Journal of Number Theory 137 (2014) 166–178

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Hodge theory and the Mordell–Weil rank of elliptic curves over extensions of function fields

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ARTICLE INFO

Article history:
Received 5 July 2013
Received in revised form 9 November 2013
Accepted 10 November 2013
Available online 3 January 2014
Communicated by David Goss

MSC:
14J27
11G40

Keywords:
Elliptic surfaces
Mordell–Weil ranks
Hodge theory

ABSTRACT

We use Hodge theory to prove a new upper bound on the ranks of Mordell–Weil groups for elliptic curves over function fields after regular geometrically Galois extensions of the base field, improving on previous results of Silverman and Ellenberg, when the base field has characteristic zero and the supports of the conductor of the elliptic curve and of the ramification divisor of the extension are disjoint.

1. Introduction

For every field F let $\overline{F}$ denote its separable closure. For every field extension $K|F$ and variety Z defined over F let Z_K denote the base change of Z to K. Let $\mathcal{C}$ be a smooth projective geometrically irreducible curve defined over a field k and let F, g denote its function field and its genus, respectively. Let $\pi : \mathcal{C}' \to \mathcal{C}$ be a finite regular geometrically Galois cover defined over k. Let F' be the function field of $\mathcal{C}'$ and let S be

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0022-314X/$ – see front matter
http://dx.doi.org/10.1016/j.jnt.2013.11.009

the reduced ramification divisor of the cover π. Let $G = \mathrm{Aut}(\mathcal{C}'_{\bar{k}}|\mathcal{C}_{\bar{k}})$ and let Σ be the image of $\mathrm{Gal}(\bar{k}|k)$ in $\mathrm{Aut}(G)$ with respect to the natural action of $\mathrm{Gal}(\bar{k}|k)$ on G.

Next we recall the definition of Ellenberg's constant. Let V be the real vector space spanned by the irreducible complex-valued characters of $G \rtimes \Sigma$, and let W be the real vector space spanned by the irreducible complex-valued characters of G. We say that a vector v in V (resp. in W) is non-negative if its inner product with each irreducible representation of $G \rtimes \Sigma$ (resp. of G) is non-negative. Let $c \in V$ be the coset character of $G \rtimes \Sigma$ attached to Σ, and let $r \in W$ be the regular character of G. Ellenberg defines the constant $\epsilon(G, \Sigma)$ as the maximum of the inner product $\langle v, c\rangle$ over all $v \in V$ such that

(i) v is non-negative;
(ii) $r - R(v)$ is non-negative, where $R : V \to W$ is the restriction map.

The region of V defined by these two conditions above is a compact polytope, so $\epsilon(G, \Sigma)$ is well-defined.

Let E be a non-isotrivial elliptic curve over F. By the Lang–Néron theorem (see [1]) the group $E(F')$ is finitely generated. Let c_E denote the degree of the conductor of E and let d_E denote the degree of the minimal discriminant of E. Our main result is the following

Theorem 1.1. *Assume that k has characteristic zero, and the supports of S and of the conductor of E are disjoint. Then*

$$\mathrm{rank}\big(E\big(F'\big)\big) \leqslant \epsilon(G, \Sigma)\big(c_E - d_E/6 + 2g - 2 + \deg(S)\big). \tag{1.1.1}$$

Let $O(G, \Sigma)$ be the cardinality of the set of orbits of G with respect to the action of Σ. It is easy to prove that $O(G, \Sigma) = \epsilon(G, \Sigma)$ when G is an abelian group (see Proposition 2.11 of [3]). Hence we have the following immediate

Corollary 1.2. *Assume that k has characteristic zero, the supports of S and of the conductor of E are disjoint, and G is abelian. Then*

$$\mathrm{rank}\big(E\big(F'\big)\big) \leqslant O(G, \Sigma)\big(c_E - d_E/6 + 2g - 2 + \deg(S)\big). \tag{1.2.1}$$

The upper bound:

$$\mathrm{rank}\big(E\big(F'\big)\big) \leqslant O(G, \Sigma)(c_E + 4g - 4), \tag{1.2.2}$$

was first proved by Silverman in [8] under the assumption that G is abelian, the cover π is unramified (i.e. S is empty), k is a number field, and a weak form of Tate's conjecture holds for $\mathcal{E}'$ where $g' : \mathcal{E}' \to \mathcal{C}'$ is the unique relatively minimal elliptic surface over

k whose generic fibre is $E_{F'}$. Later in [3] Ellenberg proved the following more general unconditional bound:

$$\operatorname{rank}\bigl(E\bigl(F'\bigr)\bigr) \leqslant \epsilon(G,\Sigma)\bigl(c_E + 4g - 4 + 2\deg(S)\bigr), \tag{1.2.3}$$

without any restriction on the group, assuming that the characteristic of k is at least 5, and without assuming that supports of S and of the conductor of E are disjoint. Note that (1.1.1) trivially implies (1.2.3) since $d_E \geqslant 12$ for every non-isotrivial elliptic curve E. In fact $c_E \leqslant d_E$, so we also have the following immediate consequence of the main result:

$$\operatorname{rank}\bigl(E\bigl(F'\bigr)\bigr) \leqslant \epsilon(G,\Sigma)\bigl(5c_E/6 + 2g - 2 + \deg(S)\bigr),$$

and hence our result is stronger than Ellenberg's bound. Moreover the bound (1.1.1) is false for fields of positive characteristic, even in the degenerate case when G is trivial, see for example [9].

The strategy for proving Theorem 1.1 follows Ellenberg's original idea; it is enough to show that the $\mathbb{C}[G]$-module $E(K) \otimes \mathbb{C}$ is the quotient of the free $\mathbb{C}[G]$-module of rank $c_E - d_E/6 + 2g - 2 + \deg(S)$, where K is the composition of the fields F' and $\bar{k}$. In the course of our proof of this fact we may assume without loss of generality that k is algebraically closed. In fact by the Lefschetz principle we may also assume that k is the field of complex numbers, which we will do from now on. However our method is different because we use de Rham cohomology instead of étale cohomology and the easy direction of the Lefschetz-$(1,1)$ theorem. (A similar idea is used in [4] by Fastenberg.) We prove the required bounds on the multiplicity of irreducible representations of G appearing in certain cohomology groups of $\mathcal{E}'$ via a simple equivariant Riemann–Roch theorem (see Theorem 2.5) and an equivariant Grothendieck–Ogg–Shafarevich formula for the Euler characteristic of constructible sheaves of complex vector spaces (Theorem 3.6) and its applications to elliptic surfaces (Theorems 3.7 and 4.4) which we think are interesting results on their own.

1.3. Contents

In the next section we compute the $\mathbb{C}[G]$-module structure of the cohomology group $H^2(\mathcal{E}', \mathcal{O}_{\mathcal{E}'})$ using essentially the same arguments as in my previous paper [6]. We use the Grothendieck–Ogg–Shafarevich formula to compute the $\mathbb{C}[G]$-module structure of the cohomology group $H^1(\mathcal{C}', R^1 g'_* \mathbb{C})$ in the third section. In the last section we combine these results and the easy direction of the Lefschetz-$(1,1)$ theorem to conclude the proof of our main result.

2. Equivariant Riemann–Roch for elliptic surfaces

Definition 2.1. In this section G will denote a finite group. Let X be a normal scheme which is of finite type over $\operatorname{Spec}(\mathbb{C})$. Assume that the finite group G acts on X on the left. Let $\mathcal{F}$ be a coherent sheaf on X. A G-linearisation on $\mathcal{F}$ is a collection $\Psi = \{\psi_g\}_{g\in G}$ of isomorphisms $\psi_g : g_*(\mathcal{F}) \to \mathcal{F}$ for every $g \in G$ such that

(i) we have $\psi_1 = \mathrm{Id}_{\mathcal{F}}$,
(ii) for every g, $h \in G$ we have $\psi_{hg} = \psi_h \circ h_*(\psi_g)$,

where $h_*(\psi_g) : (hg)_*(\mathcal{F}) = h_*(g_*(\mathcal{F})) \to h_*(\mathcal{F})$ is the direct image of the map $\psi_g : g_*(\mathcal{F}) \to \mathcal{F}$ under the action of h. We define a G-sheaf over X to be a sheaf on X equipped with a G-linearisation. A coherent G-sheaf is a coherent sheaf on X equipped with a G-linearisation Ψ such that $\psi_g : g_*(\mathcal{F}) \to \mathcal{F}$ is $\mathcal{O}_X$-linear for every $g \in G$.

Definition 2.2. Let $K(\mathbb{C}[G])$ denote the Grothendieck group of all finitely generated $\mathbb{C}[G]$-modules. For every finitely $\mathbb{C}[G]$-module M let $[M]$ denote its class in $K(\mathbb{C}[G])$. Let $f : X \to Y$ be a G-cover and let $\mathcal{F}$ be a coherent G-sheaf on X. Also assume that Y is proper over $\mathrm{Spec}(\mathbb{C})$; then for every $n \in \mathbb{N}$ the cohomology group $H^n(X, \mathcal{F})$ is a finitely generated $\mathbb{C}[G]$-module with respect to the natural $\mathbb{C}[G]$-action. Therefore the element:

$$\chi_G(X, \mathcal{F}) = \sum_{n\in\mathbb{N}} (-1)^n \big[H^n(X, \mathcal{F})\big] \in K\big(\mathbb{C}[G]\big)$$

is well-defined.

In addition to the assumptions above also suppose now that $f : X \to Y$ above is a map of smooth, projective curves over $\mathrm{Spec}(\mathbb{C})$. Let $\mathcal{L}$ be a line bundle on Y. The line bundle $f^*(\mathcal{L})$ on X is naturally equipped with the structure of a coherent G-sheaf.

Lemma 2.3. *With the same notation and assumptions as above the following equation holds in* $K(\mathbb{C}[G])$*:*

$$\chi_G\big(X, f^*(\mathcal{L})\big) = \chi_G(X, \mathcal{O}_X) + \deg(\mathcal{L})\big[\mathbb{C}[G]\big].$$

Proof. This is the content of Lemma 5.6 of [6] on page 524. $\square$

Notation 2.4. Suppose now that the map $f : X \to Y$ is the cover $\pi : \mathcal{C}' \to \mathcal{C}$ of the introduction and let Δ_E denote the discriminant of a relatively minimal elliptic surface $g : \mathcal{E} \to \mathcal{C}$ whose generic fibre is E. Assume that the ramification divisor of the cover π has support disjoint from the conductor of E. Let $g' : \mathcal{E}' \to \mathcal{C}'$ be the base change of the elliptic fibration $g : \mathcal{E} \to \mathcal{C}$ with respect to the map π. Note that the $\mathcal{C}'$-scheme $\mathcal{E}'$ is a relatively minimal regular model of the base change of E to the function field of $\mathcal{C}'$ because f does not ramify at the locus of the conductor of E. Moreover $\mathcal{E}'$ is equipped with a unique action of G such that g' is equivariant with respect to this action and the one on $\mathcal{C}'$.

Theorem 2.5. *We have*:

$$\big[H^2(\mathcal{E}',\mathcal{O}_{\mathcal{E}'})\big]=\frac{d_E}{12}\big[\mathbb{C}[G]\big]-\chi_G\big(\mathcal{C}',\mathcal{O}_{\mathcal{C}'}\big).$$

Proof. By definition we have:

$$\chi_G\big(\mathcal{E}',\mathcal{O}_{\mathcal{E}'}\big)=\big[H^0\big(\mathcal{E}',\mathcal{O}_{\mathcal{E}'}\big)\big]-\big[H^1\big(\mathcal{E}',\mathcal{O}_{\mathcal{E}'}\big)\big]+\big[H^2\big(\mathcal{E}',\mathcal{O}_{\mathcal{E}'}\big)\big]. \tag{2.5.1}$$

By Lemma 4 of [5] on page 79 the map $(g')^*:\mathrm{Pic}(\mathcal{C}')\to\mathrm{Pic}(\mathcal{E}')$ induced by Picard functoriality is an isomorphism. Because this map is equivariant with respect to the induced G-actions on $\mathrm{Pic}(\mathcal{C}')$ and $\mathrm{Pic}(\mathcal{E}')$, we get that $H^1(\mathcal{E}',\mathcal{O}_{\mathcal{E}'})\cong H^1(\mathcal{C}',\mathcal{O}_{\mathcal{C}'})$ as $\mathbb{C}[G]$-modules, since these modules are isomorphic to the tangent spaces at the zero of the abelian varieties $\mathrm{Pic}(\mathcal{E}')$ and $\mathrm{Pic}(\mathcal{C}')$, respectively. Obviously $H^0(\mathcal{E}',\mathcal{O}_{\mathcal{E}'})\cong H^0(\mathcal{C}',\mathcal{O}_{\mathcal{C}'})$ as $\mathbb{C}[G]$-modules, hence from (2.5.1) we get that

$$\big[H^2\big(\mathcal{E}',\mathcal{O}_{\mathcal{E}'}\big)\big]=\chi_G\big(\mathcal{E}',\mathcal{O}_{\mathcal{E}'}\big)-\chi_G\big(\mathcal{C}',\mathcal{O}_{\mathcal{C}'}\big). \tag{2.5.2}$$

Let $\Omega^1_{\mathcal{E}/\mathcal{C}}$, $\Omega^1_{\mathcal{E}'/\mathcal{C}'}$ denote the sheaf of relative Kähler differentials of the $\mathcal{C}$-scheme $\mathcal{E}$ and the $\mathcal{C}'$-scheme $\mathcal{E}'$, respectively. Let $\omega_{\mathcal{E}/\mathcal{C}}$, $\omega_{\mathcal{E}'/\mathcal{C}'}$ denote respectively the pull-back of $\Omega^1_{\mathcal{E}/\mathcal{C}}$ and $\Omega^1_{\mathcal{E}'/\mathcal{C}'}$ with respect to the zero section. These sheaves are line bundles on $\mathcal{C}$ and $\mathcal{C}'$, respectively. Moreover by Grothendieck's duality we have $R^1g'_*(\mathcal{O}_{\mathcal{E}'})=\omega^{\otimes -1}_{\mathcal{E}'/\mathcal{C}'}$. Because all boundary maps in the spectral sequence $H^p(\mathcal{C}',R^qg'_*(\mathcal{O}_{\mathcal{E}'}))\Rightarrow H^{p+q}(\mathcal{E}',\mathcal{O}_{\mathcal{E}'})$ are $\mathbb{C}[G]$-linear we get from the above that

$$\begin{aligned}\chi_G\big(\mathcal{E}',\mathcal{O}_{\mathcal{E}'}\big)&=\chi_G\big(\mathcal{C}',\mathcal{O}_{\mathcal{C}'}\big)-\chi_G\big(\mathcal{C}',R^1g'_*(\mathcal{O}_{\mathcal{E}'})\big)\\&=\chi_G\big(\mathcal{C}',\mathcal{O}_{\mathcal{C}'}\big)-\chi_G\big(\mathcal{C}',\omega^{\otimes -1}_{\mathcal{E}'/\mathcal{C}'}\big).\end{aligned} \tag{2.5.3}$$

Combining (2.5.2) and (2.5.3) we get that

$$\big[H^2\big(\mathcal{E}',\mathcal{O}_{\mathcal{E}'}\big)\big]=-\chi_G\big(\mathcal{C}',\omega^{\otimes -1}_{\mathcal{E}'/\mathcal{C}'}\big). \tag{2.5.4}$$

By definition Δ_E is the zero divisor of a non-zero section of $\omega^{\otimes 12}_{\mathcal{E}/\mathcal{C}}$. Therefore $\deg(\Delta_E)=12\deg(\omega_{\mathcal{E}/\mathcal{C}})$. Moreover $\omega_{\mathcal{E}'/\mathcal{C}'}=\pi^*(\omega_{\mathcal{E}/\mathcal{C}})$. Hence (2.5.4) and Lemma 2.3 imply that

$$\big[H^2\big(\mathcal{E}',\mathcal{O}_{\mathcal{E}'}\big)\big]=\frac{\deg(\Delta_E)}{12}\big[\mathbb{C}[G]\big]-\chi_G\big(\mathcal{C}',\mathcal{O}_{\mathcal{C}'}\big). \qquad\square$$

3. Equivariant Grothendieck–Ogg–Shafarevich for elliptic surfaces

Notation 3.1. Let X be a quasi-projective variety over $\mathbb{C}$. For every constructible sheaf $\mathcal{F}$ of complex vector spaces and for every $n\in\mathbb{N}$ let $H^n(X,\mathcal{F})$ denote the n-th cohomology group of $\mathcal{F}$ on X with respect to the analytical topology and let

$$\chi(X,\mathcal{F})=\sum_{n\in\mathbb{N}}(-1)^n\dim_{\mathbb{C}}\bigl(H^n(X,\mathcal{F})\bigr)\in\mathbb{N}$$

denote the Euler-characteristic of $\mathcal{F}$. Assume now that X is a smooth irreducible projective curve and let $\mathcal{F}$ be a constructible sheaf of complex vector spaces on X. There is an open subcurve $U\subseteq X$ such that $\mathcal{F}|_U$ is locally free of finite rank. We define $\operatorname{rank}(\mathcal{F})$, the rank of $\mathcal{F}$, as the rank of $\mathcal{F}|_U$. For every $x\in X(\mathbb{C})$ let $\mathcal{F}_x$ and $c_x(\mathcal{F})$ denote the stalk of $\mathcal{F}$ at x and the conductor of $\mathcal{F}$ at x given by the formula:

$$c_x(\mathcal{F})=\operatorname{rank}(\mathcal{F})-\dim_{\mathbb{C}}(\mathcal{F}_x),$$

respectively. Finally let

$$\operatorname{cond}(\mathcal{F})=\sum_{x\in X(\mathbb{C})}c_x(\mathcal{F}).$$

In this section our main tool will be the complex analytic version of the Grothendieck–Ogg–Shafarevich formula:

Theorem 3.2 *(Grothendieck–Ogg–Shafarevich). For every X and $\mathcal{F}$ as above we have*:

$$\chi(X,\mathcal{F})=\operatorname{rank}(\mathcal{F})\chi(X,\mathbb{C})-\operatorname{cond}(\mathcal{F}).$$

Proof. The theorem could be easily reduced to the Grothendieck–Ogg–Shafarevich formula for torsion constructible sheaves (see Théorème 1 of [7] on page 133) using a standard set of arguments. We omit the details. □

Lemma 3.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be constructible sheaves of complex vector spaces on X. Assume that $\mathcal{G}$ is locally free in a neighbourhood of $x\in X(\mathbb{C})$. Then*

$$c_x(\mathcal{F}\otimes\mathcal{G})=\operatorname{rank}(\mathcal{G})c_x(\mathcal{F}).$$

Proof. Because $\mathcal{G}$ is locally free in a neighbourhood of $x\in X(\mathbb{C})$, we have:

$$(\mathcal{F}\otimes\mathcal{G})_x=\mathcal{F}_x\otimes\mathcal{G}_x,\quad\text{and hence}\quad\dim_{\mathbb{C}}\bigl((\mathcal{F}\otimes\mathcal{G})_x\bigr)=\operatorname{rank}(\mathcal{G})\dim_{\mathbb{C}}(\mathcal{F}_x).$$

Since

$$\operatorname{rank}(\mathcal{F}\otimes\mathcal{G})=\operatorname{rank}(\mathcal{G})\operatorname{rank}(\mathcal{F}),$$

the claim follows immediately. □

Definition 3.4. Let G continue to denote a finite group. Let X be a quasi-projective variety over $\mathbb{C}$ and assume that G acts on X on the left. Let $\mathcal{F}$ be a constructible sheaf of complex vector spaces on X. A G-linearisation on $\mathcal{F}$ is a collection $\Psi=\{\psi_g\}_{g\in G}$ of $\mathbb{C}$-linear isomorphisms of sheaves $\psi_g:g_*(\mathcal{F})\to\mathcal{F}$ for every $g\in G$ such that

(i) we have $\psi_1 = \mathrm{Id}_{\mathcal{F}}$,
(ii) for every g, $h \in G$ we have $\psi_{hg} = \psi_h \circ h_*(\psi_g)$,

where $h_*(\psi_g) : (hg)_*(\mathcal{F}) = h_*(g_*(\mathcal{F})) \to h_*(\mathcal{F})$ is the direct image of the map $\psi_g : g_*(\mathcal{F}) \to \mathcal{F}$ under the action of h. We define a complex constructible G-sheaf over X to be a constructible sheaf of complex vector spaces on X equipped with a G-linearisation.

Definition 3.5. Let X be a quasi-projective variety over $\mathbb{C}$. Let $f : X \to Y$ be a G-cover and let $\mathcal{F}$ be a complex constructible G-sheaf on X. Then for every $n \in \mathbb{N}$ the cohomology group $H^n(X, \mathcal{F})$ is a finitely generated $\mathbb{C}[G]$-module with respect to the natural $\mathbb{C}[G]$-action. Therefore the element:

$$\chi_G(X, \mathcal{F}) = \sum_{n \in \mathbb{N}} (-1)^n \big[H^n(X, \mathcal{F})\big] \in K\big(\mathbb{C}[G]\big)$$

is well-defined.

In addition to the assumptions above also suppose now that $f : X \to Y$ above is a map of smooth, projective curves over $\mathrm{Spec}(\mathbb{C})$. Let $\mathcal{F}$ be a constructible sheaf of complex vector spaces on Y. Then $f^*(\mathcal{F})$ is naturally equipped with the structure of a complex constructible G-sheaf on X. Let $j : U \to Y$ be an open immersion and assume that $\mathcal{F}|_U$ is a locally free of finite rank. Assume that the map f does not ramify on the complement of U in Y.

Theorem 3.6. *With the same notation and assumptions as above the following equation holds in $K(\mathbb{C}[G])$:*

$$\chi_G\big(X, f^*(\mathcal{F})\big) = \mathrm{rank}(\mathcal{F})\chi_G(X, \mathbb{C}) - \mathrm{cond}(\mathcal{F})\big[\mathbb{C}[G]\big].$$

Proof. Let $R(G)$ denote the set of irreducible $\mathbb{C}$-valued characters of G. For every $\alpha \in R(G)$ let $\Delta_\alpha : K(\mathbb{C}[G]) \to \mathbb{Z}$ denote the unique homomorphism such that $\Delta_\alpha([M])$ is the multiplicity of α in the character of M for every finitely generated $\mathbb{C}[G]$-module M and let $\mathrm{rank}(\alpha)$ denote the rank of α. It will be enough to show that

$$\Delta_\alpha\big(\chi_G\big(X, f^*(\mathcal{F})\big)\big) = \mathrm{rank}(\mathcal{F})\Delta_\alpha\big(\chi_G(X, \mathbb{C})\big) - \mathrm{rank}(\alpha)\,\mathrm{cond}(\mathcal{F}) \quad \big(\forall \alpha \in R(G)\big).$$

Note that there is a unique decomposition:

$$f_*(\mathbb{C}) = \bigoplus_{\alpha \in R(G)} \mathcal{G}_\alpha^{\oplus\,\mathrm{rank}(\alpha)}$$

in the category of constructible sheaves of complex vector spaces such that for every $\alpha \in R(G)$ the rank of $\mathcal{G}_\alpha$ is equal to $\mathrm{rank}(\alpha)$ and the natural $\mathbb{C}[G]$-action on $H^0(f^{-1}(V), f^*(\mathcal{G}_\alpha))$ has character α where $V \subseteq Y$ is an open subcurve such that the restriction of f onto $f^{-1}(V)$ is unramified. Since for every $\alpha \in R(G)$ we have:

$$\Delta_\alpha\big(\chi_G\big(X, f^*(\mathcal{F})\big)\big) = \chi(Y, \mathcal{F} \otimes \mathcal{G}_\alpha),$$
$$\Delta_\alpha\big(\chi_G(X, \mathbb{C})\big) = \chi(Y, \mathcal{G}_\alpha),$$

it will be enough to show for every such α that

$$\chi(Y, \mathcal{F} \otimes \mathcal{G}_\alpha) = \operatorname{rank}(\mathcal{F})\chi(Y, \mathcal{G}_\alpha) - \operatorname{rank}(\mathcal{G}_\alpha)\operatorname{cond}(\mathcal{F}). \tag{3.6.1}$$

By Theorem 3.2 for every $\alpha \in R(G)$ we have:

$$\chi(Y, \mathcal{F} \otimes \mathcal{G}_\alpha) = \operatorname{rank}(\mathcal{F})\operatorname{rank}(\mathcal{G}_\alpha)\chi(Y, \mathbb{C}) - \operatorname{cond}(\mathcal{F} \otimes \mathcal{G}_\alpha), \tag{3.6.2}$$
$$\chi(Y, \mathcal{G}_\alpha) = \operatorname{rank}(\mathcal{G}_\alpha)\chi(Y, \mathbb{C}) - \operatorname{cond}(\mathcal{G}_\alpha). \tag{3.6.3}$$

Since for every $x \in Y(\mathbb{C})$ either $\mathcal{F}$ or G_α is locally constant on a neighbourhood of x, using Lemma 3.3 we get:

$$\operatorname{cond}(\mathcal{F} \otimes \mathcal{G}_\alpha) = \operatorname{rank}(\mathcal{G}_\alpha)\operatorname{cond}(\mathcal{F}) + \operatorname{rank}(\mathcal{F})\operatorname{cond}(\mathcal{G}_\alpha) \quad \big(\forall \alpha \in R(G)\big). \tag{3.6.4}$$

Now Eq. (3.6.1) follows from combining (3.6.4) with Eqs. (3.6.2) and (3.6.3). □

Suppose again that the map $f : X \to Y$ is the cover $\pi : \mathcal{C}' \to \mathcal{C}$ of the introduction. Let $g : \mathcal{E} \to \mathcal{C}$ and $g' : \mathcal{E}' \to \mathcal{C}'$ be as above. Then $R^1 g'_*(\mathbb{C})$ is a complex constructible G-sheaf on $\mathcal{C}'$. Assume that the ramification divisor of the cover π has support disjoint from the conductor of E and let c_E denote the degree of the conductor of E as in the introduction.

Theorem 3.7. *We have*:

$$\big[H^1\big(\mathcal{C}', R^1 g'_*(\mathbb{C})\big)\big] = c_E\big[\mathbb{C}[G]\big] - 2\chi_G\big(\mathcal{C}', \mathbb{C}\big).$$

Proof. By definition we have:

$$\chi_G\big(\mathcal{C}', R^1 g'_*(\mathbb{C})\big) = \big[H^0\big(\mathcal{C}', R^1 g'_*(\mathbb{C})\big)\big] - \big[H^1\big(\mathcal{C}', R^1 g'_*(\mathbb{C})\big)\big] + \big[H^2\big(\mathcal{C}', R^1 g'_*(\mathbb{C})\big)\big].$$

As noted in the paragraph above Lemma 1.4 of [2] on page 5 we have:

$$H^0\big(\mathcal{C}', R^1 g'_*(\mathbb{C})\big) = 0 = H^2\big(\mathcal{C}', R^1 g'_*(\mathbb{C})\big),$$

and hence

$$\chi_G\big(\mathcal{C}', R^1 g'_*(\mathbb{C})\big) = -\big[H^1\big(\mathcal{C}', R^1 g'_*(\mathbb{C})\big)\big].$$

By the proper base change theorem the complex constructible G-sheaves $R^1g'_*(\mathbb{C})$ and $\pi^*(R^1g_*(\mathbb{C}))$ are isomorphic. Therefore we may use Theorem 3.6 to get

$$\chi_G\big(\mathcal{C}', R^1g'_*(\mathbb{C})\big) = \operatorname{rank}\big(R^1g_*(\mathbb{C})\big)\chi_G\big(\mathcal{C}', \mathbb{C}\big) - \operatorname{cond}\big(R^1g_*(\mathbb{C})\big)\big[\mathbb{C}[G]\big].$$

Since $\operatorname{rank}(R^1g_*(\mathbb{C})) = 2$ by the proper base change theorem and $\operatorname{cond}(R^1g_*(\mathbb{C})) = c_E$, the claim is now clear. $\square$

4. Rank bounds and Hodge theory

Notation 4.1. For every smooth projective variety X over $\mathbb{C}$ let $\mathrm{NS}(X)$ be the Néron–Severi group of X and let

$$c_1 : \mathrm{NS}(X) \longrightarrow H^1\big(X, \Omega^1_{X/\mathbb{C}}\big)$$

be the Chern class map of de Rham cohomology. Let $s : \mathcal{C}' \to \mathcal{E}'$ be the zero section of the elliptic fibration $g' : \mathcal{E}' \to \mathcal{C}'$ and let $T(\mathcal{E}') \leqslant \mathrm{NS}(\mathcal{E}')$ be the subgroup generated by the algebraic equivalence classes of $s(\mathcal{C}')$ and the irreducible components of the fibres of g'. Finally let $T_{\mathrm{dR}}(\mathcal{E}')$ be the $\mathbb{C}$-linear span of the image of $T(\mathcal{E}')$ with respect to c_1.

Lemma 4.2. *There is a* $\mathbb{C}[G]$*-linear injection*:

$$E\big(F'\big) \otimes \mathbb{C} \longrightarrow H^1\big(\mathcal{E}', \Omega^1_{\mathcal{E}'/\mathbb{C}}\big)/T_{\mathrm{dR}}\big(\mathcal{E}'\big).$$

Proof. By the Shioda–Tate formula:

$$\mathrm{NS}\big(\mathcal{E}'\big) \otimes \mathbb{C} \cong E\big(F'\big) \otimes \mathbb{C} \oplus T\big(\mathcal{E}'\big) \otimes \mathbb{C}.$$

The claim now follows from the fact that the map

$$\mathrm{NS}\big(\mathcal{E}'\big) \otimes \mathbb{C} \longrightarrow H^1\big(\mathcal{E}', \Omega^1_{\mathcal{E}'/\mathbb{C}}\big)$$

induced by c_1 is a $\mathbb{C}[G]$-linear injection. $\square$

Proposition 4.3. *We have*:

$$\big[T_{\mathrm{dR}}\big(\mathcal{E}'\big)\big] = \big[H^0\big(\mathcal{C}', R^2g'_*(\mathbb{C})\big)\big] + \big[H^2\big(\mathcal{C}', g'_*(\mathbb{C})\big)\big].$$

Proof. For every finite set T let $\mathbb{C}[T]$ denote the $\mathbb{C}$-module of formal $\mathbb{C}$-linear combination of elements of T. When T is equipped with a left G-action $\mathbb{C}[T]$ has a natural $\mathbb{C}[G]$-module structure. Let $R \subset \mathcal{C}'(\mathbb{C})$ be the set of all points x such that the fibre of g' over x is singular. For every $x \in \mathcal{C}'(\mathbb{C})$ let C_x denote the set of irreducible components of the fibre of g' over x. For every x as above and for every irreducible component $i \in C_x$ let m_i denote the multiplicity of i. For every complex vector space V and subset $T \subseteq V$ let

$\langle T\rangle \subseteq V$ denote $\mathbb{C}$-linear span of T. Equip $\coprod_{x\in R} C_x$ with the G-action induced by the G-action on $\mathcal{E}'$. Note that the subset:

$$M = \left\{ \sum_{i\in C_x} m_i i \;\middle|\; x\in R \right\} \subset \mathbb{C}\left[\coprod_{x\in R} C_x \right]$$

is G-invariant, therefore its $\mathbb{C}$-linear span is a $\mathbb{C}[G]$-submodule. We have the following isomorphism of $\mathbb{C}[G]$-modules:

$$T_{\mathrm{dR}}(\mathcal{E}') \cong T(\mathcal{E}') \otimes \mathbb{C} \cong \mathbb{C}^{\oplus 2} \oplus \mathbb{C}\left[\coprod_{x\in R} C_x \right] / \langle M\rangle,$$

where we equip $\mathbb{C}$ with the trivial $\mathbb{C}[G]$-module structure. Because the fibres of g' are connected, for every connected open subset $V \subseteq \mathcal{C}'$ we have $H^0((g')^{-1}(V), \mathbb{C}) = \mathbb{C}$, and hence the sheaf $g'_*(\mathbb{C})$ is constant of rank 1. Consequently $H^2(\mathcal{C}', g'_*(\mathbb{C}))$ is isomorphic to the trivial $\mathbb{C}[G]$-module of dimension one.

Because the map $g : \mathcal{E} \to \mathcal{C}$ is projective, there is a closed immersion $l : \mathcal{E} \to \mathbb{P}^n_{\mathcal{C}}$ of $\mathcal{C}$-schemes for some $n \in \mathbb{N}$ where $p : \mathbb{P}^n_{\mathcal{C}} \to \mathcal{C}$ is the projective n-space over $\mathcal{C}$. The base change $l' : \mathcal{E}' \to \mathbb{P}^n_{\mathcal{C}'}$ of l with respect to π is a G-equivariant closed immersion of $\mathcal{C}'$-schemes where we equip the $\mathcal{C}'$-scheme $\mathbb{P}^n_{\mathcal{C}'}$ with the left G-action induced by the natural isomorphism between $\mathbb{P}^n_{\mathcal{C}'}$ and the base change of $\mathbb{P}^n_{\mathcal{C}}$ to $\mathcal{C}'$. With respect to this action the structure map $p' : \mathbb{P}^n_{\mathcal{C}'} \to \mathcal{C}'$ is G-equivariant. The map l' furnishes a G-equivariant $\mathbb{C}$-linear homomorphism $(l')^* : R^2p'_*(\mathbb{C}) \to R^2g'_*(\mathbb{C})$ of complex constructible G-sheaves. By the Künneth formula $R^2p'_*(\mathbb{C}) = \mathbb{C}$. For every $x \in \mathcal{C}'(\mathbb{C})$ let $(l')^*|_x : R^2p'_*(\mathbb{C})|_x \to R^2g'_*(\mathbb{C})|_x$ denote the fibre of $(l')^*$ at x. By the proper base change theorem there is a $\mathbb{C}$-linear commutative diagram:

$$\begin{tikzcd}
R^2p'_*(\mathbb{C})|_x \arrow[r, "(l')^*|_x"] \arrow[d] & R^2g'_*(\mathbb{C})|_x \arrow[d] \\
\mathbb{C} \arrow[r, "d_x"] & \mathbb{C}[C_x]
\end{tikzcd}$$

such that both vertical arrows are isomorphisms and d_x has image $\langle \sum_{i\in C_x} m_i i\rangle$. For every $x \in \mathcal{C}'(\mathbb{C})$ let $i_x : x \to \mathcal{C}'$ denote the closed immersion of the point x into $\mathcal{C}'$. Note that for every complex vector space V the direct image $(i_x)_*(V)$ is a skyscraper sheaf on $\mathcal{C}'$. By the above there is a $\mathbb{C}$-linear and G-equivariant short exact sequence:

$$0 \longrightarrow \mathbb{C} \longrightarrow R^2g'_*(\mathbb{C}) \longrightarrow \mathcal{F} \longrightarrow 0 \tag{4.3.1}$$

of complex constructible G-sheaves where

$$\mathcal{F} = \bigoplus_{x\in R} (i_x)_*\left(\mathbb{C}[C_x] \middle/ \left\langle \sum_{i\in C_x} m_i i \right\rangle \right),$$

equipped with the tautological G-action. Note that every $x \in \mathcal{C}'(\mathbb{C})$ has a contractible open neighbourhood $V \subset \mathcal{C}'$ such that the first cohomology of the constant sheaf $\mathbb{C}$ on V vanishes. Therefore the restriction of the short exact sequence (4.3.1) onto V splits. Because the sheaf $\mathcal{F}$ has finite support the short exact sequence (4.3.1) splits on $\mathcal{C}'$, too. Therefore the sequence

$$0 \longrightarrow H^0\big(\mathcal{C}', \mathbb{C}\big) \longrightarrow H^0\big(\mathcal{C}', R^2 g'_*(\mathbb{C})\big) \longrightarrow H^0\big(\mathcal{C}', \mathcal{F}\big) \longrightarrow 0$$

is also exact. We get that

$$\big[H^0\big(\mathcal{C}', R^2 g'_*(\mathbb{C})\big)\big] = [\mathbb{C}] + \left[\mathbb{C}\left[\coprod_{x \in R} C_x\right] / \langle M \rangle\right]$$

and the claim follows. □

Theorem 4.4. *We have*:

$$\big[H^1\big(\mathcal{E}', \Omega^1_{\mathcal{E}'/\mathbb{C}}\big)/T_{\mathrm{dR}}\big(\mathcal{E}'\big)\big] = (c_E - d_E/6)\big[\mathbb{C}[G]\big] - \chi_G\big(\mathcal{C}', \mathbb{C}\big).$$

Proof. The G-equivariant degenerating spectral sequence:

$$H^q\big(\mathcal{E}', \Omega^p_{\mathcal{E}'/\mathbb{C}}\big) \Rightarrow H^{p+q}\big(\mathcal{E}', \mathbb{C}\big)$$

furnishes the equation:

$$\big[H^2\big(\mathcal{E}', \mathbb{C}\big)\big] = \big[H^0\big(\mathcal{E}', \Omega^2_{\mathcal{E}'/\mathbb{C}}\big)\big] + \big[H^1\big(\mathcal{E}', \Omega^1_{\mathcal{E}'/\mathbb{C}}\big)\big] + \big[H^2\big(\mathcal{E}', \mathcal{O}_{\mathcal{E}'}\big)\big]. \tag{4.4.1}$$

By Lemma 1.4 of [2] on page 5 the G-equivariant spectral sequence:

$$H^p\big(\mathcal{C}', R^q g'_*(\mathbb{C})\big) \Rightarrow H^{p+q}\big(\mathcal{E}', \mathbb{C}\big)$$

degenerates, so we get that

$$\begin{aligned}\big[H^2\big(\mathcal{E}', \mathbb{C}\big)\big] = {} & \big[H^0\big(\mathcal{C}', R^2 g'_*(\mathbb{C})\big)\big] + \big[H^1\big(\mathcal{C}', R^1 g'_*(\mathbb{C})\big)\big] \\ & + \big[H^2\big(\mathcal{C}', R^0 g'_*(\mathbb{C})\big)\big].\end{aligned} \tag{4.4.2}$$

Combining Eqs. (4.4.1) and (4.4.2) with Proposition 4.3 we get that

$$\begin{aligned}\big[H^1\big(\mathcal{E}', \Omega^1_{\mathcal{E}'/\mathbb{C}}\big)/T_{\mathrm{dR}}\big(\mathcal{E}'\big)\big] = {} & \big[H^1\big(\mathcal{C}', R^1 g'_*(\mathbb{C})\big)\big] - \big[H^0\big(\mathcal{E}', \Omega^2_{\mathcal{E}'/\mathbb{C}}\big)\big] \\ & - \big[H^2\big(\mathcal{E}', \mathcal{O}_{\mathcal{E}'}\big)\big].\end{aligned} \tag{4.4.3}$$

Let $(\cdot)^\vee : K(\mathbb{C}[G]) \to K(\mathbb{C}[G])$ denote the unique group homomorphism such that $[M]^\vee$ is the isomorphism class of the dual $\mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ for every finitely generated $k[G]$-module M. Serre's duality furnishes a perfect pairing

$$H^0(\mathcal{E}', \Omega^2_{\mathcal{E}'/\mathbb{C}}) \times H^2(\mathcal{E}', \mathcal{O}_{\mathcal{E}'}) \to H^2(\mathcal{E}', \Omega^2_{\mathcal{E}'/\mathbb{C}}) = \mathbb{C} \tag{4.4.4}$$

which is $\mathbb{C}[G]$-linear, therefore

$$\left[H^0(\mathcal{E}', \Omega^2_{\mathcal{E}'/\mathbb{C}})\right] = \left[H^2(\mathcal{E}', \mathcal{O}_{\mathcal{E}'})\right]^\vee = \frac{d_E}{12}\left[\mathbb{C}[G]\right] - \chi_G(\mathcal{C}', \mathcal{O}_{\mathcal{C}'})^\vee \tag{4.4.5}$$

by Theorem 2.5. Because the boundary maps in the spectral sequence:

$$H^q(\mathcal{C}', \Omega^p_{\mathcal{C}'/\mathbb{C}}) \Rightarrow H^{p+q}(\mathcal{C}', \mathbb{C})$$

are $\mathbb{C}[G]$-linear we get that

$$\chi_G(\mathcal{C}', \mathbb{C}) = \chi_G(\mathcal{C}', \mathcal{O}_{\mathcal{C}'}) - \chi_G(\mathcal{C}', \Omega^1_{\mathcal{C}'/\mathbb{C}}). \tag{4.4.6}$$

Serre's duality furnishes a perfect pairing

$$H^0(\mathcal{C}', \Omega^1_{\mathcal{C}'/\mathbb{C}}) \times H^1(\mathcal{C}', \mathcal{O}_{\mathcal{C}'}) \to H^1(\mathcal{C}', \Omega^1_{\mathcal{C}'/\mathbb{C}}) = \mathbb{C}$$

which is $\mathbb{C}[G]$-linear, therefore

$$\chi_G(\mathcal{C}', \Omega^1_{\mathcal{C}'/\mathbb{C}}) = -\chi_G(\mathcal{C}', \mathcal{O}_{\mathcal{C}'})^\vee. \tag{4.4.7}$$

Combining Theorem 2.5 and (4.4.5) with Eqs. (4.4.6) and (4.4.7) we get that

$$\left[H^0(\mathcal{E}', \Omega^2_{\mathcal{E}'/\mathbb{C}})\right] + \left[H^2(\mathcal{E}', \mathcal{O}_{\mathcal{E}'})\right] = \frac{d_E}{6}\left[\mathbb{C}[G]\right] - \chi_G(\mathcal{C}', \mathbb{C}). \tag{4.4.8}$$

The claim now follows from Eqs. (4.4.3) and (4.4.8) and from Theorem 3.7. □

Proof of Theorem 1.1. We are going to use the notation of the proof of Theorem 3.6. By Lemma 4.2 it will be enough to show that

$$\Delta_\alpha\left(\left[H^1(\mathcal{E}', \Omega^1_{\mathcal{E}'/\mathbb{C}})/T_{\mathrm{dR}}(\mathcal{E}')\right]\right) \leqslant \mathrm{rank}(\alpha)\left(c_E - d_E/6 + 2g - 2 + \deg(S)\right)$$

for every $\alpha \in R(G)$. In order to do so, it will be enough to prove that

$$-\Delta_\alpha\left(\chi_G(\mathcal{C}', \mathbb{C})\right) \leqslant \mathrm{rank}(\alpha)\left(2g - 2 + \deg(S)\right)$$

for every $\alpha \in R(G)$ by Theorem 4.4. We have

$$-\Delta_\alpha\left(\chi_G(\mathcal{C}', \mathbb{C})\right) = -\chi(\mathcal{C}, \mathcal{G}_\alpha) = \mathrm{rank}(\mathcal{G}_\alpha)(2g - 2) + \mathrm{cond}(\mathcal{G}_\alpha)$$

for every $\alpha \in R(G)$ by Theorem 3.2. Also note that $c_x(\mathcal{G}_\alpha) \leqslant \mathrm{rank}(\mathcal{G}_\alpha)$ for every $x \in S$ because $\dim_{\mathbb{C}}((\mathcal{G}_\alpha)_x)$ is non-negative. Since $c_x(\mathcal{G}_\alpha) = 0$ for every $x \in X(\mathbb{C}) - S$, the claim is now clear. □

Acknowledgments

The author was partially supported by the EPSRC grants P19164 and P36794.

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