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Salem numbers from a class of star-like trees ☆

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ABSTRACT

We study the Coxeter polynomials associated with certain star-like trees. In particular, we exhibit large Salem factors of these polynomials and give convergence properties of their dominant roots.

1. Introduction

Several different methods to construct minimal polynomials of Salem numbers have been investigated in the literature (see e.g. [1,2,6,11]). Various authors associate Salem numbers with Coxeter polynomials and use this relation in order to construct Salem

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numbers (cf. for instance [3–5,7,9]). In this paper we follow the very explicit approach of Gross et al. [5] and provide precise information on the decomposition of Coxeter polynomials of certain star-like trees into irreducible factors, thereby giving estimates on the degree of the occurring Salem factor.

To be more precise, let $r, a_0, \dots, a_r \in \mathbb{N}$ such that $a_0 \geq 2, \dots, a_r \geq 2$. We consider the star-like tree $T = T(a_0, \dots, a_r)$ with $r+1$ arms of $a_0 - 1, \dots, a_r - 1$ edges, respectively. According to [9, Lemma 5] the Coxeter polynomial of $T(a_0, \dots, a_r)$ is given by

$$R_{T(a_0,\dots,a_r)}(z) = \prod_{i=0}^{r}\Big(\frac{z^{a_i}-1}{z-1}\Big)\Big(z+1-z\sum_{i=0}^{r}\frac{z^{a_i-1}-1}{z^{a_i}-1}\Big).$$

Note that R_T can be written as

$$R_T(z) = C(z)S(z), \tag{1.1}$$

where C is a product of cyclotomic polynomials and S is the minimal polynomial of a Salem number or of a quadratic Pisot number. Indeed, by the results of [10], the zeros of R_T are either real and positive or have modulus 1. The decomposition (1.1) now follows from [9, Corollaries 7 and 9, together with the remark after the latter], as these results imply that R_T has exactly one irrational real positive root of modulus greater than 1.

For Coxeter polynomials corresponding to star-like trees with three arms we are able to say much more about the factors of the decomposition (1.1). In particular, we shall prove the following result.

Theorem 1. *Let $a_0, a_1, a_2 \in \mathbb{Z}$ such that $a_2 > a_1 > a_0 > 1$ and $(a_0, a_1, a_2) \neq (2, 3, t)$ for all $t \in \{4, 5, 6\}$. Further, let $T := T(a_0, a_1, a_2)$ be the star-like tree with three arms of $a_0 - 1$, $a_1 - 1$, $a_2 - 1$ edges, and let λ be its largest eigenvalue. Then $\tau > 1$ defined by*

$$\sqrt{\tau} + 1/\sqrt{\tau} = \lambda$$

is a Salem or a quadratic Pisot root of the Coxeter polynomial R_T of T. If S is the minimal polynomial of τ then we can write

$$R_T(x) = S(x)C(x), \tag{1.2}$$

where C is a product of cyclotomic polynomials of orders bounded by $420(a_2 - a_1 + a_0 - 1)$ whose roots have multiplicity bounded by an effectively computable constant $m(a_0, a_2 - a_1)$. Thus

$$\deg S \geq \deg R_T - m(a_0, a_2 - a_1) \sum_{k \leq 420(a_2 - a_1 + a_0 - 1)} \varphi(k), \tag{1.3}$$

where φ denotes Euler's φ-function.

Remark 1.1 *(Periodicity properties of cyclotomic factors).* Gross et al. [5] study certain Coxeter polynomials and prove periodicity properties of their cyclotomic factors. Contrary to their case, our Coxeter polynomials R_T do not have the same strong separability properties (cf. Lemma 2.2). For this reason, we could not exhibit analogous results for $C(x)$, however, we obtain weaker periodicity properties in the following way.

In the setting of Theorem 1 assume that a_0 as well as $a_2 - a_1$ are constant. For convenience set $S_{a_1} = R_{T(a_0,a_1,a_2)}$ and let ζ_k be a root of unity of order k. It follows from (2.2) below (see (2.1) for the definition of P) that $S_{a_1}(\zeta_k) = 0$ if and only if $S_{a_1+k}(\zeta_k) = 0$, i.e., the fact that the k-th cyclotomic polynomial divides S_{a_1} depends only on the residue class of $a_1 \pmod k$. Therefore, setting $K := \mathrm{lcm}\{1, 2, \ldots, 420(a_2 - a_1 + a_0 - 1)\}$, the set of all cyclotomic polynomials dividing S_{a_1} is determined by the residue class of $a_1 \pmod K$.

If we determine the set $\{k : k \leq 420(a_2 - a_1 + a_0 - 1),\ S_{a_1}(\zeta_k) = 0\}$ for all $a_1 \leq K$ we thus know exactly which cyclotomic factor divides which of the polynomials S_{a_1} for $a_1 \in \mathbb{N}$. Obviously, this knowledge would allow to improve the bound (1.3).

Remark 1.2 *(Degrees of the Salem numbers).* Theorem 1 enables us to exhibit Salem numbers of arbitrarily large degree. Indeed, if a_0 and the difference $a_2 - a_1$ are kept small and $a_1 \to \infty$ then (1.3) assures that $\deg S \to \infty$. We also mention here that Gross and McMullen [6, Theorem 1.6] showed that for any odd integer $n \geq 3$ there exist infinitely many *unramified* Salem numbers of degree $2n$; recall that a Salem polynomial f is said to be unramified if it satisfies $|f(-1)| = |f(1)| = 1$. The construction pursued in this work substantially differs from ours: it is proved that every unramified Salem polynomial arises from an automorphism of an indefinite lattice.

If two of the arms of the star-like tree under consideration get longer and longer, the associated Salem numbers converge to the m-bonacci number φ_m, where m is the (fixed) length of the third arm. This is made precise in the following theorem.

Theorem 2. *Let $a_1 > a_0 \geq 2$ and $\eta \geq 1$ be given and set $a_2 = a_1 + \eta$. Then, for $a_1 \to \infty$, the Salem root $\tau(a_0, a_1, a_2)$ of the Coxeter polynomial associated with $T(a_0, a_1, a_2)$ converges to φ_{a_0}, where the degree of $\tau(a_0, a_1, a_2)$ is bounded from below by (1.3).*

Besides that, we are able to give the following result which is valid for more general star-like trees.

Theorem 3. *Let $r \geq 1$, $a_r > \cdots > a_1 > a_0 \geq 2$, and choose $k \in \{1, \ldots, r-1\}$. Then, for fixed $a_0, \ldots, a_k$ and $a_{k+1}, \ldots, a_r \to \infty$, the Salem root $\tau(a_0, \ldots, a_r)$ of the Coxeter polynomial associated with $T(a_0, \ldots, a_r)$ converges to the dominant Pisot root of*

$$Q(z) = (z + 1 - r + k) \prod_{i=0}^{k} (z^{a_i} - 1) - z \sum_{i=0}^{k} (z^{a_i - 1} - 1) \prod_{\substack{j=0 \\ j \neq i}}^{k} (z^{a_j} - 1). \tag{1.4}$$

2. Salem numbers generated by Coxeter polynomials of star-like trees

For convenience, we introduce the polynomial

$$\begin{aligned} P(z) &:= (z-1)^{r+1} R_{T(a_0,\dots,a_r)}(z) \\ &= \Big(\prod_{i=0}^{r}(z^{a_i}-1)\Big)(z+1) - z\sum_{i=0}^{r}\Big((z^{a_i-1}-1)\prod_{j=0,\, j\neq i}^{r}(z^{a_j}-1)\Big). \end{aligned} \tag{2.1}$$

Of course, like R_T, the polynomial P can be decomposed as a product of a Salem (or quadratic Pisot) factor times a factor containing only cyclotomic polynomials.

Now, we concentrate on star-like trees with three arms, i.e., we assume that $r = 2$.

Lemma 2.1. *Let $a_2 > a_1 > a_0$. Then for $T(a_0, a_1, a_2)$ the polynomial $P(z)$ reads*

$$P(z) = z^{a_1+a_2}Q(z) + z^{a_1+1}R(z) + S(z) \tag{2.2}$$

with

$$\begin{aligned} Q(z) &= z^{a_0+1} - 2z^{a_0} + 1, \\ R(z) &= z^{a_2-a_1+a_0-1} - z^{a_2-a_1} + z^{a_0-1} - 1, \\ S(z) &= -z^{a_0+1} + 2z - 1. \end{aligned}$$

Moreover,

$$\max\{\deg(Q), \deg(R), \deg(S)\} = a_2 - a_1 + a_0 - 1, \quad \deg(P) = a_0 + a_1 + a_2 + 1,$$

and the (naive) height of P equals 2.

Proof. This can easily be verified by direct computation. □

Lemma 2.2. *Let $a_2 > a_1 > a_0$ and let P as in (2.1) be associated with $T(a_0, a_1, a_2)$. Then there exists an effectively computable constant $m = m(a_0, a_2 - a_1)$ which bounds the multiplicity of every root z of P with $|z| = 1$.*

Proof. Observe, that 1 is a root of Q, R, and S. Thus, for technical reasons, we work with $\tilde{P}(z) = P(z)/(z-1)$ and, defining $\tilde{Q}(z)$, $\tilde{R}(z)$, and $\tilde{S}(z)$ analogously, we write

$$\tilde{P}(z) = z^{a_1+a_2}\tilde{Q}(z) + z^{a_1+1}\tilde{R}(z) + \tilde{S}(z).$$

Our first goal is to bound the n-th derivatives $|\tilde{P}^{(n)}(z)|$ with $|z| = 1$ away from zero. To this end we define the quantities

$$\begin{aligned}
\eta(a_0) &:= \min\{|\tilde{Q}(z)| : |z| = 1\} > 0, \\
E_n = E_n(a_0) &:= \max\{|\tilde{Q}^{(k)}(z)| : 1 \le k \le n,\ |z| = 1\}, \\
F_0 = F_0(a_0, a_2 - a_1) &:= \max\{|\tilde{R}(z)| : |z| = 1\}, \\
F_n = F_n(a_0, a_2 - a_1, n) &:= \max\{|\tilde{R}^{(k)}(z)| : 1 \le k \le n,\ |z| = 1\}, \\
G_n = G(a_0, n) &:= \max\{|\tilde{S}^{(n)}(z)| : |z| = 1\}.
\end{aligned}$$

For $n \ge 1$ one easily computes that (note that $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the Pochhammer symbol)

$$\begin{aligned}
\tilde{P}^{(n)}(z) = {} & (a_1 + a_2)_{(n)} \tilde{Q}(z) z^{a_1 + a_2 - n} + (a_1 + 1)_{(n)} \tilde{R}(z) z^{a_1 + 1 - n} \\
& + \sum_{k=0}^{n-1} \binom{n}{k} (a_1 + a_2)_{(k)} \tilde{Q}^{(n-k)}(z) z^{a_1 + a_2 - k} \\
& + \sum_{k=0}^{n-1} \binom{n}{k} (a_1 + 1)_{(k)} \tilde{R}^{(n-k)}(z) z^{a_1 + 1 - k} + \tilde{S}^{(n)}(z).
\end{aligned}$$

Now for $|z| = 1$ we estimate

$$\begin{aligned}
|\tilde{P}^{(n)}(z)| \ge {} & (a_1 + a_2)_{(n)} \eta(a_0) - 2^{-n+1} (a_1 + a_2)_{(n)} F_0 \\
& - 2^{n-1} (a_1 + a_2)_{(n-1)} E_n - 2^{n-1} (a_1 + 1)_{(n-1)} F_n - G_n \\
\ge {} & (a_1 + a_2)_{(n)} \left(\eta(a_0) - 2^{-n+1} F_0 - \frac{2^{n-1}(E_n + F_n)}{a_1 + a_2 - n + 1} - \frac{G_n}{(a_1 + a_2)_{(n)}} \right). \qquad (2.3)
\end{aligned}$$

Now we fix a_0 and the difference $a_2 - a_1$. Then we choose $n_0 = n_0(a_0, a_2 - a_1)$ such that

$$\eta(a_0) - 2^{-n_0+1} F_0 > 0.$$

In view of (2.3) there exists a constant $c = c(a_0, a_2 - a_1)$ such that for all a_1, a_2 with $a_1 + a_2 > c$ (with our fixed difference) we have $|\tilde{P}^{(n_0)}(z)| > 0$ for all z with $|z| = 1$. If, on the other hand, $a_1 + a_2 \le c$, then we have $\deg \tilde{P} \le c + a_0$. Therefore, in any case, the multiplicity of a root of $\tilde{P}$ on the unit circle is bounded by $\max(n_0, c + a_0)$ and the result follows by taking $m = \max(n_0, c + a_0) + 1$. $\square$

The following lemma is a simple special case of Mann's Theorem.

Lemma 2.3. *Let $a, b, c, p, q \in \mathbb{Z}$ such that $(p, q) \ne (0, 0)$ and a, b, c nonzero. If ζ is a root of unity such that*

$$a\zeta^p + b\zeta^q + c = 0$$

then the order of ζ divides $6 \gcd(p, q)$.

Proof. This is a special case of [8, Theorem 1]. □

For subsequent use we recall some notation and facts (used in a similar context in [5]). A *divisor* on the complex plane is a finite sum

$$D = \sum_{j \in J} a_j \cdot z_j$$

where $a_j \in \mathbb{Z} \setminus \{0\}$ and

$$\operatorname{supp}(D) := \{z_j \in \mathbb{C} : j \in J\}$$

is the support of D; D is said to be *effective* if all its coefficients are positive.

The set of all divisors on $\mathbb{C}$ forms the abelian group $\operatorname{Div}(\mathbb{C})$, and the natural evaluation map $\sigma : \operatorname{Div}(\mathbb{C}) \to \mathbb{C}$ is given by

$$\sigma(D) = \sum_{j \in J} a_j z_j.$$

A *polar rational polygon* (prp) is an effective divisor $D = \sum_{j \in J} a_j \cdot z_j$ such that each z_j is a root of unity and $\sigma(D) = 0$. In this case the order $\mathrm{o}(D)$ is the cardinality of the subgroup of $\mathbb{C} \setminus \{0\}$ generated by the roots of unity $\{z_j/z_k : j, k \in J\}$. The prp D is called *primitive* if there do not exist nonzero prp's D' and D'' such that $D = D' + D''$. In particular, the coefficients of D', D'' are positive, thus each prp can be expressed as a sum of primitive prp's.

Every polynomial $f \in \mathbb{Z}[X] \setminus \{0\}$ can be uniquely written in the form

$$f = \sum_{j \in J} \varepsilon_j a_j X^j \tag{2.4}$$

with $J \subseteq \{0, \ldots, \deg(f)\}$, $\varepsilon_j = \pm 1$ and $a_j > 0$. We call

$$\ell(f) := \operatorname{Card}(J)$$

the length of f. For $\zeta \in \mathbb{C}$ with $f(\zeta) = 0$ we define the effective divisor of f (w.r.t. ζ) by

$$Df(\zeta) := \sum_{j \in J} a_j (\varepsilon_j \zeta^j).$$

Proposition 2.4. *Let $a_2 > a_1 > a_0$ and let P as in (2.1) be associated with $T(a_0, a_1, a_2)$. If ζ is a root of unity such that $P(\zeta) = 0$ then the order of ζ satisfies*

$$\operatorname{ord}(\zeta) \leq 420(a_2 - a_1 + a_0 - 1).$$

Proof. We follow the proof of [5, Theorem 2.1] and write the polynomials Q, R, S in the form

$$Q(X) = \sum Q_j(X), \qquad R(X) = \sum R_j(X), \qquad S(X) = \sum S_j(X)$$

with finite sums of integer polynomials such that

$$DP(\zeta) = \sum_j DP_j(\zeta) = \sum_j \big(\zeta^{a_1+a_2} DQ_j(\zeta) + \zeta^{a_1+1} DR_j(\zeta) + DS_j(\zeta)\big)$$

is a decomposition of the divisor $DP(\zeta)$ into primitive polar rational polygons $DP_j(\zeta)$, thus for every j the sum

$$\zeta^{a_1+a_2} Q_j(\zeta) + \zeta^{a_1+1} R_j(\zeta) + S_j(\zeta)$$

is the evaluation of the primitive prp $DP_j(\zeta)$. Observe that in view of Lemma 2.1 the (naive) height of the polynomials Q_j, R_j, S_j does not exceed 2 since the coefficients cannot increase when performing the decomposition of a prp into primitive prp's.

Case 1: $\max\{\ell(Q_j), \ell(R_j), \ell(S_j)\} > 1$ for some j.

Let us first assume $\ell(Q_j) > 1$ for some j. The ratio of any two roots of unity occurring in $DQ_j(\zeta)$ can be written in the form $\pm\zeta^e$ with $1 \le e \le \deg(Q_j) \le \deg(Q)$. Therefore we have

$$\frac{\mathrm{ord}(\zeta)}{2\deg(Q)} \le \mathrm{o}(DP_j(\zeta)).$$

By Mann's Theorem [8], $\mathrm{o}(DP_j(\zeta))$ is bounded by the product of primes at most equal to

$$\ell(P_j) \le \ell(Q_j) + \ell(R_j) + \ell(S_j) \le \ell(Q) + \ell(R) + \ell(S) \le 3 + 4 + 3 = 10.$$

The product of the respective primes is at most $2 \cdot 3 \cdot 5 \cdot 7 = 210$. Therefore by Lemma 2.1 we find

$$\frac{\mathrm{ord}(\zeta)}{2(a_0+1)} = \frac{\mathrm{ord}(\zeta)}{2\deg(Q)} \le 210,$$

which yields

$$\mathrm{ord}(\zeta) \le 420(a_0+1).$$

Analogously, the other two cases yield

$$\mathrm{ord}(\zeta) \le 420(a_2 - a_1 + a_0 - 1) \qquad \text{or} \qquad \mathrm{ord}(\zeta) \le 420(a_0+1),$$

and we conclude

$$\operatorname{ord}(\zeta) \leq 420 \max\{a_0 + 1,\, a_2 - a_1 + a_0 - 1\} = 420(a_2 - a_1 + a_0 - 1). \tag{2.5}$$

Case 2: $\max\{\ell(Q_j), \ell(R_j), \ell(S_j)\} \leq 1$ for all j.

In this case, $DP_j(\zeta)$ is either of the form

$$DP_j(\zeta) = c_{j1}\zeta^{b_{j1}} + c_{j2}\zeta^{b_{j2}} \tag{2.6}$$

or of the form

$$DP_j(\zeta) = c_{j1}\zeta^{b_{j1}} + c_{j2}\zeta^{b_{j2}} + c_{j3}\zeta^{b_{j3}}, \tag{2.7}$$

where $c_{ji} \in \{-2, \dots, 2\}$ by Lemma 2.2. We distinguish two subcases.

Case 2.1: There exists j such that $DP_j(\zeta)$ is of the form (2.7).

In this situation $DP_j(\zeta)$ can be written more explicitly as

$$DP_j(\zeta) = c_{j1}\zeta^{a_1+a_2+\eta_1} + c_{j2}\zeta^{a_1+\eta_2} + c_{j3}\zeta^{\eta_3} \tag{2.8}$$

or

$$DP_j(\zeta) = c_{j1}\zeta^{a_1+a_2+\eta_1} + c_{j2}\zeta^{a_2+\eta_2} + c_{j3}\zeta^{\eta_3}, \tag{2.9}$$

where $\eta_1 \in \{0, a_0, a_0 + 1\}$, $\eta_2 \in \{1, a_0\}$, and $\eta_3 \in \{0, 1, a_0 + 1\}$. If $DP_j(\zeta)$ is as in (2.8) then $P_j(\zeta) = 0$ implies that

$$c_{j1}\zeta^{a_1+a_2+\eta_1-\eta_3} + c_{j2}\zeta^{a_1+\eta_2-\eta_3} + c_{j3} = 0.$$

Now, using Lemma 2.3 we gain

$$\operatorname{ord}(\zeta) \mid 6 \gcd(a_1 + a_2 + \eta_1 - \eta_3, a_1 + \eta_2 - \eta_3)$$

which yields

$$\operatorname{ord}(\zeta) \mid 6(a_2 - a_1 + \eta_1 - 2\eta_2 + \eta_3),$$

hence,

$$\operatorname{ord}(\zeta) \leq 6(2a_0 + a_2 - a_1). \tag{2.10}$$

If $DP_j(\zeta)$ is as in (2.9), by analogous arguments we again obtain (2.10).

Case 2.2: For all j the divisor $DP_j(\zeta)$ is of the form (2.6).

In this case we have to form pairs of the 10 summands of $DP(\zeta)$ to obtain the divisors $DP_j(\zeta)$. As $\ell(R) = \ell(S) = 4$ there must exist j_1, j_2 such that $\ell(R_{j_1}) = 0$ and $\ell(S_{j_2}) = 0$.

In what follows, $c_{ij} \in \{-2,\dots,2\}$, and $\eta_1, \eta_1' \in \{0, a_0, a_0+1\}$, $\eta_2, \eta_2' \in \{1, a_0\}$, and $\eta_3, \eta_3' \in \{0, 1, a_0+1\}$. Then $DP_{j_1}(\zeta)$ is of the form

$$DP_{j_1}(\zeta) = c_{j_1 1}\zeta^{a_1+a_2+\eta_1} + c_{j_1 3}\zeta^{\eta_3} \tag{2.11}$$

which yields

$$c_{j_1 1}\zeta^{a_1+a_2+\eta_1-\eta_3} + c_{j_1 3} = 0,$$

and, hence,

$$\mathrm{ord}(\zeta) \mid 2(a_1 + a_2 + \eta_1 - \eta_3). \tag{2.12}$$

For $DP_{j_2}(\zeta)$ we have two possibilities. Either we have

$$DP_{j_2}(\zeta) = c_{j_2 1}\zeta^{a_1+a_2+\eta_1'} + c_{j_2 2}\zeta^{a_1+\eta_2'}. \tag{2.13}$$

This yields

$$c_{j_2 1}\zeta^{a_2+\eta_1'-\eta_2'} + c_{j_2 2} = 0,$$

and, hence,

$$\mathrm{ord}(\zeta) \mid 2(a_2 + \eta_1' - \eta_2'). \tag{2.14}$$

The second alternative for $DP_{j_2}(\zeta)$ reads

$$DP_{j_2}(\zeta) = c_{j_2 1}\zeta^{a_1+a_2+\eta_1'} + c_{j_2 2}\zeta^{a_2+\eta_2'}. \tag{2.15}$$

This yields

$$c_{j_2 1}\zeta^{a_1+\eta_1'-\eta_2'} + c_{j_2 2} = 0,$$

and, hence,

$$\mathrm{ord}(\zeta) \mid 2(a_1 + \eta_1' - \eta_2'). \tag{2.16}$$

If P_{j_2} is of the form (2.13), then (2.12) and (2.14) yield that

$$\mathrm{ord}(\zeta) \mid 2\gcd(a_1 + a_2 + \eta_1 - \eta_3, a_2 + \eta_1' - \eta_2'),$$

hence, $\mathrm{ord}(\zeta) \mid 2(a_1 - a_2 + \eta_1 - 2\eta_1' + 2\eta_2' - \eta_3)$ and therefore

$$\mathrm{ord}(\zeta) \le 2(a_2 - a_1 + 3a_0 + 1). \tag{2.17}$$

If P_{j_2} is of the form (2.15), then (2.12) and (2.16) yield

$$\operatorname{ord}(\zeta) \mid 2\gcd(a_1 + a_2 + \eta_1 - \eta_3, a_1 + \eta_1' - \eta_2'),$$

hence, (2.17) follows again.

Summing up, the proposition is proved by combining (2.5), (2.10), and (2.17). □

Combining results from [9] with our previous considerations we can now prove the main theorem of this paper.

Proof of Theorem 1. The fact that τ is either a Salem number or a quadratic Pisot number as well as the decomposition of R_T given in (1.2) follows immediately from (1.1). The bound on the orders of the roots of the cyclotomic polynomials is a consequence of Proposition 2.4. Together with Lemma 2.2 this proposition yields the estimate (1.3) on the degree of S, where the explicitly computable constant $m(a_0, a_2 - a_1)$ is the one stated in Lemma 2.2. □

3. Convergence properties of Salem numbers generated by star-like trees

In this section we prove Theorems 2 and 3. In the following proof of Theorem 2 we denote by $M_m(x) = x^m - x^{m-1} - \cdots - x - 1$ the minimal polynomial of the m-bonacci number φ_m.

Proof of Theorem 2. Note that $Q(x) = (x-1)M_{a_0}(x)$ holds. The theorem is proved if for each $\varepsilon > 0$ the polynomial $P(x)$ has a root ζ in the open ball $B_\varepsilon(\varphi_{a_0})$ for all sufficiently large a_1. We prove this by using Rouché's Theorem. Let $\varepsilon > 0$ be sufficiently small and set $C_\varepsilon := \partial B_\varepsilon(\varphi_{a_0})$. Then $\delta := \min\{|M_{a_0}(x)| : x \in C_\varepsilon\} > 0$. Thus, on C_ε we have the following estimations.

$$\begin{aligned} |x^{a_1+1}R(x) + S(x)| &< (\varphi_{a_0} + \varepsilon)^{a_1+1} 4(\varphi_{a_0} + \varepsilon)^{\eta + a_0 - 1} + 4(\varphi_{a_0} + \varepsilon)^{a_0+1}, \\ |x^{a_1+a_2}Q(x)| &> (\varphi_{a_0} - \varepsilon)^{2a_1+\eta}(\varphi_{a_0} - 1 - \varepsilon)\delta. \end{aligned} \tag{3.1}$$

Combining these two inequalities yields

$$|P(x)| > (\varphi_{a_0} - \varepsilon)^{2a_1+\eta}(\varphi_{a_0} - 1 - \varepsilon)\delta - 4(\varphi_{a_0} + \varepsilon)^{a_1+a_0+\eta} - 4(\varphi_{a_0} + \varepsilon)^{a_0+1}. \tag{3.2}$$

Since (3.1) and (3.2) imply that for sufficiently large a_1 we have

$$|P(x)| > |x^{a_1+1}R(x) + S(x)| \qquad (x \in C_\varepsilon),$$

Rouché's Theorem yields that $P(x)$ and $x^{a_1+a_2}(x-1)M_{a_0}(x)$ have the same number of roots in $B_\varepsilon(\varphi_{a_0})$. Thus our assertion is proved. □

To prove Theorem 3 we need the following auxiliary lemma.

Lemma 3.1. *Let $r \geq 1$, $a_r > \cdots > a_1 > a_0 \geq 2$, and choose $k \in \{1, \ldots, r-1\}$. If P is as in (2.1) and Q as in (1.4) then, for fixed $a_0, \ldots, a_k$ and $a_{k+1}, \ldots, a_r$ sufficiently large, we have*

$$\#\{\xi \in \mathbb{C} : P(\xi) = 0,\ |\xi| > 1\} = \#\{\xi \in \mathbb{C} : Q(\xi) = 0,\ |\xi| > 1\} = 1.$$

Proof. First observe that

$$P(z) = z^{a_{k+1}+\cdots+a_r} Q(z) + O(z^{a_{k+2}+\cdots+a_r+\eta}) \tag{3.3}$$

for some fixed constant $\eta \in \mathbb{N}$. Since by (1.1) the polynomial P has exactly one (Salem) root outside the unit disk, it is sufficient to prove the first equality in the statement of the lemma.

We first show that Q has at least one root ξ with $|\xi| > 1$. This is certainly true for $k < r-2$ as in this case we have $|Q(0)| > 1$. For $k \in \{r-2, r-1\}$ we see that

$$Q^{(\ell)}(1) = 0, \text{ for } 0 \leq \ell < a_0 + \cdots + a_k - 1, \quad Q^{(a_0+\cdots+a_k-1)}(1) < 0.$$

As the leading coefficient of Q is positive, this implies that $Q(\xi) = 0$ for some $\xi > 1$.

Now we show that Q has at most one root ξ with $|\xi| > 1$. Assume on the contrary that there exist two distinct roots $\xi_1, \xi_2 \in \mathbb{C}$ of Q outside the closed unit circle. Applying Rouché's Theorem to $P(z)$ and $z^{a_{k+1}+\cdots+a_r} Q(z)$ shows by using (3.3) that also P has two zeros outside the closed unit circle which contradicts the fact that P is a product of a Salem polynomial and cyclotomic polynomials, see (1.1). □

Proof of Theorem 3. From Lemma 3.1 and (1.4) we derive that

$$Q(z) = C(z)T(z)z^s,$$

where $s \in \{0, 1\}$ and T is a Pisot or a Salem polynomial. To show the theorem we have to prove that T is a Pisot polynomial. It suffices to show that $C(z)T(z)$ is not self-reciprocal.

We distinguish three cases. If $k < r-2$ then $|C(0)T(0)| > 1$, hence, as this polynomial has leading coefficient 1 it cannot be self-reciprocal.

Denote the ℓ-th coefficient of the polynomial f by $[z^\ell]f(z)$. If $k = r-2$ we have $[z]C(z)T(z) = 2(-1)^{r-1}$ and $[z^{a_0+\cdots+a_{r-2}}]C(z)T(z) = -r$. As $k > 0$ we have $r > 2$ and again the polynomial cannot be self-reciprocal.

Finally for $k = r-1$ we have $[z]C(z)T(z) = 0$ and $[z^{a_0+\cdots+a_{r-1}-1}]C(z)T(z) = -r$ which again excludes self-reciprocity of $C(z)T(z)$. □

4. Concluding remarks

In this note we have studied Coxeter polynomials of star-like trees with special emphasis on star-like trees with three arms. It would be nice to extend Theorem 2 to star-like trees $T(a_0, \ldots, a_r)$ with four and more arms in order to get lower estimates on the degrees of the Salem polynomials involved in Theorem 3. In fact, the estimate on the maximal multiplicity of the irreducible factors of R_T contained in Lemma 2.2 can be carried over to star-like trees with larger values of r. Concerning Proposition 2.4, the argument based on Mann's Theorem used in order to settle Case 1 of its proof can be extended to $T(a_0, \ldots, a_r)$, however, in the situation of Case 2 we were not able to prove that the orders of the occurring roots of unity are bounded by a reasonable bound. We expect that a generalization of this case requires new ideas.

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