Journal of Number Theory 148 (2015) 451–462

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Congruences for the Fourier coefficients of the Mathieu mock theta function

Tsuyoshi Miezaki[a,*], Matthias Waldherr[b]

[a] *Faculty of Education, Art and Science, Yamagata University, Yamagata 990-8560, Japan*
[b] *Graduiertenkolleg Globale Strukturen, Gyrhofstraße 8a, 50931 Köln, Germany*

A R T I C L E   I N F O

Article history:
Received 10 November 2013
Received in revised form 29 August 2014
Accepted 15 September 2014
Available online 6 November 2014
Communicated by D. Zagier

MSC:
primary 11F30
secondary 20D08, 11F27

Keywords:
Mock theta series
Mathieu group
Moonshine
Maass forms
Modular forms

A B S T R A C T

In this paper, we study the congruences for the Fourier coefficients of the Mathieu mock theta function, which appears in the Mathieu moonshine phenomenon discovered by Eguchi, Ooguri, and Tachikawa.

1. Introduction

Let $q = e^{2\pi i\tau}$ for τ in the complex upper half plane. Then the classical modular function $j : \mathbb{H} \to \mathbb{C}$ possesses the following Fourier expansion

* Corresponding author. Fax: +81 23 628 4413.
E-mail addresses: miezaki@e.yamagata-u.ac.jp (T. Miezaki), waldherr.m@gmail.com (M. Waldherr).

http://dx.doi.org/10.1016/j.jnt.2014.09.025

$$j(\tau)=\sum_{n=-1}^{\infty}c(n)q^n=\frac{1}{q}+744+196\,884q+\cdots$$

The moonshine phenomenon discovered by McKay and precisely formulated by Conway and Norton [2] asserts that the Fourier coefficients of the j-function are related to the dimensions of the irreducible representations of the Monster group. Subsequently, the moonshine conjecture was proven by Borcherds [1]. It is remarkable that the coefficients $c(n)$ have the following congruences [5,6] for $n\geq 1$, $a\geq 1$, and $1\leq b\leq 3$:

$$n\equiv 0\begin{cases}\pmod{2^a}\\ \pmod{3^a}\\ \pmod{5^a}\\ \pmod{7^a}\\ \pmod{11^b}\end{cases}\quad\Rightarrow\quad c(n)\equiv 0\begin{cases}\pmod{2^{3a+8}}\\ \pmod{3^{2a+3}}\\ \pmod{5^{a+1}}\\ \pmod{7^a}\\ \pmod{11^b}.\end{cases}$$

In 2010, Eguchi, Ooguri, and Tachikawa [3] discovered a similar phenomenon corresponding to the Mathieu group M_{24}. To describe their observation, we introduce the following functions:

$$\theta_1(z;\tau)=-\sum_{n=-\infty}^{\infty}e^{\pi i\tau(n+\frac{1}{2})^2+2\pi i(n+\frac{1}{2})(z+\frac{1}{2})},$$

$$\mu(z;\tau)=\frac{ie^{\pi iz}}{\theta_1(z;\tau)}\sum_{n\in\mathbb{Z}}(-1)^n\frac{q^{\frac{1}{2}n(n+1)}e^{2\pi inz}}{1-q^ne^{2\pi iz}},$$

and we set $\Sigma(\tau)$ and $A(n)$ as follows:

$$\begin{aligned}\Sigma(\tau)&:=-8\sum_{z\in\{1/2,\tau/2,(1+\tau)/2\}}\mu(z;\tau)\\&=-q^{-\frac{1}{8}}\left(2-\sum_{n=1}^{\infty}A(n)q^n\right)\\&=q^{-\frac{1}{8}}\left(-2+90q+462q^2+1540q^3+4554q^4+11\,592q^5+27\,830q^6+\cdots\right).\end{aligned}$$

Then the Mathieu moonshine phenomenon is the statement that the first five coefficients appearing in the Fourier expansion divided by 2,

$$\{45,231,770,2277,5796\},$$

are equal to the dimensions of the irreducible representations of M_{24}, and the other coefficients can be written as linear combinations of the dimensions of the Mathieu group M_{24}, for example $13\,915=3520+10\,395$. The reason for this mysterious phenomenon is still unknown. In this case the dimensions of the irreducible representations no longer appear in the Fourier coefficients of a modular form but rather in a mock modular form.

Recently, the first author numerically found some congruences of the coefficients $A(n)$ of the Mathieu mock theta function [7]:

$$n \equiv \begin{cases} 1 \pmod 3 \\ 1, 3 \pmod 5 \\ 2, 3, 5 \pmod 7 \\ 2, 3, 4, 6, 9 \pmod{11} \\ \begin{cases} 4, 5, 6, 7, 9, \\ 11, 12, 15, 16, 19, 21 \end{cases} \pmod{23} \end{cases} \quad \Rightarrow \quad A(n) \equiv 0 \begin{cases} \pmod 3 \\ \pmod 5 \\ \pmod 7 \\ \pmod{11} \\ \pmod{23}. \end{cases} \tag{1}$$

Furthermore, although one expects congruences for infinitely many primes, no congruences of such a simple form had been discovered. Remarkably, Zwegers made the observation that the primes occurring in these congruences are exactly the prime factors of the order of the Mathieu group M_{24}. (We note that the primes occurring in these congruences for the j-function are characterized as being of the group $\Gamma_0(p)$, which has genus zero.) Note that the conditions on n are such that n is not congruent to minus a triangular number modulo $3, 5, 7, 11$ or 23. This is a crucial part of the proof of Proposition 4.1.

The first author was able to establish the first two congruences [7]. In principal, the same method may also be used to establish the remaining congruences. However, the proof relies heavily on computer calculations, and these turn out to be infeasible for congruences modulo 7, modulo 11, and modulo 23.

In this paper we prove the remaining congruences. The proof essentially uses a modification of the argument of [10,7], where a certain sieving operator is replaced by a twisting operator. The reason why this works is the remarkable fact that the congruences appear at residue classes modulo 7 (modulo 11 and 23) whose characteristic function modulo 7 (modulo 11 and 23) may be written in terms of the unique quadratic character modulo 7 (modulo 11 and 23). In general, for an arbitrary subset of residue classes modulo 7 (modulo 11 and 23) one would need nonquadratic characters to express the characteristic function, and our argument would break down.

Theorem 1.1. *The relations in (1) are true.*

Remark 1.1. We note that the relations in (1) are slightly different from the relations in [7]. In fact, Zwegers pointed out that the method in [7] does not work for the case for $n \equiv 2 \pmod 3$, $A(n) \equiv 0 \pmod 3$. However, Zwegers found the following proof:

$$\begin{aligned} \Sigma(\tau) &= \frac{-2\left(E_2(\tau) + 24\sum_{n=1}^{\infty}(-1)^n \frac{nq^{n(n+1)/2}}{1-q^n}\right)}{\eta(\tau)^3} \quad \text{(cf. [12])} \\ &\equiv -\frac{2}{\eta(3\tau)} \pmod 3 \\ &= -2q^{-\frac{1}{8}}\left(1 + q^3 + 2q^6 + \cdots\right). \end{aligned}$$

We can see directly that $A(n) \equiv 0 \pmod 3$ if $n \equiv 1, 2 \pmod 3$.

2. Setup

$\Sigma(\tau)$ is a mock theta series [13,10] and we call it the Mathieu mock theta function.

For the reader's convenience, we recall the notion of modular forms and harmonic weak Maass forms for an arbitrary multiplier system (for more information, the reader is referred to [10]). A function $f : \mathbb{H} \to \mathbb{C}$ is called a weakly holomorphic modular form of weight $k/2$ with respect to a congruence subgroup $\Gamma \leq SL_2(\mathbb{Z})$ and a multiplier system ν if the following conditions hold:

(1) f satisfies the modular transformation property:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \nu \begin{pmatrix} ab \\ cd \end{pmatrix} (c\tau + d)^{\frac{k}{2}} f(\tau).$$

(2) f is holomorphic on $\mathbb{H}$.
(3) f has at most linear exponential growth at the cusps.

We call a function $f : \mathbb{H} \to \mathbb{C}$ a harmonic Maass form if the second condition is replaced by the weaker condition that f is annihilated by the weight $k/2$ hyperbolic Laplacian $\Delta_{\frac{k}{2}} := -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + \frac{ik}{2}(\frac{\partial}{\partial x} + \frac{\partial}{\partial y})$. The η-multiplier $\nu_\eta : SL_2(\mathbb{Z}) \to \{z \in \mathbb{C} \mid |z| = 1\}$ is defined by

$$\nu_\eta \begin{pmatrix} ab \\ cd \end{pmatrix} := \frac{1}{\sqrt{c\tau + d}} \frac{\eta(\frac{a\tau+b}{c\tau+d})}{\eta(\tau)}.$$

The η-multiplier system and the integer powers of it are multiplier systems for all half-integral weights [10].

Let

$$g_{a,b}(\tau) := \sum_{\nu \in a+\mathbb{Z}} \nu e^{\pi i \nu^2 \tau + 2\pi i \nu b},$$

where $a, b \in \mathbb{R}$. We define $\widetilde{\mu}(u;\tau)$ and $\Sigma(z)$ as follows:

$$\widetilde{\mu}(z;\tau) := \mu(z;\tau) - \frac{i}{2} R(0;\tau)$$
$$\widetilde{\Sigma}(\tau) := 8 \sum_{z \in \{1/2, \tau/2, (1+\tau)/2\}} \widetilde{\mu}(z;\tau),$$

where $R(0;\tau)$ is defined as

$$R(0;\tau) := \int_{-\bar{\tau}}^{\infty} \frac{g_{\frac{1}{2},\frac{1}{2}}(t)}{\sqrt{-i(t+\tau)}} dt.$$

Then it follows from [13] and [10, Lemma 3.3] that the following proposition holds:

Proposition 2.1. *The function $\widetilde{\Sigma}(\tau)$ is a harmonic weak Maass form of weight* $1/2$ *with respect to the group* $SL_2(\mathbb{Z})$ *with multiplier system* ν_η^{-3}.

It follows from [13, Propositions 3.6 and 3.7] that $\mu(1/2;\tau)$, $\mu(\tau/2;\tau)$, and $\mu((1+\tau)/2;\tau)$ are the holomorphic parts of $\widetilde{\mu}(1/2;\tau)$, $\widetilde{\mu}(\tau/2;\tau)$, and $\widetilde{\mu}((1+\tau)/2;\tau)$, respectively.

3. Twisting

To define twisting, let f be a harmonic weak Maass form of weight $\frac{k}{2}$ ($k \in \mathbb{Z}$) with a Fourier expansion of the form

$$f(\tau) = \sum_{n\in\mathbb{Z}} a_n(y) q^{\frac{n}{N}}.$$

Let m be a positive integer, and let ψ be a Dirichlet character modulo m. We define the twist f_ψ by

$$f_\psi(\tau) := \sum_{n\in\mathbb{Z}} \psi(n) a_n(y) q^{\frac{n}{N}}.$$ [1]

We may express the twist as a linear combination of shifts of f. More precisely, in the situation above, we have the following proposition (for a proof see, for example, [4]):

Proposition 3.1.

$$f_\psi(\tau) = \frac{g_\psi}{m} \sum_{s \pmod{m}} \bar{\psi}(s) f\left(\tau - \frac{Ns}{m}\right) = \frac{g_\psi}{m} \sum_{s \pmod{m}} \bar{\psi}(s) f\Big|_{\frac{k}{2}} \begin{pmatrix} 1 & -\frac{Ns}{m} \\ 0 & 1 \end{pmatrix}, \tag{2}$$

where

$$g_\psi := \sum_{r \pmod{m}} e^{\frac{2\pi i r}{m}} \psi(r)$$

is the Gauss sum associated with ψ.

The following is a generalization of a lemma in the thesis of the second author and is modeled along the proof for modular forms as in [4].

Proposition 3.2. *Let f be a harmonic weak Maass form of weight $\frac{1}{2}$ for $SL_2(\mathbb{Z})$ with multiplier ν_η^{-3}. Suppose f has a Fourier expansion at ∞ in terms of $q^{\frac{1}{N}}$ with $N = 8$. Let m be a positive integer, and let ψ be a Dirichlet character modulo m. Then, f_ψ is a harmonic weak Maass form of weight $\frac{1}{2}$ on $\Gamma_0(m^2)$ and multiplier $\nu_\eta^{-3}\psi^2$.*

If m is a multiple of $N = 8$, then f_ψ is a harmonic weak Maass form on $\Gamma_0(m^2/N^2)$.

[1] In fact there is also a dependency on N, which we suppress for simplicity.

Proof. Using representation 2 it is easy to establish that f_ψ has at most exponential growth at any cusp, and it is also annihilated by $\Delta_{\frac{1}{2}}$.

To prove the transformation properties, we define

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad A_{s,m} = \begin{pmatrix} 1 & -\frac{Ns}{m} \\ 0 & 1 \end{pmatrix}$$

with $a, b, c, d, s \in \mathbb{Z}$. For any $s, s' \in \mathbb{Z}$ we have

$$A_{s,m} A A_{s',m}^{-1} = \begin{pmatrix} a - \frac{Ncs}{m} & b + \frac{Ns'a - Nsd}{m} + \frac{N^2 css'}{m^2} \\ c & d + \frac{Ncs'}{m} \end{pmatrix}.$$

Now suppose $A \in \Gamma_0(m^2)$. Then it is easy to see that $A_{s,m} A A_{s',m}^{-1} \in SL_2(\mathbb{Z})$ if $s'a - sd \equiv 0 \pmod{m}$. Since a and d are coprime, we can always choose s' in such a way that this is satisfied, namely $s' = sda^{-1} \pmod{m}$, where a^{-1} denotes the integer inverse of a modulo m.

Then we have

$$\begin{aligned} f_\psi(A\tau) &= \frac{g_\psi}{m} \sum_{s \pmod{m}} \bar{\psi}(s) f\left(A\tau - \frac{Ns}{m}\right) = \frac{g_\psi}{m} \sum_{s \pmod{m}} \bar{\psi}(s) f(A_{s,m} A\tau) \\ &= \frac{g_\psi}{m} \sum_{s \pmod{m}} \bar{\psi}(s) f\big(A_{s,m} A A_{s',m}^{-1} A_{s',m}\tau\big). \end{aligned}$$

We have seen that $A_{s,m} A A_{s',m}^{-1} \in SL_2(\mathbb{Z})$. Using the fact that f is a harmonic weak Maass form of weight $\frac{1}{2}$ with character ν_η^{-3}, we conclude

$$\begin{aligned} f_\psi(A\tau) = \frac{g_\psi}{m} \sum_{s \pmod{m}} \bar{\psi}(s) \nu_\eta^{-3} &\begin{pmatrix} a - \frac{cNs}{m} & b + \frac{Ns'a - Nsd}{m} + \frac{cN^2 ss'}{m^2} \\ c & d + \frac{cNs'}{m} \end{pmatrix} \\ &\times \sqrt{cA_{s',m}\tau + d + \frac{Ncs'}{m}} f(A_{s',m}\tau). \end{aligned}$$

We see that $\sqrt{cA_{s',m}\tau + d + \frac{cs'}{m}} = \sqrt{c\tau + d}$. Furthermore, since $N = 8$, every entry of the matrix in the argument of ν_η^3 differs from the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ by some multiple of 8. Then the lemma implies

$$\begin{aligned} f_\psi(A\tau) &= \frac{g_\psi}{m} \sum_{s \pmod{m}} \bar{\psi}(s) \nu_\eta^{-3} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sqrt{c\tau + d} f(A_{s',m}\tau) \\ &= \frac{g_\psi}{m} \nu_\eta^{-3} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sqrt{c\tau + d} \sum_{s \pmod{m}} \bar{\psi}(s) f(A_{s',m}\tau). \end{aligned}$$

With s ranging through all the residue classes modulo m, the same is true for $s' = sda^{-1}$. This change of variables then yields

$$f_\psi(A\tau) = \frac{g_\psi}{m}\nu_\eta^{-3}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\sqrt{c\tau + d}\sum_{s' \pmod m} \overline{\psi}(s')\overline{\psi}(ad^{-1})f(A_{s',m}\tau).$$

However, $\overline{\psi}(ad^{-1}) = \overline{\psi}(a)\psi(d)$, and $\psi(ad) = \psi(1 + cb) = \psi(1) = 1$. Hence, $\overline{\psi}(ad^{-1}) = \psi^2(d)$, and this implies that

$$f_\psi(A\tau) = \nu_\eta^{-3}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\psi^2(d)\sqrt{c\tau + d}f_\psi(\tau). \qquad \square$$

4. Proof of Theorem 1.1

4.1. Characters

Recall that

$$\begin{aligned} \Sigma(\tau) &= -q^{-\frac{1}{8}}\left(2 - \sum_{n=1}^{\infty} A(n)q^n\right) \\ &= -2q^{-\frac{1}{8}} + q^{\frac{1}{8}}\left(90q + 462q^2 + 1540q^3 + 4554q^4 + 11\,592q^5 + 27\,830q^6 + \cdots\right). \end{aligned}$$

Since the Fourier expansion is in terms of $q^{\frac{1}{8}}$, we immediately see that congruences

$$n \equiv \begin{cases} 1 \pmod 3 \\ 1, 3 \pmod 5 \\ 2, 3, 5 \pmod 7 \\ 2, 3, 4, 6, 9 \pmod{11} \\ \begin{cases} 4, 5, 6, 7, 9, \\ 11, 12, 15, 16, 19, 21 \end{cases} \pmod{23} \end{cases} \quad \Rightarrow \quad A(n) \equiv 0 \begin{cases} \pmod 3 \\ \pmod 5 \\ \pmod 7 \\ \pmod{11} \\ \pmod{23} \end{cases}$$

are equivalent to the fact that all coefficients appearing in the Fourier expansion in front of $q^{\frac{k}{8}}$, with

$$k \equiv \begin{cases} 7 \pmod{24} \\ 7, 23 \pmod{40} \\ 15, 23, 39 \pmod{56} \\ 15, 23, 31, 47, 51 \pmod{88} \\ \begin{cases} 31, 39, 47, 55, 71, \\ 87, 95, 119, 127, 151, 167 \end{cases} \pmod{184} \end{cases}$$

are divisible by 3, 5, 7, 11, and 23, respectively. Let us define

$$\Sigma_3(\tau) = \sum_{k\equiv 7 \pmod{24}} a(n)q^{\frac{k}{n}} = q^{-\frac{1}{8}} \sum_{n\equiv 1 \pmod 3} A(n)q^n,$$

$$\Sigma_5(\tau) = \sum_{k\equiv 7,23 \pmod{40}} a(n)q^{\frac{k}{n}} = q^{-\frac{1}{8}} \sum_{n\equiv 1,3 \pmod 5} A(n)q^n,$$

$$\Sigma_7(\tau) = \sum_{k\equiv 15,23,39 \pmod{56}} a(n)q^{\frac{k}{n}} = q^{-\frac{1}{8}} \sum_{n\equiv 2,3,5 \pmod 7} A(n)q^n,$$

$$\Sigma_{11}(\tau) = \sum_{k\equiv 15,23,31,47,71 \pmod{88}} a(n)q^{\frac{k}{n}} = q^{-\frac{1}{8}} \sum_{n\equiv 2,3,4,6,9 \pmod{11}} A(n)q^n,$$

$$\Sigma_{23}(\tau) = \sum_{\substack{k\equiv 31,39,47,55,71,\\ 87,97,119,127,151,167 \pmod{184}}} a(n)q^{\frac{k}{n}} = q^{-\frac{1}{8}} \sum_{\substack{n\equiv 4,5,6,7,9,\\ 11,12,15,16,19,21 \pmod{23}}} A(n)q^n.$$

In order to apply the proposition from Section 3, we will write the characteristic functions in terms of characters. For $p = 3, 5, 7, 11, 23$, let χ_p^{triv} denote the trivial character and χ_p the following quadratic characters modulo p (in the case 7, 11, and 23, these are the unique quadratic characters):

residue class	0	1	2
order	0	1	−1
χ_3	0	1	−1

residue class	0	1	2	3	4
order	0	1	4	4	2
χ_5	0	1	−1	−1	1

residue class	0	1	2	3	4	5	6
order	0	1	3	6	3	6	2
χ_7	0	1	1	−1	1	−1	1

residue class	0	1	2	3	4	5	6	7	8	9	10
order	0	1	10	5	5	5	10	10	10	5	2
χ_{11}	0	1	−1	1	1	1	−1	−1	−1	1	1

residue class	0	1	2	3	4	5	6	7	8	9	10
order	0	1	11	11	11	22	11	22	11	11	22
χ_{23}	0	1	1	1	1	−1	1	−1	1	1	−1

residue class	11	12	13	14	15	16	17	18	19	20	21	22
order	22	11	11	22	22	11	22	11	22	22	22	2
χ_{23}	−1	1	1	−1	−1	1	−1	1	−1	−1	−1	−1

Furthermore we consider the following characters modulo 8:

	0	1	2	3	4	5	6	7
$\chi_8^{(0)}$	0	1	0	1	0	1	0	1
$\chi_8^{(1)}$	0	1	0	1	0	−1	0	−1
$\chi_8^{(2)}$	0	1	0	−1	0	1	0	−1
$\chi_8^{(3)}$	0	1	0	−1	0	−1	0	1

Then we have

$$\frac{1}{4}\big(\chi_8^{(0)}(n) - \chi_8^{(1)}(n) - \chi_8^{(2)}(n) + \chi_8^{(3)}(n)\big) = \begin{cases} 1 & \text{if } n \equiv 7 \pmod 8, \\ 0 & \text{if } n \not\equiv 7 \pmod 8, \end{cases}$$

and

$$\frac{1}{2}\big(\chi_3^{\mathrm{triv}}(n) + \chi_3(n)\big) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 3, \\ 0 & \text{else}, \end{cases}$$

$$\frac{1}{2}\big(\chi_5^{\mathrm{triv}}(n) - \chi_5(n)\big) = \begin{cases} 1 & \text{if } n \equiv 1, 4 \pmod 5, \\ 0 & \text{else}, \end{cases}$$

$$\frac{1}{2}\big(\chi_7^{\mathrm{triv}}(n) + \chi_7(n)\big) = \begin{cases} 1 & \text{if } n \equiv 1, 2, 4 \pmod 7, \\ 0 & \text{else}, \end{cases}$$

$$\frac{1}{2}\big(\chi_{11}^{\mathrm{triv}}(n) + \chi_{11}(n)\big) = \begin{cases} 1 & \text{if } n \equiv 1, 3, 4, 5, 9 \pmod{11}, \\ 0 & \text{else}, \end{cases}$$

$$\frac{1}{2}\big(\chi_{23}^{\mathrm{triv}}(n) + \chi_{23}(n)\big) = \begin{cases} 1 & \text{if } n \equiv 1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18 \pmod{23}, \\ 0 & \text{else}. \end{cases}$$

Multiplying these characteristic functions mod p with those mod 8, we obtain the characteristic functions:

$$\begin{cases} 7 \pmod{24} \\ 7, 23 \pmod{40} \\ 15, 23, 39 \pmod{56} \\ 15, 23, 31, 47, 71, \pmod{88} \\ 31, 39, 47, 55, 71, 87, 95, 119, 127, 151, 167 \pmod{184} \end{cases} \quad \text{for} \quad \begin{cases} \bmod 3 \\ \bmod 5 \\ \bmod 7 \\ \bmod 11 \\ \bmod 23. \end{cases}$$

Proposition 4.1. *For $p = 3, 5, 7, 11, 23$, the function $\Sigma_p(\tau)$ is a weakly holomorphic modular form of weight $1/2$ with respect to ν_η^{-3} for the group $\Gamma_0(p^2)$.*

Proof. By Proposition 2.1, $\widetilde{\Sigma}(\tau)$ is a harmonic weak Maass form. We write the Fourier expansion of $\widetilde{\Sigma}(\tau)$ as

$$\widetilde{\Sigma}(z) = \sum_{n\in\mathbb{Z}} \widetilde{A}(n, y) q^{\frac{n}{8}},$$

where $\widetilde{A}(n, y)$ might also depend on the imaginary part of τ. For $p = 3, 5, 7, 11, 23$, we set $\widetilde{\Sigma}(\tau)$ as follows:

$$\widetilde{\Sigma}_3(\tau) = q^{-\frac{1}{8}} \sum_{n\equiv 1 \pmod 3} \widetilde{A}(n,y)q^n,$$
$$\widetilde{\Sigma}_5(\tau) = q^{-\frac{1}{8}} \sum_{n\equiv 1,3 \pmod 5} \widetilde{A}(n,y)q^n,$$
$$\widetilde{\Sigma}_7(\tau) = q^{-\frac{1}{8}} \sum_{n\equiv 2,3,5 \pmod 7} \widetilde{A}(n,y)q^n,$$
$$\widetilde{\Sigma}_{11}(\tau) = q^{-\frac{1}{8}} \sum_{n\equiv 2,3,4,6,9 \pmod{11}} \widetilde{A}(n,y)q^n,$$
$$\widetilde{\Sigma}_{23}(\tau) = q^{-\frac{1}{8}} \sum_{\substack{n\equiv 4,5,6,7,9,\\ 11,12,15,16,19,21 \pmod{23}}} \widetilde{A}(n,y)q^n.$$

Then by Proposition 3.2, $\widetilde{\Sigma}_p$ is a harmonic Maass form for $\Gamma_0(p^2)$. As the holomorphic part of $\widetilde{\Sigma}_p$ is Σ_p, it remains to show that the function is holomorphic, i.e., its nonholomorphic part vanishes. To prove that, we argue as in [10, Proposition 3.6] that the nonholomorphic part is

$$\frac{i}{2\sqrt{\pi}} \sum_{n\in\mathbb{Z}} (-1)^n q^{-\frac{(2n+1)^2}{8}} \Gamma\left(\frac{1}{2}, \pi\frac{(2n+1)^2}{2}y\right)$$

and is supported at Fourier coefficients of the form $q^{-\frac{(2n+1)^2}{8}}$ with $n \in \mathbb{Z}$ (we note that $\frac{i\sqrt{2}}{4\sqrt{\pi}}$, $q^{-\frac{(2n+1)^2}{4}}$, and $\Gamma(\frac{1}{2}, \pi(2n+1)^2 y)$ which appear in [10, Proposition 3.6], should be $\frac{i}{2\sqrt{\pi}}$, $q^{-\frac{(2n+1)^2}{8}}$, and $\Gamma(\frac{1}{2}, \pi\frac{(2n+1)^2}{2}y))$. For $p = 3$, it is easy to see that $24n + 7$ can never be a negative square of an odd number. Namely, we have $\Sigma_3(\tau) = \widetilde{\Sigma}_3(\tau)$. The other cases are proven in a similar way. □

4.2. Cusp behavior, eta function, and computation

The following lemma appears in [10].

Lemma 4.1. *(Cf. [10, Lemma 3.11].) Suppose* $0 \le s < m$. *Set* $\ell := gcd(cm, Nsc + am)$. *For* $r = 3, 5, 7, 11, 23$, *the first Fourier coefficient of the holomorphic part of* $\Sigma_r(\tau)$ *at the cusp* a/c *up to a nonzero constant is given by* $q^{-\ell^2/8m^2}$.

Using Lemma 4.1, one can determine upper bounds for the pole orders of Σ_3, Σ_5, Σ_7, Σ_{11}, and Σ_{23}. Using this information and a computer program, which can be found on the homepage of one of the authors

https://sites.google.com/site/tmiezaki/mmt_pole/

one can check the following lemma:

Lemma 4.2. *For $p = 3, 5, 7, 11, 23$, we have that*

$$\eta(p\tau)^{3p}\Sigma_p(\tau)$$

is a modular form on $\Gamma_0(p^2)$ of weight $\frac{3p+1}{2}$ having no poles at the cusps of $\Gamma_0(p^2)$.

The following lemma provides a variant of the Sturm bound for modular forms whose q-expansion has fractional powers of q.

Lemma 4.3. *(See [11, Lemma 4.4], [9], [8].) Suppose that f is a modular form of half-integral weight $\frac{k}{2}$ with respect to a congruence subgroup Γ and some multiplier system ν, and with Fourier expansion of the form*

$$\sum_{n=0}^{\infty} a_n q^{\frac{n}{N}}.$$

Let p be some prime number. Then, $p|a_n$ for all n if, and only if, $p|a_n$ for all n with $n \leq \frac{Nk}{24}[SL_2(\mathbb{Z}) : \Gamma]$.

In our situation, we obtain the following values for the Sturm bound.

p	3	5	7	11	23
$[SL_2(\mathbb{Z}) : \Gamma_0(p^2)]$	12	30	56	132	552
Sturm bound for Σ_p	40	160	411	1496	12 880

We have checked numerically that the divisibility holds up to the Sturm bound for each prime p for the function $\eta(p\tau)^{3p}\Sigma_p(\tau)$. This in turn implies the divisibility properties of the Fourier coefficients of $\eta(p\tau)^{3p}\Sigma_p(\tau)$ and, as a consequence, also of $\Sigma_p(\tau)$. Therefore the proof of Theorem 1.1 is completed. The Fourier coefficients can be obtained electronically from

https://sites.google.com/site/tmiezaki/mmt_coef

Acknowledgments

The authors would like to thank Dr. Sander Zwegers for his helpful discussions. The authors would also like to thank the anonymous referee for beneficial comments on an earlier version of the manuscript.

This work was supported by JSPS KAKENHI (22840003, 24740031).

References

[1] R.E. Borcherds, Monstrous moonshine and monstrous Lie superalgebras, Invent. Math. 109 (2) (1992) 405–444.

[2] J.H. Conway, S.P. Norton, Monstrous moonshine, Bull. Lond. Math. Soc. 11 (1979) 308–339.
[3] T. Eguchi, H. Ooguri, Y. Tachikawa, Notes on the $K3$ surface and the Mathieu group M_{24}, Expo. Math. 20 (1) (2011) 91–96.
[4] N. Koblitz, Introduction to Elliptic Curves and Modular Forms, Springer-Verlag, Berlin/New York, 1984.
[5] J. Lehner, Divisibility properties of the Fourier coefficients of the modular invariant $j(\tau)$, Amer. J. Math. 71 (1949) 136–148.
[6] J. Lehner, Further congruence properties of the Fourier coefficients of the modular invariant $j(\tau)$, Amer. J. Math. 71 (1949) 373–386.
[7] T. Miezaki, On the Mathieu mock theta function, Proc. Japan Acad. Ser. A Math. Sci. 88 (2) (2012) 28–30.
[8] K. Ono, The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-Series, CBMS Reg. Conf. Ser. Math., vol. 102, 2004, Published for the Conference Board of the Mathematical Sciences, Washington, DC.
[9] J. Sturm, On the congruence of modular forms, in: Number Theory, New York, 1984–1985, in: Lecture Notes in Math., vol. 1240, Springer, Berlin, 1987, pp. 275–280.
[10] M. Waldherr, On certain explicit congruences for mock theta functions, Proc. Amer. Math. Soc. 139 (2011) 865–879.
[11] M. Waldherr, Arithmetic and asymptotic properties of mock theta functions and mock Jacobi forms, Ph.D thesis, Universität zu Köln, 2012.
[12] D. Zagier, Ramanujan's mock theta functions and their applications, Asterisque 326 (2009) 143–164.
[13] S. Zwegers, Mock theta functions, Ph.D thesis, Utrecht University, 2002.