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Sharp inequalities and asymptotic series of a product related to the Euler–Mascheroni constant

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ABSTRACT

We derive an asymptotic series related to the constant e^γ , where γ denotes the Euler–Mascheroni constant. Based on the result obtained, we establish some sharp inequalities for e^γ .

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1. Introduction

The Euler–Mascheroni constant γ is defined as the limit

$$\gamma = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln n \right). \quad (1.1)$$

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The constant e^γ is important in number theory, since for example, e^γ equals the following limit, where p_n is the n th prime number:

$$e^\gamma = \lim_{n \rightarrow \infty} \frac{1}{\ln p_n} \prod_{j=1}^n \frac{p_j}{p_j - 1}. \quad (1.2)$$

This restates the third of Mertens' theorems (see [11]). The numerical value of e^γ is:

$$e^\gamma = 1.78107241799019798523650410310717954916964521430343 \dots$$

There is a the curious radical representation

$$e^\gamma = \left(\frac{2}{1}\right)^{1/2} \left(\frac{2^2}{1 \cdot 3}\right)^{1/3} \left(\frac{2^3 \cdot 4}{1 \cdot 3^3}\right)^{1/4} \left(\frac{2^4 \cdot 4^4}{1 \cdot 3^6 \cdot 5}\right)^{1/5} \dots, \quad (1.3)$$

where the n th factor is

$$\left(\prod_{k=0}^n (k+1)^{(-1)^{k+1} \binom{n}{k}}\right)^{1/(n+1)}.$$

The product (1.3), first discovered in 1926 by Ser [7], was rediscovered in [6,9,10].

It is known [5, p. 14] that

$$e^\gamma = \prod_{k=1}^{\infty} \frac{e^{\frac{1}{k}}}{1 + \frac{1}{k}}.$$

For $n \in \mathbb{N} := \{1, 2, \dots\}$, let

$$P_n = \prod_{k=1}^n \frac{e^{\frac{1}{k}}}{1 + \frac{1}{k}}. \quad (1.4)$$

The first aim of the present paper is to give a pair of recurrence relations for determining the constants $\lambda_\ell \equiv \lambda_\ell(r)$ and $\mu_\ell \equiv \mu_\ell(r)$ such that

$$P_n \sim e^\gamma \left(1 + \sum_{\ell=1}^{\infty} \frac{\lambda_\ell}{(n + \mu_\ell)^{2\ell-1}}\right)^{1/r}, \quad n \rightarrow \infty,$$

where $r \neq 0$ is any given real number. The second aim of this paper is to determine the best possible constants a and b such that

$$P_n \left(1 + \frac{1}{2(n+a)}\right) \leq e^\gamma < P_n \left(1 + \frac{1}{2(n+b)}\right)$$

holds for all $n \in \mathbb{N}$.

2. Lemmas

The gamma function may be defined for $x > 0$ by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt.$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \Gamma'(x)/\Gamma(x)$, is called the psi (or digamma) function. The psi function has the integer values [1, p. 258, Equation (6.3.2)]:

$$\psi(1) = -\gamma \quad \text{and} \quad \psi(n+1) = -\gamma + \sum_{k=1}^n \frac{1}{k} \quad (2.1)$$

and the recurrence formula [1, p. 258, Equation (6.3.5)]:

$$\psi(x+1) = \psi(x) + \frac{1}{x}. \quad (2.2)$$

Lemma 1. *Let P_n be defined in (1.4). Then*

$$P_n = \frac{e^{\psi(n+1)+\gamma}}{n+1}. \quad (2.3)$$

Note that (2.3) can be proved by induction, we omit it.

By mainly using the partition function, Chen [3] established Lemma 2 below. We here introduce the following set of partitions of an integer $n \in \mathbb{N}$:

$$\mathcal{A}_n := \{(k_1, k_2, \dots, k_n) \in \mathbb{N}_0^n : k_1 + 2k_2 + \dots + nk_n = n\}, \quad (2.4)$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In number theory, the partition function $p(n)$ represents the number of possible partitions of $n \in \mathbb{N}$, which is to say the number of distinct ways of representing n as a sum of natural numbers (with order irrelevant). By convention every $p(0) = 1$ and $p(n) = 0$ for every n negative integers. For more information on the partition function $p(n)$, please refer to [12] and the references therein. The first several values of the partition function $p(n)$ are (starting with $p(0) = 1$) (see [8]):

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, \dots$$

It is easy to see that the cardinality of the set \mathcal{A}_n is equal to the partition function $p(n)$.

Lemma 2. (See [3].) *Let $t \in \mathbb{R}$ and let r be a given nonzero real number. The following asymptotic expansion holds true:*

$$\frac{e^{r\psi(x+t)}}{x^r} \sim 1 + \sum_{j=1}^{\infty} \frac{b_j(r, t)}{x^j}, \quad x \rightarrow \infty, \quad (2.5)$$

with the coefficients $b_j(r, t)$ ($j \in \mathbb{N}$) given by

$$b_j(r, t) = (-1)^j \sum_{(k_1, k_2, \dots, k_j) \in \mathcal{A}_j} \frac{(-r)^{k_1+k_2+\dots+k_j}}{k_1!k_2!\dots k_j!} \left(\frac{B_1(t)}{1}\right)^{k_1} \left(\frac{B_2(t)}{2}\right)^{k_2} \dots \left(\frac{B_j(t)}{j}\right)^{k_j}, \quad (2.6)$$

where \mathcal{A}_n are given in (2.4) and $B_n(t)$ denote the Bernoulli polynomials defined by the following generating function:

$$\frac{xe^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!}. \quad (2.7)$$

Elezović [4, Theorem 2.2] gave a recursive relation for determining the coefficients $b_j(r, t)$ in (2.5).

Note that the Bernoulli numbers B_n are defined by $B_n := B_n(0)$ in (2.7).

Setting $(x, t) = (n+1, 0)$ in (2.5) yields

$$\frac{e^{r\psi(n+1)}}{(n+1)^r} \sim 1 + \sum_{j=1}^{\infty} \frac{b_j}{(n+1)^j}, \quad n \rightarrow \infty, \quad (2.8)$$

with the coefficients $b_j \equiv b_j(r, 0)$ ($j \in \mathbb{N}$) given by

$$b_j = (-1)^j \sum_{(k_1, k_2, \dots, k_j) \in \mathcal{A}_j} \frac{(-r)^{k_1+k_2+\dots+k_j}}{k_1!k_2!\dots k_j!} \left(\frac{B_1}{1}\right)^{k_1} \left(\frac{B_2}{2}\right)^{k_2} \dots \left(\frac{B_j}{j}\right)^{k_j}. \quad (2.9)$$

The choice $r = 1$ in (2.8) yields

$$\begin{aligned} \frac{e^{\psi(n+1)}}{n+1} &\sim 1 - \frac{1}{2(n+1)} + \frac{1}{24(n+1)^2} + \frac{1}{48(n+1)^3} + \frac{23}{5760(n+1)^4} \\ &\quad - \frac{17}{3840(n+1)^5} - \frac{10\,099}{2\,903\,040(n+1)^6} + \dots, \quad n \rightarrow \infty. \end{aligned} \quad (2.10)$$

The choice $r = -1$ in (2.8) yields

$$\begin{aligned} \frac{e^{\psi(n+1)}}{n+1} &\sim \left(1 + \frac{1}{2(n+1)} + \frac{5}{24(n+1)^2} + \frac{1}{16(n+1)^3} + \frac{47}{5760(n+1)^4} \right. \\ &\quad \left. - \frac{1}{2304(n+1)^5} + \frac{8713}{2\,903\,040(n+1)^6} + \dots \right)^{-1}, \quad n \rightarrow \infty. \end{aligned} \quad (2.11)$$

Lemma 3. For $x \geq 1$,

$$e^{\psi(x)} < x - \frac{1}{2} + \frac{1}{24x} + \frac{1}{48x^2} + \frac{23}{5760x^3}. \quad (2.12)$$

Proof. Define the function $F(x)$ by

$$F(x) = \psi(x) - \ln \left(x - \frac{1}{2} + \frac{1}{24x} + \frac{1}{48x^2} + \frac{23}{5760x^3} \right).$$

It follows from the known result (see [2, Theorem 9]) that

$$\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} < \psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3}, \quad x > 0. \quad (2.13)$$

Differentiating $F(x)$ and applying the first inequality in (2.13), we obtain that for $x \geq 1$,

$$\begin{aligned} F'(x) &= \psi'(x) - \frac{3(1920x^4 - 80x^2 - 80x - 23)}{x(5760x^4 - 2880x^3 + 240x^2 + 120x + 23)} \\ &> \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} - \frac{3(1920x^4 - 80x^2 - 80x - 23)}{x(5760x^4 - 2880x^3 + 240x^2 + 120x + 23)} \\ &= \frac{3825x^3 - 125x^2 - 120x - 23}{30x^5(5760x^4 - 2880x^3 + 240x^2 + 120x + 23)} > 0. \end{aligned}$$

Hence, $F(x)$ is strictly increasing for $x \geq 1$, and we have

$$F(x) < \lim_{t \rightarrow \infty} F(t) = 0, \quad x \geq 1.$$

The proof of Lemma 3 is complete. \square

From (2.12), we obtain

$$\begin{aligned} 2 \left(x - e^{\psi(x)} \right)^2 &> 2 \left(\frac{1}{2} - \frac{1}{24x} - \frac{1}{48x^2} - \frac{23}{5760x^3} \right)^2 \\ &= \frac{(2880x^3 - 240x^2 - 120x - 23)^2}{16\,588\,800x^6}, \quad x \geq 1. \end{aligned} \quad (2.14)$$

It follows from the second inequality in (2.13) that

$$0 < x\psi'(x) - 1 < \frac{1}{2x} + \frac{1}{6x^2}, \quad x > 0. \quad (2.15)$$

The proof of Theorem 3.2 makes use of the inequalities (2.12), (2.14) and (2.15).

3. The main results

Theorem 3.1. *Let $r \neq 0$ be any given real number. The sequence P_n has the following asymptotic series expansion:*

$$P_n \sim e^\gamma \left(1 + \sum_{\ell=1}^{\infty} \frac{\alpha_\ell}{(n+1+\beta_\ell)^{2\ell-1}} \right)^{1/r}, \quad n \rightarrow \infty, \quad (3.1)$$

where $\alpha_\ell \equiv \alpha_\ell(r)$ and $\beta_\ell \equiv \beta_\ell(r)$ are given by a pair of recurrence relations

$$\alpha_\ell = b_{2\ell-1} - \sum_{k=1}^{\ell-1} \alpha_k \beta_k^{2\ell-2k} \binom{2\ell-2}{2\ell-2k}, \quad \ell \geq 2 \quad (3.2)$$

and

$$\beta_\ell = -\frac{1}{(2\ell-1)\alpha_\ell} \left\{ b_{2\ell} + \sum_{k=1}^{\ell-1} \alpha_k \beta_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1} \right\}, \quad \ell \geq 2, \quad (3.3)$$

with $\alpha_1 = b_1$ and $\beta_1 = -\frac{b_2}{b_1}$. Here b_j are given in (2.9).

Proof. Write (3.1) as

$$\frac{e^{r\psi(n+1)}}{(n+1)^r} \sim 1 + \sum_{j=1}^{\infty} \frac{\alpha_j}{(n+1)^{2j-1}} \left(1 + \frac{\beta_j}{n+1} \right)^{-2j+1}. \quad (3.4)$$

Direct computation yields

$$\begin{aligned} & \sum_{j=1}^{\infty} \frac{\alpha_j}{(n+1)^{2j-1}} \left(1 + \frac{\beta_j}{n+1} \right)^{-2j+1} \\ & \sim \sum_{j=1}^{\infty} \frac{\alpha_j}{(n+1)^{2j-1}} \sum_{k=0}^{\infty} \binom{-2j+1}{k} \frac{\beta_j^k}{(n+1)^k} \\ & \sim \sum_{j=1}^{\infty} \frac{\alpha_j}{(n+1)^{2j-1}} \sum_{k=0}^{\infty} (-1)^k \binom{k+2j-2}{k} \frac{\beta_j^k}{(n+1)^k} \\ & \sim \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} \alpha_{k+1} \beta_{k+1}^{j-k-1} (-1)^{j-k-1} \binom{j+k-1}{j-k-1} \frac{1}{(n+1)^{j+k}}, \end{aligned}$$

which can be written as

$$\begin{aligned} & \sum_{j=1}^{\infty} \frac{\alpha_j}{(n+1)^{2j-1}} \left(1 + \frac{\beta_j}{n+1} \right)^{-2j+1} \\ & \sim \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{\lfloor \frac{j+2}{2} \rfloor} \alpha_k \beta_k^{j-2k+1} (-1)^{j-1} \binom{j-1}{j-2k+1} \right\} \frac{1}{(n+1)^j}. \quad (3.5) \end{aligned}$$

It follows from (3.4) and (3.5) that

$$\frac{e^{r\psi(n+1)}}{(n+1)^r} \sim 1 + \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{\lfloor \frac{j+2}{2} \rfloor} \alpha_k \beta_k^{j-2k+1} (-1)^{j-1} \binom{j-1}{j-2k+1} \right\} \frac{1}{(n+1)^j}. \quad (3.6)$$

Equating coefficients of the term $(n+1)^{-j}$ on the right sides of (2.8) and (3.6), we obtain

$$b_j = \sum_{k=1}^{\lfloor \frac{j+2}{2} \rfloor} \alpha_k \beta_k^{j-2k+1} (-1)^{j-1} \binom{j-1}{j-2k+1}, \quad j \in \mathbb{N}. \quad (3.7)$$

Setting $j = 2\ell - 1$ and $j = 2\ell$ in (3.7), respectively, yields

$$b_{2\ell-1} = \sum_{k=1}^{\ell} \alpha_k \beta_k^{2\ell-2k} \binom{2\ell-2}{2\ell-2k} \quad (3.8)$$

and

$$\begin{aligned} b_{2\ell} &= - \sum_{k=1}^{\ell+1} \alpha_k \beta_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1} \\ &= - \sum_{k=1}^{\ell} \alpha_k \beta_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1} - \alpha_{\ell+1} \beta_{\ell+1}^{-1} \binom{2\ell-1}{-1} \\ &= - \sum_{k=1}^{\ell} \alpha_k \beta_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1}. \end{aligned} \quad (3.9)$$

For $\ell = 1$, from (3.8) and (3.9) we obtain

$$\alpha_1 = b_1(r, 0) \quad \text{and} \quad \beta_1 = -\frac{b_2(r, 0)}{\alpha_1} = -\frac{b_2(r, 0)}{b_1(r, 0)},$$

and for $\ell \geq 2$ we have

$$b_{2\ell-1} = \sum_{k=1}^{\ell-1} \alpha_k \beta_k^{2\ell-2k} \binom{2\ell-2}{2\ell-2k} + \alpha_\ell$$

and

$$b_{2\ell} = - \sum_{k=1}^{\ell-1} \alpha_k \beta_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1} - (2\ell-1) \alpha_\ell \beta_\ell.$$

We then obtain the recurrence relations (3.2) and (3.3). The proof is complete. \square

For $r = 1$ in (3.1), we give explicit numerical values of some first terms of α_ℓ and β_ℓ by using the recurrence relations (3.2) and (3.3).

$$\alpha_1 = b_1(1, 0) = -\frac{1}{2}, \quad \beta_1 = -\frac{b_2(1, 0)}{b_1(1, 0)} = -\frac{\frac{1}{24}}{-\frac{1}{2}} = \frac{1}{12},$$

$$\alpha_2 = b_3(1, 0) - \alpha_1\beta_1^2 = \frac{1}{48} - \left(-\frac{1}{2}\right) \cdot \left(\frac{1}{12}\right)^2 = \frac{7}{288},$$

$$\beta_2 = -\frac{b_4(1, 0) + \alpha_1\beta_1^3}{3\alpha_2} = -\frac{\frac{23}{5760} + \left(-\frac{1}{2}\right) \cdot \left(\frac{1}{12}\right)^3}{3 \cdot \left(\frac{7}{288}\right)} = -\frac{16}{315},$$

$$\begin{aligned} \alpha_3 &= b_5(1, 0) - \alpha_1\beta_1^4 - 6\alpha_2\beta_2^2 \\ &= -\frac{17}{3840} - \left(-\frac{1}{2}\right) \cdot \left(\frac{1}{12}\right)^4 - 6 \cdot \left(\frac{7}{288}\right) \cdot \left(-\frac{16}{315}\right)^2 = -\frac{104\,057}{21\,772\,800}, \end{aligned}$$

$$\begin{aligned} \beta_3 &= -\frac{b_6(1, 0) + \alpha_1\beta_1^5 + 10\alpha_2\beta_2^3}{5\alpha_3} \\ &= -\frac{-\frac{10\,099}{2\,903\,040} + \left(-\frac{1}{2}\right) \cdot \left(\frac{1}{12}\right)^5 + 10 \cdot \left(\frac{7}{288}\right) \cdot \left(-\frac{16}{315}\right)^3}{5 \cdot \left(-\frac{104\,057}{21\,772\,800}\right)} = -\frac{57\,818\,693}{393\,335\,460}. \end{aligned}$$

Write (3.1) as

$$P_n \sim e^\gamma \left(1 + \sum_{\ell=1}^{\infty} \frac{\lambda_\ell}{(n + \mu_\ell)^{2\ell-1}}\right)^{1/r}, \quad n \rightarrow \infty, \quad (3.10)$$

where

$$\lambda_\ell := \alpha_\ell \quad \text{and} \quad \mu_\ell := 1 + \beta_\ell. \quad (3.11)$$

For $r = 1$ in (3.10), we obtain

$$P_n \sim e^\gamma \left(1 - \frac{\frac{1}{2}}{n + \frac{13}{12}} + \frac{\frac{7}{288}}{(n + \frac{299}{315})^3} - \frac{\frac{104\,057}{21\,772\,800}}{(n + \frac{335\,516\,767}{393\,335\,460})^5} + \cdots\right), \quad n \rightarrow \infty. \quad (3.12)$$

For $r = -1$ in (3.1), we find

$$P_n \sim e^\gamma \left(1 + \frac{\frac{1}{2}}{n + \frac{7}{12}} - \frac{\frac{7}{288}}{(n + \frac{194}{315})^3} + \frac{\frac{130\,937}{21\,772\,800}}{(n + \frac{322\,138\,717}{494\,941\,860})^5} - \cdots\right)^{-1}, \quad n \rightarrow \infty. \quad (3.13)$$

The formula (3.13) motivated us to establish the following theorem.

Theorem 3.2. For $n \in \mathbb{N}$, we have

$$P_n \left(1 + \frac{1}{2(n+a)} \right) \leq e^\gamma < P_n \left(1 + \frac{1}{2(n+b)} \right) \quad (3.14)$$

with the best possible constants

$$a = \frac{3 - 4e^{\gamma-1}}{4e^{\gamma-1} - 2} = 0.61061795\dots \quad \text{and} \quad b = \frac{7}{12} = 0.58333333\dots \quad (3.15)$$

Proof. The inequality (3.14) can be written as

$$a \geq f(n) > b, \quad n \geq 1,$$

where

$$f(x) = \frac{1}{2 \left(\frac{x+1}{e^{\psi(x+1)}} - 1 \right)} - x.$$

Direct computation yields

$$f(1) = \frac{3 - 4e^{\gamma-1}}{4e^{\gamma-1} - 2}.$$

By using the asymptotic series expansion (3.13), we find that

$$\lim_{n \rightarrow \infty} f(n) = \frac{7}{12}.$$

In order to prove the Theorem 3.2, it suffices to show that the sequence $\{f(n)\}$ is strictly decreasing. Differentiation yields

$$-2 \left((x+1) - e^{\psi(x+1)} \right)^2 f'(x) =: g(x+1),$$

where

$$\begin{aligned} g(x) &= 2 \left(x - e^{\psi(x)} \right)^2 - e^{\psi(x)} \left(x\psi'(x) - 1 \right) \\ &> \frac{(2880x^3 - 240x^2 - 120x - 23)^2}{16\,588\,800x^6} - \left(x - \frac{1}{2} + \frac{1}{24x} + \frac{1}{48x^2} + \frac{23}{5760x^3} \right) \left(\frac{1}{2x} + \frac{1}{6x^2} \right) \\ &= \frac{3\,276\,529 + 8\,281\,200(x-2) + 7\,434\,240(x-2)^2 + 2\,862\,720(x-2)^3 + 403\,200(x-2)^4}{16\,588\,800x^6}. \end{aligned}$$

By using the inequalities (2.12), (2.14) and (2.15), we deduce that $g(x) > 0$, for every $x \geq 2$. We then obtain

$$f'(x) < 0, \quad x \geq 1.$$

So, the sequence $\{f(n)\}$ is strictly decreasing. The proof is complete. \square

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