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A Note on Schmidt's Subspace Type Theorem with Moving Hyperplanes[1]

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Abstract

In this paper, we show a Schmidt's subspace type theorem with moving hyperplanes, in which these moving hyperplanes are assumed to be non-degenerate. As the application of this result, we discuss the number of integer solutions to a sequence of decomposable form inequalities.

Keywords: Diophantine approximation, Schmidt's subspace theorem, moving targets, decomposable form inequality
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1. Introduction

Schmidt's subspace theorem is a very powerful tool from Diophantine approximation, which has some significant applications to Diophantine equations, Diophantine geometry, and various other things. To state subspace theorem, we first recall some standard notations and definitions in number theory.

Let k be a number field of degree $[k:\mathbb{Q}]$. Denote by M_k the set of places (i.e., equivalence classes of absolute values) of k. We choose the normalized absolute value $|\ \ |_v$ such that $|\ \ |_v = |\ \ |$ on $\mathbb{Q}$ (the standard absolute value) if $v \in M_k$ is archimedean. For a non-archimedean place v, the absolute value $|\ \ |_v$ is defined such that $|p|_v = p^{-1}$ if v lies above the rational prime p. Let M_k^∞ be the set of archimedean places of k, and let M_k^0 be the set of non-archimedean places of k.

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Denote by k_v the completion of k with respect to v. For $v \in M_k^\infty$, let

$$\| \ \|_v = \begin{cases} | \ |^{1/[k:\mathbb{Q}]} & \text{if } k_v = \mathbb{R}, \\ | \ |^{2/[k:\mathbb{Q}]} & \text{if } k_v = \mathbb{C}. \end{cases}$$

For $v \in M_k^0$ (with respect to the prime p), let $\| \ \|_v = | \ |_v^{[k_v:\mathbb{Q}_p]/[k:\mathbb{Q}]}$, where $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic absolute value. The norm $\| \ \|_v$ satisfies the following properties:

(i) $\|x\|_v \geq 0$, with equality if and only if $x = 0$;

(ii) $\|xy\|_v = \|x\|_v \|y\|_v$ for all $x, y \in k$;

(iii) $\|x_1 + \cdots + x_n\|_v \leq n^{N_v} \max\{\|x_1\|_v, \ldots, \|x_n\|_v\}$ for all $x_1, \ldots, x_n \in k$, $n \in \mathbb{N}$, where

$$N_v = \begin{cases} 0 & \text{if } v \text{ is non-archimedean,} \\ 1/[k:\mathbb{Q}] & \text{if } v \text{ is real,} \\ 2/[k:\mathbb{Q}] & \text{if } v \text{ is complex;} \end{cases}$$

(iv) for any $x \in k \setminus \{0\}$, we have the **product formula**: $\prod_{v \in M_k} \|x\|_v = 1$.

For $v \in M_k$, we also extend $\| \ \|_v$ to an absolute value on the algebraic closure $\bar{k}_v$. Let S be a finite subset of M_k containing M_k^∞. An element $x \in k$ is said to be a S-**integer** if $\|x\|_v \leq 1$ for each $v \in M_k \setminus S$. Denote by $\mathcal{O}_S$ the set of S-integers. The unit of $\mathcal{O}_S$ is called S-**unit**. The set of all S-units forms a multiplicative group which is denoted by $\mathcal{O}_S^*$.

For $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}^n(k)$, let $\|\mathbf{x}\|_v = \max_{0 \leq i \leq n} \|x_i\|_v$. Moreover, define the **logarithmic height of x** by

$$h(\mathbf{x}) = \sum_{v \in M_k} \log \|\mathbf{x}\|_v.$$

By the product formula, its definition is independent of the choice of the representations. Let $H \subset \mathbb{P}^n(k)$ be a hyperplane, and let $L = a_0x_0 + \cdots + a_nx_n$ be the linear form defining H with $a_0, \ldots, a_n \in k$. Set $\|L\|_v = \max_{0 \leq i \leq n} \|a_i\|_v$. We also define the height of H (or L) as

$$h(H) = h([a_0 : \cdots : a_n]) = \sum_{v \in M_k} \log \|L\|_v.$$

The **Weil function with respect to** H is given by, for any $v \in M_k$ and $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}^n(k) \setminus H$,

$$\lambda_{H,v}(\mathbf{x}) = \log \frac{\|\mathbf{x}\|_v \cdot \|L\|_v}{\|a_0x_0 + \cdots + a_nx_n\|_v}.$$

Let $S \subset M_k$ be a finite set containing M_k^∞. Define

$$m_S(H, \mathbf{x}) = \sum_{v \in S} \lambda_{H,v}(\mathbf{x}), \quad N_S(H, \mathbf{x}) = \sum_{v \notin S} \lambda_{H,v}(\mathbf{x}).$$

Then, by the product formula,

$$m_S(H, \mathbf{x}) + N_S(H, \mathbf{x}) = h(\mathbf{x}) + h(H)$$

holds for all $\mathbf{x}$ with $L(\mathbf{x}) \neq 0$.

The following version of Schmidt's subspace theorem is given by Vojta [12].

Theorem A. (Schmidt's subspace theorem) Let k be a number field and let $S \subset M_k$ be a finite set containing M_k^∞. Let $H_1, \dots, H_q$ be hyperplanes in $\mathbb{P}^n(k)$ with defining linear forms $L_1, \dots, L_q$. Assume that $H_1, \dots, H_q$ are in general position, which means any $n+1$ linear forms in $\{L_1, \dots, L_q\}$ are linearly independent over k. Then, for every $\varepsilon > 0$, there exists a finite collection $\mathcal{V}$ of proper subspaces of $\mathbb{P}^n(k)$ such that the inequality

$$\sum_{j=1}^{q} m_S(H_j, \mathbf{x}) \leq (n + 1 + \varepsilon) h(\mathbf{x})$$

holds for all points $\mathbf{x} \in \mathbb{P}^n(k) - \bigcup_{V \in \mathcal{V}} V$.

Due to the work of Osgood, Lang, Vojta, etc., people have started to realize that there is a close relationship between Nevanlinna theory in complex analysis and Diophantine approximation. Especially, Vojta formulated an analogue of these two theories by giving a so-called dictionary (see [12]). Via Vojta's dictionary, Schmidt's subspace theorem corresponds to Cartan's second main theorem for holomorphic curves in Nevanlinna theory (see [1]). As the generalization of Cartan's second main theorem, Ru and Stoll [8] proved a second main theorem with slowly moving hyperplanes. Motivated by this result, Ru and Vojta [9] obtained the Schmidt's subspace theorem with moving hyperplanes.

Definition 1.1. Let Λ be an infinite index.

(i) A **moving hyperplane** indexed by Λ in $\mathbb{P}^n(k)$ is a map H defined by $\alpha (\in \Lambda) \mapsto H(\alpha)$, where $H(\alpha)$ is a hyperplane in $\mathbb{P}^n(k)$.

(ii) Let $\mathcal{H} := \{H_1, \dots, H_q\}$ be a family of moving hyperplanes indexed by Λ. For each $j = 1, \dots, q$ and $\alpha \in \Lambda$ choose $a_{j,0}(\alpha), \dots, a_{j,n}(\alpha) \in k$ such that $H_j(\alpha)$ is defined by the linear form $L_j(\alpha) = a_{j,0}(\alpha)x_0 + \cdots + a_{j,n}(\alpha)x_n$.

Then a subset $A \subseteq \Lambda$ is said to be **coherent** with respect to $\mathcal{H}$ if, for every polynomial

$$P \in k[x_{1,0}, \ldots, x_{1,n}, \ldots, x_{q,0}, \ldots, x_{q,n}]$$

that is homogeneous in $x_{j,0}, \ldots, x_{j,n}$ for each $j = 1, \ldots, q$, either

$$P(a_{1,0}(\alpha), \ldots, a_{1,n}(\alpha), \ldots, a_{q,0}(\alpha), \ldots, a_{q,n}(\alpha))$$

vanishes for all $\alpha \in A$ or it vanishes for only finitely many $\alpha \in A$. (For the existence of an infinite coherent subset $A \subseteq \Lambda$ with respect to $\mathcal{H}$, see Lemma 1.1 in [9] or Lemma 2.1 in [3].)

(iii) For a subset $A \subseteq \Lambda$, we define $\mathcal{R}_A^0$ as the set of equivalence classes of pairs (C, a), where $C \subseteq A$ is a cofinite subset, $a : C \to k$ is a map, and the equivalence relation is defined by $(C, a) \sim (C', a')$ if there exists $C'' \subseteq C \cap C'$ such that C'' is a cofinite subset of A and $a|_{C''} = a'|_{C''}$. This is a ring containing k as its subring.

(iv) If A is coherent with respect to $\mathcal{H}$, and $a_{j,t}(\alpha) \neq 0$ for all but finitely many $\alpha \in A$, then $a_{j,l}/a_{j,t}$ defines an element of $\mathcal{R}_A^0$. Moreover, by coherence, the subring of $\mathcal{R}_A^0$ generated by all such elements is an integral domain. We define $\mathcal{R}_A$ as the field of fractions of that integral domain. It is clear that if $B \subseteq A$ then B is also coherent and $\mathcal{R}_B \subseteq \mathcal{R}_A$.

In [9], Ru and Vojta proved the follows.

Theorem B. (Schmidt's subspace theorem with moving hyperplanes) Let k be a number field and let $S \subset M_k$ be a finite set containing M_k^∞. Let Λ be an infinite index set and let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be a set of moving hyperplanes indexed by Λ in $\mathbb{P}^n(k)$. Let $\mathbf{x} = [x_0 : \cdots : x_n] : \Lambda \to \mathbb{P}^n(k)$ be a sequence of points. Assume that

(i) for all $\alpha \in \Lambda$, $H_1(\alpha), \ldots, H_q(\alpha)$ are in general position;

(ii) $\mathbf{x}$ is linearly non-degenerate with respect to $\mathcal{H}$, which means, for each infinite coherent subset $A \subseteq \Lambda$ with respect to $\mathcal{H}$, $x_0|_A, \ldots, x_n|_A$ are linearly independent over $\mathcal{R}_A$;

(iii) $h(H_j(\alpha)) = o(h(\mathbf{x}(\alpha)))$ for all $j = 1, \ldots, q$ (that is, for all $\delta > 0$, $h(H_j(\alpha)) \leq \delta h(\mathbf{x}(\alpha))$ for all but finitely many $\alpha \in \Lambda$).

Then, for every $\varepsilon > 0$, there exists an infinite index subset $A \subseteq \Lambda$ such that

$$\sum_{j=1}^{q} m_S(H_j(\alpha), \mathbf{x}(\alpha)) \leq (n + 1 + \varepsilon) h(\mathbf{x}(\alpha))$$

holds for all $\alpha \in A$.

As a generalization of Ru and Wong's result [11], they also obtained a Schmidt's subspace type theorem for moving targets without condition (ii) in Theorem B.

Theorem C. [9] Let $k, S, \Lambda, \mathcal{H}, H_1, \ldots, H_q$, and $\mathbf{x}$ be as in the first three sentences of the above theorem. Suppose that

(i) for each $\alpha \in \Lambda$, $H_1(\alpha), \ldots, H_q(\alpha)$ are in general position;

(ii) $h(\mathbf{x}(\alpha))$ is unbounded and $h(H_j(\alpha)) = o(h(\mathbf{x}(\alpha)))$ for all $j = 1, \ldots, q$;

(iii) $\mathbf{x}(\alpha) \notin H_j(\alpha)$ for all $j = 1, \ldots, q$ and $\alpha \in \Lambda$.

Then, for every $\varepsilon > 0$, there exists an infinite index subset $A \subseteq \Lambda$ such that

$$\sum_{j=1}^{q} m_S(H_j(\alpha), \mathbf{x}(\alpha)) \leq (2n + \varepsilon)h(\mathbf{x}(\alpha))$$

holds for all $\alpha \in A$.

In 2005, Chen and Ru [2] proved a generalized Schmidt's subspace type theorem for fixed hyperplanes, which is the counterpart of a second main theorem given in [10].

Definition 1.2. Let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be a family of (fixed) hyperplanes in $\mathbb{P}^n(k)$ and $\mathcal{L} = \{L_1, \ldots, L_q\}$ be the set of pairwise linearly independent linear forms defining $H_1, \ldots, H_q$. We say $\mathcal{H}$ (or $\mathcal{L}$) is **non-degenerate** if

(i) $\dim(\mathcal{L})_k = n + 1$, where $(\mathcal{L})_k$ is the linear span of $\mathcal{L}$ over k;

(ii) for each proper nonempty subset $\mathcal{L}_1 \subset \mathcal{L}$, $(\mathcal{L}_1)_k \cap (\mathcal{L} \setminus \mathcal{L}_1)_k \cap \mathcal{L} \neq \emptyset$.

Theorem D. [2] Let k be a number field and let $S \subset M_k$ be a finite set containing M_k^∞. Let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be a set of hyperplanes in $\mathbb{P}^n(k)$. Assume that $\mathcal{H}$ is non-degenerate. Then, for every $\varepsilon > 0$,

$$(1 - \varepsilon)h(\mathbf{x}) \leq \sum_{j=1}^{q} n(2n - 1)N_S(H_j, \mathbf{x}) + O(1)$$

holds for every $\mathbf{x} \in \mathbb{P}^n(k)$ with $L_j(\mathbf{x}) \neq 0$ for all $j = 1, \ldots, q$.

Motivated by the development in Nevanlinna theory (see [5]), Liu [7] improved Theorem D as follows.

Theorem E. Assume that $\mathcal{H}$ is non-degenerate. Then, for every $\varepsilon > 0$,

$$(1 - \varepsilon)h(\mathbf{x}) \leq \sum_{j=1}^{q} N_S(H_j, \mathbf{x}) + O(1)$$

holds for every $\mathbf{x} \in \mathbb{P}^n(k)$ with $L_j(\mathbf{x}) \neq 0$ for all $j = 1, \ldots, q$.

We note that, in [5, 10], the second main theorems are established for moving hyperplanes. The purpose of this paper is to extend Theorem E to moving hyperplanes and give its application on decomposable form inequalities.

Our main results are stated as follows.

Theorem 1.1. Let k be a number field and let $S \subset M_k$ be a finite set containing M_k^∞. Let Λ be an infinite index set and let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be a set of moving hyperplanes indexed by Λ in $\mathbb{P}^n(k)$. Let $\mathbf{x} = [x_0 : \cdots : x_n] : \Lambda \to \mathbb{P}^n(k)$ be an infinite collection of points in $\mathbb{P}^n(k)$ (which implies $h(\mathbf{x}(\alpha))$ is unbounded). Assume that

(i) $\{H_1(\alpha), \ldots, H_q(\alpha)\}$ is non-degenerate for each $\alpha \in \Lambda$;

(ii) $h(H_j(\alpha)) = o(h(\mathbf{x}(\alpha)))$ for all $j = 1, \ldots, q$;

(iii) $L_j(\alpha)(\mathbf{x}(\alpha)) \neq 0$ for all $j = 1, \ldots, q$ and $\alpha \in \Lambda$.

Then, for every $\varepsilon > 0$, there exists an infinite index subset $A \subseteq \Lambda$ such that

$$\sum_{j=1}^{q} m_S(H_j(\alpha), \mathbf{x}(\alpha)) \leq (q - 1 + \varepsilon) h(\mathbf{x}(\alpha))$$

or

$$(1 - \varepsilon) h(\mathbf{x}(\alpha)) \leq \sum_{j=1}^{q} N_S(H_j(\alpha), \mathbf{x}(\alpha))$$

holds for all $\alpha \in A$.

2. Proof of Theorem 1.1

Since $\mathbf{x}(\alpha)$ is a collection of infinitely many points in $\mathbb{P}^n(k)$, we may assume $\mathbf{x}(\alpha)$ are pairwise distinct on Λ. There exists an infinite index subset $A \subseteq \Lambda$ which is coherent with respect to $\mathcal{H}$. If B is any infinite subset of A, then B is still coherent. Therefore, in the proof, we may freely pass to infinite subsets which are still denote by A for simplicity. From the assumption (ii) and $h(\mathbf{x}(\alpha)) \to \infty$, we also note that, for each $a \in \mathcal{R}_A$ and $v \in M_k$,

$$\log \|a(\alpha)\|_v \leq \sum_{v' \in M_k} \log^+ \|a(\alpha)\|_{v'} = h(a(\alpha)) \leq o(h(\mathbf{x}(\alpha))) \text{ for all } \alpha \in A.$$

Set $\mathcal{L} = \{L_1(\alpha), \ldots, L_q(\alpha)\}$ be the set of linear forms defining $\mathcal{H}$. By the assumption (i), for any $\alpha \in A$, we have $\dim(\mathcal{L}(\alpha))_k = n + 1$ and $(\mathcal{L}_1(\alpha))_k \cap (\mathcal{L}(\alpha) \setminus \mathcal{L}_1(\alpha))_k \cap \mathcal{L}(\alpha) \neq \emptyset$ for any nonempty subset $\mathcal{L}_1(\alpha) \subset \mathcal{L}(\alpha)$. Since there are only finitely many choices of $n+1$ linear forms in $\mathcal{L}$. We can find an infinite subset A and $\{j_1,\ldots,j_{n+1}\}\subset\{1,\ldots,q\}$ such that

$$L_{j_1}(\alpha),\ldots,L_{j_{n+1}}(\alpha)$$

are linearly independent over k for every $\alpha\in A$. For any subset $\mathcal{L}_1\subset\mathcal{L}$, by the same reason, we can find an integer m with $1\le m\le n+1$ such that $\dim(\mathcal{L}_1(\alpha))_k=m$ for each α. Moreover, we can find $\{l_1,\ldots,l_t\}\subset\{1,\ldots,q\}$ such that

$$(\mathcal{L}_1(\alpha))_k\cap(\mathcal{L}(\alpha)\setminus\mathcal{L}_1(\alpha))_k\cap\mathcal{L}(\alpha)=\{L_{l_1}(\alpha),\ldots,L_{l_t}(\alpha)\}$$

for each $\alpha\in A$.

By the above argument, we can pick out an infinite subset A satisfying:
i) $(\mathcal{L}(\alpha))_k=(L_{j_1}(\alpha),\ldots,L_{j_{n+1}}(\alpha))_k$ for all $\alpha\in A$, where $j_1,\ldots,j_{n+1}$ are independent of α;
ii) for any $\mathcal{L}_1\subset\mathcal{L}$, $(\mathcal{L}_1(\alpha))_k\cap(\mathcal{L}(\alpha)\setminus\mathcal{L}_1(\alpha))_k\cap\mathcal{L}(\alpha)=\{L_{l_1}(\alpha),\ldots,L_{l_t}(\alpha)\}$ for all $\alpha\in A$, where $l_1,\ldots,l_t$ and t depend only on $\mathcal{L}_1$.

By coherence of A, for each j, there exists $a_{j,i_j}(\alpha)$, one of the coefficients in $L_j(\alpha)$, such that $a_{j,i_j}(\alpha)\neq 0$ for all but finitely many $\alpha\in A$. Fix this $a_{j,i_j}(\alpha)$ and set

$$\widetilde{L}_j(\alpha)=\frac{a_{j,0}(\alpha)}{a_{j,i_j}(\alpha)}x_0+\cdots+\frac{a_{j,n}(\alpha)}{a_{j,i_j}(\alpha)}x_n.$$

Note that $\frac{a_{j,i}}{a_{j,i_j}}:=(\{\alpha\in A|a_{j,i_j}(\alpha)\neq 0\},\alpha\mapsto\frac{a_{j,0}(\alpha)}{a_{j,i_j}(\alpha)})$ defines an element of $\mathcal{R}_A$. Let $\widetilde{a}_{j,i}:=\frac{a_{j,i}}{a_{j,i_j}}$ for $i=0,\ldots,n$ and $\widetilde{L}_j:=\widetilde{a}_{j,0}x_0+\cdots+\widetilde{a}_{j,n}x_n$ be the linear form with coefficients in $\mathcal{R}_A$. Hence, $\widetilde{\mathcal{L}}=\{\widetilde{L}_1,\ldots,\widetilde{L}_q\}$ satisfies

(i) $\widetilde{\mathcal{L}}$ is pairwise linearly independent over $\mathcal{R}_A$;

(ii) $\dim(\widetilde{\mathcal{L}})_{\mathcal{R}_A}=n+1$;

(iii) for any proper nonempty subset $\widetilde{\mathcal{L}}_1\subset\widetilde{\mathcal{L}}$, $(\widetilde{\mathcal{L}}_1)_{\mathcal{R}_A}\cap(\widetilde{\mathcal{L}}\setminus\widetilde{\mathcal{L}}_1)_{\mathcal{R}_A}\cap\widetilde{\mathcal{L}}\neq\emptyset$.

Lemma 2.1. (Lemma 2.5 in [7] or Lemma 3.1 in [13]) There exists an integer $u\ge 1$ and disjoint subsets $I_1,\ldots,I_u$ of $\{\widetilde{L}_j\}_{j=1}^q$ with the following properties:
(a) $\{\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\}_{\widetilde{L}_j\in I_1}$ is minimal and $\{\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\}_{\widetilde{L}_j\in I_k}$ is linearly independent over $\mathcal{R}_A$ for $2\le k\le u$, where $\{\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\}_{\widetilde{L}_j\in I_1}$ is minimal means $\{\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\}_{\widetilde{L}_j\in I_1}$ is linearly dependent over $\mathcal{R}_A$ but each

nonempty proper subset of $\{\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\}_{\widetilde{L}_j\in I_1}$ is linearly independent over $\mathcal{R}_A$.

(b) u is the minimal positive integer such that

$$\Big(\{\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\}_{\widetilde{L}_j\in\bigcup_{k=1}^u I_k}\Big)_{\mathcal{R}_A}=\Big(\{\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\}_{j=1}^q\Big)_{\mathcal{R}_A}.$$

(c) For each l with $2\le l\le u$, there exists $c_j\in\mathcal{R}_A\setminus\{0\}$ such that

$$\sum_{\widetilde{L}_j\in I_l}c_j\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\in\Big(\{\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\}_{\widetilde{L}_j\in\bigcup_{k=1}^{l-1}I_k}\Big)_{\mathcal{R}_A}.$$

Proof. Since $\widetilde{L}_1\in(\widetilde{\mathcal{L}}\setminus\{\widetilde{L}_1\})_{\mathcal{R}_A}$, this implies that

$$\widetilde{L}_1(\alpha)(\mathbf{x}(\alpha))|_A\in\Big(\{\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\}_{j=2}^q\Big)_{\mathcal{R}_A}.$$

We can choose a subset I_1 of $\widetilde{\mathcal{L}}$ containing $\widetilde{L}_1$ such that $\{\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\}_{\widetilde{L}_j\in I_1}$ is minimal. Assume that $I_1=\{\widetilde{L}_1,\dots,\widetilde{L}_{t_1}\}$. Then there exists $c_j\in\mathcal{R}_A$, $1\le j\le t_1-1$, and $c_{t_1}=-1$ such that

$$\sum_{j=1}^{t_1}c_j\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A=0.$$

If $(\{\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\}_{\widetilde{L}_j\in I_1})_{\mathcal{R}_A}=(\{\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\}_{j=1}^q)_{\mathcal{R}_A}$, by taking $u=1$, then the proof is finished.

Otherwise, there is an $\widetilde{L}\in(I_1)_{\mathcal{R}_A}\cap(\widetilde{\mathcal{L}}\setminus I_1)_{\mathcal{R}_A}\cap\widetilde{\mathcal{L}}$. Then one of the following two cases holds:

(i) $\widetilde{L}\in\widetilde{\mathcal{L}}\setminus I_1$, and we may assume that $\widetilde{L}=\widetilde{L}_{t_1+1}\in(I_1)_{\mathcal{R}_A}$, i.e.,

$$\widetilde{L}_{t_1+1}(\alpha)(\mathbf{x}(\alpha))|_A\in\Big(\{\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\}_{\widetilde{L}_j\in I_1}\Big)_{\mathcal{R}_A}.$$

Put $I_2=\{\widetilde{L}_{t_1+1}\}$ and $c_{t_1+1}=1$.

(ii) $\widetilde{L}\in I_1$, since $\widetilde{L}\in(\widetilde{\mathcal{L}}\setminus I_1)_{\mathcal{R}_A}$, there exists a subset of $\widetilde{\mathcal{L}}\setminus I_1$, which may be assumed to be $\{\widetilde{L}_{t_1+1},\dots,\widetilde{L}_{t_2}\}$, such that $\{\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\}_{j=t_1+1}^{t_2}$ is linearly independent over $\mathcal{R}_A$ and

$$\widetilde{L}(\alpha)(\mathbf{x}(\alpha))|_A=\sum_{j=t_1+1}^{t_2}c_j\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A$$

with $c_j \in \mathcal{R}_A \setminus \{0\}$ for $j = t_1+1, \ldots, t_2$. Set $I_2 = \{\widetilde{L}_{t_1+1}, \ldots, \widetilde{L}_{t_2}\}$.

If $(\{\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\}_{\widetilde{L}_j \in I_1 \cup I_2})_{\mathcal{R}_A} = (\{\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\}_{j=1}^q)_{\mathcal{R}_A}$, then the proof is finished; Otherwise, repeating the above argument, we would get another subset I_3. By continuing this process, since $\dim(\widetilde{\mathcal{L}})_{\mathcal{R}_A} = n+1$ is finite, there exists an u and subsets $I_1, \ldots, I_u$ satisfying the assertions (a)–(c) of the lemma. $\square$

Remark 2.1. We may assume that the cardinality of I_l satisfies $\sharp I_1 \geq 3$ and $\sharp I_l \geq 2$ for $2 \leq l \leq u$.

(a) If $\sharp I_l = 1$, for some l with $2 \leq l \leq u$, i.e., $I_l = \{\widetilde{L}_{t_l}\}$, $\widetilde{L}_{t_l} \in \widetilde{\mathcal{L}}$, then $\widetilde{L}_{t_l}(\alpha)(\mathbf{x}(\alpha)) \in (\{\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\}_{\widetilde{L}_j \in \bigcup_{k=1}^{l-1} I_k})_{\mathcal{R}_A}$, so

$$\left(\{\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\}_{\widetilde{L}_j \in \bigcup_{k=1}^{l} I_k}\right)_{\mathcal{R}_A} = \left(\{\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\}_{\widetilde{L}_j \in \bigcup_{k=1}^{l-1} I_k}\right)_{\mathcal{R}_A},$$

and we can always delete I_l from $\{I_1, \ldots, I_u\}$, hence $\sharp I_l \geq 2$ for any l.

(b) If $\sharp I_1 = 2$, then $\widetilde{L}_1(\alpha)(\mathbf{x}(\alpha))|_A$ and $\widetilde{L}_2(\alpha)(\mathbf{x}(\alpha))|_A$ are linearly dependent over $\mathcal{R}_A$. If $u \geq 2$, we replace I_1 by $\{\widetilde{L}_1\} \cup I_2$ which is minimal. Otherwise, $(\widetilde{L}_1(\alpha)(\mathbf{x}(\alpha))|_A)_{\mathcal{R}_A} = (\{\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\}_{\widetilde{L}_j \in \widetilde{\mathcal{L}}})_{\mathcal{R}_A}$, which means that there exists $c_{k,l} \in \mathcal{R}_A$ such that

$$\widetilde{L}_k(\alpha)(\mathbf{x}(\alpha))|_A = c_{k,l}\widetilde{L}_l(\alpha)(\mathbf{x}(\alpha))|_A \tag{1}$$

for any $1 \leq k < l \leq q$. Since $\dim(\widetilde{\mathcal{L}})_{\mathcal{R}_A} = n+1$, (1) implies $\frac{x_i(\alpha)}{x_0(\alpha)}|_A \in \mathcal{R}_A$ for $i = 1, \ldots, n$. (Here, without loss of generality, we may assume $x_0(\alpha) \neq 0$.) Hence $h(\mathbf{x}(\alpha)) \leq \sum_{i=0}^n h(\frac{x_i(\alpha)}{x_0(\alpha)}) \leq o(h(\mathbf{x}(\alpha)))$, which is a contradiction.

For an integer N with $1 \leq N \leq u$, put $I := \bigcup_{k=1}^N I_k = \{\widetilde{L}_1, \ldots, \widetilde{L}_{t_N}\}$, $\sharp I = t_N$. By the assumption (iii), we consider

$$\mathbf{X}_I(\alpha) = [\widetilde{L}_1(\alpha)(\mathbf{x}(\alpha)) : \cdots : \widetilde{L}_{t_N}(\alpha)(\mathbf{x}(\alpha))], \ \alpha \in A,$$

which is the point in $\mathbb{P}^{\sharp I-1}(k)$, and $\{\mathbf{X}_I(\alpha)|\alpha \in A\}$ can be regarded as a map $\mathbf{X}_I : A \to \mathbb{P}^{\sharp I-1}(k)$. Now, we show the following conclusion for $\mathbf{X}_I(\alpha)$.

Lemma 2.2. For every $\varepsilon > 0$, there exists an infinite index subset A such that

$$\sum_{\widetilde{L}_j \in I} \sum_{v \in S} \log \frac{\|\mathbf{X}_I(\alpha)\|_v}{\|\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))\|_v} \leq (\sharp I - 1 + \varepsilon) h(\mathbf{X}_I(\alpha)) + o(h(\mathbf{x}(\alpha))) \tag{2}$$

holds for all $\alpha \in A$.

Proof. We now prove Lemma 2.2 by induction on N.

If $N=1$, then $I = I_1 = \{\widetilde{L}_1, \ldots, \widetilde{L}_{t_1}\} := I_0 \cup \{\widetilde{L}_{t_1}\}$. By the construction in the proof of Lemma 2.1, $\{\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\}_{\widetilde{L}_j \in I_1}$ is minimal and

$$\widetilde{L}_{t_1}(\alpha)(\mathbf{x}(\alpha)) = \sum_{j=1}^{t_1-1} c_j \widetilde{L}_j(\alpha)(\mathbf{x}(\alpha)),\ c_j \in \mathcal{R}_A \setminus \{0\}.$$

Take

$$\mathbf{X}'_{I_0}(\alpha) = [c_1\widetilde{L}_1(\alpha)(\mathbf{x}(\alpha)) : c_2\widetilde{L}_2(\alpha)(\mathbf{x}(\alpha)) : \cdots : c_{t_1-1}\widetilde{L}_{t_1-1}(\alpha)(\mathbf{x}(\alpha))]$$

which is a map from A to $\mathbb{P}^{t_1-2}(k)$. Let

$$H_j = \{x_{j-1} = 0\},\ j = 1, \ldots, t_1 - 1, \text{ and } H_{t_1} = \{x_0 + \cdots + x_{t_1-2} = 0\}$$

be fixed hyperplanes in $\mathbb{P}^{t_1-2}(k)$. By applying Schmidt's subspace theorem to points $\mathbf{X}'_{I_0}(\alpha)$ and H_j, $1 \le j \le t_1$, there is a finite collection $\mathcal{V}$ of proper linear subspaces of $\mathbb{P}^{t_1-2}(k)$ such that

$$\begin{aligned} &\sum_{j=1}^{t_1-1}\sum_{v\in S} \log \frac{\|\mathbf{X}'_{I_0}(\alpha)\|_v}{\|c_j\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))\|_v} + \sum_{v\in S} \log \frac{\|\mathbf{X}'_{I_0}(\alpha)\|_v}{\|\widetilde{L}_{t_1}(\alpha)(\mathbf{x}(\alpha))\|_v} \\ \le\ & (t_1 - 1 + \varepsilon) h(\mathbf{X}'_{I_0}(\alpha)) \end{aligned} \tag{3}$$

for all α such that $\mathbf{X}'_{I_0}(\alpha) \notin \bigcup_{V\in\mathcal{V}} V$. We note that $\{c_j\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))|_A\}_{j=1}^{t_1-1}$ is linearly independent over k. Hence, by passing to an infinite index subset satisfying $\mathbf{X}'_{I_0}(\alpha) \notin \bigcup_{V\in\mathcal{V}} V$, we may assume (3) holds for all $\alpha \in A$.

Now, we compare $h(\mathbf{X}'_{I_0}(\alpha))$ and $h(\mathbf{X}_{I_1}(\alpha))$. By

$$\begin{aligned} \|\mathbf{X}'_{I_0}(\alpha)\|_v &\le \max_{1\le i\le t_1-1} \|c_i\|_v \cdot \max_{1\le i\le t_1-1} \|\widetilde{L}_i(\alpha)(\mathbf{x}(\alpha))\|_v \\ &\le \max_{1\le i\le t_1-1} \|c_i\|_v \cdot \|\mathbf{X}_{I_1}(\alpha)\|_v, \end{aligned}$$

it yields

$$h(\mathbf{X}'_{I_0}(\alpha)) \le h(\mathbf{X}_{I_1}(\alpha)) + o(h(\mathbf{x}(\alpha))). \tag{4}$$

On the other hand, by $\widetilde{L}_{t_1}(\alpha)(\mathbf{x}(\alpha)) = \sum_{j=1}^{t_1-1} c_j\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))$, we have

$$\|\widetilde{L}_{t_1}(\alpha)(\mathbf{x}(\alpha))\|_v \le (t_1-1)^{N_v}\|\mathbf{X}'_{I_0}(\alpha)\|_v \le c\|\mathbf{X}'_{I_0}(\alpha)\|_v$$

for all $v \in S$, where c is a positive constant. Together with

$$\max_{1\le i\le t_1-1} \|\widetilde{L}_i(\alpha)(\mathbf{x}(\alpha))\|_v \le \max_{1\le i\le t_1-1} \left\|\frac{1}{c_i}\right\|_v \cdot \|\mathbf{X}'_{I_0}(\alpha)\|_v,$$

we deduce

$$\|\mathbf{X}_{I_1}(\alpha)\|_v = \max_{1\le i\le t_1} \|\widetilde{L}_i(\alpha)(\mathbf{x}(\alpha))\|_v \le \max\left\{\max_{1\le i\le t_1-1}\left\|\frac{1}{c_i}\right\|_v, c\right\} \cdot \|\mathbf{X}'_{I_0}(\alpha)\|_v,$$

which implies that

$$\sum_{v\in S} \log \|\mathbf{X}_{I_1}(\alpha)\|_v \le \sum_{v\in S} \log \|\mathbf{X}'_{I_0}(\alpha)\|_v + o(h(\mathbf{x}(\alpha))).$$

Therefore,

$$\begin{aligned}\sum_{j=1}^{t_1}\sum_{v\in S} \log \frac{\|\mathbf{X}_{I_1}(\alpha)\|_v}{\|\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))\|_v} &\le \sum_{j=1}^{t_1-1}\sum_{v\in S} \log \frac{\|\mathbf{X}'_{I_0}(\alpha)\|_v}{\|c_j\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))\|_v} \\ &+ \sum_{v\in S} \log \frac{\|\mathbf{X}'_{I_0}(\alpha)\|_v}{\|\widetilde{L}_{t_1}(\alpha)(\mathbf{x}(\alpha))\|_v} + o(h(\mathbf{x}(\alpha))). \end{aligned} \tag{5}$$

Combining (3), (4) and (5), we derive that

$$\sum_{j=1}^{t_1}\sum_{v\in S} \log \frac{\|\mathbf{X}_{I_1}(\alpha)\|_v}{\|\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))\|_v} \le (t_1 - 1 + \varepsilon)h(\mathbf{X}_{I_1}(\alpha)) + o(h(\mathbf{x}(\alpha))).$$

Lemma 2.2 is proved for $N = 1$.

Let us assume that Lemma 2.2 holds for some integer N with $1 \le N < u$. Let $I_{N+1}^c := \bigcup_{k=1}^{N} I_k$ and $I := \bigcup_{k=1}^{N+1} I_k = I_{N+1}^c \cup I_{N+1}$.

By induction, we obtain

$$\sum_{\widetilde{L}_j\in I_{N+1}^c}\sum_{v\in S} \log \frac{\|\mathbf{X}_{I_{N+1}^c}(\alpha)\|_v}{\|\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))\|_v} \le (t_N - 1 + \varepsilon)h(\mathbf{X}_{I_{N+1}^c}(\alpha)) + o(h(\mathbf{x}(\alpha))), \tag{6}$$

where $t_N = \sharp I_{N+1}^c$. By (c) of Lemma 2.1, set

$$\widetilde{L}(\alpha)(\mathbf{x}(\alpha)) := \sum_{j=t_N+1}^{t_{N+1}} c_j\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha)), \quad c_j \in \mathcal{R}_A \setminus \{0\}, \tag{7}$$

and we have $\widetilde{L}(\alpha)(\mathbf{x}(\alpha)) \in (\{\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))\}_{\widetilde{L}_j \in I_{N+1}^c})_{\mathcal{R}_A}$. There exists $c_j' \in \mathcal{R}_A$ such that

$$\widetilde{L}(\alpha)(\mathbf{x}(\alpha)) = \sum_{\widetilde{L}_j \in I_{N+1}^c} c_j' \widetilde{L}_j(\alpha)(\mathbf{x}(\alpha)). \tag{8}$$

Take

$$\mathbf{X}'_{I_{N+1}}(\alpha) = [c_{t_N+1}\widetilde{L}_{t_N+1}(\alpha)(\mathbf{x}(\alpha)) : \cdots : c_{t_{N+1}}\widetilde{L}_{t_{N+1}}(\alpha)(\mathbf{x}(\alpha))] \in \mathbb{P}^{\sharp I_{N+1}-1}(k).$$

Let

$$H_j = \{x_j = 0\},\ j = 0, \ldots, \sharp I_{N+1} - 1, \text{ and } H_{\sharp I_{N+1}} = \{x_0 + \cdots + x_{\sharp I_{N+1}-1} = 0\}$$

be fixed hyperplanes in $\mathbb{P}^{\sharp I_{N+1}-1}(k)$. Applying Schmidt's subspace theorem for $\mathbf{X}'_{I_{N+1}}(\alpha)$ and H_j, $0 \leq j \leq \sharp I_{N+1}$, we have

$$\begin{aligned} & \sum_{\widetilde{L}_j \in I_{N+1}} \sum_{v \in S} \log \frac{\|\mathbf{X}'_{I_{N+1}}(\alpha)\|_v}{\|c_j \widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))\|_v} + \sum_{v \in S} \log \frac{\|\mathbf{X}'_{I_{N+1}}(\alpha)\|_v}{\|\widetilde{L}(\alpha)(\mathbf{x}(\alpha))\|_v} \\ \leq\ & (\sharp I_{N+1} + \varepsilon) h(\mathbf{X}'_{I_{N+1}}(\alpha)) \end{aligned} \tag{9}$$

for all $\alpha \in A$. Denote by $\mathbf{X}_{I_{N+1}}(\alpha) = [\widetilde{L}_{t_N+1}(\alpha)(\mathbf{x}(\alpha)) : \cdots : \widetilde{L}_{t_{N+1}}(\alpha)(\mathbf{x}(\alpha))]$ and note that $\|\mathbf{X}_{I_{N+1}}(\alpha)\|_v \leq \max_{\widetilde{L}_i \in I_{N+1}} \|\frac{1}{c_i}\|_v \cdot \|\mathbf{X}'_{I_{N+1}}(\alpha)\|_v$, which leads to

$$\sum_{v \in S} \log \frac{\|\mathbf{X}_{I_{N+1}}(\alpha)\|_v}{\|\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))\|_v} \leq \sum_{v \in S} \log \frac{\|\mathbf{X}'_{I_{N+1}}(\alpha)\|_v}{\|c_j \widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))\|_v} + o(h(\mathbf{x}(\alpha)))$$

for $\widetilde{L}_j \in I_{N+1}$, and

$$\sum_{v \in S} \log \frac{\|\mathbf{X}_{I_{N+1}}(\alpha)\|_v}{\|\widetilde{L}(\alpha)(\mathbf{x}(\alpha))\|_v} \leq \sum_{v \in S} \log \frac{\|\mathbf{X}'_{I_{N+1}}(\alpha)\|_v}{\|\widetilde{L}(\alpha)(\mathbf{x}(\alpha))\|_v} + o(h(\mathbf{x}(\alpha))).$$

Further, similar to (4), $h(\mathbf{X}'_{I_{N+1}}(\alpha)) \leq h(\mathbf{X}_{I_{N+1}}(\alpha)) + o(h(\mathbf{x}(\alpha)))$. Thus, (9) can be rewritten as

$$\begin{aligned} & \sum_{\widetilde{L}_j \in I_{N+1}} \sum_{v \in S} \log \frac{\|\mathbf{X}_{I_{N+1}}(\alpha)\|_v}{\|\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))\|_v} + \sum_{v \in S} \log \frac{\|\mathbf{X}_{I_{N+1}}(\alpha)\|_v}{\|\widetilde{L}(\alpha)(\mathbf{x}(\alpha))\|_v} \\ \leq\ & (\sharp I_{N+1} + \varepsilon) h(\mathbf{X}_{I_{N+1}}(\alpha)) + o(h(\mathbf{x}(\alpha))). \end{aligned} \tag{10}$$

It follows from (6) and (10) that

$$\sum_{\widetilde{L}_j\in I}\sum_{v\in S}\log\frac{\|\mathbf{X}_I(\alpha)\|_v}{\|\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))\|_v}\le(\sharp I^c_{N+1}-1+\varepsilon)h(\mathbf{X}_{I^c_{N+1}}(\alpha))$$
$$+\sharp I^c_{N+1}\sum_{v\in S}\log\frac{\|\mathbf{X}_I(\alpha)\|_v}{\|\mathbf{X}_{I^c_{N+1}}(\alpha)\|_v}+(\sharp I_{N+1}+\varepsilon)h(\mathbf{X}_{I_{N+1}}(\alpha))$$
$$+\sharp I_{N+1}\sum_{v\in S}\log\frac{\|\mathbf{X}_I(\alpha)\|_v}{\|\mathbf{X}_{I_{N+1}}(\alpha)\|_v}-\sum_{v\in S}\log\frac{\|\mathbf{X}_{I_{N+1}}(\alpha)\|_v}{\|\widetilde{L}(\alpha)(\mathbf{x}(\alpha))\|_v}+o(h(\mathbf{x}(\alpha))).\quad(11)$$

Now, we consider the right hand side of (11). Firstly, we show that

$$\sum_{v\in S}\log\frac{\|\mathbf{X}_{I_{N+1}}(\alpha)\|_v}{\|\widetilde{L}(\alpha)(\mathbf{x}(\alpha))\|_v}+o(h(\mathbf{x}(\alpha)))\ge\sum_{v\in S}\log\frac{\|\mathbf{X}_I(\alpha)\|_v}{\|\mathbf{X}_{I^c_{N+1}}(\alpha)\|_v}.\qquad(12)$$

For every $\alpha\in A$, let $S_{1,\alpha}=\{v\in S:\|\mathbf{X}_I(\alpha)\|_v=\|\mathbf{X}_{I^c_{N+1}}(\alpha)\|_v\}$ and $S_{2,\alpha}=S\setminus S_{1,\alpha}$. For any $v\in S$, by (7) and (8), there is some positive constant c such that

$$\|\widetilde{L}(\alpha)(\mathbf{x}(\alpha))\|_v\le c\max_{\widetilde{L}_j\in I_{N+1}}\|c_j\|_v\cdot\|\mathbf{X}_{I_{N+1}}(\alpha)\|_v$$

and

$$\|\widetilde{L}(\alpha)(\mathbf{x}(\alpha))\|_v\le c\max_{\widetilde{L}_j\in I^c_{N+1}}\|c'_j\|_v\cdot\|\mathbf{X}_{I^c_{N+1}}(\alpha)\|_v$$

for all $\alpha\in A$. It follows that

$$\sum_{v\in S_{1,\alpha}}\log\frac{\|\mathbf{X}_{I_{N+1}}(\alpha)\|_v}{\|\widetilde{L}(\alpha)(\mathbf{x}(\alpha))\|_v}+o(h(\mathbf{x}(\alpha)))\ge 0=\sum_{v\in S_{1,\alpha}}\log\frac{\|\mathbf{X}_I(\alpha)\|_v}{\|\mathbf{X}_{I^c_{N+1}}(\alpha)\|_v}\qquad(13)$$

and

$$\begin{aligned}\sum_{v\in S_{2,\alpha}}\log\frac{\|\mathbf{X}_{I_{N+1}}(\alpha)\|_v}{\|\widetilde{L}(\alpha)(\mathbf{x}(\alpha))\|_v}+o(h(\mathbf{x}(\alpha)))&\ge\sum_{v\in S_{2,\alpha}}\log\frac{\|\mathbf{X}_{I_{N+1}}(\alpha)\|_v}{\|\mathbf{X}_{I^c_{N+1}}(\alpha)\|_v}\\&=\sum_{v\in S_{2,\alpha}}\log\frac{\|\mathbf{X}_I(\alpha)\|_v}{\|\mathbf{X}_{I^c_{N+1}}(\alpha)\|_v}.\quad(14)\end{aligned}$$

Hence, by (13) and (14), we proved (12). Next, we note that

$$(\sharp I^c_{N+1}-1+\varepsilon)h(\mathbf{X}_{I^c_{N+1}}(\alpha))+(\sharp I^c_{N+1}-1)\sum_{v\in S}\log\frac{\|\mathbf{X}_I(\alpha)\|_v}{\|\mathbf{X}_{I^c_{N+1}}(\alpha)\|_v}$$

$$\le (\sharp I_{N+1}^c - 1 + \varepsilon)h(\mathbf{X}_{I_{N+1}^c}(\alpha)) + (\sharp I_{N+1}^c - 1)\sum_{v\in M_k}\log\frac{\|\mathbf{X}_I(\alpha)\|_v}{\|\mathbf{X}_{I_{N+1}^c}(\alpha)\|_v}$$
$$\le (\sharp I_{N+1}^c - 1)h(\mathbf{X}_I(\alpha)) + \varepsilon h(\mathbf{X}_{I_{N+1}^c}(\alpha)) \le (\sharp I_{N+1}^c - 1 + \varepsilon)h(\mathbf{X}_I(\alpha)). \quad (15)$$

Similarly,

$$\begin{aligned}&(\sharp I_{N+1} + \varepsilon)h(\mathbf{X}_{I_{N+1}}(\alpha)) + \sharp I_{N+1}\sum_{v\in S}\log\frac{\|\mathbf{X}_I(\alpha)\|_v}{\|\mathbf{X}_{I_{N+1}}(\alpha)\|_v}\\ \le\ &(\sharp I_{N+1} + \varepsilon)h(\mathbf{X}_I(\alpha)).\end{aligned} \quad (16)$$

(11), (12), (15) and (16) yield

$$\begin{aligned}&\sum_{\widetilde{L}_j\in I}\sum_{v\in S}\log\frac{\|\mathbf{X}_I(\alpha)\|_v}{\|\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))\|_v}\\ \le\ &(\sharp I_{N+1} + \varepsilon)h(\mathbf{X}_I(\alpha)) + (\sharp I_{N+1}^c - 1 + \varepsilon)h(\mathbf{X}_I(\alpha)) + o(h(\mathbf{x}(\alpha)))\\ =\ &(\sharp I - 1 + 2\varepsilon)h(\mathbf{X}_I(\alpha)) + o(h(\mathbf{x}(\alpha))).\end{aligned}$$

Therefore, Lemma 2.2 is proved. □

We continue the proof of Theorem 1.1. From Lemma 2.2, we can pick $N = u$, $I = \bigcup_{k=1}^u I_k$.

By the fact $\dim(\widetilde{\mathcal{L}})_{\mathcal{R}_A} = n+1$, there exist $n+1$ elements of $\widetilde{\mathcal{L}}$, we may assume they are $\widetilde{L}_1, \dots, \widetilde{L}_{n+1}$, which are linearly independent. By solving the linear system

$$\widetilde{a}_{j,0}x_0(\alpha) + \cdots + \widetilde{a}_{j,n}x_n(\alpha) = \widetilde{L}_j(\alpha)(\mathbf{x}(\alpha)),\ 1 \le j \le n+1,$$

we obtain

$$x_i(\alpha) = c_{i,1}\widetilde{L}_1(\alpha)(\mathbf{x}(\alpha)) + \cdots + c_{i,n+1}\widetilde{L}_{n+1}(\alpha)(\mathbf{x}(\alpha)),\ 0 \le i \le n,$$

with $c_{i,j} \in \mathcal{R}_A$. By (b) of Lemma 2.1, $x_i(\alpha) = \sum_{\widetilde{L}_j\in I} c'_{i,j}\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))$, $0 \le i \le n$, with $c'_{i,j} \in \mathcal{R}_A$. Clearly, there is a positive constant c such that $\|x_i(\alpha)\|_v \le c \cdot \max_{\widetilde{L}_j\in I}\|c'_{i,j}\|_v \cdot \|\mathbf{X}_I(\alpha)\|_v$, $0 \le i \le n$, for all $\alpha \in A$. We have

$$\sum_{v\in S}\log\frac{\|\mathbf{x}(\alpha)\|_v}{\|\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))\|_v} \le \sum_{v\in S}\log\frac{\|\mathbf{X}_I(\alpha)\|_v}{\|\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))\|_v} + o(h(\mathbf{x}(\alpha))),\ \widetilde{L}_j \in I. \quad (17)$$

On the other hand, $\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha)) = \sum_{i=0}^{n} \widetilde{a}_{i,j} x_i(\alpha)$, $\widetilde{L}_j \in I$. Similarly,

$$\|\mathbf{X}_I(\alpha)\|_v \le c \cdot \max_{0\le i\le n, \widetilde{L}_j \in I} \|\widetilde{a}_{i,j}\|_v \cdot \|\mathbf{x}(\alpha)\|_v$$

for some positive c, which means that

$$h(\mathbf{X}_I(\alpha)) \le h(\mathbf{x}(\alpha)) + o(h(\mathbf{x}(\alpha))). \tag{18}$$

By (2), (17) and (18), we have

$$\begin{aligned}
\sum_{\widetilde{L}_j\in I}\sum_{v\in S} \log \frac{\|\mathbf{x}(\alpha)\|_v}{\|\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))\|_v} &\le \sum_{\widetilde{L}_j\in I}\sum_{v\in S} \log \frac{\|\mathbf{X}_I(\alpha)\|_v}{\|\widetilde{L}_j(\alpha)(\mathbf{x}(\alpha))\|_v} + o(h(\mathbf{x}(\alpha))) \\
&\le (\sharp I - 1 + \varepsilon) h(\mathbf{X}_I(\alpha)) + o(h(\mathbf{x}(\alpha))) \\
&\le (\sharp I - 1 + \varepsilon) h(\mathbf{x}(\alpha)) + o(h(\mathbf{x}(\alpha))).
\end{aligned}$$

This is equivalent to

$$\sum_{j=1}^{q} m_S(H_j(\alpha), \mathbf{x}(\alpha)) \le (q - 1 + \varepsilon) h(\mathbf{x}(\alpha)) + o(h(\mathbf{x}(\alpha))) \text{ for } \alpha \in A.$$

3. Decomposable form inequalities

In this section, we will give an application of our main theorem on the decomposable form inequalities.

Let S be a finite subset of M_k containing M_k^∞. For $\mathbf{x} = (x_0, \dots, x_m) \in k^{m+1}$, we also define the S-**height** as

$$H_S(\mathbf{x}) = \prod_{v\in S} \|\mathbf{x}\|_v,$$

and the **logarithmic** S-**height** as $h_S(\mathbf{x}) = \log H_S(\mathbf{x})$. If $\mathbf{x} \in \mathcal{O}_S^{m+1} \setminus \{\mathbf{0}\}$, then $H_S(\mathbf{x}) \ge 1$ and $H_S(\alpha \mathbf{x}) = H_S(\mathbf{x})$ for all $\alpha \in \mathcal{O}_S^*$.

Let $F(x_0, \dots, x_m)$ be a homogeneous polynomial in $m + 1 (\ge 2)$ variables with coefficients in k. For each finite set S of places of k containing M_k^∞, and for given two positive real numbers c and λ, we consider the solutions of the inequality

$$0 < \prod_{v\in S} \|F(x_0, \dots, x_m)\|_v \le c H_S^\lambda(x_0, \dots, x_m) \text{ for } (x_0, \cdots, x_m) \in \mathcal{O}_S^{m+1}. \tag{19}$$

If $\mathbf{x}$ is a solution of (19), then so is $\mathbf{x}' = \eta\mathbf{x}$ for every $\eta \in \mathcal{O}_S^*$. Such solutions $\mathbf{x}$, $\mathbf{x}'$ are called **$\mathcal{O}_S^*$-proportional**.

Definition 3.1. Let $F(x_0, \ldots, x_m)$ be a form (homogeneous polynomial) in $m + 1 (\geq 2)$ variables with coefficients in k. F is **decomposable** if it can be factorized into linear factors over some finite extension of k.

As a consequence of Theorem C, Győry and Ru [6] studied integer solution of a sequence of decomposable form inequalities.

Theorem F. [6] Let q, m be positive integers. Let c, λ be real numbers with $c > 0$, $\lambda < q - 2m$ and k' be a finite extension of k. For $n = 1, 2, \ldots$, let $F_n(\mathbf{x}) = F_n(x_0, \ldots, x_m) \in \mathcal{O}_S[\mathbf{x}]$ denote a decomposable form of degree q which can be factorized into linear factors over k', and suppose that these factors are in general position for each n. Then there does not exist an infinite sequence of $\mathcal{O}_S^*$-nonproportional $\mathbf{x}_n \in \mathcal{O}_S^{m+1}$, $n = 1, 2, \ldots$, for which

$$0 < \prod_{v \in S} \|F_n(\mathbf{x}_n)\|_v \leq c H_S^{\lambda}(\mathbf{x}_n)$$

and

$$h(F_n) = o(h(\mathbf{x}_n)) \text{ if } h(\mathbf{x}_n) \to \infty \text{ as } n \to \infty.$$

Using Theorem E, Liu [7] proved a conjecture posed in [2] as follows.

Theorem G. Let q, m be positive integers. Let c, λ be real numbers with $c > 0$, $\lambda < 1$ and k' be a finite extension of k. Let $F(x_0, \ldots, x_m)$ be a decomposable form of degree q with coefficients in $\mathcal{O}_S$ which can be factorized into linear factors over k', and suppose that these factors are non-degenerate (over k'). Then (19) has only finitely many $\mathcal{O}_S^*$-nonproportional solutions.

Now, we will show Theorem G remains valid for the sequences of decomposable form inequalities.

Theorem 3.1. Let q, m be positive integers. Let c, λ be real numbers with $c > 0$, $\lambda < 1$ and k' be a finite extension of k. For $n = 1, 2, \ldots$, let $F_n(\mathbf{x}) = F_n(x_0, \ldots, x_m) \in \mathcal{O}_S[\mathbf{x}]$ denote a decomposable form of degree q which can be factorized into linear factors over k', and suppose that these factors are non-degenerate (over k') for each n. Then there does not exist an infinite sequence of $\mathcal{O}_S^*$-nonproportional $\mathbf{x}_n \in \mathcal{O}_S^{m+1}$, $n = 1, 2, \ldots$, for which

$$0 < \prod_{v \in S} \|F_n(\mathbf{x}_n)\|_v \leq c H_S^{\lambda}(\mathbf{x}_n) \tag{20}$$

and

$$h(F_n) = o(h(\mathbf{x}_n)) \text{ if } h(\mathbf{x}_n) \to \infty \text{ as } n \to \infty. \tag{21}$$

Proof. Let $S' \subset M_{k'}$ consist of the extension of the places of S to k', then every S-integer in k is also an S'-integer in k'. Moreover, we have $H_S(\mathbf{x}_n) = H_{S'}(\mathbf{x}_n)$ and $\prod_{v\in S} \|F_n(\mathbf{x}_n)\|_v = \prod_{w\in S'} \|F_n(\mathbf{x}_n)\|_w$ for $\mathbf{x}_n \in \mathcal{O}_S^{m+1}$. So (20) is preserved when we work on k'. Therefore, for simplicity, we assume that $k' = k$.

Assume that there is an infinite sequence $\mathbf{x}_n = (x_{0,n}, \ldots, x_{m,n}) \in \mathcal{O}_S^{m+1}$ which satisfies (20).

First we consider the case that the values $h(\mathbf{x}_n)$ are bounded. Without loss of generality, we may assume that $x_{0,n} \neq 0$ for each n. Then $h(\mathbf{x}_n/x_{0,n})$ are bounded and this implies that there are infinitely many n's such that $\mathbf{x}_n = x_{0,n}\mathbf{x}_0$ for some $\mathbf{x}_0 \in k^{m+1}$. For these n's we deduce from (20) that

$$0 < (\prod_{v\in S} \|x_{0,n}\|_v)^q \prod_{v\in S} \|F_n(\mathbf{x}_0)\|_v \leq c(\prod_{v\in S} \|x_{0,n}\|_v)^\lambda H_S^\lambda(\mathbf{x}_0)$$

and hence $\prod_{v\in S} \|x_{0,n}\|_v$ are bounded. Since $x_{0,n} \in \mathcal{O}_S$, it follows that there are infinitely many n's for which $x_{0,n} = \eta_n x_0'$ with some $\eta_n \in \mathcal{O}_S^*$ and fixed $x_0' \in \mathcal{O}_S$ (see [4]). This implies that for these n's the $\mathbf{x}_n$ considered above are $\mathcal{O}_S^*$-proportional, which is a contradiction.

Next we consider the case that $h(\mathbf{x}_n)$ are not bounded. We may assume that $h(\mathbf{x}_n) \to \infty$, $n \to \infty$, and $\mathbf{x}_n/x_{0,n}$ is distinct for each n. Then, by assumption, (21) also holds. Further it follows that $H_S(\mathbf{x}_n) \to \infty$ as $n \to \infty$. By $\max_j h(L_{j,n}) \leq h(F_n) + O(1)$ and (21), we have $h(L_{j,n}) = o(h(\mathbf{x}_n))$ for each j. On the other hand, by Theorem 1.1, there is an infinite subsequence of $\{\mathbf{x}_n\}$, which may be denoted by $\{\mathbf{x}_n\}$ itself, such that

$$\sum_{v\in S}\sum_{j=1}^{q} \log \frac{\|\mathbf{x}_n\|_v \cdot \|L_{j,n}\|_v}{\|L_{j,n}(\mathbf{x}_n)\|_v} \leq (q-1+\varepsilon)h(\mathbf{x}_n) \leq \log H_S^{(q-1+\varepsilon)}(\mathbf{x}_n).$$

Hence

$$\frac{H_S(\mathbf{x}_n)^q \cdot \prod_{v\in S}\prod_{j=1}^q \|L_{j,n}\|_v}{\prod_{v\in S} \|F_n(\mathbf{x}_n)\|_v} \leq H_S^{(q-1+\varepsilon)}(\mathbf{x}_n),$$

where $\prod_{v\in S}\prod_{j=1}^q \|L_{j,n}\|_v \geq c' \prod_{v\in S} \|F_n\|_v \geq c'$ for some positive constant c'.

Furthermore, it follows from (20) that

$$H_S(\mathbf{x}_n)^q \quad \leq \quad \frac{1}{c'}\prod_{v\in S} \|F_n(\mathbf{x}_n)\|_v \cdot H_S^{(q-1+\varepsilon)}(\mathbf{x}_n) \leq \frac{c}{c'} H_S^{(\lambda+q-1+\varepsilon)}(\mathbf{x}_n).$$

Hence $H_S^{(1-\lambda-\varepsilon)}(\mathbf{x}_n) \leq c/c'$, this is a contradiction. $\square$

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