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Asymptotics of class number and genus for abelian extensions of an algebraic function field

Kenneth Ward

Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA

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ABSTRACT

Among abelian extensions of a congruence function field, an asymptotic relation of class number and genus is established: namely, for such extensions with class number h, genus g, and field of constants $\mathbb{F}$, that $\ln h \sim g \ln |\mathbb{F}|$. The proof is completely classical, employing well known results from congruence function field theory. This gives an answer to a question of E. Inaba.

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1. Introduction

Let K be a congruence function field with genus g_K and class number h_K. The study of the asymptotic behavior of class number and genus for congruence function fields dates to a result of E. Inaba [7], which established, for a natural number m, that among congruence function fields K with a fixed choice of finite constant field $\mathbb{F}_q$ and an element $x \in K$ that satisfies $[K : \mathbb{F}_q(x)] \leqslant m$,

$$\lim_{g_K \to \infty} \frac{\ln h_K}{g_K \ln |\mathbb{F}_q|} = 1. \tag{1.1}$$

E-mail address: kward@math.okstate.edu.

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In his paper Inaba remarked that he was not aware of whether this relation remains true if the bound involving m is removed. As noted by K. Iwasawa in Inaba's article, the requirement that m be fixed resembles R. Brauer's first result on the Brauer–Siegel theorem for algebraic number fields [1]. Results similar to that of Inaba also appear in the work of I. Luthar and S. Gogia [13] and M. Tsfasman [17].

Much later, M. Madan and D. Madden [14] noted, for congruence function fields K with a fixed choice of constant field $\mathbb{F}_q$ and an element $x \in K \backslash \mathbb{F}_q$, that Inaba's method yields

$$\lim_{\frac{[K:\mathbb{F}_q(x)]}{g_K} \to 0} \frac{\ln h_K}{g_K \ln |\mathbb{F}_q|} = 1. \tag{1.2}$$

One may observe that (1.2) loosely resembles the condition required in R. Brauer's second paper [2] on the Brauer–Siegel theorem, with an exception: in addition to requiring for an extension L of the rational numbers with discriminant d that $[L:\mathbb{Q}]/\ln d$ tends to zero, it was necessary for Brauer to assume that the extension $L/\mathbb{Q}$ be normal. It is difficult to surmount both of these requirements in the case of number fields as a result of the connection to the Riemann hypothesis [10]. However, Brauer's result may be extended to abelian number fields without any relative requirement on discriminant growth; for example, see the concise argument of S.R. Louboutin [12].

The objective of this paper is to establish the analogue to Brauer's theorem for finite abelian extensions of *any* choice of congruence function field. In fact, P. Lam-Estrada and G.D. Villa-Salvador [9] have already noted that, by a result of D. Hayes [6], the relation (1.1) holds among the cyclotomic extensions of the field of rational functions $\mathbb{F}_q(T)$. The objective is met by means of two theorems. For what follows, let $\mathbb{F}_K$ denote the constant field of K.

Theorem 1. *Let K be a congruence function field. It holds that*

$$\liminf_{g_K \to \infty} \frac{\ln h_K}{g_K \ln |\mathbb{F}_K|} \geqslant 1.$$

The bound attained in the proof of Theorem 1 is effective. Furthermore, Theorem 1 makes no requirement that K be a finite abelian extension of a congruence function field. It is for the proof of the upper bound that abelian structure is essential.

Theorem 2. *Let F be a congruence function field. Let K be a finite abelian extension of F. It holds that*

$$\limsup_{g_K \to \infty} \frac{\ln h_K}{g_K \ln |\mathbb{F}_K|} \leqslant 1.$$

The method of proof for Theorem 2 in this paper is unique to the abelian case; indeed, the required properties are violated within the simplest class of non-abelian extensions of a congruence function field for which the genus may become large: finite, geometric, tamely ramified, and solvable extensions of $\mathbb{F}_q(T)$. This is a consequence of the possibility of slow growth of the genus [5]. Also, unlike Theorem 1, the bound attained in the proof of Theorem 2 is ineffective. As a corollary of Theorems 1 and 2, one obtains the main result of this paper.

Corollary. *Let F be a congruence function field. Let K be a finite abelian extension of F. It holds that*

$$\lim_{g_K \to \infty} \frac{\ln h_K}{g_K \ln |\mathbb{F}_K|} = 1.$$

2. The lower bound

The proof of Theorem 1 proceeds as follows.

1. Count the number of monic irreducible polynomials of a given degree with coefficients in $\mathbb{F}_K$ via Möbius inversion;
2. For $x \in K \backslash \mathbb{F}_K$, compare the number of places of a given degree in K to the number of places of the same degree in $\mathbb{F}_K(x)$ via Möbius inversion and Riemann's hypothesis;
3. Obtain a lower bound for the number of integral divisors of degree $2g_K$ in K via the Riemann–Roch theorem.

This proof follows closely Inaba's original method in [7]. The first step is a basic result in field theory [11].

Lemma 1. *Let $x \in K \backslash \mathbb{F}_K$. For each $m \in \mathbb{N}$, let $\psi(m)$ be the number of monic irreducible elements of $\mathbb{F}_K[x]$ of degree in x equal to m. Let μ be the Möbius function. It holds for each $m \in \mathbb{N}$ that*

$$\psi(m) = \frac{1}{m} \sum_{d|m} \mu\left(\frac{m}{d}\right) |\mathbb{F}_K|^d.$$

The second step of the proof follows a method known to H. Reichardt [16]. The basic principle is as follows. For K, let $\mathbb{P}_K$ denote the collection of places and d_K the degree function on divisors. For each $m \in \mathbb{N}$, let

$$N_m = \left|\left\{\mathfrak{P} \in \mathbb{P}_K \;\middle|\; d_K(\mathfrak{P}) = m\right\}\right|.$$

Also, let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and $u = |\mathbb{F}_K|^{-s}$. One may write the zeta function $\zeta_K(s)$ of K as

$$\zeta_K(s) = \prod_{\mathfrak{P} \in \mathbb{P}_K} \left(1 - \frac{1}{|\mathbb{F}_K|^{d_K(\mathfrak{P})s}}\right)^{-1} = \prod_{k=1}^{\infty} \left(1 - u^k\right)^{-N_k}. \tag{2.1}$$

Let $x \in K \backslash \mathbb{F}_K$. For $\mathbb{F}_K(x)$, let $\mathbb{P}_0$ denote the collection of places, d_0 the degree function on divisors, $\zeta_0(s)$ the zeta function, and

$$n_m = \left|\left\{\mathfrak{p} \in \mathbb{P}_0 \;\middle|\; d_0(\mathfrak{p}) = m\right\}\right|.$$

Application of the logarithmic derivative to both (2.1) and the analogous identity for $\zeta_0(s)$ yields that

$$\frac{\zeta_K'(s)}{\zeta_K(s)} - \frac{\zeta_0'(s)}{\zeta_0(s)} = -\ln|\mathbb{F}_K| \sum_{m=1}^{\infty} \left(\sum_{d|m} d(N_d - n_d)\right) u^m. \tag{2.2}$$

Let $P_K(s) = (1-u)(1-|\mathbb{F}_K|u)\zeta_K(s)$. It is well known [3] that there exist $\omega_1, \ldots, \omega_{2g_K} \in \mathbb{C}$ with

$$P_K(s) = \prod_{i=1}^{2g_K} (1 - \omega_i u). \tag{2.3}$$

Furthermore, as $\mathrm{Re}(s) > 1$, one has that $P_K(s) = \zeta_K(s)/\zeta_0(s)$. From (2.2) and (2.3), it follows that

$$\sum_{d|m} d(N_d - n_d) = -\sum_{i=1}^{2g_K} \omega_i^m. \tag{2.4}$$

By Riemann's hypothesis, it holds for each $i = 1, \dots, 2g_K$ that $|\omega_i| = |\mathbb{F}_K|^{\frac{1}{2}}$. By Möbius inversion, one then obtains from (2.4) the following lemma.

Lemma 2. *Let $x \in K \backslash \mathbb{F}_K$. For each $m \in \mathbb{N}$, it holds that*

$$|N_m - n_m| \leqslant 4g_K |\mathbb{F}_K|^{\frac{m}{2}}.$$

Proof of Theorem 1. For a divisor class C of K, let $l_K(C)$ denote the dimension over $\mathbb{F}_K$ of the Riemann–Roch space for any element of C. If C is of degree equal to $2g_K$, the Riemann–Roch theorem gives that $l_K(C) = g_K + 1$. Thus the number of integral divisors A_{2g_K} of K of degree $2g_K$ satisfies

$$A_{2g_K} = h_K\left(\frac{|\mathbb{F}_K|^{g_K+1} - 1}{|\mathbb{F}_K| - 1}\right). \tag{2.5}$$

By (2.5) and Lemmas 1 and 2, one obtains that

$$\begin{aligned}
h_K\left(\frac{|\mathbb{F}_K|^{g_K+1} - 1}{|\mathbb{F}_K| - 1}\right) &\geqslant N_{2g_K} \geqslant n_{2g_K} - 4g_K|\mathbb{F}_K|^{g_K} = \psi(2g_K) - 4g_K|\mathbb{F}_K|^{g_K} \\
&\geqslant \frac{|\mathbb{F}_K|^{2g_K}}{2g_K} - \left|\frac{1}{2g_K}\sum_{\substack{d|2g_K\\ d<2g_K}} \mu\left(\frac{2g_K}{d}\right)|\mathbb{F}_K|^d\right| - 4g_K|\mathbb{F}_K|^{g_K} \\
&\geqslant \frac{|\mathbb{F}_K|^{2g_K}}{2g_K} - \sum_{\substack{d|2g_K\\ d<2g_K}} |\mathbb{F}_K|^d - 4g_K|\mathbb{F}_K|^{g_K} \\
&\geqslant \frac{|\mathbb{F}_K|^{2g_K}}{2g_K} - \sum_{d=1}^{g_K}|\mathbb{F}_K|^d - 4g_K|\mathbb{F}_K|^{g_K} \\
&\geqslant \frac{|\mathbb{F}_K|^{2g_K}}{2g_K} - (4g_K + 2)|\mathbb{F}_K|^{g_K}.
\end{aligned} \tag{2.6}$$

By (2.6), if g_K is large enough, it holds for any possible value of $|\mathbb{F}_K|$ that

$$h_K \geqslant \frac{(|\mathbb{F}_K| - 1)|\mathbb{F}_K|^{g_K-1}}{4g_K}. \tag{2.7}$$

It may be assumed that $g_K > 0$. As $|\mathbb{F}_K| \geqslant 2$, by application of the logarithm to (2.7), one obtains for g_K large enough that

$$\frac{\ln h_K}{g_K \ln|\mathbb{F}_K|} \geqslant \frac{\ln(|\mathbb{F}_K| - 1)}{g_K \ln|\mathbb{F}_K|} + 1 - \frac{1}{g_K} - \frac{\ln 4g_K}{g_K \ln|\mathbb{F}_K|} \geqslant 1 - \frac{1 + \ln 4g_K}{g_K \ln 2}. \tag{2.8}$$

The result follows. □

3. The abelian case

For this section, let F also denote a congruence function field. Though more involved, the proof of Theorem 2 is also quite basic. The essential steps are as follows.

1. Establish the upper bound of Theorem 2 for K with a condition on the growth of the genus via ramification theory and Riemann's inequality;
2. Obtain an upper bound for the degree of a finite, geometric, unramified, and abelian extension H/F via global class field theory [8];
3. Obtain a lower bound for the degree of the different of a finite abelian extension K/F via higher ramification theory and the Hasse–Arf theorem [15];
4. Derive a contradiction for a sequence that violates the statement of Theorem 2 via the Riemann–Roch theorem, Riemann's hypothesis, and the Riemann–Hurwitz formula.

Throughout this section, the notation of Section 2 is assumed. The first part of this proof is similar to the method of Madden and Madan in [14]. For a divisor class C of K of degree n, one has by Riemann's inequality that $l_K(C) \geqslant n - g_K + 1$. Thus the number of integral divisors A_n of K of degree n satisfies

$$A_n \geqslant h_K \left(\frac{|\mathbb{F}_K|^{n-g_K+1} - 1}{|\mathbb{F}_K| - 1} \right). \tag{3.1}$$

Let $s \in \mathbb{R}$ with $s > 1$ and $x \in K \backslash \mathbb{F}_K$. By (3.1), it follows that

$$\begin{aligned} \zeta_K(s) &= \sum_{n=0}^{\infty} \frac{A_n}{|\mathbb{F}_K|^{ns}} \geqslant \sum_{n=g_K}^{\infty} \frac{A_n}{|\mathbb{F}_K|^{ns}} \geqslant \sum_{n=g_K}^{\infty} h_K \left(\frac{|\mathbb{F}_K|^{n-g_K+1} - 1}{|\mathbb{F}_K| - 1} \right) \frac{1}{|\mathbb{F}_K|^{ns}} \\ &= \frac{h_K}{|\mathbb{F}_K|^{g_K s}} \sum_{n=g_K}^{\infty} \frac{|\mathbb{F}_K|^{n-g_K+1} - 1}{|\mathbb{F}_K| - 1} \frac{1}{|\mathbb{F}_K|^{(n-g_K)s}} = \frac{h_K}{|\mathbb{F}_K|^{g_K s}} \zeta_0(s). \end{aligned} \tag{3.2}$$

Let $\mathfrak{p} \in \mathbb{P}_0$, and let $\mathfrak{P}_1, \ldots, \mathfrak{P}_r$ be those places of K that lie above $\mathfrak{p}$. For each $i = 1, \ldots, r$, let $e(\mathfrak{P}_i|\mathfrak{p})$ denote the ramification index and $f(\mathfrak{P}_i|\mathfrak{p})$ the relative degree of $\mathfrak{P}_i|\mathfrak{p}$. By ramification theory (see for example [18]), it holds that

$$\sum_{i=1}^{r} e(\mathfrak{P}_i|\mathfrak{p}) f(\mathfrak{P}_i|\mathfrak{p}) = \big[K : \mathbb{F}_K(x)\big]. \tag{3.3}$$

As $K/\mathbb{F}_K(x)$ is a geometric extension, it follows from (3.3) that

$$\begin{aligned} \zeta_K(s) &= \prod_{\mathfrak{P} \in \mathbb{P}_K} \left(1 - \frac{1}{|\mathbb{F}_K|^{d_K(\mathfrak{P})s}} \right)^{-1} \\ &\leqslant \prod_{\mathfrak{p} \in \mathbb{P}_0} \left(1 - \frac{1}{|\mathbb{F}_K|^{d_0(\mathfrak{p})s}} \right)^{-[K:\mathbb{F}_K(x)]} = \zeta_0(s)^{[K:\mathbb{F}_K(x)]}. \end{aligned} \tag{3.4}$$

By (3.2) and (3.4), one obtains that

$$\frac{h_K}{|\mathbb{F}_K|^{g_K s}} \leqslant \zeta_0(s)^{[K:\mathbb{F}_K(x)]-1}. \tag{3.5}$$

As $|\mathbb{F}_K| \geqslant 2$ and $\zeta_0(s) \leqslant \zeta_{\mathbb{F}_2(T)}(s)$, application of the logarithm to (3.5) yields that

$$\frac{\ln h_K}{g_K \ln |\mathbb{F}_K|} \leqslant s + \frac{([K:\mathbb{F}_K(x)] - 1) \ln \zeta_{\mathbb{F}_2(T)}(s)}{g_K \ln 2}. \tag{3.6}$$

Let ε be fixed and positive, and let $s = 1 + \frac{\varepsilon}{2}$. If the quantity $[K : \mathbb{F}_K(x)]/g_K$ is chosen to be sufficiently close to zero, it follows from (3.6) that

$$\frac{\ln h_K}{g_K \ln |\mathbb{F}_K|} < 1 + \varepsilon.$$

The following lemma and the first step of the proof of Theorem 2 have therefore been established.

Lemma 3. *Let $x \in K \backslash \mathbb{F}_K$. It holds that*

$$\limsup_{\frac{[K:\mathbb{F}_K(x)]}{g_K} \to 0} \frac{\ln h_K}{g_K \ln |\mathbb{F}_K|} \leqslant 1.$$

By the reciprocity map of global class field theory, a maximal finite, geometric, unramified, and abelian extension of a congruence function field F is of degree h_F. From this fact, one obtains the following lemma and the second step of the proof of Theorem 2.

Lemma 4. *Let H/F be a finite, geometric, unramified, and abelian extension. It holds that $[H : F] \leqslant h_F$.*

The third step of the proof of Theorem 2 follows a method known to G. Frey et al. [4]. Let K/F be a finite abelian extension. Let $\mathfrak{p} \in \mathbb{P}_F$ and $\mathfrak{P} \in \mathbb{P}_K$ with $\mathfrak{P}|\mathfrak{p}$. For each $n = 0, 1, 2, \ldots$, let $G_n(\mathfrak{P}|\mathfrak{p})$ denote the nth ramification group of $\mathfrak{P}|\mathfrak{p}$. Also, let $\alpha_{\mathfrak{P}|\mathfrak{p}}$ denote the differential exponent of $\mathfrak{P}|\mathfrak{p}$. First, by ramification theory, one obtains that

$$\alpha_{\mathfrak{P}|\mathfrak{p}} = \sum_{n=0}^{\infty} \bigl(\bigl|G_n(\mathfrak{P}|\mathfrak{p})\bigr| - 1\bigr). \tag{3.7}$$

Let $k(\mathfrak{P}|\mathfrak{p})$ denote the number of ramification jumps of $\mathfrak{P}|\mathfrak{p}$. By the Hasse–Arf theorem, it follows from (3.7) that

$$\alpha_{\mathfrak{P}|\mathfrak{p}} \geqslant \frac{1}{2} k(\mathfrak{P}|\mathfrak{p}) e(\mathfrak{P}|\mathfrak{p}). \tag{3.8}$$

Second, let $K_{\mathfrak{P}}$ and $F_{\mathfrak{P}}$ denote the completion of K, respectively F, according to $\mathfrak{P}$, respectively $\mathfrak{p}$. Let $\mathfrak{P}$ be identified with its maximal ideal, $\vartheta_{\mathfrak{P}}$ denote the valuation ring for $\mathfrak{P}$, and $\pi_{\mathfrak{P}}$ be an element prime for $\mathfrak{P}$. As $K_{\mathfrak{P}}/F_{\mathfrak{P}}$ is abelian, the action of $\mathrm{Gal}(K_{\mathfrak{P}}/F_{\mathfrak{P}})$ is trivial on each element in the image of each injection

$$\psi_0 : G_0(\mathfrak{P}|\mathfrak{p})/G_1(\mathfrak{P}|\mathfrak{p}) \to (\vartheta_{\mathfrak{P}}/\mathfrak{P})^*, \quad \psi_0(\sigma) = \frac{\sigma(\pi_{\mathfrak{P}})}{\pi_{\mathfrak{P}}}$$

and, for each $n \in \mathbb{N}$,

$$\psi_n : G_n(\mathfrak{P}|\mathfrak{p})/G_{n+1}(\mathfrak{P}|\mathfrak{p}) \to \mathfrak{P}^n/\mathfrak{P}^{n+1}, \quad \psi_n(\sigma) = \frac{\sigma(\pi_{\mathfrak{P}})}{\pi_{\mathfrak{P}}} - 1.$$

Identifying $\mathfrak{p}$ with its maximal ideal and denoting by $\vartheta_{\mathfrak{P}}$ the valuation ring of $\mathfrak{p}$, it follows that

$$e(\mathfrak{P}|\mathfrak{p}) \leqslant |\vartheta_{\mathfrak{p}}/\mathfrak{p}|^{k(\mathfrak{P}|\mathfrak{p})}. \tag{3.9}$$

Finally, observing that the fixed field of the product of the inertia groups $G_0(\mathfrak{P}|\mathfrak{p})$ over all $\mathfrak{p} \in \mathbb{P}_F$ is simply the maximal unramified extension of F in K, one obtains the following result as a consequence of (3.8) and (3.9).

Lemma 5. *Let K/F be a finite abelian extension. Let $H_{K/F}$ denote the maximal unramified extension of F in K. It follows that the different $\mathfrak{D}_{K/F}$ satisfies*

$$d_K(\mathfrak{D}_{K/F}) \geqslant \frac{[K:F]}{2\ln|\mathbb{F}_F|}\bigl(\ln[K:F] - \ln[H_{K/F}:F]\bigr).$$

Proof of Theorem 2. Consider a sequence $\{K_n\}_{n\in\mathbb{N}}$ with K_n/F a finite abelian extension for each $n \in \mathbb{N}$ and unbounded sequence of genera $\{g_{K_n}\}_{n\in\mathbb{N}}$. Furthermore, suppose that there exists a positive $\delta \in \mathbb{R}$ with, for each $n \in \mathbb{N}$, $\ln h_{K_n}/(g_{K_n}\ln|\mathbb{F}_{K_n}|) \geqslant 1+\delta$. Let $x \in F\backslash\mathbb{F}_F$. By Lemma 3, there exists a positive $\varepsilon \in \mathbb{R}$ with, for each $n \in \mathbb{N}$,

$$\frac{[K_n : \mathbb{F}_{K_n}(x)]}{g_{K_n}} \geqslant \varepsilon. \tag{3.10}$$

Let $s \in \mathbb{C}$, $u = |\mathbb{F}_K|^{-s}$, and $n \in \mathbb{N}$. Let $P_{K_n}(s)$ be defined as in Section 2. As noted in (2.3), there exist $\omega_1, \dots, \omega_{2g_{K_n}}$ so that

$$P_{K_n}(s) = \prod_{i=1}^{2g_{K_n}} (1-\omega_i u). \tag{3.11}$$

By Riemann's hypothesis, one has for each $i = 1, \dots, 2g_{K_n}$ that $|\omega_i| = |\mathbb{F}_{K_n}|^{\frac{1}{2}}$. Also, it is well known [3] that $P_{K_n}(0) = h_{K_n}$. From (3.11), one obtains that

$$h_{K_n} = P_{K_n}(0) = \bigl|P_{K_n}(0)\bigr| = \prod_{i=1}^{2g_{K_n}} |1-\omega_i| \leqslant \bigl(1+|\mathbb{F}_{K_n}|^{\frac{1}{2}}\bigr)^{2g_{K_n}}. \tag{3.12}$$

It may be assumed for each $n \in \mathbb{N}$ that $g_{K_n} > 0$. Application of the logarithm to (3.12) yields that

$$\frac{\ln h_{K_n}}{g_{K_n}\ln|\mathbb{F}_{K_n}|} \leqslant \frac{2\ln(1+|\mathbb{F}_{K_n}|^{\frac{1}{2}})}{\ln|\mathbb{F}_{K_n}|}. \tag{3.13}$$

By (3.13), it follows that the field

$$\mathbb{E} = \prod_{n\in\mathbb{N}} \mathbb{F}_{K_n} \tag{3.14}$$

is finite. By the definition of $\mathbb{E}$ in (3.14), it follows for each $n \in \mathbb{N}$ that the extension $\mathbb{E}K_n/\mathbb{E}F$ is geometric. By the Riemann–Hurwitz formula and Lemmas 4 and 5, one obtains that

$$\begin{aligned}
\frac{g_{\mathbb{E}K_n}}{[\mathbb{E}K_n : \mathbb{E}F]} &\geqslant g_{\mathbb{E}F} - 1 + \frac{1}{2[\mathbb{E}K_n : \mathbb{E}F]} d_{\mathbb{E}K_n}(\mathfrak{D}_{\mathbb{E}K_n/\mathbb{E}F}) \\
&\geqslant g_{\mathbb{E}F} - 1 + \frac{1}{4\ln|\mathbb{E}|}\bigl(\ln[\mathbb{E}K_n : \mathbb{E}F] - \ln[H_{\mathbb{E}K_n/\mathbb{E}F} : \mathbb{E}F]\bigr) \\
&\geqslant g_{\mathbb{E}F} - 1 + \frac{1}{4\ln|\mathbb{E}|}\bigl(\ln[\mathbb{E}K_n : \mathbb{E}F] - \ln h_{\mathbb{E}F}\bigr).
\end{aligned} \tag{3.15}$$

By basic function field theory, it holds that $[\mathbb{E}K_n : \mathbb{E}(x)] = [K_n : \mathbb{F}_{K_n}(x)]$. As the sequence of genera $\{g_{K_n}\}_{n\in\mathbb{N}}$ is unbounded, it follows from (3.10) that the sequence $\{[\mathbb{E}K_n : \mathbb{E}F]\}_{n\in\mathbb{N}}$ is also unbounded. However, by the Riemann–Roch theorem, one obtains for each $n \in \mathbb{N}$ that $g_{\mathbb{E}K_n} = g_{K_n}$. By (3.10) and (3.15), it follows that the sequence $\{[\mathbb{E}K_n : \mathbb{E}F]\}_{n\in\mathbb{N}}$ is bounded. This is a contradiction. The result follows. $\square$

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