General Section

Epsilon factors of representations of finite general linear groups

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A B S T R A C T

We define epsilon factors for irreducible representations of finite general linear groups using Macdonald's correspondence. These epsilon factors satisfy multiplicativity, and are expressible as products of Gauss sums. The tensor product epsilon factors are related to the Rankin-Selberg gamma factors, by which we prove that the Rankin-Selberg gamma factors can be written as products of Gauss sums. The exterior square epsilon factors relate the Jacquet-Shalika exterior square gamma factors and the Langlands-Shahidi exterior square gamma factors for level zero supercuspidal representations. We prove that these exterior square factors coincide in a special case.

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1. Introduction

Long before the local Langlands correspondence was established by Harris-Taylor [6] and Henniart [7], Macdonald had already established a correspondence between irreducible representations of $\mathrm{GL}_n(k)$, k a finite field, and inertia equivalence classes of

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admissible tamely ramified n-dimensional Weil-Deligne representations of W_K, where K is a non-archimedean local field with residue field k and W_K is the Weil group of K. This correspondence matches epsilon factors. In view of [17, (A.1)], Macdonald's correspondence is the restriction of the local Langlands correspondence to level zero representations.

Let $\mathfrak{o}$ be the ring of integers of K, $\mathfrak{p} \subset \mathfrak{o}$ be its prime ideal. Let q be the size of k. Let ψ be an additive character of K of conductor $\mathfrak{p}$, i.e., ψ is trivial on $\mathfrak{p}$ but not on $\mathfrak{o}$. Thus, it descends to a non-trivial additive character ψ on k. Let dx be the Haar measure on K normalized such that $\mathfrak{p}$ has volume $q^{-\frac{1}{2}}$. If ϕ is a tamely ramified n-dimensional Weil-Deligne representation of W_K corresponding to the irreducible representation π of $\mathrm{GL}_n(k)$, then the match of epsilon factors asserts that

$$\epsilon(\pi, \psi) = \epsilon_0(\phi, \psi, dx).$$

Here $\epsilon(\pi, \psi)$ is the Godement-Jacquet epsilon factor of π defined in [10], and $\epsilon_0(\phi, \psi, dx)$ is the arithmetic epsilon factor defined by Deligne in [3]. We note that operations such as direct sums, tensor products, exterior powers and symmetric powers preserve tame ramification of Weil-Deligne representations of W_K. Thus in the spirit of Macdonald's correspondence, we define various ϵ_0-factors of representations over the finite field with respect to these operations in a way that they match the arithmetic ϵ_0-factors of Deligne, see Definition 3.2. The immediate benefit of such definition is that ϵ_0-factors over finite fields inherit good properties from the arithmetic ϵ_0-factors of Deligne. The most important ones are multiplicativity and being expressible as products of Gauss sums.

ϵ_0-factors over finite fields agree with those gamma factors coming from integral representations. In Section 4, we write down explicitly the formulas for the finite ϵ_0-factors with respect to the tensor product operation in terms of Gauss sums. Then we show that they are equal to the corresponding Rankin-Selberg gamma factors defined in [15,12,19], up to some effective constants. Since ϵ_0-factors can be written as products of Gauss sums, we then prove in Corollary 4.5 that the Rankin-Selberg gamma factors are also products of Gauss sums, which answers [14, Conjecture 2.2].

Section 5 is similar to Section 4, only that we analyze the exterior square epsilon factors in this section. The exterior square epsilon factors are also multiplicative and are products of Gauss sums. We will not repeat the proofs of these statements, since the techniques are demonstrated in Section 4. The exterior square epsilon factor is equal, up to some constant c_f, to the exterior square gamma factor defined in [21]. Unfortunately, the constants c_f are not effective. We conjecture that $c_f = 1$, which is roughly equivalent to the statement that the two exterior square epsilon factors coming from the Jacquet-Shalika integral representation and the Langlands-Shahidi method coincide.

2. Tamely ramified representations

Let K be a non-archimedean local field with residue field k of size q. Let $\mathfrak{o}$ be the ring of integers of K and $\mathfrak{p} = (\varpi)$ be the maximal ideal of $\mathfrak{o}$, where ϖ is a fixed uniformizer. Then k is isomorphic to $\mathfrak{o}/\mathfrak{p}$. Set $\overline{K}$ and $\overline{k}$ as the separable closures of K and k respectively. We have a short exact sequence

$$1 \to I \to \mathrm{Gal}(\overline{K}/K) \to \mathrm{Gal}(\overline{k}/k) \cong \widehat{\mathbb{Z}} \to 1,$$

where I is called the inertia subgroup, and $\widehat{\mathbb{Z}} = \varprojlim_m \mathbb{Z}/m\mathbb{Z}$ is the inverse limit of the Galois groups of the finite degree field extensions of k. The pro-p subgroup P of I (where p is the characteristic of k) is the wild inertia subgroup of K. Taking the preimage of $\mathbb{Z} \subset \widehat{\mathbb{Z}}$, we obtain another short exact sequence involving the Weil group W of K:

$$1 \to I \to W(\overline{K}/K) \to \mathbb{Z} \to 1.$$

For convenience, we set $W = W(\overline{K}/K)$. Let $K^{un} \subset \overline{K}$ be the maximal unramified extension of K. Let $F \in \mathrm{Gal}(K^{un}/K)$ be a geometric Frobenius element, i.e., it is the inverse image of an automorphism of $\overline{k}/k$, also denoted by F, defined by $F(x^q) = x$ for $x \in \overline{k}$. Then $W = I \rtimes \langle F \rangle$. We define a norm $\|\cdot\|$ on W by setting

$$\|i\| = 1, \|F\| = q^{-1},$$

where $i \in I$.

A Weil-Deligne representation of W is a pair $\phi = (\rho, N)$ satisfying the following conditions:

(1) $\rho : W \to \mathrm{GL}(V)$ is a finite dimensional representation on V over $\mathbb{C}$, such that $\rho(w)$ is semisimple for $w \in W$, and $\ker(\rho)$ contains an open subgroup of I;
(2) $N : V \to V$ is nilpotent, and $\rho(w)N\rho(w)^{-1} = \|w\| \cdot N$ for $w \in W$.

The *degree* (or *dimension*) of ϕ is set to be $\dim \rho$. If $\ker(\rho)$ contains I (resp. P), then ρ and ϕ are said to be *unramified* (resp. *tamely ramified*). Two Weil-Deligne representations $\phi = (\rho, N)$ and $\phi' = (\rho', N')$ are *equivalent* if there exists a linear isomorphism $\alpha : V \to V'$ such that the following two diagrams commute for all $w \in W$, where V and V' are the underlying vector spaces of ρ and ρ' respectively.

$$
\begin{array}{ccc}
V & \xrightarrow{\alpha} & V' \\
{\scriptstyle\rho(w)}\downarrow & & \downarrow{\scriptstyle\rho'(w)} \\
V & \xrightarrow{\alpha} & V'
\end{array}
\qquad\qquad
\begin{array}{ccc}
V & \xrightarrow{\alpha} & V' \\
{\scriptstyle N}\downarrow & & \downarrow{\scriptstyle N'} \\
V & \xrightarrow{\alpha} & V'
\end{array}
$$

Similarly, ϕ and ϕ' are said to be *I-equivalent* if the above diagrams commute with ρ and ρ' replaced by their restrictions to I with some linear isomorphism $\alpha : V \to V'$. Following

the notations in [10, Section 3], we set $\Phi^t(\mathrm{GL}_n)$ to be the set of equivalence classes of tamely ramified Weil-Deligne representations of W of degree n, and set $\Phi^t_I(\mathrm{GL}_n)$ to be the set of I-equivalence classes of tamely ramified Weil-Deligne representations of W of degree n.

For a Weil representation $\rho : W \to \mathrm{GL}(V)$, Deligne [3, Section 4 and 5] defined the epsilon factors $\epsilon(\rho, \psi, dx)$ and $\epsilon_0(\rho, \psi, dx)$ associated to it, where ψ is a non-trivial additive character of K and dx is an arbitrary Haar measure on K. These two epsilon factors are non-zero constants related by

$$\epsilon(\rho, \psi, dx) = \epsilon_0(\rho, \psi, dx) \det\left(-F, V^I\right)^{-1}, \tag{1}$$

where V^I is the maximal subspace of V on which $\rho(I)$ acts trivially. For a Weil-Deligne representation $\phi = (\rho, N)$, following [3, 8.12], we can also define the epsilon factors associated to it by

$$\epsilon_0(\phi, \psi, dx) = \epsilon_0(\rho, \psi, dx), \tag{2}$$

and

$$\epsilon(\phi, \psi, dx) = \epsilon_0(\rho, \psi, dx) \det\left(-F, V^I_N\right)^{-1}, \tag{3}$$

where V^I_N is the null space of $N : V^I \to V^I$ (N preserves V^I because $\rho(i)N\rho(i)^{-1} = \|i\| \cdot N = N$ for $i \in I$).

These epsilon factors are in general hard to made explicit. Deligne computed ϵ_0 for the case where ρ is tamely ramified.

Theorem 2.1 ([3, Section 5.16]). *Let $\rho : W = W(\overline{K}/K) \to \mathrm{GL}(V)$ be a tamely ramified representation, i.e., $\rho(P) \equiv 1$. Then*

(1) ρ is a direct sum of induced representations of tamely ramified characters $\chi_i : K^\times_i \to \mathbb{C}^\times$ of unramified extensions K_i/K, i.e.,

$$\rho = \sum_{i=1}^r \mathrm{Ind}^W_{W(\overline{K}/K_i)} \chi_i,$$

where $i = 1, \cdots, r$ for some integer r, and χ_i is treated as a character of $W(\overline{K}/K_i)$ via the natural map $W(\overline{K}/K_i) \to W(\overline{K}/K_i)^{ab} \cong K^\times_i$ from local class field theory, normalized so that F is sent to ϖ.

(2) Let ψ be an additive character of K of conductor $\mathfrak{p}$, that is, ψ is trivial on $\mathfrak{p}$ but not on $\mathfrak{o}$. Then ψ can be treated as a character ψ_k of k via the isomorphism $\mathfrak{o}/\mathfrak{p} \to k$. Let dx be the Haar measure on K, normalized such that $\int_\mathfrak{p} dx = q^{-\frac{1}{2}}$. Then

$$\epsilon_0(\rho, \psi, dx) = (-1)^{\dim V} q^{-\frac{\dim V}{2}} \prod_{i=1}^{r} \tau(\widetilde{\chi}_i, \psi_{k_i}),$$

where k_i is the residue field of K_i, $\psi_{k_i} = \psi_k \circ \mathrm{Tr}_{k_i/k}$ is an additive character of k_i, $\widetilde{\chi}_i$ is the multiplicative character of $k_i^{\times}$ defined by χ_i, and $\tau(\widetilde{\chi}_i, \psi_{k_i})$ is the Gauss sum of $\widetilde{\chi}_i$ with respect to the additive character ψ_{k_i} defined by

$$\tau(\widetilde{\chi}_i, \psi_{k_i}) = -\sum_{x \in k_i^{\times}} \widetilde{\chi}_i\left(x^{-1}\right) \psi_{k_i}(x).$$

These epsilon factors are equal for two representations in the same equivalence class. They might not be equal for two representations that are only in the same I-equivalence class. However, an immediate corollary of the above theorem is that $\epsilon_0(\phi, \psi, dx)$ is well-defined on $\Phi_I^t(\mathrm{GL}_n)$, since the Frobenius element F is not involved in the explicit formula of $\epsilon_0(\rho, \psi, dx)$.

Corollary 2.2. *Let $\phi = (\rho, N)$ and $\phi' = (\rho', N')$ be two tamely ramified Weil-Deligne representations of W such that $\rho_I \cong \rho'_I$. Then for any Haar measure dx on K,*

$$\epsilon_0(\phi, \psi, dx) = \epsilon_0(\phi', \psi, dx).$$

In particular, the above equality holds if ϕ and ϕ' are I-equivalent.

Tamely ramified Weil-Deligne representations are parameterized in [10, Section 3]. We now recall this parameterization. A *partition* of a non-negative integer n is a tuple $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ and $|\lambda| := \sum_{i=1}^{r} \lambda_i = n$. We define $\mathfrak{n}(\lambda) := r$ to be the number of parts of λ. For convenience, we sometimes write $\lambda = (\lambda_i)$ and ignore the ordering of λ_i's. For example, if $\lambda = (\lambda_i)$ and $\mu = (\mu_j)$ are partitions of n and m respectively, then $\lambda + \mu := (\lambda_i; \mu_j)$, the concatenation of these two partitions, is a partition of $n + m$ and $\lambda \cdot \mu := (\lambda_i \mu_j)$ is a partition of nm. We denote the *empty partition* of 0 by (). We set $\mathcal{P}_n$ to be the set of partitions of n and $\mathcal{P} = \bigcup_{n \geq 0} \mathcal{P}_n$.

Let k_n be the (unique) field extension of k of degree n in $\overline{k}$. Let Γ_n be the character group of $k_n^{\times}$. For $m|n$, the norm map $N_{n,m} : k_n^{\times} \to k_m^{\times}$ induces an embedding $N_{n,m} : \Gamma_m \to \Gamma_n$. $\{\Gamma_n\}$ forms a directed system under these norm maps, and we define

$$\Gamma = \varinjlim \Gamma_n.$$

The Frobenius element F acts on Γ by $F\gamma = \gamma^q$ for $\gamma \in \Gamma$. We identify Γ_n with the subgroup $\{\gamma \in \Gamma : F^n \gamma = \gamma\}$ of Γ. If f is an F-orbit in Γ, we define the *degree* $d(f)$ of f to be the cardinality of f. Thus if $\gamma \in f$, then $\gamma \in \Gamma_{d(f)}$.

Let $P_n(\Gamma)$ be the set of partition-valued functions $\lambda : \Gamma \to \mathcal{P}$ such that

(i) $\lambda \circ F = \lambda$, i.e., λ is constant on the F-orbits;

(ii) $\sum_{\gamma \in \Gamma} |\lambda(\gamma)| = n$ (Since $|\lambda(\gamma)| \geq 0$, it is actually a finite sum).

If $\lambda \in P_n(\Gamma)$, then since $\lambda \circ F = \lambda$, it makes sense to define $\lambda(f) := \lambda(\gamma)$ for any F-orbit f and any $\gamma \in f$. Let $\gamma \in \Gamma_n$ be a multiplicative character of $k_n^\times$ and let ψ_n be the additive character of k_n defined by ψ. In Theorem 2.1, we have defined the Gauss sum of γ with respect to ψ_n:

$$\tau(\gamma, \psi_n) = -\sum_{x \in k_n^\times} \gamma\left(x^{-1}\right) \psi_n(x).$$

We observe that $\tau(F(\gamma), \psi_n) = \tau(\gamma, \psi_n)$: since $\psi_n(x^q) = \psi \circ \mathrm{Tr}_{k_n/k}(x^q) = \psi \circ \mathrm{Tr}_{k_n/k}(x) = \psi_n(x)$ for $x \in k_n$, and $x \mapsto x^q$ is an automorphism of $k_n^\times$, we have

$$\tau(F(\gamma), \psi_n) = \tau(\gamma^q, \psi_n) = -\sum_{x \in k_n^\times} \gamma\left(x^{-1}\right)^q \psi_n(x)$$

$$= -\sum_{x \in k_n^\times} \gamma\left(x^q\right)^{-1} \psi_n(x) = -\sum_{x \in k_n^\times} \gamma\left(x^q\right)^{-1} \psi_n(x^q)$$

$$= -\sum_{x \in k_n^\times} \gamma\left(x\right)^{-1} \psi_n(x) = \tau(\gamma, \psi_n).$$

Therefore, we can define the Gauss sum of an F-orbit. Let f be an F-orbit of $\gamma \in \Gamma_n$, we define

$$\tau(f, \psi_n) := \tau(\gamma, \psi_n).$$

Theorem 2.3 *([10]). There is a natural bijection from $P_n(\Gamma)$ onto $\Phi_I^t(\mathrm{GL}_n)$.*

Given $\lambda \in P_n(\Gamma)$, we use $\phi_\lambda = (\rho_\lambda, N_\lambda)$ to denote a representative in the corresponding I-equivalence class of tamely ramified Weil-Deligne representations. The underlying vector space of ρ_λ is denoted by V_λ. If there is no ambiguity, we will use ϕ_λ to denote its I-equivalence class. If λ is supported on a (single) degree n F-orbit f, i.e., $\lambda(\gamma) = (1)$ for $\gamma \in f$ and $\lambda(\gamma) = ()$ otherwise, then we will sometimes use $\phi_f = (\rho_f, N_f)$ to mean $\phi_\lambda = (\rho_\lambda, N_\lambda)$. We will also use V_f for V_λ in this case.

With Theorem 2.3 and the notations introduced above, we can restate Theorem 2.1 as follows. See [10, Section 3] for more details.

Theorem 2.4. *Let $\lambda \in P_n(\Gamma)$ and $\phi_\lambda = (\rho_\lambda, N_\lambda)$ be as above.*

(1) Let $(\rho_\lambda)_I$ be the restriction of ρ_λ to I. Then

$$(\rho_\lambda)_I = \sum_{\gamma \in \Gamma} |\lambda(\gamma)| \gamma = \sum_{f \in F \backslash \Gamma} |\lambda(f)| \sum_{\gamma \in f} \gamma,$$

where $F \backslash \Gamma$ is the set of F-orbits in Γ, and the sums above are direct sums.

(2) *Let ψ, ψ_k and dx be as in Theorem 2.1. For an F-orbit f in Γ, let $\psi_{d(f)} = \psi_k \circ \mathrm{Tr}_{k_{d(f)}/k}$ be the additive character of $k_{d(f)}$ defined by ψ_k. Then*

$$\epsilon_0(\phi_\lambda, \psi, dx) = (-1)^n q^{-\frac{n}{2}} \prod_f \tau(f, \psi_{d(f)})^{|\lambda(f)|}.$$

Let $\gamma \in \Gamma_n$ and let f be the F-orbit of γ, i.e., f is the set $\{\gamma, F\gamma, \cdots, F^{n-1}\gamma\}$ with duplicated elements removed. The size of f, or equivalently the degree of f, is not necessary n. Let $m = d(f)$, then $m|n$. In this case, γ is actually an element in the subgroup $\Gamma_m \subset \Gamma_n$, where the embedding of Γ_m into Γ_n is given by the norm map $N_{n,m}: k_n \to k_m$. We introduce the notion of the *F-set generated by γ with respect to Γ_n* to be the (multi-)set $\{\gamma, F\gamma, \cdots, F^{n-1}\gamma\}$, possibly with duplicated elements. Let h be the F-set generated by γ with respect to Γ_n, then h consists of $\frac{n}{m}$ copies of f. Since the elements of h are in the same F-orbit, we can define the Gauss sum for h to be

$$\tau(h, \psi_n) := \tau(\alpha, \psi_n),$$

for any $\alpha \in h$.

Theorem 2.5 *(Hasse-Davenport, [3, 5.12]). Let $m|n$ and let $\gamma \in \Gamma_m$ be a character of $k_m^\times$, such that its F-orbit f is of degree m. Let h be the F-set generated by γ with respect to Γ_n. Then $\tau(h, \psi_n) = \tau(f, \psi_m)^{\frac{n}{m}}$.*

Let

$$G = \mathrm{Gal}(\overline{K}/K) \supset G_0 = I \supset G_1 = P \supset G_2 \supset \cdots \supset G_N = 1$$

be the filtration of the ramification subgroups [16]. For a Weil-Deligne representation $\phi = (\rho, N)$, where $\rho : W \to \mathrm{GL}(V)$ for some complex vector space V, we define its *Artin conductor* $a(\phi)$ by

$$a(\phi) = \sum_{j=0}^{N} \dim\left(V/V^{G_j}\right) \frac{|\rho(G_j)|}{|\rho(G_0)|},$$

where V^{G_j} is the subspace of $\rho(G_j)$-fixed vectors in V. We note that by definition, $\ker \rho$ contains an open subgroup of $G_0 = I$, and since I is compact, the image $\rho(G_0)$ of G_0 is finite. Hence so are the other $\rho(G_j)$ for $j = 1, \ldots, N$, as $G_j \subset G_0$. Isolating the $j = 0$ term in $a(\phi)$, we can write

$$a(\phi) = \dim\left(V/V^I\right) + b(\phi),$$

where $b(\phi)$ is called the *Swan conductor* of ϕ. Deligne defines $\epsilon(s, \phi, \psi, dx)$, which is an indispensable ingredient in the local Langlands correspondence, by the formula

$$\epsilon(s, \phi, \psi, dx) = \epsilon(\phi, \psi, dx) q^{(\dim V - a(\phi))s}.$$

See [3, 5.5 and 8.18] for details, and note that $n(\psi) = -1$ for our choice of ψ.

Let $\lambda \in P_n(\Gamma)$ and $\phi_\lambda = (\rho_\lambda, N_\lambda)$ be the tamely ramified Weil-Deligne representation under the bijection in Theorem 2.3. Since $\rho_\lambda(P)$ acts trivially on V_λ, we have $\dim\left(V_\lambda/V_\lambda^P\right) = 0$, so $b(\phi_\lambda) = 0$. Thus in this case, $a(\phi) = \dim\left(V_\lambda/V_\lambda^I\right) = \dim V_\lambda - \dim V_\lambda^I$. Therefore,

$$\epsilon(s, \phi_\lambda, \psi, dx) = \epsilon(\phi_\lambda, \psi, dx) q^{\left(\dim V_\lambda^I\right)s} = \epsilon(\phi_\lambda, \psi, dx) q^{|\lambda(1)|s}, \qquad (4)$$

where $\lambda(1)$ is the partition of λ evaluated at the trivial character 1 of $k^\times$.

Finally, we want to discuss when $\epsilon(\phi_\lambda, \psi, dx) = \epsilon_0(\phi_\lambda, \psi, dx)$. From Equation (2) and Equation (3), we have

$$\epsilon(\phi_\lambda, \psi, dx) = \epsilon_0(\phi_\lambda, \psi, dx) \det\left(-F, V_{N_\lambda}^I\right)^{-1}.$$

Thus, we want to know when $\det\left(-F, (V_\lambda)_{N_\lambda}^I\right)$ equals 1. An obvious sufficient condition is $V_\lambda^I = 0$.

Proposition 2.6. *Let $\lambda \in P_n(\Gamma)$ and let $\phi_\lambda = (\rho_\lambda, N_\lambda)$ be the corresponding I-equivalence class of a tamely ramified Weil-Deligne representation. Then $V_\lambda^I = 0$ if and only if $\lambda(1) = ()$.*

Proof. Since we only concentrate on the I-fixed subspace, it makes sense to work with I-equivalence classes directly. By Theorem 2.4,

$$(\rho_\lambda)_I = \sum_{\gamma \in \Gamma} |\lambda(\gamma)| \gamma.$$

Therefore, V_λ has a non-trivial I-fixed vector if and only if $\lambda(1) \neq ()$. $\quad\square$

Corollary 2.7. *Let $\lambda \in P_n(\Gamma)$ be such that $\lambda(1) = ()$, i.e., λ is not supported on 1. Then*

$$\epsilon(s, \phi_\lambda, \psi, dx) = \epsilon(\phi_\lambda, \psi, dx) = \epsilon_0(\phi_\lambda, \psi, dx).$$

Proof. The first equality comes from Equation (4) and $\lambda(1) = ()$. Since $V_\lambda^I = 0$ by Proposition 2.6, $(V_\lambda)_{N_\lambda}^I = 0$, so $\det\left(-F, (V_\lambda)_{N_\lambda}^I\right) = 1$. Then the second equality follows. $\quad\square$

3. Macdonald's correspondence and epsilon factors

From now on, we fix an additive character ψ on K of conductor $\mathfrak{p}$, and let dx be the Haar measure on K such that $\int_{\mathfrak{p}} dx = q^{-\frac{1}{2}}$. dx is self dual with respect to ψ. We also use ψ to denote the additive character on k defined by ψ via the isomorphism $k \cong \mathfrak{o}/\mathfrak{p}$.

Let k_n be the (unique) field extension of k of degree n. Let ψ_n be the additive character on k_n defined by $\psi_n = \psi \circ \mathrm{Tr}_{k_n/k}$.

Let $\Pi(\mathrm{GL}_n(k))$ be the set of equivalence classes of irreducible representations of $\mathrm{GL}_n(k)$. Green [5] established a natural bijection between $\Pi(\mathrm{GL}_n(k))$ and $P_n(\Gamma)$, see also [10, Section 1]. If $\lambda \in P_n(\Gamma)$, we denote by $\pi_\lambda \in \Pi(\mathrm{GL}_n(k))$ the corresponding (equivalence class of the) irreducible representation. If λ is supported only on a (single) orbit f of degree n, we will also write π_f for π_λ.

In light of Green's correspondence and Theorem 2.3, Macdonald obtained the following

Theorem 3.1 (*Macdonald's correspondence, [10]*). *There is a canonical bijection*

$$\mathcal{M} : \Pi(\mathrm{GL}_n(k)) \to \Phi_I^t(\mathrm{GL}_n)$$

such that $\mathcal{M}(\pi_\lambda) = \phi_\lambda$ for $\lambda \in P_n(\Gamma)$ and

$$\epsilon(\pi_\lambda, \psi) = \epsilon_0(\phi_\lambda, \psi, dx), \tag{5}$$

where $\epsilon(\pi, \psi)$ is the Godement-Jacquet epsilon factor defined in [10, Section 2].

Macdonald's correspondence $\mathcal{M}$ and Equation (5) inspire the following definition of epsilon factors:

Definition 3.2. Let $\pi \in \Pi(\mathrm{GL}_n(k))$, and let r be an operation on Weil-Deligne representations of W that preserves tame ramification, for example r can be direct sum, tensor product, exterior power or symmetric power. We define a non-zero constant $\epsilon_0(\pi, r, \psi)$ associated to the pair (π, r) by

$$\epsilon_0(\pi, r, \psi) := \epsilon_0(r(\mathcal{M}(\pi)), \psi, dx),$$

Remark 3.3.

(1) There is exactly one Haar measure dx on K such that $\int_{\mathfrak{p}} dx = q^{-\frac{1}{2}}$, but there are $q-1$ different additive characters of conductor $\mathfrak{p}$ on K. Therefore, we keep ψ and omit dx in the notation $\epsilon_0(\pi, r, \psi)$, though we have fixed ψ and dx at the beginning of the section. One should keep in mind that the definition of $\epsilon_0(\pi, r, \psi)$ depends on the choices of both ψ and dx.

(2) If $r = id$ is the identity operation, then Equation (5) is the same as

$$\epsilon(\pi, \psi) = \epsilon_0(\pi, id, \psi),$$

where $\epsilon(\pi, \psi)$ is the Godement-Jacquet epsilon factor of π.

(3) If r is either the direct sum or the tensor product, then it is a binary operator, so the corresponding epsilon factors should receive as input two representations. For example, when $r = \oplus$ is the direct sum, it should be understood that the epsilon factors are defined for two representations $\pi_1 \in \Pi(\mathrm{GL}_n(k))$ and $\pi_2 \in \Pi(\mathrm{GL}_m(k))$ and we write

$$\epsilon_0(\pi_1 \boxplus \pi_2, \psi) := \epsilon_0(\pi_1, \pi_2, \oplus, \psi) = \epsilon_0(\mathcal{M}(\pi_1) \oplus \mathcal{M}(\pi_2), \psi).$$

And for $r = \otimes$ the tensor product, we write

$$\epsilon_0(\pi_1 \times \pi_2, \psi) := \epsilon_0(\pi_1, \pi_2, \otimes, \psi) = \epsilon_0(\mathcal{M}(\pi_1) \otimes \mathcal{M}(\pi_2), \psi).$$

(4) If r is one of these four operations listed in Definition 3.2, then it preserves tame ramification. That is to say, if ϕ is tamely ramified Weil-Deligne representation of W (or a pair of tamely ramified Weil-Deligne representations when r is the direct sum or the tensor product), then so is $r(\phi)$.

It follows immediately from the definition of $\epsilon_0(\pi, r, \psi)$ that epsilon factors of irreducible representations of general linear groups over k enjoy the same good properties as those of epsilon factors of tamely ramified Weil-Deligne representations over K. For example, they satisfy multiplicativity [3, (5.2)], and they can be written as products of Gauss sums by Theorem 2.1 or Theorem 2.4. To illustrate these properties, we will look at the case where r is the identity operator for the rest of this section. For simplicity of notations, we define

$$\epsilon_0(\pi, \psi) := \epsilon_0(\pi, id, \psi) = \epsilon_0(\mathcal{M}(\pi), \psi).$$

Let $\lambda \in P_n(\Gamma)$ and $\mu \in P_m(\Gamma)$. Then by multiplicativity of ϵ_0-factors of Weil-Deligne representations,

$$\epsilon_0(\pi_\lambda \boxplus \pi_\mu, \psi) = \epsilon_0(\phi_\lambda \oplus \phi_\mu, \psi) = \epsilon_0(\phi_\lambda, \psi)\epsilon_0(\phi_\mu, \psi) = \epsilon_0(\pi_\lambda, \psi)\epsilon_0(\pi_\mu, \psi).$$

This is the multiplicativity property for ϵ_0-factors of irreducible representations of general linear groups over k. On the other hand, from Theorem 2.4 and Definition 3.2, we can express $\epsilon_0(\pi_\lambda, \psi)$ as a product of Gauss sums:

$$\epsilon_0(\pi_\lambda, \psi) = \epsilon_0(\phi_\lambda, \psi) = (-1)^n q^{-\frac{n}{2}} \prod_f \tau(f, \psi_{d(f)})^{|\lambda(f)|}.$$

There are two ways to express $\epsilon_0(\pi_\lambda \boxplus \pi_\mu, \psi)$ as a product of Gauss sums. The first way is to use the multiplicativity of ϵ_0-factors:

$$\epsilon_0(\pi_\lambda \boxplus \pi_\mu, \psi) = \epsilon_0(\pi_\lambda, \psi)\epsilon_0(\pi_\mu, \psi)$$

$$= (-1)^{n+m} q^{-\frac{n+m}{2}} \prod_f \tau(f, \psi_{d(f)})^{|\lambda(f)|+|\mu(f)|}.$$

On the other hand, by the explicit correspondence between $P_n(\Gamma)$ and $\Phi_I^t(\mathrm{GL}_n)$ as in [10, Section 3], we see that $\phi_\lambda \oplus \phi_\mu$ is a tamely ramified representation of degree $n + m$ corresponding to $\nu = \lambda + \mu \in P_{n+m}(\Gamma)$. Here, we have for $\gamma \in \Gamma$,

$$\nu(\gamma) = \lambda(\gamma) + \mu(\gamma),$$

where addition above is the concatenation of two partitions. Thus,

$$\epsilon_0(\pi_\lambda \boxplus \pi_\mu, \psi) = \epsilon_0(\phi_\nu, \psi) = (-1)^{n+m} q^{-\frac{n+m}{2}} \prod_f \tau(f, \psi_{d(f)})^{|\nu(f)|}.$$

These two ways agree because $|\nu(f)| = |\lambda(f)| + |\mu(f)|$. Moreover, since $\phi_\nu = \phi_\lambda \oplus \phi_\mu$, we can define the addition $\pi_\lambda \boxplus \pi_\mu$ to be π_ν.

4. Tensor product epsilon factors

This section is devoted to the tensor product epsilon factors $\epsilon_0(\pi_\lambda \times \pi_\mu, \psi)$. First of all, we show that they are equal to products of Gauss sums. Secondly, we relate them to the Rankin-Selberg gamma factors defined in [15,12,19]. Their relation to the Rankin-Selberg gamma factors can be used to solve a conjecture made by Nien and Zhang [14, Conjecture 2.2].

4.1. Gauss sums

In terms of Green's classification, irreducible *cuspidal* representations of $\mathrm{GL}_n(k)$ are in a one-to-one correspondence with F-orbits in Γ of degree n. In other words, if π is an irreducible cuspidal representation of $\mathrm{GL}_n(k)$, then it corresponds to $\lambda \in P_n(\Gamma)$ supported on a (single) degree n F-orbit f, i.e., $\lambda(\gamma) = (1)$ if $\gamma \in f$ and $\lambda(\gamma) = ()$ otherwise. In this case, we will use π_f instead of π_λ, and we will define ϕ_f and ρ_f by $\mathcal{M}(\pi_f) = \phi_f = (\rho_f, 0)$. Note that the nilpotent map corresponding to π_f is 0, since π_f is cuspidal, and the corresponding ϕ_f is irreducible.

Theorem 4.1. *Let π_λ and π_μ be two irreducible representations parameterized by $\lambda \in P_n(\Gamma)$ and $\mu \in P_m(\Gamma)$ respectively. Then*

$$\epsilon_0(\pi_\lambda \times \pi_\mu, \psi) = \prod_{f,g} \epsilon_0(\pi_f \times \pi_g, \psi)^{|\lambda(f)||\mu(g)|},$$

where f and g run over all the F-orbits in Γ.

Proof. Let $\phi_\lambda = (\rho_\lambda, N_\lambda) = \mathcal{M}(\pi_\lambda)$ and $\phi_\mu = (\rho_\mu, N_\mu) = \mathcal{M}(\pi_\mu)$. Then by Theorem 2.4,

$$(\rho_\lambda)_I = \sum_{\gamma \in \Gamma} |\lambda(\gamma)| \gamma = \sum_{f \in F \backslash \Gamma} |\lambda(f)| \sum_{\gamma \in f} \gamma,$$

and

$$(\rho_\mu)_I = \sum_{\gamma \in \Gamma} |\mu(\gamma)| \gamma = \sum_{g \in F \backslash \Gamma} |\mu(g)| \sum_{\gamma \in g} \gamma.$$

Let $\phi = (\rho, N) = \phi_\lambda \otimes \phi_\mu$. Then $\rho = \rho_\lambda \otimes \rho_\mu$, and

$$\rho_I = (\rho_\lambda)_I \otimes (\rho_\mu)_I = \sum_{f,g} |\lambda(f)||\mu(g)| \sum_{\gamma_1 \in f, \gamma_2 \in g} \gamma_1 \gamma_2 = \sum_{f,g} |\lambda(f)||\mu(g)| (\rho_f \otimes \rho_g)_I.$$

Here $(\rho_f, 0)$ are $(\rho_g, 0)$ correspond to the cuspidal representations π_f and π_g respectively. The last equality above comes from

$$(\rho_f \otimes \rho_g)_I = (\rho_f)_I \otimes (\rho_g)_I = \sum_{\gamma_1 \in f} \gamma_1 \cdot \sum_{\gamma_2 \in g} \gamma_2 = \sum_{\gamma_1 \in f, \gamma_2 \in g} \gamma_1 \gamma_2.$$

Therefore, we have by Corollary 2.2 and multiplicativity of ϵ_0 [3, (5.2)] (see also the example at the end of Section 3),

$$\epsilon_0(\phi, \psi) = \prod_{f,g} \epsilon_0(\rho_f \otimes \rho_g, \psi)^{|\lambda(f)||\mu(g)|}.$$

Now by Definition 3.2,

$$\epsilon_0(\pi_\lambda \times \pi_\mu, \psi) = \epsilon_0(\phi, \psi) = \prod_{f,g} \epsilon_0(\rho_f \otimes \rho_g, \psi)^{|\lambda(f)||\mu(g)|}$$

$$= \prod_{f,g} \epsilon_0(\pi_f \times \pi_g, \psi)^{|\lambda(f)||\mu(g)|}. \quad \Box$$

With Theorem 4.1, we can reduce the study of $\epsilon_0(\pi_\lambda \times \pi_\mu, \psi)$ to that of $\epsilon_0(\pi_f \times \pi_g, \psi)$ for F-orbits f and g in Γ. Let $n \geq m$ be two positive integers. Let f be an F-orbit of degree n. Fix an $\alpha \in f$, then $\alpha \in \Gamma_n$ and $f = \{\alpha^{q^i} : i = 0, 1, \ldots, n-1\}$. Similarly, let $g = \{\beta^{q^j} : j = 0, 1, \ldots, m-1\}$ be an F-orbit of degree m for some $\beta \in \Gamma_m$. Let π_f and π_g be the corresponding cuspidal representations of $\mathrm{GL}_n(k)$ and $\mathrm{GL}_m(k)$ respectively. With these notations, we have the following

Theorem 4.2. *Let $d = \gcd(n, m)$ be the greatest common divisor of n and m, and let $l = \frac{nm}{d}$ be their least common multiple. Then*

$$\epsilon_0(\pi_f \times \pi_g, \psi) = (-1)^{nm} q^{-\frac{nm}{2}} \prod_{i=0}^{d-1} \tau(\alpha \beta^{q^i}, \psi_l),$$

where $\alpha\beta^{q^i} \in \Gamma_l$ is defined by

$$(\alpha\beta^{q^i})(x) = \alpha\left(N_{l,n}(x)\right)\beta\left(\left(N_{l,m}(x)\right)^{q^i}\right),$$

for $x \in k_l^\times$.

Proof. Let $\phi_f = (\rho_f, N_f) = \mathcal{M}(\pi_f)$ and $\phi_g = (\rho_g, N_g) = \mathcal{M}(\pi_g)$. By Theorem 2.4, we have

$$(\rho_f)_I = \sum_{i=0}^{n-1} \alpha^{q^i} \text{ and } (\rho_g)_I = \sum_{j=0}^{m-1} \beta^{q^j},$$

where $(\rho_f)_I, (\rho_g)_I$ are the restrictions to the inertia subgroup I of ρ_f, ρ_g respectively. If we set $\phi = (\rho, N) = \phi_f \otimes \phi_g \in \Phi_I^t(\mathrm{GL}_{nm})$, then

$$\rho_I = (\rho_f \otimes \rho_g)_I = (\rho_f)_I \otimes (\rho_g)_I = \left(\sum_{i=0}^{n-1} \alpha^{q^i}\right) \otimes \left(\sum_{j=0}^{m-1} \beta^{q^j}\right) = \sum_{i,j} \alpha^{q^i}\beta^{q^j}.$$

We claim that these $\alpha^{q^i}\beta^{q^j}$'s are elements of Γ_l, and that they are partitioned into the d F-sets (might not be distinct) $h_0, h_1, \ldots, h_{d-1}$, where h_i is the F-set generated by $\alpha\beta^{q^i}$ with respect to Γ_l. On one hand, each h_i is of size l, so there are $ld = nm$ characters in $h_0, h_1, \ldots, h_{d-1}$ in total. On the other hand, $\alpha^{q^i}\beta^{q^j} \in h_\delta$ if and only if there exists an integer t such that $\alpha^{q^i}\beta^{q^j} = \alpha^{q^t}\beta^{q^{\delta+t}}$. Since $\alpha^{q^n} = \alpha$ and $\beta^{q^m} = \beta$, $\alpha^{q^i}\beta^{q^j} \in h_\delta$ if and only if the following system of congruence equations has a solution:

$$\begin{cases} t \equiv i \mod n \\ t \equiv j - \delta \mod m \end{cases}.$$

This system has a solution if and only if $i \equiv j - \delta \mod d$, or equivalently $\delta \equiv j - i \mod d$. Therefore, for each $\alpha^{q^i}\beta^{q^j}$, there exists exactly one $\delta \in \{0, 1, \ldots, d-1\}$ such that $\alpha^{q^i}\beta^{q^j} \in h_\delta$, i.e., the only one satisfying $\delta \equiv j - i \mod d$. Since the size of $\{\alpha^{q^i}\beta^{q^j} : 0 \le i \le n-1, 0 \le j \le m-1\}$ is nm and each of these appears in h_δ for some $\delta \in \{0, 1, \ldots, d-1\}$, our claim follows from a counting argument.

For each $i = 0, 1, \ldots, d-1$, h_i consists of t_i copies of some F-orbit f_i of degree $\frac{l}{t_i}$. Thus, by Theorem 2.4 and Theorem 2.5, we have

$$\epsilon_0(\phi, \psi) = (-1)^{nm} q^{-\frac{nm}{2}} \prod_{i=0}^{d-1} \tau(f_i, \psi_{\frac{l}{t_i}})^{t_i} = (-1)^{nm} q^{-\frac{nm}{2}} \prod_{i=0}^{d-1} \tau(h_i, \psi_l).$$

Thus, by the definition of $\tau(h_i, \psi_l)$, we have

$$\epsilon_0(\pi_f \times \pi_g, \psi) = \epsilon_0(\phi, \psi) = (-1)^{nm} q^{-\frac{nm}{2}} \prod_{i=0}^{d-1} \tau(\alpha\beta^{q^i}, \psi_l). \quad \square$$

If $\gcd(n,m) = 1$, then $\epsilon_0(\pi_f \times \pi_g, \psi)$ is just a multiple of a Gauss sum $\tau(\alpha\beta, \psi_{nm})$, where $\alpha\beta \in \Gamma_{nm}$ is defined by $(\alpha\beta)(x) = \alpha(N_{nm,n}(x))\beta(N_{nm,m}(x))$ for $x \in k_{nm}^\times$. In particular if $m = 1$, we have

$$\epsilon_0(\pi_f \times \beta, \psi) = (-1)^n q^{-\frac{n}{2}} \tau(\alpha\beta, \psi_n).$$

4.2. Rankin-Selberg gamma factors

Let $n > m$ be two positive integers. Let f and g be two F-orbits in Γ such that $d(f) = n$ and $d(g) = m$. Then π_f and π_g are two irreducible cuspidal representations of $\mathrm{GL}_n(k)$ and $\mathrm{GL}_m(k)$ respectively. Let $\phi_f = (\rho_f, 0)$ and $\phi_g = (\rho_g, 0)$ be the tamely ramified Weil-Deligne representations of W corresponding to π_f and π_g respectively. Let Π_f be the level zero supercuspidal representation of $\mathrm{GL}_n(K)$ constructed from the pair $(\pi_f, 1)$, see [19, Section 2.1]. That is to say,

$$\Pi_f = \mathrm{ind}_{K^\times \cdot \mathrm{GL}_n(\mathfrak{o})}^{\mathrm{GL}_n(K)} (\chi_f \cdot \pi_f),$$

where π_f is a representation of $\mathrm{GL}_n(\mathfrak{o})$ via the natural map $\mathrm{GL}_n(\mathfrak{o}) \to \mathrm{GL}_n(k)$, $\chi_f : K^\times \to \mathbb{C}^\times$ is the character defined on $\mathfrak{o}^\times$ via the central character of π_f, and on ϖ by $\chi_f(\varpi) = 1$, $(\chi_f \cdot \pi_f)(z \cdot k) = \chi_f(z)\pi_f(k_0)$ for $z \in K^\times$ and $k_0 \in \mathrm{GL}_n(\mathfrak{o})$, and ind is compact induction. Similarly, let Π_g be the level zero supercuspidal representation of $\mathrm{GL}_m(K)$ constructed from the pair $(\pi_g, 1)$. The choices of Π_f and Π_g here are not essential, we can choose any level zero supercuspidal representations Π_f and Π_g constructed from π_f and π_g respectively.

Let $\gamma(s, \Pi_f \times \Pi_g, \psi)$ and $\epsilon(s, \Pi_f \times \Pi_g, \psi)$ be the Rankin-Selberg factors defined in [8], and let $\gamma(\pi_f \times \pi_g, \psi)$ be the Rankin-Selberg gamma factor defined in [15,12] (see also [19] for a different normalization). $\gamma(\pi_f \times \pi_g, \psi)$ is related to $\gamma(s, \Pi_f \times \Pi_g, \psi)$ by

Theorem 4.3 ([14, Theorem 3.11], [19, Theorem 3.1]). *Let* ω_g *be the central character of* π_g. *Then*

$$\omega_g(-1)^{n-1} \gamma(s, \Pi_f \times \Pi_g, \psi) = q^{\frac{m(n-m-1)}{2}} \gamma(\pi_f \times \pi_g, \psi).$$

Next, we will show that $\gamma(\pi_f \times \pi_g, \psi)$ is also equal to $\epsilon_0(\pi_f \times \pi_g, \psi)$ up to some constant.

Theorem 4.4.

$$\gamma(\pi_f \times \pi_g, \psi) = q^{-\frac{m(n-m-1)}{2}} \omega_g(-1)^{n-1} \epsilon_0(\pi_f \times \pi_g, \psi).$$

Proof. Since $n > m$ and Π_f and Π_g are supercuspidal, $L(s, \Pi_f \times \Pi_g) = 1$. Similarly, $L\left(1 - s, \Pi_f^\vee \times \Pi_g^\vee\right) = 1$, where $\Pi_f^\vee$ and $\Pi_g^\vee$ are the contragredient representations of Π_f, Π_g respectively. Thus, we have

$$\gamma(s, \Pi_f \times \Pi_g, \psi) = \epsilon(s, \Pi_f \times \Pi_g, \psi). \tag{6}$$

We have from [17, Proposition 1 in A.2] the following commutative diagram:

$$
\begin{array}{ccc}
\Pi_0(\mathrm{GL}_n(K)) & \xrightarrow{LLC_0} & \Phi^t(\mathrm{GL}_n) \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\
\Pi_0(\mathrm{GL}_n(k)) & \xrightarrow{\mathcal{M}} & \Phi^t_I(\mathrm{GL}_n)
\end{array}
,
$$

where $\Pi_0(\mathrm{GL}_n(K))$ is the set of equivalence classes of level zero supercuspidal representations of $\mathrm{GL}_n(K)$, $\Pi_0(\mathrm{GL}_n(k))$ is the set of equivalence classes of cuspidal representations of $\mathrm{GL}_n(k)$, LLC_0 is the restriction of the local Langlands correspondence to level zero supercuspidal representations, $\mathcal{M}$ is Macdonald's correspondence, p_1 is the map sending a level zero supercuspidal representation to the cuspidal representation from which it was constructed, and p_2 is the canonical projection.

Therefore, there exist $\phi_f = (\rho_f, 0) \in \Phi^t(\mathrm{GL}_n)$ and $\phi_g = (\rho_g, 0) \in \Phi^t(\mathrm{GL}_m)$ in the I-equivalence classes of tamely ramified Weil-Deligne representations parameterized by f and g respectively that are images of Π_f and Π_g under the local Langlands correspondence. Since the local Langlands correspondence matches epsilon factors of pairs [6, Corollary VII.2.17], we get

$$\epsilon(s, \Pi_f \times \Pi_g, \psi) = \epsilon(s, \phi_f \otimes \phi_g, \psi, dx). \tag{7}$$

Note that it is important to normalize dx in Equation (7) to be self dual with respect to ψ, which is exactly what we do.

Since ρ_f and ρ_g are tamely ramified, $\rho_f \otimes \rho_g$ is also tamely ramified and

$$(\rho_f \otimes \rho_g)_I = \sum_{\gamma_1 \in f, \gamma_2 \in g} \gamma_1 \gamma_2.$$

Since ρ_f and ρ_g are of different degrees, $\gamma_1 \gamma_2$ is not fixed by the action of F, so $\gamma_1 \gamma_2$ in the summand above will never be the trivial character of k. In other words, if $\lambda' \in P_n(\Gamma)$ is the partition-valued function on Γ corresponding to $\phi_f \otimes \phi_g$ under Theorem 2.3, then $\lambda'(1) = ()$. Therefore, by Corollary 2.7, we have $\epsilon(s, \phi_f \otimes \phi_g, \psi, dx) = \epsilon_0(\phi_f \otimes \phi_g, \psi)$. Thus, we conclude the theorem from Theorem 4.2, Theorem 4.3, Equation (6) and Equation (7). $\quad\square$

As a corollary of Theorem 4.2 and Theorem 4.4, we get

Corollary 4.5. *Using the notations of Theorem 4.2 and Theorem 4.4, we have*

$$\gamma(\pi_f \times \pi_g, \psi) = (-1)^{nm} q^{-nm + \frac{m(m+1)}{2}} \beta(-1)^{n-1} \prod_{i=0}^{d-1} \tau(\alpha \beta^{q^i}, \psi_l).$$

Remark 4.6. After this paper was written, the preprint [18] of Yang appeared, in which a different proof of Corollary 4.5 is given.

Proof. It is a direct consequence of Theorem 4.2 and Theorem 4.4, with an observation that ω_g, the central character of π_g, is $\beta|_{k^\times}$, i.e., the restriction of β to $k^\times$. □

The appearance of $\beta(-1)^{n-1}$ in Corollary 4.5 is due to the different choices of normalization of the Rankin-Selberg gamma factors in [8] and [12]. The result when $m = 1$ has already been proven in [13, Theorem 1.1]. The corollary also answers [14, Conjecture 2.2]. Nien and Zhang conjecture that

$$\gamma(\pi_f \times \pi_g, \psi) = (-1)^{nm-m+1} q^{-nm+\frac{m(m+1)}{2}} \alpha(-1)^{m-1} \beta(-1)^{n-1} \tau(\alpha\beta, \psi_{nm}), \quad (8)$$

which in general is not correct. If n and m are coprime, then their conjecture is correct, possibly up to a sign. However, if they are not coprime, one can find examples in which Nien-Zhang's conjecture does not compute gamma factors correctly, as illustrated in the following example.

Example. Let $k = \mathbb{F}_3$, the field of 3 elements. Let $\psi : k \to \mathbb{C}^\times$ be defined by $\psi(x) = e^{\frac{2\pi i x}{3}}$. Let $n = 4$ and $m = 2$. $l = \mathrm{lcm}(n, m) = 4$ and $d = \gcd(n, m) = 2$. Let ξ be a root of $x^4 + 2x^3 + 2 = 0$ and $\zeta = \xi^{10}$ be a root of $x^2 + 2x + 2 = 0$. Then $k_4 = k(\xi)$ and $k_2 = k(\zeta)$, and ξ, ζ are generators of the cyclic groups $k_4^\times$ and $k_2^\times$ respectively. Let α be the multiplicative character of k_4 defined by $\alpha(\xi) = e^{\frac{33\pi i}{20}}$ and β be the multiplicative character of k_2 defined by $\beta(\zeta) = e^{\frac{\pi i}{4}}$. Let f be the F-orbit of α, then $f = \{\alpha, \alpha^3, \alpha^9, \alpha^{27}\}$ with evaluations at ξ being

$$\{e^{\frac{33\pi i}{20}}, e^{\frac{19\pi i}{20}}, e^{\frac{17\pi i}{20}}, e^{\frac{11\pi i}{20}}\}.$$

Similarly, if $g = \{\beta, \beta^3\}$ is the F-orbit of β with evaluations at ζ being $\{e^{\frac{\pi i}{4}}, e^{\frac{3\pi i}{4}}\}$. Then π_f is an irreducible cuspidal representation of $\mathrm{GL}_4(k)$ and π_g is an irreducible cuspidal representation of $\mathrm{GL}_2(k)$.

Gauss sums can be computed easily with the help of SageMath. We computed the right hand side of the equation in Corollary 4.5:

$$(-1)^{nm} q^{-nm+\frac{m(m+1)}{2}} \beta(-1)^{n-1} \prod_{i=0}^{d-1} \tau(\alpha\beta^{q^i}, \psi_l) = -\frac{2}{9} + \frac{\sqrt{5}}{9} i.$$

The right hand side of Equation (8) is

$$(-1)^{nm-m+1} q^{-nm+\frac{m(m+1)}{2}} \alpha(-1)^{m-1} \beta(-1)^{n-1} \tau(\alpha\beta, \psi_{nm}) = -\frac{1}{27} - \frac{4\sqrt{5}}{27} i.$$

To compute $\gamma(\pi_f \times \pi_g, \psi)$, we use the formula involving Bessel functions, see [13, Proposition 2.8]:

$$\gamma(\pi_f \times \pi_g, \psi) = \sum_{h \in U_m \backslash \mathrm{GL}_m(k)} B_{f,\psi} \begin{pmatrix} 0 & I_{n-m} \\ h & 0 \end{pmatrix} B_{g,\psi^{-1}}(h), \tag{9}$$

where $U_m \subset \mathrm{GL}_m(k)$ is the standard maximal unipotent subgroup, $B_{f,\psi}$ is the normalized Bessel function of π_f with respect to ψ and $B_{g,\psi^{-1}}$ is the normalized Bessel function of π_g with respect to ψ^{-1}. To be more explicit, the algorithm runs by the following steps. First, we compute the trace of π_f and π_g using [4, (6.1)]. Secondly, we compute the normalized Bessel functions using [4, Proposition 4.5]. Finally, we compute the gamma factor using Equation (9). We implement the algorithm using SageMath and get

$$\gamma(\pi_f \times \pi_g, \psi) = -\frac{2}{9} + \frac{\sqrt{5}}{9} i.$$

5. Exterior square epsilon factors

In this section, we perform a similar analysis for $\epsilon_0(\pi, \wedge^2, \psi)$ as in Section 4. Since some techniques have already been presented in the previous section, we will present only the core of the proofs of some of the theorems.

Let $\lambda \in P_n(\Gamma)$, and π_λ be the corresponding irreducible representation of $\mathrm{GL}_n(k)$. Suppose that the support of λ is $\{f_1, f_2, \ldots, f_t\}$, where the f_i's are F-orbits in Γ, i.e., $\lambda(\gamma)$ is a non-empty partition if and only if $\gamma \in f_i$ for some i. $\epsilon_0(\pi_\lambda, \wedge^2, \psi)$ satisfies the following multiplicativity theorem.

Theorem 5.1. *Denote by π_{f_i} the (equivalence class of the) irreducible cuspidal representation corresponding to the F-orbit f_i as before. Let $n_i = |\lambda(f_i)|$. Then*

$$\epsilon_0(\pi_\lambda, \wedge^2, \psi) = \prod_{1 \leq i < j \leq t} \epsilon_0(\pi_{f_i} \times \pi_{f_j}, \psi)^{n_i n_j} \cdot \prod_{i=1}^{t} \epsilon_0(\pi_{f_i} \times \pi_{f_i}, \psi)^{\binom{n_i}{2}} \cdot \prod_{i=1}^{t} \epsilon_0(\pi_{f_i}, \wedge^2, \psi)^{n_i}.$$

Proof. Similar to the proof of Theorem 4.1. By definitions,

$$\epsilon_0(\pi_\lambda, \wedge^2, \psi) = \epsilon_0(\wedge^2(\phi_\lambda), \psi) = \epsilon_0(\wedge^2(\rho_\lambda), \psi).$$

Since ρ_λ is tamely ramified, so is $\wedge^2(\rho)$. Thus, by Corollary 2.2,

$$\epsilon_0(\wedge^2(\rho_\lambda), \psi) = \epsilon_0(\wedge^2(\rho_\lambda)_I, \psi) = \epsilon_0\left(\wedge^2\left((\rho_\lambda)_I\right), \psi\right).$$

By Theorem 2.4, we have

$$(\rho_\lambda)_I = \sum_{f \in F \backslash \Gamma} |\lambda(f)| \sum_{\gamma \in f} \gamma,$$

and for any F-orbit f,

$$(\rho_f)_I = \sum_{\gamma \in f} \gamma.$$

Therefore

$$(\rho_\lambda)_I = \sum_f |\lambda(f)|(\rho_f)_I.$$

Now the theorem follows by repeatedly using

$$\epsilon_0(\wedge^2(\rho_1 + \rho_2), \psi) = \epsilon_0(\rho_1 \otimes \rho_2, \psi)\epsilon_0(\wedge^2(\rho_1), \psi)\epsilon_0(\wedge^2(\rho_2), \psi),$$

and Definition 3.2. □

By Theorem 4.2, we know how to compute the tensor product epsilon factors in terms of Gauss sums. Thus in order to compute $\epsilon_0(\pi_\lambda, \wedge^2, \psi)$, we just need to compute $\epsilon_0(\pi_f, \wedge^2, \psi)$ in light of Theorem 5.1.

Theorem 5.2. *Let $f = \{\alpha^{q^i} : i = 0, 1, \ldots, n-1\}$ be an F-orbit of degree n for some $\alpha \in \Gamma_n$. Let $m = \lfloor \frac{n}{2} \rfloor$ be the biggest integer smaller or equal to $\frac{n}{2}$. Then*

$$\epsilon_0(\pi_f, \wedge^2, \psi) = (-1)^{\binom{n}{2}} q^{-\frac{1}{2}\binom{n}{2}} \tau(\alpha^{1+q^m}, \psi_d) \prod_{i=1}^{m-1} \tau(\alpha^{1+q^i}, \psi_n),$$

where $d = n$ if n is odd and $d = m$ if n is even.

Proof. Similar to the proof of Theorem 4.2. The key is that

$$\left(\wedge^2(\rho_f)\right)_I = \sum_{1 \le i < j \le n} \alpha^{q^i + q^j} = \sum_{j=1}^m \sum_{\beta \in h_j} \beta,$$

where h_j is the F-set generated by α^{1+q^j} with respect to Γ_n if $1 \le j < \frac{n}{2}$, and if $n = 2m$ is even, then h_m is the F-set generated by α^{1+q^m} with respect to Γ_m (Note that when $n = 2m$ is even, $\alpha^{q^{2m}} = \alpha$ and $(\alpha^{1+q^m})^{q^m} = \alpha^{q^m + q^{2m}} = \alpha^{q^m + 1} = \alpha^{1+q^m}$ so $\alpha^{1+q^m} \in \Gamma_m$). It can be checked that these h_j's, for $1 \le j \le m$, form a partition of $\{\alpha^{q^i + q^j} : 1 \le i < j \le n\}$. □

Let $f = \{\alpha^{q^i} : i = 0, 1, \ldots, n-1\}$ be an F-orbit in Γ_n for some $\alpha \in \Gamma_n$. Assume that α is not trivial when restricted to $k_m^\times$ if $n = 2m$, or equivalently, α^{1+q^m} is not the trivial character. This assumption assures that the cuspidal representation π_f does not have non-trivial Shalika vectors [21, Section 2.3.3]. The authors define the exterior square gamma factor $\gamma(\pi_f, \wedge^2, \psi)$ in [21], and show that

$$\gamma(\pi_f, \wedge^2, \psi) = \gamma_{JS}(s, \Pi_f, \wedge^2, \psi), \tag{10}$$

where, as before, Π_f is a level zero supercuspidal representation constructed from π_f, and $\gamma_{JS}(s, \Pi_f, \wedge^2, \psi)$ is the Jacquet-Shalika gamma factor from [9,11,1]. It is natural to ask what is the relation between $\epsilon_0(\pi_f, \wedge^2, \psi)$ and $\gamma(\pi_f, \wedge^2, \psi)$, which might enable us to express $\gamma(\pi_f, \wedge^2, \psi)$ as a product of Gauss sums. Unfortunately, we don't have the matching between $\epsilon_{JS}(s, \Pi_f, \wedge^2, \psi)$ and $\epsilon(s, \wedge^2(\phi_f), \psi)$ under the local Langlands correspondence. Thus, the arguments in Theorem 4.4 can not go through. We want to mention that $\epsilon_{LS}(s, \Pi_f, \wedge^2, \psi)$, the exterior square epsilon factor coming from Langlands-Shahidi method, does match $\epsilon(s, \wedge^2(\phi_f), \psi)$. This is the work of Cogdell, Shahidi and Tsai [2]. But we don't know whether or not the two exterior square epsilon factors coming from the Jacquet-Shalika integral representation and the Langlands-Shahidi method are the same.

Nevertheless, since $\gamma(\pi_f, \wedge^2, \psi)$ and $\epsilon_0(\pi, \wedge^2, \psi)$ are constants, there must exist a constant c_f depending on f such that

$$\gamma(\pi_f, \wedge^2, \psi) = c_f \, \epsilon_0(\pi_f, \wedge^2, \psi)$$

$$= c_f \cdot (-1)^{\binom{n}{2}} q^{-\frac{1}{2}\binom{n}{2}} \tau(\alpha^{1+q^m}, \psi_d) \prod_{i=1}^{m-1} \tau(\alpha^{1+q^i}, \psi_n).$$

We know from [21] that $|\gamma(\pi_f, \wedge^2, \psi)| = 1$ and that the absolute value of a Gauss sum $\tau(\alpha, \psi_n)$ is $q^{\frac{n}{2}}$ if α is not trivial. Therefore, by taking absolute values on both sides of the equation above, we get $|c_f| = 1$. We conjecture that $c_f = 1$. For $n = 2$, $\gamma(\pi_f, \wedge^2, \psi)$ degenerates into a Godement-Jacquet gamma factor of the central character ω_f of π_f, thus $c_f = 1$ is a consequence of Corollary 4.5. For $n = 3$, $\gamma(\pi_f, \wedge^2, \psi)$ has a nice expression in terms of the Bessel function of π_f [21], which can be manipulated into the 3×1 Rankin-Selberg gamma factor $\overline{\gamma(\pi_f \times \omega_f^{-1}, \psi)}$. Thus $c_f = 1$ is again a consequence of Corollary 4.5. We don't know how to prove $c_f = 1$ for $n \geq 4$. Let Π_f be any level zero representation constructed from π_f. We know that $\epsilon_{JS}(s, \pi_f, \wedge^2, \psi) = \gamma(\pi_f, \wedge^2, \psi)$ [21]. On the other hand, by Corollary 2.7 and [2], $\epsilon_{LS}(s, \Pi_f, \wedge^2, \psi) = \epsilon_0(\pi_f, \wedge^2, \psi)$. Therefore, $c_f = 1$ is equivalent to $\epsilon_{JS}(s, \Pi_f, \wedge^2, \psi) = \epsilon_{LS}(s, \Pi_f, \wedge^2, \psi)$.

The above discussion requires $\alpha|_{k_m^\times} \neq 1$ if $n = 2m$, because $\gamma(\pi_f, \wedge^2, \psi)$ is not defined in this special case. Interestingly, if we assume $n = 2m$ and $\alpha|_{k_m^\times} = 1$, we can show by computations that $\epsilon_{JS}(s, \Pi_f, \wedge^2, \psi) = \epsilon_{LS}(s, \Pi_f, \wedge^2, \psi)$ for any level zero supercuspidal representation Π_f constructed from π_f.

Let $n = 2m$ and $\alpha|_{k_m^\times} = 1$. We have computed in [21] that

$$\epsilon_{JS}(s, \Pi_f, \wedge^2, \psi) = \omega_{\Pi_f}(\varpi)^{-1} q^{m(s-\frac{1}{2})s},$$

where ω_{Π_f} is the central character of Π_f. Let ϕ_f be the image of Π_f under the local Langlands correspondence. Let $W = \wedge^2(\phi_f)$. Then from Equation (4),

$$\epsilon(s, \wedge^2(\phi_f), \psi) = \epsilon(W, \psi) q^{\left(\dim W^I\right)s} = \epsilon_0(W, \psi) \det(-F, W^I)^{-1} q^{\left(\dim W^I\right)s}.$$

We computed that $\epsilon_0(W, \psi) = -q^{-\frac{m}{2}}$ using Theorem 5.2 and computations of the Gauss sums, $\det(-F, W^I) = -\omega_{\Pi_f}(\varpi)$ using the local Langlands correspondence, and $\dim W^I = m$ by counting the multiplicity of the trivial character in $W_I = \sum_{1 \le i < j \le n} \alpha^{q^i + q^j}$, where W_I is the restriction of W to I. See also [20, Section 4.4] for details. Therefore,

$$\epsilon(s, \wedge^2(\phi_f), \psi) = \omega_{\Pi_f}(\varpi)^{-1} q^{m(s - \frac{1}{2})s} = \epsilon_{JS}(s, \Pi_f, \wedge^2, \psi).$$

We know $\epsilon_{LS}(s, \Pi_f, \wedge^2, \psi) = \epsilon(s, \wedge^2(\phi_f), \psi)$ from [2], so

$$\epsilon_{JS}(s, \Pi_f, \wedge^2, \psi) = \epsilon_{LS}(s, \Pi_f, \wedge^2, \psi).$$

CRediT authorship contribution statement

Rongqing Ye: Conceptualization, Formal analysis, Investigation, Software, Validation, Writing - original draft. **Elad Zelingher:** Conceptualization, Formal analysis, Investigation, Software, Validation, Writing - review & editing.

References

[1] J.W. Cogdell, N. Matringe, The functional equation of the Jacquet-Shalika integral representation of the local exterior-square L-function, Math. Res. Lett. 22 (3) (2015) 697–717.

[2] J.W. Cogdell, F. Shahidi, T.-L. Tsai, Local Langlands correspondence for GL_n and the exterior and symmetric square ε-factors, Duke Math. J. 166 (11) (2017) 2053–2132.

[3] P. Deligne, Les constantes des équations fonctionnelles des fonctions L, Lect. Notes Math. 349 (1973) 501–597.

[4] S.I. Gel'fand, Representations of the full linear group over a finite field, Math. USSR Sb. 12 (13) (1970) 13–39.

[5] J.A. Green, The characters of the finite general linear groups, Trans. Am. Math. Soc. 80 (1955) 402–447.

[6] M. Harris, R. Taylor, The Geometry and Cohomology of Some Simple Shimura Varieties, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.

[7] G. Henniart, Une preuve simple des conjectures de Langlands pour $\mathrm{GL}(n)$ sur un corps p-adique, Invent. Math. 139 (2) (2000) 439–455.

[8] H. Jacquet, I.I. Piatetskii-Shapiro, J. Shalika, Rankin-Selberg convolutions, Am. J. Math. 105 (2) (1983) 367–464.

[9] H. Jacquet, J. Shalika, Exterior square L-functions, in: Automorphic Forms, Shimura Varieties, and L-Functions, Vol. II, Ann Arbor, MI, 1988, in: Perspect. Math., vol. 11, Academic Press, Boston, MA, 1990, pp. 143–226.

[10] I.G. Macdonald, Zeta functions attached to finite general linear groups, Math. Ann. 249 (1) (1980) 1–15.

[11] N. Matringe, Linear and Shalika local periods for the mirabolic group, and some consequences, J. Number Theory 138 (2014) 1–19.

[12] C. Nien, A proof of the finite field analogue of Jacquet's conjecture, Am. J. Math. 136 (3) (2014) 653–674.

[13] C. Nien, $n \times 1$ local gamma factors and Gauss sums, Finite Fields Appl. 46 (2017) 255–270.

[14] C. Nien, L. Zhang, Converse theorem meets Gauss sums (with an appendix by Zhiwei Yun), arXiv: 1806.04850, 2018.

[15] E.-A. Roditty, On gamma factors and Bessel functions for representations of general linear groups over finite fields, Master's thesis, Tel Aviv University, 2010.

[16] J.-P. Serre, Local Fields, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York-Berlin, 1979. Translated from the French by Marvin Jay Greenberg.

[17] A.J. Silberger, E.-W. Zink, Explicit Shintani base change and the Macdonald correspondence for characters of $GL_n(k)$, J. Algebra 319 (10) (2008) 4147–4176.

[18] C. Yang, Gamma factors of level zero supercuspidal representations, arXiv:1903.11497, 2019.

[19] R. Ye, Rankin-Selberg gamma factors of level zero representations of GL_n, Forum Math. 31 (2) (2019) 503–516.

[20] R. Ye, Explicit formulas for local factors of supercuspidal representations of GL_n and their applications, PhD thesis, The Ohio State University, 2019.

[21] R. Ye, E. Zelingher, Exterior square gamma factors for cuspidal representations of GL_n: finite field analogs and level zero representations, arXiv:1807.04816, 2018.