheta)/2$ which has not been covered by the work of Blomer-Harcos [3].

1. Introduction

Let f be a primitive (either holomorphic or Maass) cusp form of Hecke eigenvalues $\lambda_f(n)$, level P and χ be a primitive Dirichlet character modulo M. The L-function attached to the twist $f\otimes\chi$ is defined as

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https://doi.org/10.1016/j.jnt.2019.03.005

$$L(f \otimes \chi, s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)\chi(n)}{n^s},$$

at least for $\mathrm{Re}(s)$ sufficiently large. It is of great interest to establish a subconvexity bound for $L(f \otimes \chi, s)$ when both P and M are allowed to vary. Many methods have developed in recent years to produce successively strengthened level aspect hybrid subconvexity for twisted L-functions (not merely twisted by a Dirichlet character χ) when the levels of the forms are varying at different rate, say satisfying $P \sim M^{\eta}$, up to a constant $\eta > 0$ (see for instance the related works [1,3,12,14,15]). In an interesting paper Blomer and Harcos [3] firstly achieved a hybrid subconvexity for L-function $L(f \otimes \chi, s)$ simultaneously in all parameters of the form f and the conductor of χ, invoking the idea in Bykovskii [5]. Particularly if one supposes P and M are co-prime positive integers, they may show that

$$L(1/2, f \otimes \chi) \ll \mathcal{Q}^{\frac{1}{4}+\varepsilon} \left(\mathcal{Q}^{-\frac{1}{8(2+\eta)}} + \mathcal{Q}^{-\frac{1-\eta}{4(2+\eta)}} \right) \tag{1.1}$$

for $0 < \eta < 1$, where $f \in \mathcal{B}_k^*(P)$ or $\mathcal{B}_\lambda^*(P)$ (see §2 for definitions) and $\mathcal{Q} = PM^2$ is the size of the (arithmetic) conductor $\mathcal{Q}(f \otimes \chi)$ of the L-function $L(s, f \otimes \chi)$ (see [16, Chapter 7]). More recently K. Aggarwal, Y. Jo and K. Nowland [1] showed that

$$L(1/2, f \otimes \chi) \ll \mathcal{Q}^{\frac{1}{4} - \frac{2-5\eta}{20(2+\eta)} + \varepsilon} \tag{1.2}$$

for $0 < \eta < 2/5$, where $f \in \mathcal{B}_k^*(P)$ and M is a square-free positive integer.

It seems reasonable to ask how large of the exponent η to be ensured to produce a subconvexity, and further how to obtain a sharp subconvexity bound for a fixed η (note that the bound (1.2) is always weaker than (1.1)). In this paper we continue this theme by studying the average of the second moment of $L(1/2, f \otimes \chi)$ over a family of forms. We are able to establish the following bound for the average of the second moment.

Theorem 1.1. *Let M be a positive square-free integer and P be a prime such that $(P, M) = 1$. Let $k \geq 2$ be an even integer. Let h be a smooth function, compactly supported on $[1/2, 5/2]$ with bounded derivatives. Set $\mathcal{Q} = PM^2$. Then for $X \leq \mathcal{Q}^{1/2+\varepsilon}$ and any newform $f \in \mathcal{B}_k^*(P)$ (or $\mathcal{B}_\lambda^*(P)$) we have*

$$\sideset{}{^*}\sum_{\chi \bmod M} \frac{1}{\varphi^*(M)} \left| \sum_n \psi_f(n)\chi(n)h(n/X) \right|^2 \ll_\varepsilon X\sqrt{P}\mathcal{Q}^{\varepsilon} \left(\frac{1}{P^{\frac{1}{2}}} + \frac{P^{\frac{1}{3}}}{M^{\frac{1}{2}-\vartheta}} \right), \tag{1.3}$$

where $\varepsilon > 0$ is arbitrary, $\psi_f(n)$ denotes the n-th Fourier coefficient of the form f, χ runs over the primitive characters modulo M, and $\varphi^*(M) = \sum_{ab=M} \varphi(a)\mu(b)$ is the number of primitive characters modulo M, with $\varphi(n)$ being the Euler's totient function. Here ϑ is the current known approximation towards the Generalized Ramanujan Conjecture, which is zero if f is a holomorphic cusp form, and not greater than $7/64$ if f is a Maass cusp form.

As an immediate consequence we obtain a subconvexity bound when $P \sim M^\eta$ with $0 < \eta < 3/2 - 3\vartheta$.

Corollary 1.2. *Let $M, P, k, \mathcal{Q}$ be as in Theorem 1.3. Let $\eta = \frac{\log P}{\log M}$. Then for any newform $f \in \mathcal{B}_k^*(P)$ (or $\mathcal{B}_\lambda^*(P)$) we have*

$$L(1/2, f \otimes \chi) \ll \mathcal{Q}^{\frac{1}{4}+\varepsilon} \left(\mathcal{Q}^{-\frac{\eta}{4(2+\eta)}} + \mathcal{Q}^{-\frac{3-6\vartheta-2\eta}{12(2+\eta)}} \right). \tag{1.4}$$

Remark 1.3. a) Compared to (1.1) the estimate (1.4) is stronger whenever $\max\{6\vartheta, 1/2\} < \eta < (3-6\vartheta)/2$. b) The second moment method behaves like a harmonic detection process. One may seek the harmonic extraction by introducing the amplifier $\sum_{l \le L} \alpha_l \chi(l)$ for some real sequence $\{\alpha_l : l \le L\}$ in the sum of (1.3) by the amplification method of [7]. However the choice of L is closely related to the level of the form f and the conductor of the character χ (see for instance [4, Section 5]) which in turn imposes an extra constraint on the parameter η in Theorem 1.1. One can find the moment computation without amplification suffices to get a better exponent when at least two of the objects are varying, as in the present work. See [13, Section 2] for details.

Our main general result, Theorem 1.1, will then follow from the following bound for the average of the shifted convolution sum, as shown in Theorem 1.4 below. In our setting the 'well' structure of the summation condition of the convolution sum enable us to get a saving by equipping with the large sieve relative to a trivial application of the Weil bound for individual Kloosterman sums. This advantage would finally make the exponent η go beyond 1 (it should, in view of the previous description, be $3/2 - 3\vartheta$). Such improvement will allow us to average over the larger level family (increasing the degree of freedom of the basis of newforms) to produce a subconvexity when we are doing moment average. When we are faced with seeking non-trivial bounds for the shifted convolution sums, explicitly determining the dependence on the levels of the forms, on the other hand, is also one of the points, which does not follow easily from any of the current works (see for instance [7–9] for comparison).

Theorem 1.4. *Let l be a positive integer and $X, Y \ge 1$. Let $F(x, y)$ be a smooth function supported on $[1/2, 5/2] \times [1/2, 5/2]$ with partial derivatives satisfying*

$$x^i y^j \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial y^j} F\left(\frac{x}{X}, \frac{y}{Y}\right) \ll Z Z_x^i Z_y^j$$

for some $Z > 0$ and $Z_x, Z_y \ge 1$. Let $k \ge 2$ be an even integer. For any non-zero integer r and newforms $f_1, f_2 \in \mathcal{B}_k^*(P)$ (or $\mathcal{B}_\lambda^*(P)$) we define

$$\mathcal{S}_{f_1,f_2}(X, Y) = \sum_{r \ne 0} \sum_{m=n+rl} \psi_{f_1}(n) \psi_{f_2}(m) F\left(\frac{n}{X}, \frac{m}{Y}\right). \tag{1.5}$$

Then we have

$$\mathcal{S}_{f_1,f_2}(X,Y) \ll (XYP)^{\varepsilon} \frac{ZXYP^{5/6}}{l^{1/2-\vartheta}} \max(Z_x, Z_y)^{12} \frac{\max(X,Y)^{1/2}}{\min(X,Y)^{3/2}} \left(1 + \sqrt{\frac{\max(X,Y)}{lP}}\right) \tag{1.6}$$

uniformly for l*, where the implied constant depends only on* ε *and the spectral parameters of the forms. In particular if* $X = Y$ *and* $Z_x = Z_y$ *the bound in* (1.6) *is non-trivial for* $l \leq \min\left\{ \left(\frac{X^6}{P^5}\right)^{\frac{1}{3(1+2\vartheta)}}, \left(\frac{X^3}{P^2}\right)^{\frac{1}{6\vartheta}} \right\}$.

2. Preliminaries

2.1. Automorphic forms

We will recall some fundamental facts about cusp forms (see for instance Iwaniec's book [16]). Let $k \geq 2$ be an even integer and $N > 0$ be an integer. We denote by $\mathcal{S}_k(N)$ the vector space of holomorphic cusp forms on $\Gamma_0(N)$ with trivial nebentypus and weight k. For any $f \in \mathcal{S}_k(N)$ one has a Fourier expansion

$$f(z) = \sum_{n \geq 1} \psi_f(n) n^{\frac{k-1}{2}} e(nz)$$

for $\mathrm{Im}(z) > 0$. Here $e(z)$ means $e^{2\pi iz}$ for any $z \in \mathbb{C}$. Analogously we denote by $\mathcal{S}_\lambda(N)$ the vector space of Maass forms on $\Gamma_0(N)$ with trivial nebentypus, weight 0 and eigenvalue $\lambda = 1/4 + r^2 > 1/4$ (so that $r \in \mathbb{R}$). Then for any $f \in \mathcal{S}_\lambda(N)$ one has a Fourier expansion

$$f(z) = 2\sqrt{|y|} \sum_{n \neq 0} \psi_f(n) K_{ir}(2\pi|ny|) e(nx),$$

where $z = x + iy$ and K_{ir} denotes the K-Bessel function.

$\mathcal{S}_k(N)$ and $\mathcal{S}_\lambda(N)$ are finite dimensional Hilbert spaces which can be equipped with the Petersson inner products

$$\langle f_1, f_2 \rangle = \int_{\Gamma_0(N)\backslash\mathbb{H}} f_1(z)\overline{f_2(z)} y^{k-2} \mathrm{d}x\mathrm{d}y$$

and

$$\langle f_1, f_2 \rangle = \int_{\Gamma_0(N)\backslash\mathbb{H}} f_1(z)\overline{f_2(z)} \frac{\mathrm{d}x\mathrm{d}y}{y^2},$$

respectively. We recall the Hecke operators $\{T_n\}$ with $(n, N) = 1$ which satisfy the multiplicativity relation

$$T_n T_m = \sum_{d|(n,m)} T_{\frac{nm}{d^2}}. \tag{2.7}$$

The adjoint of T_n with respect to the Petersson inner products is itself, hence T_n is normal. One can find an orthogonal basis $\mathcal{B}_k(N)$ ($\mathcal{B}_\lambda(N)$ respectively) of $\mathcal{S}_k(N)$ ($\mathcal{S}_\lambda(N)$ respectively) consisting of common eigenfunctions of all the Hecke operators T_n with $(n, N) = 1$. For each $f \in \mathcal{B}_k(N)$ or $\mathcal{B}_\lambda(N)$, denote by $\lambda_f(n)$ the n-th Hecke eigenvalue which satisfies

$$T_n f(z) = \lambda_f(n) f(z)$$

for all $(n, N) = 1$. From (2.7) one has

$$\psi_f(m)\lambda_f(n) = \sum_{d|(n,m)} \psi_f\left(\frac{mn}{d^2}\right)$$

for any $m, n > 1$ with $(n, N) = 1$. In particular $\psi_f(1)\lambda_f(n) = \psi(n)$ if $(n, N) = 1$. Therefore

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(n,m)} \lambda_f\left(\frac{mn}{d^2}\right) \tag{2.8}$$

if $(mn, N) = 1$.

The Hecke eigenbasis $\mathcal{B}_k(N)$ ($\mathcal{B}_\lambda(N)$ respectively) also contains a subset of newforms $\mathcal{B}_k^*(N)$ ($\mathcal{B}_\lambda^*(N)$ respectively), those forms which are simultaneous eigenfunctions of all the Hecke operators T_n for any $n \geq 1$ and normalized to have first Fourier coefficient $\psi_f(1) = 1$. The elements of $\mathcal{B}_k^*(N)$ and $\mathcal{B}_\lambda^*(N)$ are usually called primitive forms.

We will need the following general Voronoï-type summation formula which is Theorem A.4 [18].

Lemma 2.1. *Let $k \geq 2$ be an even integer and $N > 0$ be an integer. Let $f \in B_k^*(N)$ (or $B_\lambda^*(N)$) be a newform. For $(a, q) = 1$ set $N_2 := N/(N, q)$. If $h \in \mathbb{C}^\infty(\mathbb{R}^{\times,+})$ is a Schwartz class function vanishing in a neighbourhood of zero, then there exists a complex number ϖ of modulus one, which depends on a, q and f, and a newform $f^\star \in B_k^*(N)$ (or $B_\lambda^*(N)$) such that*

$$\begin{aligned}\sum_n \lambda_f(n) e\left(\frac{an}{q}\right) h(n) =& \frac{2\pi\varpi}{q\sqrt{N_2}} \sum_n \lambda_{f^\star}(n) e\left(-\frac{\overline{aN_2}n}{q}\right) \int_0^\infty h(\xi) J_f\left(\frac{4\pi\sqrt{n\xi}}{q\sqrt{N_2}}\right) \mathrm{d}\xi \\ &+ \frac{2\pi\varpi}{q\sqrt{N_2}} \sum_n \lambda_{f^\star}(n) e\left(\frac{\overline{aN_2}n}{q}\right) \int_0^\infty h(\xi) K_f\left(\frac{4\pi\sqrt{n\xi}}{q\sqrt{N_2}}\right) \mathrm{d}\xi.\end{aligned}$$

In this formula,

- *if f is holomorphic of weight k then*

$$J_f(x) = 2\pi i^k J_{k-1}(x), \quad K_f(x) = 0.$$

- *if f is a Maass form with eigenvalue $\lambda = 1/4 + r^2$ then*

$$J_f(x) = \frac{-\pi}{\sin(\pi ir)}(J_{2ir}(x) - J_{-2ir}(x)), \quad K_f(x) = 4\cosh(\pi r)K_{2ir}(x).$$

2.2. A Wilton-type bound

We have the following Wilton-type bound involving the level of the cusp forms (see for instance [24, Lemma 2.1]). The uniform estimate for exponential sums associated with f in term of the sup-norm $\| f \|_\infty$ was first proved in [9]. For any $f \in \mathcal{B}_k^*(N)$ (or $\mathcal{B}_\lambda^*(N)$) with square-free level N, it shown in [10] that $\| f \|_\infty \ll N^{1/3+\varepsilon}$ for any $\varepsilon > 0$.

Lemma 2.2. *Let $X \geq 2$ and h be a smooth function, compactly supported on $[1/2, 5/2]$ with bounded derivatives. Then for any $\alpha \in \mathbb{R}$ and any newform $f \in \mathcal{B}_k^*(N)$ (or $\mathcal{B}_\lambda^*(N)$) with square-free level N, we have*

$$\sum_{n\leq X} \lambda_f(n)e(n\alpha)h\left(\frac{n}{X}\right) \ll X^{\frac{1}{2}}N^{\frac{1}{3}}(XN)^\varepsilon,$$

where the implied constant depends on ε and the parameter k (or λ).

2.3. Bessel functions

We recall some properties of Bessel functions which can be found in the Appendix of [4] and [15, Section 2].

Lemma 2.3. *For any complex number s we have*

$$(x^s J_s(x))' = x^s J_{s-1}(x), \quad (x^s K_s(x))' = -x^s K_{s-1}(x).$$

Denote by H_s the J_s or K_s. Then for any $a > 0$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(a\sqrt{x}\right)^{s+1} H_{s+1}(a\sqrt{x}) = \pm(a^2/2)\left(a\sqrt{x}\right)^s H_s(a\sqrt{x}), \tag{2.9}$$

where the sign $\pm$ is positive if $H_s = J_s$, and negative if $H_s = K_s$. Moreover for non-negative integers j there exist polynomials $Q_i, i \leq j$ of degree i such that

$$\begin{aligned} x^j H_s^{(j)}(a\sqrt{x}) = {} & Q_j(s)(a\sqrt{x})^j H_{s-j}(a\sqrt{x}) + Q_{j-1}(s)(a\sqrt{x})^{s-j+1} H_{j-1}(a\sqrt{x}) \\ & + \cdots + Q_0(s)H_s(a\sqrt{x}). \end{aligned} \tag{2.10}$$

Lemma 2.4. *Let $k \geq 2$ be an even integer. For any newform $f \in B_k^*(N)$ (or $B_\lambda^*(N)$) and non-negative integers j we have*

$$x^j (J_f)^{(j)}(x) \ll_{j,k} \frac{x}{(1+x)^{3/2}}$$

if $f \in B_k^*(N)$, and

$$x^j (J_f)^{(j)}(x) \ll_{j,r} \frac{1}{(1+x)^{1/2}}$$

if $f \in B_\lambda^*(N)$. Furthermore

$$K_f(x) \ll_\varepsilon \begin{cases} (1+|r|)^\varepsilon, & 0 < x \leq 1 + \pi|r|; \\ e^{-x}x^{-1/2}, & x > 1 + \pi|r|. \end{cases}$$

Lemma 2.5. *Let $k \geq 2$ be an even integer and $r > 0$ be an integer. Let P, q be positive integers. Take $Q > 1$ and $X, Y \geq 1$. Let $E(x, y)$ as in (3.19). For any $a, b > 0$ and newforms $f_1, f_2 \in \mathcal{B}_k^*(N)$ (or $\mathcal{B}_\lambda^*(N)$) define*

$$I_{J,J}(a,b) = \int_0^\infty \int_0^\infty E(x,y) J_{f_1}(4\pi a\sqrt{x}) J_{f_2}(4\pi b\sqrt{y}) \mathrm{d}x\mathrm{d}y, \tag{2.11}$$

and $I_{J,K}(a,b)$ to be the integral $I_{J,J}(a,b)$ with the second Bessel function J_{f_1} (resp. J_{f_2}) replaced by K_{f_1} (resp. K_{f_2}) in (2.11). Similarly we define $I_{K,J}(a,b)$ and $I_{K,K}(a,b)$. Denote by $I_{*,*}$ any element belonging to $\{I_{J,J}, I_{J,K}, I_{K,J}, I_{K,K}\}$. Then for any non-negative integers i and j we have

$$I_{*,*}(a,b) \ll_{i,j,\nu_{f_1},\nu_{f_2}} \mathcal{E}_0 \tag{2.12}$$

with

$$\mathcal{E}_0 := \frac{ZXY}{\left(1 + a\sqrt{X}\right)^{1/2}\left(1 + b\sqrt{Y}\right)^{1/2}} \left[\frac{Z_x}{a\sqrt{X}}\right]^i \left[\frac{Z_y}{b\sqrt{Y}}\right]^j,$$

where $\nu_{f_1}, \nu_{f_2} = \pm 2ir$ or $k-1$.

Proof. We only consider $I_{J,J}$ since the proofs for $I_{J,K}, I_{K,J}, I_{K,K}$ are similar. Integrating by parts once in x and together with the property (2.9) gives

$$I_{J,J}(a,b) = \pm\frac{1}{2\pi a} \int_0^\infty \int_0^\infty \left(\sqrt{x}\frac{\partial}{\partial x}E(x,y) - \frac{\nu_{f_1} E(x,y)}{2\sqrt{x}}\right) J_{\nu_{f_1}+1}(4\pi a\sqrt{x}) J_{\nu_{f_2}}(4\pi b\sqrt{y}) \mathrm{d}x\mathrm{d}y.$$

Note that $E^{(i,j)}(x,y) \ll_{i,j} \frac{Z_x^i Z_y^j}{X^i Y^j}$. Thus combining with Lemma 2.4 one arrives at (2.12) with $i=1$ and $j=0$. Repeated integration by parts would then establish (2.12) for all i and j. $\square$

2.4. A large sieve inequality

Let $S(m,n;c)$ be the classical Kloosterman sum. We have the following form of the large sieve inequality (see [6, Theorem 9]):

Lemma 2.6. *Let r, q, D be positive integers with q and D co-prime. Let C, M, N be positive real numbers and g be real-valued infinitely differentiable function with support in $[M, 2M] \times [N, 2N] \times [C, 2C]$ such that*

$$\frac{\partial^{i+j+k}}{\partial m^i \partial n^j \partial c^k} g(m,n,c) \ll_{i,j,k} \frac{Z^{i+j+k}}{M^i N^j C^k}$$

for all $i, j, k \geq 0$. Let

$$X_q := \frac{\sqrt{qMN}}{C\sqrt{r}}.$$

Then for any $\varepsilon > 0$ and complex sequences $\mathbf{a} = \{a_m\}$, $\mathbf{b} = \{b_n\}$ one has

$$\begin{aligned}
&\sum_m \sum_n \sum_{\substack{c \bmod D \\ (c,r)=1}} a_m b_n \frac{S(qm, \pm n\overline{r}; c)}{c} g(m,n,c) \\
&\ll_\varepsilon (qDMNZC)^\varepsilon q^\vartheta \sqrt{r} \frac{1 + X_q^{-2\vartheta}}{Z + X_q} \left(Z + X_q + \sqrt{\frac{M}{rD}} \right) \\
&\quad \times \left(Z + X_q + \sqrt{\frac{N}{rD}} \right) Z^9 \|a\|_2 \|b\|_2.
\end{aligned}$$

2.5. Circle method

We will now briefly recall a version of the circle method which has been investigated by Jutila (see [19] and [20]). In this paper we will employ Jutila's variation of the circle method with an important new input - the 'conductor lowering mechanism' (see [21], [22] or [23]). As shows in Lemma 2.7, Jutila's circle method provides a smooth bump function of q as $q \to \infty$, which is more convenient for us to implement the large sieve method later.

Lemma 2.7. *Let $Q \geq 2$ be a large parameter to be chosen later. Let w be a non-oscillating smooth function supported on $[Q/2, 5Q/2]$ with values in $[0,1]$, which equals to 1 on $[Q, 2Q]$ and satisfies that $w^{(i)} \ll_i Q^{-i}$ for any $i \geq 0$. For any set $S \subset \mathbb{R}$, let I_S denote* *the associated characteristic function, i.e.* $I_S(x) = 1$ *for* $x \in S$ *and* 0 *otherwise. We also set*

$$\tau := Q^{-1}, \quad L := \sum_{q=1}^{\infty} w(q)\varphi(q) \tag{2.13}$$

and define

$$I(\alpha) = \frac{1}{2\tau L} \sum_{q=1}^{\infty} w(q) \sideset{}{^*}\sum_{a \bmod q} I_{[\frac{a}{q}-\tau, \frac{a}{q}+\tau]}(\alpha).$$

Assume that $L \gg_{\varepsilon} Q^{2-\varepsilon}$ *for any* $\varepsilon > 0$*. We then have*

$$\int_0^1 |1 - I(\alpha)|^2 \, \mathrm{d}\alpha \ll_{\varepsilon} Q^{-1+\varepsilon}.$$

3. Proof of Theorem 1.4

For any given integer $l \geq 2$ and $X, Y \geq 1$, denote by $\mathcal{S}_{f_1,f_2}(X, Y; r)$ the inner sum over n, m in (1.5). Hence we write $S_{f_1,f_2}(X, Y)$ as

$$S_{f_1,f_2}(X, Y) = \sum_{r} \mathcal{S}_{f_1,f_2}(X, Y; r), \tag{3.14}$$

where clearly r satisfies $r \ll \max(X, Y)/l$. Appealing to $\delta(n, 0)$, the Dirac symbol at 0, one may write

$$\mathcal{S}_{f_1,f_2}(X, Y; r) = \sum_{n} \sum_{m} \lambda_{f_1}(n)\lambda_{f_2}(m) F\left(\frac{n}{X}, \frac{m}{Y}\right) \delta(n - m + rl, 0). \tag{3.15}$$

Notice that for any positive integer K we have

$$\delta(\tau, 0) = \mathcal{C}_{K,\tau}\delta(\tau/K, 0),$$

where $\mathcal{C}_{K,\tau}$ is equal to 1 or 0 according as $K|\tau$ or not. We then have

$$\delta(\tau, 0) = \frac{1}{K} \int_0^1 \sum_{b \bmod K} e\left(\frac{b\tau}{K}\right) e\left(\alpha\frac{\tau}{K}\right) \mathrm{d}\alpha.$$

Thus with the option that $\tau = n - m + rl$ and $K = P$ we obtain a approximation to $\mathcal{S}_{f_1,f_2}(X, Y; r)$ by Lemma 2.7:

$$\begin{aligned}\tilde{\mathcal{S}}_{f_1,f_2}(X,Y;r) =& \frac{1}{P}\int_0^1 \sum_{b \bmod P} \sum_n \sum_m \lambda_{f_1}(n)\lambda_{f_2}(m) F\left(\frac{n}{X},\frac{m}{Y}\right) e\left(\frac{b(n-m+rl)}{P}\right) \\ & \times I(\alpha) e\left(\alpha \frac{n-m+rl}{P}\right) \mathrm{d}\alpha.\end{aligned} \tag{3.16}$$

Lemma 3.1. *We have*

$$\mathcal{S}_{f_1,f_2}(X,Y;r) = \tilde{\mathcal{S}}_{f_1,f_2}(X,Y;r) + O_{f_1,f_2,\varepsilon}\left(\frac{ZX^{1/2}YP^{1/3}Z_xZ_y(XYP)^{\varepsilon}}{\sqrt{Q}}\right). \tag{3.17}$$

Proof. By partial summation it follows that

$$\begin{aligned}&\left|\mathcal{S}_{f_1,f_2}(X,Y;r) - \tilde{\mathcal{S}}_{f_1,f_2}(X,Y;r)\right| \\ &\leq \frac{1}{XYP} \sum_{b \bmod P} \int_0^{\infty}\int_0^{\infty}\int_0^1 \left|F^{(1,1)}\left(\frac{x}{X},\frac{y}{Y}\right)\right| \cdot |1-I(\alpha)| \\ &\times \left|\sum_{n\leq x} \lambda_{f_1}(n) e\left(n(\alpha+b)/P\right)\right| \cdot \left|\sum_{m\leq y} \lambda_{f_2}(m)\left(-m(\alpha+b)/P\right)\right| \mathrm{d}\alpha \mathrm{d}x \mathrm{d}y.\end{aligned}$$

For the middle sum we have $\sum_{n\leq x} \lambda_{f_1}(n)e(\alpha n) \ll_{\varepsilon} x^{1/2}P^{1/3}(XP)^{\varepsilon}$ by Lemma 2.2. Thus applying Cauchy's inequality we now arrive at the expression on the right hand side is

$$\begin{aligned}\leq \frac{ZZ_xZ_yP^{1/3+\varepsilon}}{X^{1/2}Y} \int_0^{\infty}\int_0^{\infty} \left|F^{(1,1)}\left(\frac{x}{X},\frac{y}{Y}\right)\right| \cdot \left(\int_0^1 |1-I(\alpha)|^2 \mathrm{d}\alpha\right)^{1/2} \\ \times \left(\int_0^1 \left|\sum_{m\leq y} \lambda_{f_2}(m) e(-m(\alpha+b)/P)\right|^2 \mathrm{d}\alpha\right)^{1/2} \mathrm{d}x\mathrm{d}y\end{aligned}$$

which is bounded by

$$\ll \frac{(XYP)^{\varepsilon} ZX^{1/2}YP^{1/3}Z_xZ_y}{\sqrt{Q}}$$

by Lemma 2.7, together with the basic estimate $\sum_{n\leq y} \lambda_{f_2}^2(n) \ll y$. $\square$

Now we proceed towards the estimation of $\tilde{\mathcal{S}}_{f_1,f_2}(X,Y;r)$. Using the definition of the approximating function $I(\alpha)$, we get

$$\begin{aligned}\tilde{\mathcal{S}}_{f_1,f_2}(X,Y;r) =& \frac{1}{PL}\sum_{q=1}^{\infty} w(q) \sideset{}{^*}\sum_{a \bmod q} \sum_{b \bmod P} e\left(\frac{rl(a+qb)}{qP}\right) \sum_n \lambda_{g_1}(n) e\left(\frac{n(a+bq)}{qP}\right) \\ &\times \sum_m \lambda_{g_2}(m) e\left(-\frac{m(a+bq)}{qP}\right) E(n,m),\end{aligned} \tag{3.18}$$

where

$$E(x,y) = F\left(\frac{x}{X},\frac{y}{Y}\right)\frac{1}{2\tau}\int_{-\tau}^{\tau} e(\alpha(x-y+rl)/P)\mathrm{d}\alpha \tag{3.19}$$

with τ, L being as in (2.13). We write $\gamma = a + bq$, so that $\tilde{\mathcal{S}}_{f_1,f_2}(X,Y;r)$ reduces to

$$\frac{1}{PL}\sum_{q=1}^{\infty} w(q) \sum_{\substack{\gamma \bmod qP \\ (\gamma,q)=1}} e\left(\frac{rl\gamma}{qP}\right) \sum_n \lambda_{f_1}(n) e\left(\frac{n\gamma}{qP}\right) \sum_m \lambda_{f_2}(m) e\left(-\frac{m\gamma}{qP}\right) E(n,m).$$

Our focus thus turns to investigating the cancellations of the averages involving the Fourier coefficients and the harmonics. The Voronoï summation formula would be put into use in an effort to obtain the transformation formula with the Kloosterman sums and smooth weights involved. To construct individual Kloosterman sums we would identify two situations $(\gamma, P) = 1$ or not. We will now proceed by considering these two cases.

3.1. Case 1: $(\gamma, P) = 1$

In this case one has $(\gamma, qP) = 1$. Thus $\tilde{\mathcal{S}}_{f_1,f_2}(X,Y;r)$ becomes

$$\begin{aligned}\tilde{\mathcal{S}}_{f_1,f_2}(X,Y;r) = \frac{1}{PL}\sum_q \sideset{}{^*}\sum_{\gamma \bmod qP} e\left(\frac{rl\gamma}{qP}\right) \sum_n \lambda_{f_1}(n) e\left(\frac{n\gamma}{qP}\right) \\ \times \sum_m \lambda_{f_2}(m) e\left(-\frac{m\gamma}{qP}\right) G(q,n,m),\end{aligned}$$

where $G(q,n,m) = w(q)E(n,m)$. Applying the Voronoï summation formula (Lemma 2.1) to sums over n and m respectively yields, up to a constant factor,

$$\begin{aligned}\tilde{\mathcal{S}}_{f_1,f_2}(X,Y;r) =& \frac{1}{P^3L}\sum_{q\geq 1}\frac{1}{q^2}\sum_n\sum_m \lambda_{f_1}(n)\lambda_{f_2}(m)\{S(rl,m-n;qP)G_{J,J}(q,n,m) \\ &+ S(rl,n+m;qP)G_{J,K}(q,n,m) + S(rl,-n-m;qP)G_{K,J}(q,n,m) \\ &+ S(rl,n-m;qP)G_{K,K}(q,n,m)\},\end{aligned} \tag{3.20}$$

where

$$G_{J,J}(q,n,m)=\int_0^\infty\int_0^\infty G(q,x,y)J_{f_1}\left(\frac{4\pi\sqrt{xn}}{qP}\right)J_{f_2}\left(\frac{4\pi\sqrt{ym}}{qP}\right)\mathrm{d}x\mathrm{d}y,$$

$G_{J,K}$, $G_{K,J}$ and $G_{K,K}$ are defined similarly as $G_{J,J}$. Moreover by Lemma 2.5 one easily finds that every element belonging to $\{G_{J,J},G_{J,K},G_{K,J},G_{K,K}\}$ is negligibly small unless

$$n\ll\frac{Z_x^2q^2P^2}{X}(XYP)^\varepsilon,\quad m\ll\frac{Z_y^2q^2P^2}{Y}(XYP)^\varepsilon \tag{3.21}$$

for any $\varepsilon>0$.

We may write $\tilde{\mathcal{S}}_{f_1,f_2}(X,Y;r)$ as a sum of four terms $\tilde{\mathcal{S}}^{\pm,\pm}_{f_1,f_2}(X,Y;r)$ say, depending on the signs of m,n in the Kloosterman sums in (3.20). Correspondingly we denote by $\tilde{\mathcal{S}}^{\pm,\pm}_{f_1,f_2}(X,Y)$ the contributions of these terms when summing with respect to r, upon recalling (3.14). In what follows we only consider the case where $r>0$ for the sums $\tilde{\mathcal{S}}^{\pm,\pm}_{f_1,f_2}(X,Y)$, the argument for the situation where $r<0$ following similarly. We only treat $\tilde{\mathcal{S}}^{+,-}_{f_1,f_2}(X,Y)$, upon noting that the argument of the other terms (that is, $\tilde{\mathcal{S}}^{-,+}_{f_1,f_2}$, $\tilde{\mathcal{S}}^{+,+}_{f_1,f_2}$ and $\tilde{\mathcal{S}}^{-,-}_{f_1,f_2}$) can follow similarly with it.

By dyadic subdivision we may decompose $G_{J,J}(q,n,m)$ in the q variable such that

$$G_{J,J}(q,n,m)=\sum_{Q^\star\geq 1}G_{J,J;Q^\star}(q,n,m) \tag{3.22}$$

with $G_{J,J;Q^\star}$ being a smooth function of q supported on $q\sim Q^\star$, where $Q^\star$ runs through the powers of 2 independently and satisfies that $Q/2\leq Q^\star\leq 5Q/2$. Analogously, for the sum over r in $\tilde{\mathcal{S}}^{+,-}_{f_1,f_2}(X,Y)$, we introduce another smooth partition of unity and break the sum into dyadic segments of size R. Hence we recast $\tilde{\mathcal{S}}^{+,-}_{f_1,f_2}(X,Y)$ as

$$\tilde{\mathcal{S}}^{+,-}_{f_1,f_2}(X,Y)=\frac{1}{PL}\sum_{Q^\star}\sum_n\sum_m\lambda_{f_1}(n)\lambda_{f_2}(m)\sum_{c\bmod P}\sum_r\frac{S(rl,m-n;c)}{c}\Xi(m,n,r,c), \tag{3.23}$$

where

$$\Xi(m,n,r,c)=G_{J,J;Q^\star}(c/P,n,m)\eta_R(r)/c. \tag{3.24}$$

Here η_R is a smooth function supported on $[R/2,5R/2]$. Notice that R must be of size at most $\max(X,Y)/l$ by the congruence condition and $m\leq S_1,n\leq S_2$ with

$$S_1:=\frac{(Z_xQ^\star P)^2}{X}(XYP)^\varepsilon\quad\text{and}\quad S_2:=\frac{(Z_yQ^\star P)^2}{Y}(XYP)^\varepsilon$$

for any $\varepsilon>0$. Write $m-n=h$. Then we split the inner quadruple sum in (3.23) as the "diagonal term" D_0 and the "off-diagonal" term D_1, where

$$D_0(X,Y,Q^\star)=\sum_n\sum_{m=n}\lambda_{f_1}(n)\lambda_{f_2}(m)\sum_{c \bmod P}\sum_r\frac{S(rl,0;c)}{c}\Xi(m,n,r,c),$$

and

$$D_1(X,Y,Q^\star)=\sum_{c \bmod P}\sum_r\sum_{h\neq 0}\frac{S(rl,h;c)}{c}\sum_{m-n=h}\lambda_{f_1}(n)\lambda_{f_2}(m)\Xi(m,n,r,c).$$

We now turn to the estimations of these two terms as follows.

3.1.1. Treatment of D_0

Using the identity for Ramanujan sum

$$S(n,0;c)=\sum_{c'|(n,c)}\frac{\mu(c/c')}{c'}$$

so that we write $c=c'c''$ with $c'|lr$ and $P|c'c''$. Changing variable $r\to c'r/(c',l)$ the inner sum over c in D_0 can be rewritten as

$$\sum_{c''}\frac{\mu(c'')}{c''}\sum_{\frac{P}{(P,c'')}|c'}\Xi(n,n,c'r/(c',l),c'c''),$$

whence

$$D_0\ll\frac{ZXYR(S_1+S_2)}{PQ^\star}\sum_{c''}\frac{\mu(c'')}{c''}\sum_{\substack{\frac{P}{(P,c'')}|c'\\ c'c''\sim P}}\frac{(P,c'')}{P}\ll ZXYRQ^\star\max(Z_x,Z_y). \tag{3.25}$$

This contributes $\tilde{\mathcal{S}}^{+,-}_{f_1,f_2}(X,Y)$ an amount

$$\ll\frac{(XYP)^\varepsilon ZXY\max(Z_x,Z_y)\max(X,Y)}{PQl}, \tag{3.26}$$

upon noting that $L\gg Q^{2-\varepsilon}$.

3.1.2. Treatment of D_1

Using the idea in [2] (namely partial summation) to separate the variables m,n involving the Fourier coefficients $\lambda_{f_1}(n),\lambda_{f_2}(m)$ and the smooth weight $\Xi(m,n,r,c)$, one might rewrite D_1 as

$$D_1(X,Y,Q^\star)=\int_1^\infty D_1(y)\mathrm{d}y, \tag{3.27}$$

where

$$D_1(y) = \sum_{c \bmod P} \sum_r \sum_{h \neq 0} b_{h,y} \frac{S(rl,h;c)}{c} g_y(r,h,c) \tag{3.28}$$

with

$$b_{h,y} := \sum_{\substack{m \leq y \\ m-n=h}} \lambda_{f_1}(n)\lambda_{f_2}(m) \quad \text{and} \quad g_y(r,h,c) = \frac{\partial}{\partial y}\Xi(y, y-h, r, c). \tag{3.29}$$

As in (3.22), one may proceed by decomposing $g_y(r,h,c)$ in the h and c variables dyadically writing

$$g_y(r,h,c) = \sum_{H,C \geq 1} g_{y;H,C}(r,h,c), \tag{3.30}$$

with $g_{y;H,C}$ being a smooth function of h and c, supported on $h \sim H$ and $c \sim C$, respectively. Recalling (3.24), it is clear from Lemma 2.5 that $g_{y;H,C}$ and all its partial derivatives are very small unless

$$C \sim PQ^\star, \quad y \leq S_1, \quad H \leq S_1 + S_2. \tag{3.31}$$

In such truncated range we may thus deduce the bound for the partial derivatives of Ξ as follows.

Lemma 3.2. *Under (3.31), for any non-negative integers i, j, k, l and any $\varepsilon > 0$ one has*

$$m^i n^j r^k c^l \Xi^{(i,j,k,l)}(m,n,r,c) \ll (XYP)^\varepsilon \mathcal{E} \left(Z_x + Z_y\right)^{i+j}, \tag{3.32}$$

where $\mathcal{E} := \frac{ZXY}{C}$.

Proof. Applying the recurrence relation (2.10) in Lemma 2.3 to the m and n variables one easily see that

$$m^i n^j r^k c^l \Xi^{(i,j,k,l)}(m,n,r,c) \ll \frac{\partial^k}{\partial r^k}\eta_R(r) \int_0^\infty \int_0^\infty \frac{\partial^l}{\partial c^l}\left(\frac{G(c/P,x,y)}{c}\right) W_1^i W_2^j$$
$$\times J_{\nu_{f_2}-i}\left(\frac{4\pi\sqrt{ym}}{c}\right) J_{\nu_{f_1}-j}\left(\frac{4\pi\sqrt{xn}}{c}\right) \mathrm{d}x\mathrm{d}y,$$

where $W_1 := \frac{4\pi\sqrt{ym}}{c}$ and $W_2 := \frac{4\pi\sqrt{xn}}{c}$. Note that the partial derivative $\frac{\partial^l}{\partial c^l} G(c/P,x,y) \ll_l C^{-l}$, whence

$$\begin{aligned} m^i n^j r^k c^l \Xi^{(i,j,k,l)}(m,n,r,c) &\ll (XYP)^\varepsilon \mathcal{E} \left(1 + \frac{\sqrt{nX}}{C} + \frac{\sqrt{mY}}{C}\right)^{i+j} \\ &\ll (XYP)^\varepsilon \mathcal{E} \left(Z_x + Z_y\right)^{i+j}. \quad \square \end{aligned}$$

Consequently we infer that the smooth weight $g_y(r,h,c)$ in (3.30) satisfies

$$\begin{aligned} r^i h^j c^k g_y^{(i,j,k)}(r,h,c) &= r^i h^j c^k \left\{\Xi^{(j+1,0,i,k)}(y, y-h, r, c) + \Xi^{(j,1,i,k)}(y, y-h, r, c)\right\} \\ &\ll \frac{\mathcal{E}(XYP)^\varepsilon}{y} \left(Z_x + Z_y\right)^{i+j} \end{aligned}$$

for all $i, j, k \geq 0$ and any $\varepsilon > 0$. Now an application of Lemma 2.6 gives that

$$\begin{aligned} D_1(y) \ll &\frac{\mathcal{E}(XYP)^\varepsilon}{y} l^\vartheta \left(1 + \frac{\sqrt{lRH}}{C}\right)^{-1-2\vartheta} \\ &\times \tilde{Z}^9 \left(\tilde{Z} + \frac{\sqrt{lRH}}{C} + \sqrt{\frac{R}{P}}\right)\left(\tilde{Z} + \frac{\sqrt{lRH}}{C} + \sqrt{\frac{H}{P}}\right) R^{\frac{1}{2}} \parallel b_y \parallel_2, \end{aligned}$$

where $\tilde{Z} = Z_x + Z_y$ and meanwhile one sees that by Lemma 2.2

$$\parallel b_y \parallel_2 \ll P^{1/3} y^{1/2} S_2^{1/2},$$

so that (3.27) shows that

$$\begin{aligned} &\sum_{Q^\star \geq 1} D_1(X, Y, Q^\star) \\ &\ll \sup_{Q^\star \leq Q} \max_{\substack{H \leq S_1 + S_2 \\ C \sim Q^\star}} P^{1/3} l^\vartheta \mathcal{E}(XYP)^\varepsilon \left(1 + \frac{\sqrt{lRH}}{C}\right)^{-1-2\vartheta} \\ &\quad \times \tilde{Z}^9 \left(\tilde{Z} + \frac{\sqrt{lRH}}{C} + \sqrt{\frac{R}{P}}\right)\left(\tilde{Z} + \frac{\sqrt{lRH}}{C} + \sqrt{\frac{H}{P}}\right) R^{1/2} S_1^{1/2} S_2^{1/2} \\ &\ll (XYP)^\varepsilon ZXYP^{-2/3} l^\vartheta R^{1/2} \left(1 + \sqrt{R/P}\right) \tilde{Z}^{10} \sup_{Q^\star \leq Q} \frac{1}{Q^\star} \cdot \\ &\quad \times \max_{H \leq \frac{\{\max(Z_x, Z_y) Q^\star P\}^2}{\min(X,Y)} (XYP)^\varepsilon} \left(1 + \frac{\sqrt{lRH}}{PQ^\star}\right)^{-2\vartheta} \left(\tilde{Z} + \frac{\sqrt{lRH}}{PQ^\star} + \sqrt{\frac{H}{P}}\right) S_1^{1/2} S_2^{1/2} \\ &\ll (XYP)^\varepsilon \frac{ZXYP^{11/6} Q^2}{l^{1/2-\vartheta}} \left(1 + \sqrt{\max(X,Y)/lP}\right) \\ &\quad \times \max(Z_x, Z_y)^{12} \frac{\max(X,Y)^{1/2}}{\min(X,Y)^{3/2}}. \end{aligned}$$

This will thus contribute $\tilde{\mathcal{S}}^{+,-}_{f_1,f_2}(X,Y)$ in (3.23) an amount, upon recalling (3.23) and $L \gg Q^{2-\varepsilon}$,

$$\ll (XYP)^{\varepsilon} \frac{ZXYP^{5/6}}{l^{1/2-\vartheta}} \max(Z_x, Z_y)^{12} \frac{\max(X,Y)^{1/2}}{\min(X,Y)^{3/2}} \left(1 + \sqrt{\frac{\max(X,Y)}{lP}}\right). \quad (3.33)$$

3.2. Case 2: $P|\gamma$

In this case, changing variable γ to γP, we are led to the equality $\tilde{\mathcal{S}}_{f_1,f_2}(X,Y;r) = \tilde{\mathcal{S}}^{\flat}_{f_1,f_2}(X,Y;r) - \tilde{\mathcal{S}}^{\natural}_{f_1,f_2}(X,Y;r)$ with

$$\begin{aligned}&\tilde{\mathcal{S}}^{\flat}_{f_1,f_2}(X,Y;r)\\ &\quad := \frac{1}{PL}\sum_{q}\ \sideset{}{^*}\sum_{\gamma \bmod q} e\left(\frac{rl\gamma}{q}\right)\sum_{n}\lambda_{f_1}(n)e\left(\frac{n\gamma}{q}\right)\sum_{m}\lambda_{f_2}(m)e\left(-\frac{m\gamma}{q}\right)G(q,n,m),\end{aligned}$$

and

$$\begin{aligned}&\tilde{\mathcal{S}}^{\natural}_{f_1,f_2}(X,Y;r)\\ &\quad := \frac{1}{PL}\sum_{P|q}\ \sideset{}{^*}\sum_{\gamma \bmod q} e\left(\frac{rl\gamma}{q}\right)\sum_{n}\lambda_{f_1}(n)e\left(\frac{n\gamma}{q}\right)\sum_{m}\lambda_{f_2}(m)e\left(-\frac{m\gamma}{q}\right)G(q,n,m).\end{aligned}$$

Correspondingly we denote by $\tilde{\mathcal{S}}^{\flat}_{f_1,f_2}(X,Y)$ and $\tilde{\mathcal{S}}^{\natural}_{f_1,f_2}(X,Y)$ the contributions when summing with respect to r outside for these two terms. It suffices to treat $\tilde{\mathcal{S}}^{\flat}_{f_1,f_2}(X,Y;r)$, the argument of $\tilde{\mathcal{S}}^{\natural}_{f_1,f_2}(X,Y;r)$ will follow similarly. To do so we follow the line of the argument of dealing with $\tilde{\mathcal{S}}_{f_1,f_2}(X,Y;r)$ in Case 1.

Hence after using the Voronoï summation formula one sees that the estimation of $\tilde{\mathcal{S}}^{\flat}_{f_1,f_2}(X,Y;r)$ is boiled down to bounding

$$\frac{1}{P^2L}\sum_{q\geq 1}\frac{1}{q^2}\sum_{n}\sum_{m}\lambda_{g_1}(n)\lambda_{g_2}(m)S(rl,\overline{P}(m-n);q)G^{\flat}_{J,J}(q,n,m), \quad (3.34)$$

where

$$G^{\flat}_{J,J}(q,n,m) = \int_0^{\infty}\int_0^{\infty} G(q,x,y)J_{f_1}\left(\frac{4\pi\sqrt{xn}}{q\sqrt{P}}\right)J_{f_2}\left(\frac{4\pi\sqrt{ym}}{q\sqrt{P}}\right)\mathrm{d}x\mathrm{d}y.$$

We observe that $G^{\flat}_{J,J}(q,n,m)$ is negligibly small unless

$$n \ll \frac{Z_x^2q^2P}{X}(XYP)^{\varepsilon}, \quad m \ll \frac{Z_y^2q^2P}{Y}(XYP)^{\varepsilon} \quad (3.35)$$

for any $\varepsilon > 0$. Decomposing dyadically the r variable $r \sim R$, and the q variable $q \sim Q^\star$ of the sum $G^\flat_{J,J}(q,n,m)$ such that $G^\flat_{J,J}(q,n,m) = \sum_{Q^\star \geq 1} G^\flat_{J,J;Q^\star}(q,n,m)$, we see that

$$\tilde{\mathcal{S}}^\flat_{f_1,f_2}(X,Y) = \frac{1}{P^2L} \sum_{Q^\star} \sum_n \sum_m \lambda_{f_1}(n)\lambda_{f_2}(m) \sum_c \sum_r \frac{S(rl, \overline{P}(m-n); c)}{c} \Xi^\flat(m,n,r,c), \tag{3.36}$$

where $\Xi^\flat(m,n,r,c) = G^\flat_{J,J;Q^\star}(c,n,m)\eta_R(r)/c$. Here the smooth function $\eta_R(r)$ is as before, $R \leq \max(X,Y)/l$, and $Q/2 \leq Q^\star \leq 5Q/2$.

Argue as in Case 1, one can split the inner quadruple sum in (3.36) as two parts (that is, the "diagonal term" and the "off-diagonal" term) $D^\flat_0$ and $D^\flat_1$, say. For $D^\flat_0$, one might verify that it is dominated by the upper-bound of D_0 in (3.25), thereby the contribution to $\tilde{\mathcal{S}}^\flat_{f_1,f_2}(X,Y)$ is less than the bound (3.26). For $D^\flat_1$, one has (analogous to (3.27))

$$D^\flat_1 = \int_1^\infty D^\flat_1(y)\mathrm{d}y,$$

where

$$D^\flat_1(y) = \sum_c \sum_r \sum_{h \neq 0} b_{h,y} \frac{S(rl, h\overline{P}; c)}{c} g^\flat_y(r,h,c)$$

with the function $g^\flat_y$ denoting the g_y-function associated with $\Xi^\flat$ in (3.29). By dyadic subdivision one sees that the integrand $D^\flat_1(y)$ is

$$\ll \max_{H',C'} \sum_c \sum_r \sum_{h \neq 0} b_{h,y} \frac{S(rl, h\overline{P}; c)}{c} g^\flat_{y;H',C'}(r,h,c),$$

where $g^\flat_{y;H',C'}(r,h,c)$ is a smooth function supported on $h \sim H', c \sim C'$, and satisfies that

$$g^\flat_y(r,h,c) = \sum_{H',C' \geq 1} g^\flat_{y;H',C'}(r,h,c).$$

Now an application of Lemma 2.5 shows that $g^\flat_{y;H',C'}$ and all its partial derivatives are very small unless

$$C' \sim Q^\star, \quad y \leq S'_1, \quad r \sim R, \quad H' \leq S'_1 + S'_2,$$

where

$$S'_1 = \frac{(Z_xQ^\star)^2P}{X}(XYP)^\varepsilon \quad \text{and} \quad S'_2 = \frac{(Z_yQ^\star)^2P}{Y}(XYP)^\varepsilon$$

for any $\varepsilon > 0$. Moreover it can be easily seen that

$$r^i h^j c^k \frac{\partial^{i+j+k}}{\partial r^i \partial h^j \partial c^k} g^{\flat}_{y;H',C'}(r,h,c) \ll \frac{\mathcal{E}'(XYP)^{\varepsilon}}{y} (Z_x + Z_y)^{i+j}$$

with $\mathcal{E}' = ZXY/C'$ for any non-negative integers i, j, k and any $\varepsilon > 0$. Hence, under these circumstances above, Lemma 2.6 together with Lemma 2.2 may yield

$$\begin{aligned} &\sum_{Q^\star \geq 1} D_1^{\flat}(X,Y,Q^\star) \\ &\ll \sup_{Q^\star \leq Q} \max_{\substack{H' \leq S_1' + S_2' \\ C' \leq Q^\star}} P^{1/3}\mathcal{E}'(XYP)^{\varepsilon} l^{\vartheta}\sqrt{P}\left(1 + \frac{\sqrt{lRH'}}{C'\sqrt{P}}\right)^{1-2\vartheta} \\ &\quad \times \tilde{Z}^9 \left(\tilde{Z} + \frac{\sqrt{lRH'}}{C'\sqrt{P}} + \sqrt{\frac{R}{P}}\right)\left(\tilde{Z} + \frac{\sqrt{lRH'}}{C'\sqrt{P}} + \sqrt{\frac{H'}{P}}\right) R^{1/2} {S_1'}^{1/2} {S_2'}^{1/2} \\ &\ll (XYP)^{\varepsilon} \frac{ZXYP^{11/6}Q^2}{l^{1/2-\vartheta}} \max(Z_x, Z_y)^{12} \frac{\max(X,Y)^{1/2}}{\min(X,Y)^{3/2}} \left(1 + \sqrt{\frac{\max(X,Y)}{lP}}\right) \end{aligned} \tag{3.37}$$

which is of the same order as that for $D_1(X, Y, Q^\star)$. We thus conclude the contribution from the "off-diagonal" term $D_1^{\flat}$ to $\tilde{\mathcal{S}}^{\flat}_{f_1,f_2}(X,Y)$ in (3.36) is bounded by the estimate (3.33) in Case 1.

Combining with (3.17), (3.26) and (3.33) and choosing Q to be sufficiently large, say $Q = (XYPM)^{100}$, Theorem 1.4 follows immediately.

4. Proof of Theorem 1.1

In this section we are concerned about the sum

$$\sum_{\chi \bmod M}^{*} \frac{1}{\varphi^*(M)} \left| \sum_n \psi_f(n)\chi(n)h(n/X) \right|^2, \tag{4.38}$$

where $X \leq \mathcal{Q}^{1/2+\varepsilon}$ for any $\varepsilon > 0$, and χ runs over the primitive characters modulo M. Note that the trivial bound of (4.38) is $O(X^2/M)$ which would lead to the convexity bound for any individual $L(1/2, f \otimes \chi)$. To get cancellation we expand the square deriving that

$$\begin{aligned} (4.38) = \sum_n \psi_f^2(n) h^2\left(\frac{n}{X}\right) + \frac{1}{\varphi^*(M)} \sum_{l|M} \varphi(l)\mu\left(\frac{M}{l}\right) \\ \times \sum_{r \neq 0} \sum_{m=n+rl} \psi_f(n)\psi_f(m) h\left(\frac{n}{X}\right) h\left(\frac{m}{X}\right), \end{aligned} \tag{4.39}$$

where we have used the relation

$$\sum_{\chi \bmod M}^{*} \chi(m)\overline{\chi}(n) = \sum_{\substack{ab=M \\ a|(m-n)}} \varphi(a)\mu(b)$$

for any m, n with $(mn, M) = 1$ (see for instance [11]). The first term on the right hand side of (4.39) is trivially $O_\varepsilon(X\mathcal{Q}^\varepsilon)$ for any $\varepsilon > 0$; while by Theorem 1.4 the second term is bounded by

$$\ll \frac{1}{\varphi^*(M)} \sum_{l|M} \varphi(l) \frac{(XP)^\varepsilon XP^{\frac{5}{6}}}{l^{\frac{1}{2}-\vartheta}} \left(1 + \sqrt{\frac{X}{lP}}\right) \ll \frac{(XP)^\varepsilon XP^{\frac{5}{6}}}{\varphi^*(M)} \sum_{l|M} \frac{\varphi(l)}{l^{\frac{1}{2}-\vartheta}}$$
$$\ll X\sqrt{P}\mathcal{Q}^\varepsilon \cdot \frac{P^{\frac{1}{3}}}{M^{\frac{1}{2}-\vartheta}},$$

and hence Theorem 1.1.

5. Proof of Corollary 1.2

Let us recall that the approximate functional equation (see for instance [17, Chapter 5]) implies that

$$L(1/2, f \otimes \chi) \ll \mathcal{Q}^\varepsilon \sup_{X \leq \mathcal{Q}^{1/2+\varepsilon}} \frac{|S(X)|}{\sqrt{X}},$$

where $S(X)$ are sums of the type

$$S(X) = \sum_n \lambda_f(n)\chi(n)h\left(\frac{n}{X}\right)$$

for some smooth function h compactly supported on $[1/2, 5/2]$ with bounded derivatives. By Theorem 1.1 we pick up only one term when summing over the family of the primitive characters getting

$$\sup_{X \leq \mathcal{Q}^{1/2+\varepsilon}} \frac{S(X)}{\sqrt{X}} \ll \mathcal{Q}^{\frac{1}{4}+\varepsilon} \left(\frac{1}{P^{\frac{1}{4}}} + \frac{P^{\frac{1}{6}}}{M^{\frac{1}{4}-\frac{\vartheta}{2}}}\right).$$

Replacing all occurrences of P by M^η we obtain Corollary 1.2.

Acknowledgments

The authors are very grateful to the referee for careful reading of the earlier draft and valuable suggestions. The first author is supported in part by Startup Foundation of XAUT (256081703) and NSFC (413618022). The second author is supported in part by NSFC (61402335), the China Scholarship Council, SRPF-SXPED (17JK0266) and NSF-WNU (17ZRRC01).

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