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Bounding differences in Jager Pairs

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ABSTRACT

Symmetrical subdivisions in the space of Jager Pairs for continued fractions-like expansions will provide us with bounds on their differences. Results will also apply to the classical regular and backwards continued fractions expansions, which are realized as special cases.

1. Introduction

Given a real number r and a rational number, written as the unique quotient $\frac{p}{q}$ of the relatively prime integers p and $q > 0$, our fundamental object of interest from diophantine approximation is the *approximation coefficient* $\theta(r, \frac{p}{q}) := q^2|r - \frac{p}{q}|$. Small approximation coefficients suggest high quality approximations, combining accuracy with simplicity. For instance, the error in approximating π using $\frac{355}{113} = 3.1415920353982$ is smaller than the error of its decimal expansion to the fifth digit $3.14159 = \frac{314\,159}{100\,000}$. Since the former rational also has a much smaller denominator, it is of far greater quality than the latter. Indeed $\theta(\pi, \frac{355}{113}) < 0.0341$ whereas $\theta(\pi, \frac{314\,159}{100\,000}) > 26\,535$.

Since adding integers to fractions does not change their denominators, we have $\theta(r, \frac{p}{q}) = \theta(r - \lfloor r \rfloor, \frac{p}{q} - \lfloor r \rfloor)$, where $\lfloor r \rfloor$ is the largest integer smaller than or equal to r (a.k.a. the floor of r), allowing us to restrict our attention to the unit interval. Expanding an irrational initial seed $x_0 \in (0, 1) - \mathbb{Q}$ as an infinite regular continued fraction

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$$x_0 = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}}$$

provides us with the unique symbolic representation via the sequence $\{b_n\}_1^\infty$ of positive integers, known as the partial quotients or *digits of expansion* of x_0. For all $n \geqslant 0$, we label the approximation coefficient associated with the *convergent*

$$\frac{p_0}{q_0} := \frac{0}{1}, \qquad \frac{p_n}{q_n} := \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\cdots + \cfrac{1}{b_n}}}}, \quad n \geqslant 1$$

by θ_n and refer to the sequence $\{\theta_n\}_1^\infty$ as the *sequence of approximation coefficients*.

Much work has been done with this sequence, from its inception in the classical era to more recent excursion which establishes interesting connections with ergodic theory and hyperbolic geometry. For a concise description of some classical results concerning this sequence, refer to the introduction section of [2]. A more thorough treatment can be found in [3, Chapter 5]. Another well-known continued fraction theory is the *backward continued fraction* expansion:

$$x_0 = 1 - \cfrac{1}{b_1 + 1 - \cfrac{1}{b_2 + 1 - \cfrac{1}{b_3 + \cdots}}}$$

leading to a new unique sequence of digits, hence new sequences of convergents and approximation coefficients.

The focus of this paper is the *space of Jager Pairs*

$$\Gamma(x_0) := \bigl\{\bigl(\theta_{n-1}(x_0), \theta_n(x_0)\bigr),\ x_0 \in (0,1) - \mathbb{Q},\ n \geqslant 1\bigr\}.$$

The spaces corresponding to the regular and backwards continued fraction expansions were initially introduced and studied in [7] and [5] respectively. We are going to reveal an elegant symmetrical internal structure for both these spaces. Our approach is to treat the regular and continued fraction expansions as limiting cases for the two families of one parameter continued fraction-like expansions, first introduced by Haas and Molnar. Using simple plane geometry, we will provide in Corollary 4.4 upper and lower bounds for the growth rate of the associated sequence of approximation coefficients. For instance, knowing a priori that $b_2 = b_3 = 1$ and $b_4 = 3$ are the digits of the classical regular continued fraction expansion will allow us to obtain the bounds $|\theta_2 - \theta_1| < \frac{\sqrt{2}}{3}$ and $\frac{2\sqrt{2}}{7} < |\theta_3 - \theta_2| < \frac{3\sqrt{2}}{5}$.

2. Preliminaries

This section is a paraphrased summary of excerpts from [4,5], given for sake of completeness. In general, the fractional part of Möbius transformation which maps $[0,1]$ onto $[0,\infty]$ leads to expansion of real numbers as continued fractions. To characterize all these transformations, we recall that Möbius transformations are uniquely determined by their values for three distinct points. Thus, we will need to introduce a parameter for the image of an additional point besides 0 and 1, which we will naturally take to be ∞. Since our maps fix the real line, the image of ∞, denoted by $-k$, can take any value within the set of negative real numbers. We let $m \in \{0,1\}$ equal zero for orientation reversing transformations, i.e. $0 \mapsto \infty$, $1 \mapsto 0$ and let m equal one for orientation preserving transformations, i.e. $0 \mapsto 0$, $1 \mapsto \infty$. Conclude that all such transformations must be the extension of the maps $x \mapsto \frac{k(1-m-x)}{x-m}$, $k > 0$, mapping $(0,1)$ homeomorphically to $(0,\infty)$, to the extended complex plane. The maps $T_{(m,k)} : [0,1) \to [0,1)$, $0 \mapsto 0$,

$$T_{(m,k)}(x) = \frac{k(1-m-x)}{x-m} - \left\lfloor \frac{k(1-m-x)}{x-m} \right\rfloor, \quad x > 0$$

are called *Gauss-like* and *Rényi-like* for $m = 0$ and $m = 1$ respectively.

We expand the initial seed $x_0 \in (0,1)$ as an (m,k)-continued fraction using the following iteration process:

1. Set $n := 1$.
2. If $x_{n-1} = 0$, stop and exit.
3. Set the *digit* and *future* of x_0 at time n to be $a_n := \lfloor \frac{k(1-m-x_{n-1})}{x_{n-1}-m} \rfloor \in \mathbb{Z}_{\geqslant 0}$ and $x_n := \frac{k(1-m-x_{n-1})}{x_{n-1}-m} - a_n \in [0,1)$. Increase n by one and go to step 2.

We thus have

$$x_n = T_{(m,k)}(x_{n-1}) = \frac{k(1-m-x_{n-1})}{x_{n-1}-m} - a_n$$

so that

$$x_{n-1} = m + \frac{k(1-2m)}{a_n + k + x_n}.$$

Therefore, this iteration scheme leads to the expansion of the initial seed x_0 as

$$x_0 = m + \frac{k(1-2m)}{a_1 + k + x_1} = m + \frac{k(1-2m)}{a_1 + k + m + \frac{k(1-2m)}{a_2+k+x_2}} = \cdots.$$

Remark 2.1. The classical $k = 1$ cases lead to the regular and backwards continued fractions expansions for $m = 0$ and $m = 1$ respectively. However, due to the definition of the map $T_{(m,k)}$ the digits of expansion will be smaller by one than their classical representation. For instance, plugging $k = 1$ into

$$\cfrac{k}{0 + k + \cfrac{k}{1 + k + \cfrac{k}{2+k}}} = \frac{k^2+4k+2}{k^2+5k+4}$$

yields the fraction

$$\frac{7}{10} = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3}}}$$

and plugging $k=1$ into

$$=1-\cfrac{k}{0+k+1-\cfrac{k}{1+k+1-\cfrac{k}{2+k}}}=\frac{k+4}{k^3+3k^2+5k+4}$$

yields the fraction

$$\frac{5}{13}=1-\cfrac{1}{1+1-\cfrac{1}{2+1-\cfrac{1}{3}}}.$$

For the classical regular and backwards continued fraction expansions this iteration process eventually terminates precisely when the initial seed x_0 is a rational number. Analogously, we denote the countable set of all numbers in the interval with finite (m,k)-expansion by $\mathbb{Q}_{(m,k)}$. For all $a\in\mathbb{Z}_{\geqslant 0}$, the cylinder set

$$\Delta_a:=\left(\frac{(1-m)k+ma}{a+k+1-m},\frac{(1-m)k+m(a+1)}{a+k+m}\right) \tag{1}$$

is defined such that $x_0\in\Delta_a$ if and only if $a_1=a$. More generally, we have

$$x_n\in\Delta_a \quad\Leftrightarrow\quad a_{n+1}=a,\quad n\geqslant 0, \tag{2}$$

that is, the restriction of the map $T_{(m,k)}$ to Δ_a is a homeomorphism onto $(0,1)$. When x_0 is an (m,k)-irrational, define the *past* of x_0 at time $n\geqslant 1$ to be $Y_1:=m-k-[a_1]_{(m,k)}$ and

$$Y_n:=m-k-a_n-[a_{n-1},\dots,a_1]_{(m,k)}\in(m-k-a_n-1,m-k-a_n),\quad n\geqslant 2. \tag{3}$$

Then for all initial seeds $x_0\in(0,1)-\mathbb{Q}_{(m,k)}$ and $n\geqslant 1$, we have $(x_n,Y_n)\in\Omega_{(m,k)}:=(0,1)\times(-\infty,m-k)$. We call the set $\Omega_{(m,k)}$ the *space of dynamic pairs*.

The sequence of approximation coefficients $\{\theta_n(x_0)\}_1^\infty$ for the (m,k)-expansion is defined just like the classical object, i.e. $\theta_n(x_0):=q_n^2|x_0-\frac{p_n}{q_n}|$, where $\frac{p_n}{q_n}=[a_1,\dots,a_n]_{(m,k)}\in\mathbb{Q}_{(m,k)}$ are the appropriate convergents. The sequence of approximation coefficients relates to the future and past sequences of x_0 using the identity

$$\theta_n(x_0)=\frac{1}{x_{n+1}-Y_{n+1}},$$

first proven for the classical regular continued fraction case in 1921 by Perron [8]. Each of the continuous maps

$$\Psi_{(m,k)}:\Omega_{(m,k)}\to\mathbb{R}^2,\quad (x,y)\mapsto\left(\frac{1}{x-y},\frac{(m-x)(m-y)}{(2m-1)k(x-y)}\right),$$
$$(\theta_n,\theta_{n+1})=\Psi_{(m,k)}(x_{n+1},Y_{n+1}), \tag{4}$$

is a homeomorphism onto its image

$$\Gamma_{(m,k)} := \Psi_{(m,k)}(\Omega_{(m,k)}).$$

Furthermore, the two families $\{\Psi_{(0,k)}\}$ and $\{\Psi_{(1,k)}\}$ are known to be equicontinuous. The corresponding *space of Jager Pairs*

$$\big\{\big(\theta_n(x_0), \theta_{n+1}(x_0)\big)\colon x_0 \in (0,1) - \mathbb{Q}_{(m,k)},\ n \geqslant 1\big\}$$

is a dense subset of $\Gamma_{(m,k)}$. In our approach, we will require the parameter k to be greater than one, treating the classical regular and backwards continued fraction expansions as the limit of the (m,k)-expansions. As $k \to 1^+$. The choice of taking the limit from the right is not arbitrary, for the treatment of the $0 < k < 1$ cases exhibits certain pathologies, see [1, Section 3.3].

3. The finer structure for the space of Jager Pairs

Fix $m \in \{0,1\}$, $k \in (1,\infty)$, and an initial seed $x_0 \in (0,1) - \mathbb{Q}_{(m,k)}$. In order to ease the notation, we will omit the subscripts $\square_{(m,k)}$ from now on. For all $a \in \mathbb{Z}_{\geqslant 0}$, define the regions

$$P_{(m,k,a)} = P_a := (0,1) \times (m-a-k-1, m-a-k)$$

and

$$F_{(m,k,a)} = F_a := \Delta_a \times (-\infty, m-k),$$

where Δ_a is the cylinder set (1). Using the identity (2) and the definition (3) of Y_n, we see that for all $n \geqslant 1$ we have

$$(x_{n+1}, y_{n+1}) \in P_a \cap F_b \quad \Leftrightarrow \quad a_{n+1} = a \quad \text{and} \quad a_{n+2} = b.$$

We label the images of each of these subsets of Ω under Ψ by $P_a^\#$ and $F_a^\#$. Using formula (4) and the fact that Ψ is a homeomorphism, we see that for all $n \geqslant 1$, we have

$$(\theta_n, \theta_{n+1}) \in P_a^\# \cap F_b^\# \quad \Leftrightarrow \quad a_{n+1} = a \quad \text{and} \quad a_{n+2} = b. \tag{5}$$

Next, let $p_{(m,k,a)} = p_a$ be the open horizontal line segment $(0,1) \times \{m-a-k\}$, let $f_{(m,k,a)} = f_a$ be the open vertical ray $\{\frac{(1-m)k+m}{a+k}\} \times (-\infty, m-k)$ and let $p_a^\#$ and $f_a^\#$ be their image under Ψ. Since both the collections $\{P_a \cup p_a\}_{a \in \mathbb{Z}_{\geqslant 0}}$ and $\{F_a \cup f_a\}_{a \in \mathbb{Z}_{\geqslant 0}}$ partition Ω, the image of their intersections under Ψ, $\{(P_a^\# \cup p_a^\#) \cap (F_b^\# \cup f_b^\#)\}_{a,b \in \mathbb{Z}_{\geqslant 0}}$ will partition Γ. We will call each member of this refined partition a *subdivision*.

Proposition 3.1. *For all $k \in \mathbb{R} > 1$ and $a, b \in \mathbb{Z}^+$ the region $P_a^\# \cap F_b^\#$ is the open interior of the quadrangle in $\mathbb{E}^2$ with vertices*

$$\left(\frac{b+k}{(1-2m)k + (a+k)(b+k)}, \frac{a+k}{(1-2m)k + (a+k)(b+k)}\right),$$

$$\left(\frac{b+k}{(1-2m)k+(a+k+1)(b+k)}, \frac{a+k+1}{(1-2m)k+(a+k+1)(b+k)}\right),$$
$$\left(\frac{b+k+1}{(1-2m)k+(a+k)(b+k+1)}, \frac{a+k}{(1-2m)k+(a+k)(b+k+1)}\right)$$

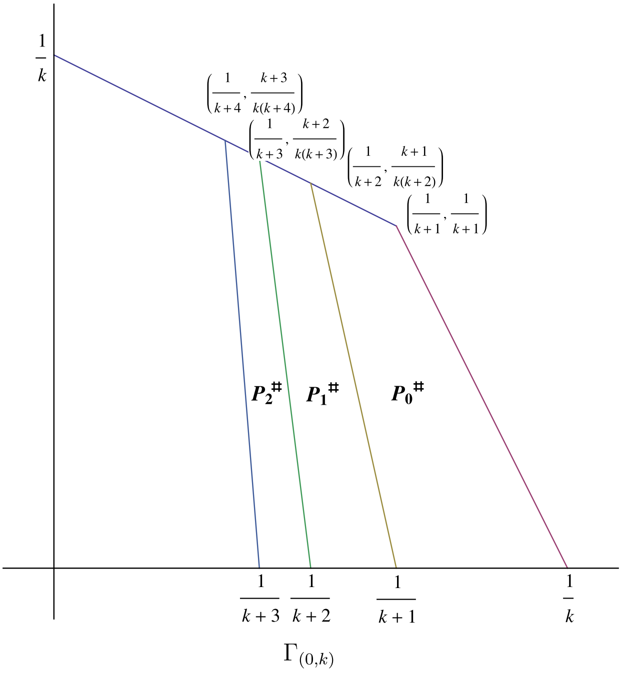
and

$$\left(\frac{b+k+1}{(1-2m)k+(a+k+1)(b+k+1)}, \frac{a+k+1}{(1-2m)k+(a+k+1)(b+k+1)}\right).$$

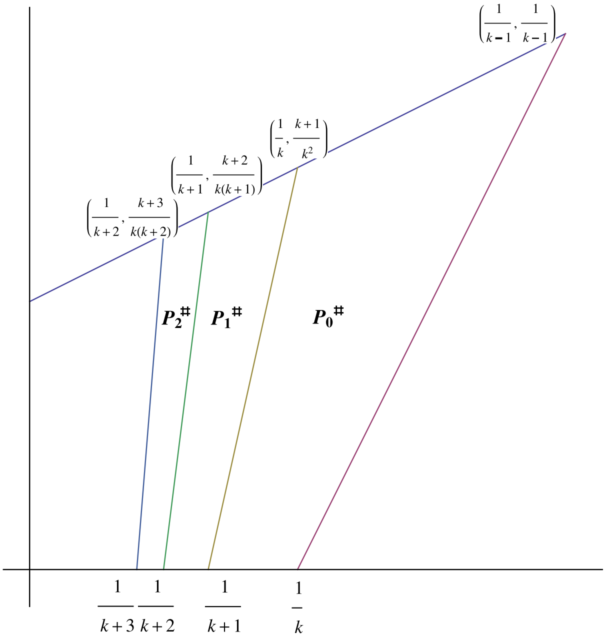
Proof. Fix $a \in \mathbb{Z}^+$ and for every $(x, y_0) \in p_a$ i.e. $x \in (0,1)$, $y_0 := m-k-a$, set $(u, v) := \Psi(x, y_0)$. Using the definition (4) of Ψ, we write $u = \frac{1}{x-y_0} = \frac{1}{x+a+k-m}$. Then, as x tends from 0 to 1, u tends from $\frac{1}{a+k-m}$ to $\frac{1}{a+k+1-m}$. Since $m - x = m - y - \frac{1}{u}$, we express v in terms of u as

$$\begin{aligned} v &= \frac{(m-x)(m-y_0)}{(2m-1)k(x-y_0)} = \frac{(m-x)(m-y_0)u}{(2m-1)k} \\ &= \frac{a+k}{(2m-1)k}\big((a+k)u-1\big) \end{aligned}$$

so that as x tends from 0 to 1, v tends from $\frac{m(a+k)}{k(a+k-m)}$ to $\frac{(1-m)(k+a)}{k(k+a+1-m)}$. Since $\Psi : \Omega \to \Gamma$ is a homeomorphism, we see that $p_a^\#$ is an open segment of the line $(a+k)^2u + (1-2m)kv = a+k$ between the points $(\frac{1}{a+k-m}, \frac{m(a+k)}{k(a+k-m)})$ and $(\frac{1}{a+k+1-m}, \frac{(1-m)(k+a)}{k(a+k+1-m)})$.



$\Gamma_{(0,k)}$



$\Gamma_{(1,k)}$

Next, fix $b \in \mathbb{Z}^+$ and set $x_0 := \frac{(1-m)k+mb}{b+k}$. We use the definition (4) of Ψ to first write $m - y = m - x + \frac{1}{u}$ and then express v in terms of u as

$$\begin{aligned} v &= \frac{u}{(2m-1)k}(m - x_0)\left(m - x_0 + \frac{1}{u}\right) \\ &= \frac{1}{b+k}\left(\frac{(2m-1)k}{b+k}u + 1\right). \end{aligned} \tag{6}$$

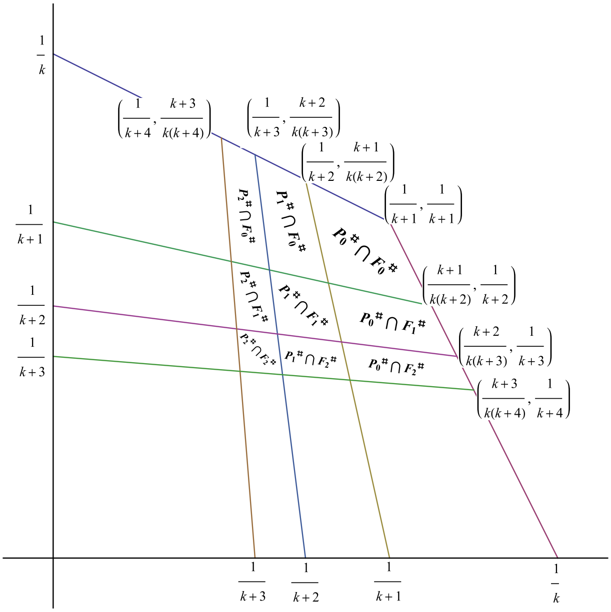
Since Ψ is a homeomorphism, this allows us to conclude that Ψ maps f_b to an open segment of a line in the uv plane, which is the reflection of the line $p_b^\#$ along the diagonal $u - v = 0$. In particular, $P_a^\# \cap F_b^\#$ is the region interior to the quadrangle with vertices $p_a^\# \cap f_b^\#$, $p_{a+1}^\# \cap f_b^\#$, $p_a^\# \cap f_{b+1}^\#$ and $p_{a+1}^\# \cap f_{b+1}^\#$. But $(x_0, y_0) = p_a \cap f_b$, so that $(u_0, v_0) := \Psi(x_0, y_0) = p_a^\# \cap f_b^\#$. The definition (4) of Ψ and formula (6) now yield

$$u_0 = \frac{1}{x_0 - y_0} = \frac{b+k}{(1-2m)k + (a+k)(b+k)}$$

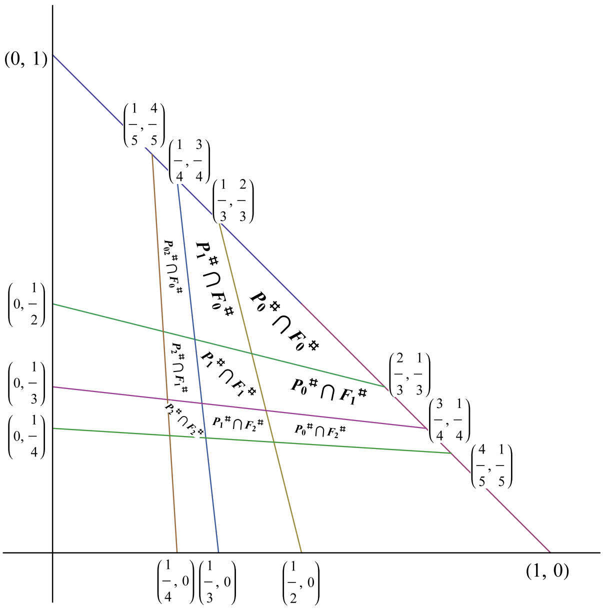
and

$$v_0 = \frac{a+k}{(1-2m)k + (a+k)(b+k)}$$

as desired. $\square$



$\Gamma_{(0,k)}$ revisited

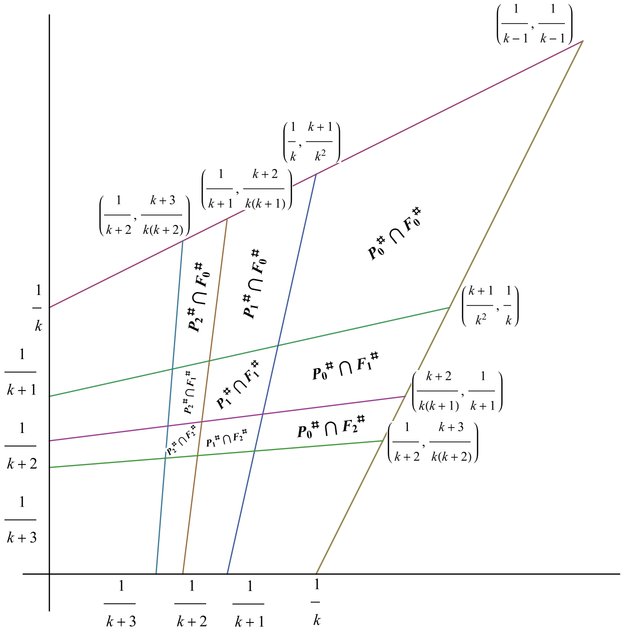


$\Gamma_{(0,1)}$

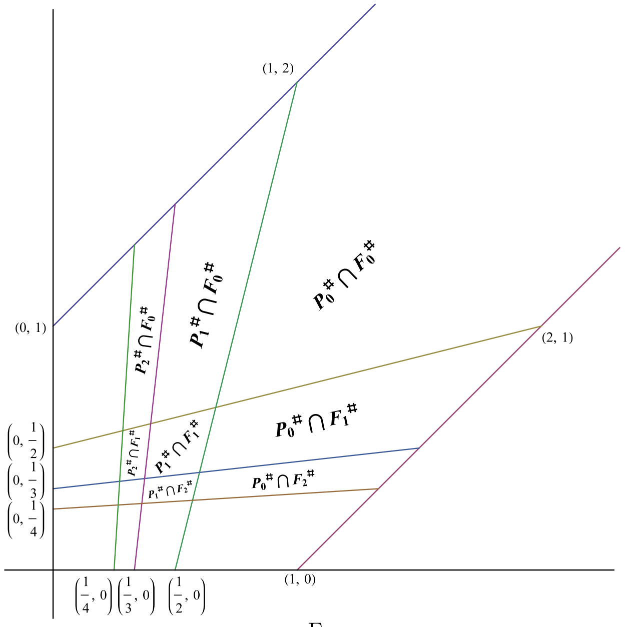
Remark 3.2. This result was first proved for the classical regular continued fractions expansion, corresponding to the parameters $m = 0$ and $k = 1$ by Jager and Kraaikamp [6] (see also [3, Exercise 5.3.4]). For this case, the region $P_0^\# \cap F_0^\#$ degenerates to the interior of the triangle with vertices $(\frac{1}{3}, \frac{2}{3})$, $(\frac{2}{3}, \frac{1}{3})$ and $(\frac{2}{5}, \frac{2}{5})$. In these papers, the regions $P_a^\#$ and $F_a^\#$ are denoted by V_b^* and H_b^* respectively, where $b = a + 1$. We diverge from their notational choice to better illustrate the dynamical structure at hand:

the point $(\theta_{n-1}, \theta_n) \in P_a^{\#}$ precisely when the first digit a_n in Y_n, the past of x_0 at time n, is a. Similarly, the point $(\theta_{n-1}, \theta_n) \in F_a^{\#}$ precisely when the first digit a_{n+1} in x_n, the future representation of x_0 at time n, is a.

When $m = 1$, we obtain the depicted spaces of Jager Pairs:



$\Gamma_{(1,k)}$ revisited



$\Gamma_{(1,1)}$

The picture for the classical backwards continued fractions $k=1$, is obtained after we let $k \to 1^+$ and use the equicontinuity of the family $\{\Psi_{(1,k)}\}_{k>1}$. For this case, the region $P_0^\# \cap F_0^\#$ expands to the unbounded region in the uv plane, which is the intersection of the regions $u-v>1$, $v-u<1$, $4u-v>2$ and $4v-u>2$.

4. Bounding growth rate in Jager Pairs

This symmetry in the subdivisions will allow us to provide uniform bounds for the rate of growth of the sequence $\{\theta_n\}_1^\infty$, which is the sequence whose members are the Jager Pair differences $\{|\theta_{n+1}-\theta_n|\}_1^\infty$. But first, we already know that $\Gamma_{(1,1)}$ is the unbounded region in the first quadrant of the uv plane bounded by the lines $u-v<1$ and $v-u<1$, hence we obtain at once:

Theorem 4.1. *In general, the sequence of approximation coefficients associated with the classical backwards continued fraction expansion $m=k=1$ has no uniform upper bound. However its growth rate is uniformly bounded by* 1.

Our main result is:

Theorem 4.2. *Given $m \in \{0,1\}$, $k \in \mathbb{R} > 1$, $x_0 \in (0,1) - \mathbb{Q}_{(m,k)}$ and $N \geqslant 1$, we write $l := \min\{a_{N+1}, a_{N+2}, a_{N+3}\}$ and $L := \max\{a_{N+1}, a_{N+2}, a_{N+3}\}$, where a_n is the digit at time n in the (m,k)-expansion for x_0. Then the inequality*

$$(\theta_{N+1}-\theta_N)^2 + (\theta_{N+2}-\theta_{N+1})^2 < 2\left(\frac{L-l+1}{(1-2m)k+(l+k)(L+k+1)}\right)^2$$

is sharp. Furthermore, if $L-l>1$, then the inequality

$$(\theta_{N+1}-\theta_N)^2 + (\theta_{N+2}-\theta_{N+1})^2 > 2\left(\frac{L-l}{(1-2m)k+(l+k+1)(L+k)}\right)^2$$

is sharp.

In order to prove this theorem, we will first prove:

Lemma 4.3. *Given $m \in \{0,1\}$, $k \in \mathbb{R} > 1$, $x_0 \in (0,1) - \mathbb{Q}_{(m,k)}$ and $N \geqslant 1$, write $a := \min\{a_{N+1}, a_{N+2}\}$, $A := \max\{a_{N+1}, a_{N+2}\}$, $b := \min\{a_{N+2}, a_{N+3}\}$ and $B := \max\{a_{N+2}, a_{N+3}\}$. Then the inequality*

$$\begin{aligned}
&(\theta_{N+1}-\theta_N)^2 + (\theta_{N+2}-\theta_{N+1})^2 \\
&\quad < \left(\frac{B+k+1}{(1-2m)k+(a+k)(B+k+1)} - \frac{b+k}{(1-2m)k+(b+k)(A+k+1)}\right)^2 \\
&\qquad + \left(\frac{A+k+1}{(1-2m)k+(b+k)(A+k+1)} - \frac{a+k}{(1-2m)k+(a+k)(B+k+1)}\right)^2
\end{aligned}$$

is sharp. Furthermore, if $\max\{A-a, B-b\} > 1$, *then the inequality*

$$\begin{aligned}
&(\theta_{N+1}-\theta_N)^2 + (\theta_{N+2}-\theta_{N+1})^2 \\
&\quad > \left(\frac{B+k}{(1-2m)k+(a+k+1)(B+k)} - \frac{b+k+1}{(1-2m)k+(b+k+1)(A+k)}\right)^2
\end{aligned}$$

$$+\left(\frac{A+k}{(1-2m)k+(b+k+1)(A+k)}-\frac{a+k+1}{(1-2m)k+(a+k+1)(B+k)}\right)^2$$

is sharp.

Proof. Let $R_{(m,k,a,b,A,B)}=R$ be the set

$$\bigcup_{a\leqslant i\leqslant A,\ b\leqslant j\leqslant B}\left(P_i^{\#}\cap F_j^{\#}\right).$$

Formula (5) yields $(\theta_N,\theta_{N+1})\in P_{a_{N+1}}^{\#}\cap F_{a_{N+2}}^{\#}\subset R$ and $(\theta_{N+1},\theta_{N+2})\in P_{a_{N+2}}^{\#}\cap F_{a_{N+3}}^{\#}\subset R$. From Proposition 3.1, we see that R is the interior of the bounded convex quadrangle, consisting of sections from the line segments $p_a^{\#}$, $p_{A+1}^{\#}$, $f_b^{\#}$ and $f_{B+1}^{\#}$. Unless $\max\{A-a,B-b\}\leqslant 1$, we also let $r_{(m,k,a,b,A,B)}=r$ be the non-empty set

$$\bigcup_{a+1\leqslant i\leqslant A-1,\ b+1\leqslant j\leqslant B-1}\left(P_i^{\#}\cap F_j^{\#}\right),$$

so that

$$\left\{(\theta_N,\theta_{N+1}),(\theta_{N+1},\theta_{N+2})\right\}\subset R-r. \tag{7}$$

From the same proposition, we see that $r\subset R$ is the interior of the convex quadrangle, consisting of sections from the segments $p_{a+1}^{\#}$, $p_A^{\#}$, $f_{b+1}^{\#}$ and $f_B^{\#}$, that is, the quadrangle r is obtained from peeling the outer layer of R's subdivisions.

When $m=0$, let $(u_0,v_0):=p_A^{\#}\cap f_{b+1}^{\#}$ and $(U_0,V_0):=p_{a+1}^{\#}\cap f_B^{\#}$. Then

$$(u,v)\in P_A^{\#}\cap F_b^{\#}\quad\Rightarrow\quad u<u_0\quad\text{and}\quad v>v_0$$

and

$$(u,v)\in P_a^{\#}\cap F_B^{\#}\quad\Rightarrow\quad u>U_0\quad\text{and}\quad v<V_0.$$

When $m=1$, let $(u_1,v_1):=p_A^{\#}\cap f_B^{\#}$ and $(U_1,V_1):=p_{a+1}^{\#}\cap f_{b+1}^{\#}$. Then

$$(u,v)\in P_a^{\#}\cap F_b^{\#}\quad\Rightarrow\quad u<u_0\quad\text{and}\quad v<v_0$$

and

$$(u,v)\in P_A^{\#}\cap F_B^{\#}\quad\Rightarrow\quad u>U_0\quad\text{and}\quad v>V_0.$$

In tandem with formula (7), this allows us to conclude that, in either case, we have

$$\begin{aligned}\operatorname{diam}(r)^2&<d\big((\theta_N,\theta_{N+1}),(\theta_{N+1},\theta_{N+2})\big)\\&=(\theta_N-\theta_{N+1})^2+(\theta_{N+1}-\theta_{N+2})^2<\operatorname{diam}(R)^2,\end{aligned} \tag{8}$$

where d is the familiar distance formula between two points in $\mathbb{E}^2$ and $\operatorname{diam}(R)$ stands for the Euclidean diameter of the region R.

The diameter of the interior of a convex quadrangle in $\mathbb{E}^2$ with opposite acute angles is the length of the diagonal connecting the vertices corresponding to these acute angles. To verify this simple observation from plane geometry, observe that from the continuity of the Euclidean distance formula, the diameter of a convex polygon is either a side or a diagonal. The diagonal connecting the two acute angles will lie opposite to an obtuse angle in either one of the induced triangles obtained. Since for any triangle, the longest side lies opposite to the largest angle, we conclude that this diagonal is longer than all of the four sides in this convex quadrangle. To verify it is longer than the other diagonal, consider one of the four induced triangles obtained after drawing both diagonals and use the same opposite angle argument.

When $m=0$ and $c\in\mathbb{Z}^+$, the slope of the line containing $p_c^\#$ lies in $(-\infty,-1)$ whereas the slope of the line containing $f_c^\#$ lies in $(-1,0)$. Conclude that both R and r have opposite acute angles, so that their diameters are the length of the diagonal connecting the vertices $p_a^\#\cap f_{B+1}^\#$ with $p_{A+1}^\#\cap f_b^\#$ and $p_{a+1}^\#\cap f_B^\#$ with $p_A^\#\cap f_{b+1}^\#$ respectively. Hence

$$\operatorname{diam}(\mathrm{R})=\mathrm{d}\big(p_a^\#\cap f_{B+1}^\#,p_{A+1}^\#\cap f_b^\#\big)$$

and

$$\operatorname{diam}(\mathrm{r})=\mathrm{d}\big(p_{a+1}^\#\cap f_B^\#,p_A^\#\cap f_{b+1}^\#\big).$$

When $m=1$, the slope of the line containing $p_c^\#$ lies in $(1,\infty)$ whereas the slope of the line containing $f_c^\#$ lies in $(0,1)$. Conclude that both R and r have opposite acute angles, so that their diameters are the length of the diagonal connecting the vertices $p_a^\#\cap f_b^\#$ with $p_{A+1}^\#\cap f_{B+1}^\#$ and $p_{a+1}^\#\cap f_{b+1}^\#$ with $p_A^\#\cap f_B^\#$ respectively. Hence

$$\operatorname{diam}(\mathrm{R})=\mathrm{d}\big(p_a^\#\cap f_b^\#,p_{A+1}^\#\cap f_{B+1}^\#\big)$$

and

$$\operatorname{diam}(\mathrm{r})=\mathrm{d}\big(p_{a+1}^\#\cap f_{b+1}^\#,p_A^\#\cap f_B^\#\big).$$

In either case, we have

$$\begin{aligned}\operatorname{diam}(\mathrm{R})^2={}&\left(\frac{B+k+1}{(1-2m)k+(A+k+1)(B+k+1)}-\frac{b+k}{(1-2m)k+(a+k)(b+k)}\right)^2\\&+\left(\frac{A+k+1}{(1-2m)k+(A+k+1)(B+k+1)}-\frac{a+k}{(1-2m)k+(a+k)(b+k)}\right)^2\end{aligned}$$

and

$$\begin{aligned}\operatorname{diam}(\mathrm{r})^2={}&\left(\frac{B+k}{(1-2m)k+(A+k)(B+k)}-\frac{b+k+1}{(1-2m)k+(a+k+1)(b+k+1)}\right)^2\\&+\left(\frac{A+k}{(1-2m)k+(A+k)(B+k)}-\frac{a+k+1}{(1-2m)k+(a+k+1)(b+k+1)}\right)^2.\end{aligned}$$

Formula (8) now yields the desired inequalities and the density of the space of Jager Pairs in $\Gamma_{(m,k)}$ establishes their sharpness. $\square$

Proof of Theorem 4.2. For all integers a, b, A and B with $l \leqslant a \leqslant A \leqslant L$ and $l \leqslant b \leqslant B \leqslant L$, we have

$$\frac{A+k+1}{(1-2m)k+(b+k)(A+k+1)} \leqslant \frac{L+k+1}{(1-2m)k+(l+k)(L+k+1)}$$

and

$$\frac{l+k}{(1-2m)k+(l+k)(L+k+1)} \leqslant \frac{a+k}{(1-2m)k+(a+k)(B+k+1)},$$

so that

$$\begin{aligned} 0 &\leqslant \frac{L-l}{(1-2m)k+(l+k+1)(L+k)} \\ &\leqslant \frac{A+k+1}{(1-2m)k+(b+k)(A+k+1)} - \frac{a+k}{(1-2m)k+(a+k)(B+k+1)} \\ &\leqslant \frac{L+k+1}{(1-2m)k+(l+k)(L+k+1)} - \frac{l+k}{(1-2m)k+(l+k)(L+k+1)} \\ &= \frac{L-l+1}{(1-2m)k+(l+k)(L+k+1)}. \end{aligned}$$

This argument remains identical after we exchange a for b and A for B. Furthermore, setting $a = b := l$ and $A = B := L$ shows that we cannot replace these weak inequalities with strict ones. The lemma will now provide us with the result. □

This theorem will enable us to provide bounds for the rate of growth for the sequence of approximation coefficients between time n and $n+1$, assuming we know the bounds for the digits of expansion at times $n+1$, $n+2$ and $n+3$, as expressed in the following corollary:

Corollary 4.4. *Assuming the hypothesis of the theorem, we have the inequality*

$$\max\{|\theta_{N+1}-\theta_N|, |\theta_{N+2}-\theta_{N+1}|\} < \sqrt{2}\left(\frac{L-l+1}{(1-2m)k+(l+k)(L+k+1)}\right).$$

Furthermore, if $L-l>1$, we also have the inequality

$$\min\{|\theta_{N+1}-\theta_N|, |\theta_{N+2}-\theta_{N+1}|\} > \sqrt{2}\left(\frac{L-l}{(1-2m)k+(l+k+1)(L+k)}\right).$$

These results extend to the classical $k=1$ cases with one exception. The upper bound statement does not apply for the $m=1$ case when $a_{n+2}=0$ and either $a_{n+1}=0$ or $a_{n+3}=0$.

Proof. When $k>1$, the proof is an immediate consequence of the theorem. Plugging in $m=0$ and letting $k \to 1^+$ establishes the result for the classical Gauss case $m=0$, $k=1$. When $m=1$, we let $k \to 1^+$ and use the equicontinuity of the family $\{\Psi_{(1,k)}\}_{k>1}$ to obtain the result, once we exclude the possibility of the unbounded region $P_0^\# \cap F_0^\#$ from belonging to the region R, when computing the upper bound portion. This happens precisely when either $a_{n+1}=a_{n+2}=0$ or $a_{n+2}=a_{n+3}=0$. □

Remark 4.5. The bounds given in the last section of the introduction are readily verified once we remember to translate the digits $b_2=b_3=1$ and $b_4=3$ in the classical regular continued fraction expansion to $a_2=a_3=0$ and $a_4=2$ in the $(0,1)$-expansion.

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