Journal of Number Theory 163 (2016) 61–74

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Joint universality and generalized strong recurrence with rational parameter

Łukasz Pańkowski [a,b,*,1]

[a] *Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland*
[b] *Graduate School of Mathematics, Nagoya University, Nagoya, 464-8602, Japan*

ARTICLE INFO

Article history:
Received 5 September 2015
Received in revised form 13 November 2015
Accepted 13 November 2015
Available online 8 January 2016
Communicated by David Goss

MSC:
primary 11M41

Keywords:
Strong recurrence
Self-approximation
Riemann zeta function
Riemann hypothesis

ABSTRACT

We prove that, for every rational $d \neq 0, \pm 1$ and every compact set $K \subset \{s \in \mathbb{C} : 1/2 < \mathrm{Re}(s) < 1\}$ with connected complement, any analytic non-vanishing functions f_1, f_2 on K can be approximated, uniformly on K, by the shifts $\zeta(s + i\tau)$ and $\zeta(s + id\tau)$, respectively. As a consequence we deduce that the set of τ satisfying $|\zeta(s + i\tau) - \zeta(s + id\tau)| < \varepsilon$ uniformly on K has a positive lower density for every $d \neq 0$.

* Correspondence to: Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland.
E-mail address: lpan@amu.edu.pl.
[1] The author is an International Research Fellow of the Japan Society for the Promotion of Science and the work was partially supported by (JSPS) KAKENHI grant no. 26004317 and the grant no. 2013/11/B/ST1/02799 from the National Science Centre.

http://dx.doi.org/10.1016/j.jnt.2015.11.005
0022-314X/

1. Introduction

In 1981 Bagchi [1] discovered an interesting connection between the Riemann Hypothesis and Voronin's universality theorem (see [18]) for the Riemann zeta function $\zeta(s)$. Namely, he proved that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > \frac{1}{2}$ if and only if for every compact set $K \subset \{s \in \mathbb{C} : \frac{1}{2} < \mathrm{Re}(s) < 1\}$ with connected complement and every $\varepsilon > 0$ we have

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \max_{s\in K} |\zeta(s+i\tau) - \zeta(s)| < \varepsilon\right\} > 0, \tag{1}$$

where meas$\{\cdot\}$ denotes the real Lebesgue measure. In the language of topological dynamics (see [6]) (1) is called the *strong recurrence* property for the Riemann zeta function.

Bagchi's observation was extended to the case of Dirichlet L-functions by himself in [2] and [3], and to the case of general universal L-functions, for which the Generalized Riemann Hypothesis is expected, in [17, Theorem 8.4].

Nakamura [10] suggested the following related problem: find all d such that for every compact set $K \subset \{s \in \mathbb{C} : \frac{1}{2} < \mathrm{Re}(s) < 1\}$ with connected complement and every $\varepsilon > 0$ we have

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \max_{s\in K} |\zeta(s+i\tau) - \zeta(s+id\tau)| < \varepsilon\right\} > 0. \tag{2}$$

This property can be called *generalized strong recurrence* with parameter d. However, it should be noted that sometimes in the literature it is called also the self-approximation property with parameter d. Using this notion Bagchi's result states that the Riemann Hypothesis is equivalent to the generalized strong recurrence property for $\zeta(s)$ with parameter $d = 0$.

Nakamura, in the same paper, gave the partial answer to this question by proving that (2) holds if d is algebraic irrational. He also observed that the generalized strong recurrence property holds for almost all real parameters d. His result was improved by the author in [13] to all irrational parameter d. The positive answer for non-zero rational d was claimed by Garunkštis [5] and Nakamura [11]. Unfortunately, their arguments have a gap, which was pointed out by Nakamura and Pańkowski [12] and partially filled, in the same paper, for all non-zero rational $d = \frac{a}{b}$ with $\gcd(a,b) = 1$ and $|a - b| \neq 1$.

The crucial step in the proof of the generalized strong recurrence property with parameter d is to show that the following set

$$\{\log p : p \text{ is prime}\} \cup \{d \log p : p \text{ is prime}\} \tag{3}$$

is linearly independent over $\mathbb{Q}$. This was proved for all algebraic irrational d and for almost all d by Nakamura [10]. Moreover, by using the six exponential theorem from the theory of transcendental numbers, the author noticed in [13] that for a given irrational d only a finite number of primes p can possibly be involved in the linear dependence of (3). This allowed to prove the following joint universality theorem, which easily implies the generalized strong recurrence property. It was showed by Nakamura for algebraic irrational d and by the author for all irrational d.

Theorem A. *Let d be irrational, $K \subset \{s \in \mathbb{C} : \frac{1}{2} < \mathrm{Re}(s) < 1\}$ be a compact set with connected complement and f, g be continuous non-vanishing functions on K, which are analytic in the interior of K. Then, for every $\varepsilon > 0$, we have*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \begin{array}{l} \max_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon \\ \max_{s\in K} |\zeta(s+id\tau) - g(s)| < \varepsilon \end{array}\right\} > 0. \tag{4}$$

The above joint universality theorem is also related to the following open problem introduced by Andreas Weiermann in 2008 during the conference "New Directions in the Theory of Universal Zeta- and L-Functions" in Wuřzburg:

Assume a, b are transcendental and algebraically independent and functions f, g satisfy the assumptions of the universality theorem. Can we find one single real τ such that f is approximated by $\zeta(s + ia\tau)$ and g is approximated by $\zeta(s + ib\tau)$ and both approximations are uniformly as usual?

Thus, Theorem A implies that the above open problem is true even for linearly independent real non-zero numbers a, b.

The case when d is rational is more delicate, since one can easily observe that the set (3) is linearly dependent over $\mathbb{Q}$, even if we exclude a finite number of primes. So, in order to prove (2) for rational $d = \frac{a}{b}$ with $|a - b| \neq 1$ and $\gcd(a, b) = 1$, Nakamura and the author [12] proved only that (4) holds for one common function $f = g$ depending on d. Moreover, by the lack of linear independence over $\mathbb{Q}$ of (3), it was expected that Theorem A with arbitrary given functions f, g cannot hold for rational d. However, in the present paper we introduce the approach which allows to overcome the fact that (3) is not linearly independent over $\mathbb{Q}$ and we prove the following joint universality theorem. This theorem solves completely Weiermann's problem, since it is obvious that we cannot expect a positive answer if $a = b$ or $a = -b$ by the fact that $\zeta(\overline{s}) = \overline{\zeta(s)}$.

Theorem 1.1. *Let $a, b \in \mathbb{Z} \setminus \{0\}$ with $\frac{a}{b} \neq \pm 1$. Assume that $K \subset \{s \in \mathbb{C} : \frac{1}{2} < \mathrm{Re}(s) < 1\}$ is a compact set with connected complement and f_a, f_b are non-vanishing continuous on K and analytic in the interior of K. Then, for every $\varepsilon > 0$, we have*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \max_{c\in\{a,b\}} \max_{s\in K} |\zeta(s+ic\tau) - f_c(s)| < \varepsilon\right\} > 0.$$

Remark 1.2. It should be mentioned that the above theorem can be easily generalized to the wide class of L-functions, for which a universality theorem is proved by Voronin's approach (for example for a wide class introduced in [8, Chapter VII, Section 3.1] or [7, Section 3]).

Moreover, the above theorem together with Theorem A can be treated as a new method how to approximate more than one function by certain modifications of one zeta or L-function. Indeed, the above results say that if we desire to approximate two analytic non-vanishing functions f, g by a given L-function $L(s)$ it suffices to consider the shifts $L(s + i\tau)$, $L(s + id\tau)$, where d is non-zero real number $\neq \pm 1$. A different well-known method of this kind is to consider twists of $L(s)$ with pairwise non-equivalent Dirichlet characters (see [17, Theorem 12.8]).

As an immediate consequence of Theorem 1.1 we obtain the following corollary.

Corollary 1.3. *Let $d \neq 0, \pm 1$ be a rational number. Assume that $K \subset \{s \in \mathbb{C} : \frac{1}{2} < \mathrm{Re}(s) < 1\}$ is a compact set with connected complement and f, g are non-vanishing continuous on K and analytic in the interior of K. Then, for every $\varepsilon > 0$, we have*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \begin{array}{l} \max_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \\ \max_{s\in K} |\zeta(s + id\tau) - g(s)| < \varepsilon \end{array} \right\} > 0.$$

Obviously, taking $f \equiv g$ in the above corollary proves (2) for all rational $d \neq 0, \pm 1$. However, for $d = 1$ the inequality (2) holds trivially, and the generalized strong recurrence property for $d = -1$ is implied by $\overline{\zeta(s + i\tau)} = \zeta(\overline{s} - i\tau)$ and the fact that, by Voronin's theorem, we have $\max_{s\in K\cup K_c} |\zeta(s + i\tau) - 1| < \varepsilon$, where $K_c := \{\overline{s} : s \in K\}$. Therefore, the following result holds.

Theorem 1.4. *Let $d \neq 0$ be a real number and $K \subset \{s \in \mathbb{C} : \frac{1}{2} < \mathrm{Re}(s) < 1\}$ be a compact set with connected complement. Then, for every $\varepsilon > 0$, we have*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \max_{s\in K} |\zeta(s + i\tau) - \zeta(s + id\tau)| < \varepsilon \right\} > 0.$$

The above theorem reduces Nakamura's problem to the case when $d = 0$, which, as we mentioned before, is equivalent to the Riemann Hypothesis.

2. Denseness lemma

The so-called denseness lemma (see Lemma 2.4 below) plays a crucial role in the proof of our main theorem, and, essentially, contains the main idea of this paper how to overcome the lack of linear independence of (3). In order to prove it we need the following lemmas concerning analytic functions of exponential type.

Lemma 2.1. *(See [9, Lemma 6].) Let $G(z)$ be an analytic function satisfying*

$$0 \not\equiv G(z) = \sum_{m=0}^{\infty} \frac{\alpha_m}{m!} z^m, \qquad |\alpha_m| < A^m$$

for some positive constant A. Let $c_1 > 0$ and N_1 be a positive integer. Then there exist a positive c_2 and a positive integer $N_2 > N_1$ such that for any sufficiently large x the *interval* $[x, x + c_1 x^{-N_1}]$ *contains a subinterval* I *of length* $|I| \geq c_2 x^{-N_2}$ *such that* $G(t)$ *has no zeros on* I*.*

Lemma 2.2. *Let* $U \subset \mathbb{C}$ *be a simply connected bounded smooth Jordan domain such that its closure* $\overline{U}$ *is a subset of* D*. Assume that for all* $s \in U$ *we have* $1/2 < \sigma_1 < \mathrm{Re}\, s < \sigma_2 < 1$ *and* g_1*,* g_2 *are non-zero elements of the Bergman space* $B^2(U) = \{f \in L^2(U) : f$ *is holomorphic on* $U\}$*. For* $z \in \mathbb{C}$ *we put*

$$G_j(z) = \iint\limits_U e^{-sz}\overline{g_j(s)}d\sigma dt, \qquad j = 1, 2.$$

Then for every η *with* $0 < \eta < \pi/2$ *there exist a sequence* x_n *tending to* ∞ *and intervals* $I_n \subset [x_n, x_n + 1]$ *of length* $|I_n| \geq Bx_n^{-N}$ $(N > 0,\ B := B(U, \eta) > 0)$*, such that for all* $t \in I_n$ *we have*

$$|G_1(t)| \gg e^{-\sigma_2 x_n}$$

and, moreover, the argument of $G_1(t)$ *and* $G_2(t)$ *on* I_n *varies less than* η*.*

Proof. Firstly, let us find a sequence x_n. Notice that $G_1 \not\equiv 0$, since otherwise, taking the n-th derivative of G_1 at point $z = 0$ and using the fact that the linear space of polynomials is dense in the Bergman space (see for example [15, Theorem 7.2.2]), we get the contradiction with $g \neq 0$. On the other hand, $G_1(z) \ll e^{c|z|}$ for some positive constant c depending on U, and for sufficiently small $\omega = \omega(U) > 0$ and for all complex z with $|\arg(-z)| \leq \omega$ we have

$$|e^{\sigma_2 z} G_1(z)| \ll 1.$$

Hence, by [7, Lemma 3], which proof based on the Phragmén–Lindelöf principle, there exists a real sequence x_n tending to ∞ such that

$$|G_1(x_n)| \gg e^{-\sigma_2 x_n}.$$

Let us fix n and put $x = x_n$. As in the proof of [7, Lemma 4] one can prove that for $t \in [x, x + 1]$ and every $C > 0$ we have

$$G_1(t) = P(t) + O(e^{-Ct}),$$

where $P(t)$ is a polynomial of degree $\ll x$.

Let $x_0 \in [x, x + 1]$ be such that $|P(x_0)| = \max_{x \leq t \leq x+1} |P(t)|$. Then by Markoff's inequality (see e.g. [16]) we have

$$\max_{x \leq t \leq x+1} |P'(t)| \ll x^2 |P(x_0)|$$

and hence for $t \in [x, x+1]$ satisfying $|t - x_0| \le \frac{B_0}{x_0^2}$ with sufficiently small $B_0 > 0$ we have

$$\begin{aligned} |P(x_0)| - |P(t)| &\le |P(t) - P(x_0)| \le \max_{x \le t \le x+1} |P'(t)||t - x_0| \ll B_0 |P(x_0)| \\ &\le \sin\left(\frac{\eta}{2}\right) |P(x_0)|, \end{aligned} \tag{5}$$

so

$$|P(t)| \ge \left(1 - \sin\left(\frac{\eta}{2}\right)\right) |P(x_0)| \ge \left(1 - \sin\left(\frac{\eta}{2}\right)\right) |P(x)|.$$

Therefore, for $t \in I_0 := [x, x+1] \cap [x_0 - B_0/x_0^2, x_0 + B_0/x_0^2]$ it holds

$$\begin{aligned} \left(1 - \sin\left(\frac{\eta}{2}\right)\right) |G_1(x)| &\le \left(1 - \sin\left(\frac{\eta}{2}\right)\right) |P(x)| + O\left(e^{-Cx}\right) \\ &\le |P(t)| + O\left(e^{-Cx}\right) \le |G_1(t)| + O\left(e^{-Cx}\right) \end{aligned}$$

and hence

$$|G_1(t)| \ge \left(1 - \sin\left(\frac{\eta}{2}\right)\right) |G_1(x)| + O\left(e^{-Cx}\right) \gg e^{-\sigma_2 x}.$$

Using again (5) we get that

$$\begin{aligned} \left| \frac{G_1(t)}{G_1(x_0)} - 1 \right| &\le \frac{|P(t) - P(x_0)| + O(e^{-Cx})}{|G_1(x_0)|} \le \frac{\sin\left(\frac{\eta}{2}\right) |P(x_0)| + O(e^{-Cx})}{|G_1(x_0)|} \\ &\le \sin\left(\frac{\eta}{2}\right) + O(e^{-(C - \sigma_2)x}). \end{aligned}$$

Thus $\left| \arg \frac{G_1(t)|}{G_1(x_0)} \right| \le \eta$ on I_0 for sufficiently large x.

Now, we use Lemma 2.1 to find subinterval I of I_0 such that $\arg G_2(t)$ varies at most η on I. The fact that g_2 is analytic implies (see the proof of [8, Lemma 7.1]) that

$$G_2(z) = \sum_{m=0}^{\infty} \frac{\alpha_m}{m!} z^m, \qquad |\alpha_m| < A^m$$

for some $A > 0$. Moreover, since $g_2 \ne 0$, we have $G_2 \not\equiv 0$.

Let us define $\beta_m = \frac{\alpha_m + \overline{\alpha_m}}{2}$, $\gamma_m = \frac{\alpha_m - \overline{\alpha_m}}{2i}$ and put

$$G_2^1(z) = \sum_{m=0}^{\infty} \frac{\beta_m}{m!} z^m, \qquad G_2^2(z) = \sum_{m=0}^{\infty} \frac{\gamma_m}{m!} z^m.$$

Then for any real t we have $G_2^1(t) = \operatorname{Re} G_2(t)$ and $G_2^2(t) = \operatorname{Im} G_2(t)$.

Now, by Lemma 2.1 to $G_2^1(z)$ and $G_2^2(z)$, we can find subinterval $I_1 \subset I$ of length $|I_1| \geq B_1 x^{-N_1}$ such that $\operatorname{Re} G_2(t)$ and $\operatorname{Im} G_2(t)$ have no zeros on I_1. Therefore, there exists $k_1 \in \{0,1,2,3\}$ such that for $t \in I_1$ we have

$$\frac{k_1}{2}\pi \leq \arg G_2(t) \leq \frac{k_1+1}{2}\pi,$$

so the argument of $G_2(t)$ on I_1 varies less than $\pi/2$.

Next, repeating the above argument for $G_3(z) = \exp(-\frac{k_1 \pi i}{2} - \frac{\pi i}{4})G_2(z)$ instead of G_2, gives that there is a subinterval $I_2 \subset I_1$ of lenght $|I_2| \geq B_2 x^{-M_2}$ such that for $t \in I_2$ we have

$$\frac{k_1}{2}\pi + \frac{k_2}{4}\pi \leq \arg G_2(t) \leq \frac{k_1}{2}\pi + \frac{k_2+1}{4}\pi$$

for suitable $k_2 \in \{0,1\}$, and hence the argument of $G_2(t)$ on I_2 varies less than $\pi/4$.

Thus, applying this reasoning sufficiently many times we can prove that there is an interval $I \subset I_0$ of length $|I| \geq Bx^{-N}$ such that the argument of $G_2(t)$ varies less than η, and the proof is complete. $\square$

In the sake of simplicity, for a finite set M of prime numbers and for real numbers θ_p, $p \in M$, define

$$\zeta_M(s,(\theta_p) = \prod_{p\in M}\left(1 - \frac{e(\theta_p)}{p^s}\right)^{-1},$$

where, as usual, $e(t) = \exp(2\pi i t)$. Moreover, let us call an open bounded subset U of $\mathbb{C}$ admissible when for every sufficiently small positive ε the set $U_\varepsilon := \{s \in \mathbb{C} : \exists_{s_0\in U}|s-s_0| < \varepsilon\}$ has connected complement.

Now we are ready to formulate and prove the denseness lemma, which proof based on the following generalization of the classical Riemann rearrangement theorem.

Lemma 2.3. *(See [14].) Let H be a real Hilbert space and let $u_n \in H$ be such that $\sum_{n=1}^\infty \|u_n\|^2 < \infty$. Assume that for every $e \in H$ with $\|e\| = 1$ the series $\sum_{n=1}^\infty \langle u_n, e\rangle$ are conditionally convergent after suitable permutation of terms. Then, for every $v \in H$ there exists a permutation (n_k) such that $\sum_{k=1}^\infty u_{n_k} = v$.*

Lemma 2.4. *Let U be an admissible set satisfying $\overline{U} \subset \{s \in \mathbb{C} : 1/2 < \operatorname{Re}(s) < 1\}$ and $a, b \in \mathbb{Z}\setminus\{0\}$ with $a \neq \pm b$. Then there exists a sequence θ_p of real numbers indexed by primes such that for any analytic non-vanishing functions f_a, f_b on $\overline{U}$, $\varepsilon > 0$ and $y > 0$ there exists a finite set of primes M containing all primes $p \leq y$ such that*

$$\max_{c\in\{a,b\}}\max_{s\in\overline{U}}|\zeta_M(s,(c\theta_p)) - f_c(s)| < \varepsilon.$$

Proof. Let U_1 be a simply connected smooth Jordan domain such that U_1 is admissible, f_a, f_b are analytic non-vanishing on $\overline{U_1}$ and $\overline{U} \subset U_1 \subset \overline{U_1} \subset \{s \in \mathbb{C} : \sigma_1 < \mathrm{Re}(s) < \sigma_2\}$ for suitable σ_1, σ_2 with $1/2 < \sigma_1 < \sigma_2 < 1$. For suitable analytic g_a, g_b we have $f_c = \exp g_c$ for $c \in \{a, b\}$.

Without loss of generality we can assume that $|a| < |b|$. Define

$$u_p(s) = \left(-\log\left(1 - \frac{e(a\theta_p)}{p^s}\right), -\log\left(1 - \frac{e(b\theta_p)}{p^s}\right)\right)$$

and

$$u_p^*(s) = \left(\frac{e(a\theta_p)}{p^s}, \frac{e(b\theta_p)}{p^s}\right),$$

where $\theta_{p_n} = \frac{n}{l}$, p_n denotes the n-th prime number and l is a positive integer depending on a, b, which we choose later.

We shall use Lemma 2.3 for the real Hilbert space $B^2(U_1) \times B^2(U_1)$ with the inner product given by

$$\langle \phi, \psi \rangle = \sum_{j=1}^{2} \mathrm{Re} \iint\limits_{U_1} \phi_j(s)\overline{\psi_j(s)} d\sigma dt$$

for $\phi = (\phi_1, \phi_2)$, $\psi = (\psi_1, \psi_2)$.

We are going to prove that for any $\phi = (\phi_1, \phi_2) \in B^2(U_1) \times B^2(U_1)$ with $||\phi|| = 1$ there exists a permutation of the series $\sum_p u_p(s)$, which converges to ϕ. Then, using the fact that $|f(z)| \leq \frac{||f||}{\sqrt{\pi}\,\mathrm{dist}(z, \partial U_1)}$ for any analytic function f and z lying in the interior of U_1 (see [4, Chapter III, Lemma 1.1]), we get that approximation with respect to L^2 norm on U_1 implies uniform approximation on U, provided $\overline{U} \subset U_1$, which completes the proof.

Obviously, since $\mathrm{Re}(s) > \sigma_1 > 1/2$ for every $s \in \overline{U_1}$, we have $\sum_p ||u_p(s)||^2 < \infty$. Hence it suffices to prove that there are two permutations of the series $\sum_p \langle u_p(s), \phi\rangle$ tending to $+\infty$ and $-\infty$, respectively. In fact, we show only the existence of a permutation of the series, which diverges to $+\infty$, since the case $-\infty$ is similar and can be left to the reader. Moreover, let us observe that it is sufficient to prove it for $u_p^*(s)$ instead of $u_p(s)$, since $\sum_p (u_p(s) - u_p^*(s))$ converges absolutely for $\mathrm{Re}(s) > 1/2$.

The case when $\phi_1 = 0$ or $\phi_2 = 0$ can be treated in the same way, so without loss of generality assume that $\phi_2 = 0$. Then we have to show that some permutation of the series

$$\sum_p \langle u_p^*, \phi \rangle = \sum_p \mathrm{Re}\, e(a\theta_p) \iint\limits_{U_1} \frac{1}{p^s}\overline{\phi_1(s)} d\sigma dt$$

diverges to $+\infty$.

By Lemma 2.2 we can show that there are infinitely many intervals $I = [x, x + Bx^{-N}]$ with $B, N > 0$ such that $G(\log p) = \iint_{U_1} p^{-s}\overline{\phi_1(s)}d\sigma dt \gg \exp(-\sigma_2 x)$ and $|\arg G(\log p) - \omega_1| \leq \frac{\pi}{4}$ for suitable $\omega_1 \in [-\pi, \pi]$, provided $\log p \in I$. Hence for sufficiently large $l > 0$ there is an integer k with $0 \leq k < l$ such that $\arg e(ak/l)G(\log p) \in [-\pi/3, \pi/3]$, which implies that, if $\log p_n \in I$ and $n \equiv k \bmod l$, then $\operatorname{Re} e(e\theta_p)G(\log p) \geq c_1 \exp(-\sigma_2 x)$ for some $c_1 > 0$. This together with $\sigma_2 < 1$ and the fact that the number of primes p_n satisfying $\log p_n \in I$ and $n \equiv k \bmod l$ is $\gg e^x/x^{N+2}$ shows that there is a permutation (n_k) such that $\sum_k \langle u^*_{p_{n_k}}, \phi\rangle = +\infty$.

Next let us consider the case when $\phi_1 \neq 0$ and $\phi_2 \neq 0$. We have to show that there is a permutation (n_k) such that

$$\sum_k \langle u^*_{p_{n_k}}, \phi\rangle = \sum_k \operatorname{Re} e(a\theta_{p_{n_k}})G_1(\log p_{n_k}) + \operatorname{Re} e(b\theta_{p_{n_k}})G_2(\log p_{n_k}) = +\infty,$$

where

$$G_j(z) = \iint\limits_{U_1} e^{-sz}\overline{\phi_j(s)}d\sigma dt, \qquad j = 1, 2.$$

Again, by Lemma 2.2, we see that there exist infinitely many intervals $I = [x, x + Bx^{-N}]$ with $B, N > 0$ such that

$$G_1(t) \gg e^{-\sigma_2 x}, \qquad t \in I$$

and for every $\eta > 0$ there are $\omega_1, \omega_2 \in [-\pi, \pi]$ such that

$$|\arg G_j(t) - \omega_j| \leq \eta, \qquad t \in I,\ j = 1, 2.$$

We shall show that for sufficiently large l there is k with $0 \leq k < l$ such that for $t \in I$ we have

$$\arg e(ak/l)G_1(t) \in \left[-\frac{\pi}{2} + \eta, \frac{\pi}{2} - \eta\right] \qquad \text{and} \qquad \arg e(bk/l)G_2(t) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \tag{6}$$

Then for p_n satisfying $\log p_n \in I$ and $n \equiv k \bmod l$ we have

$$\operatorname{Re} e(a\theta_{p_n})G_1(\log p_n) \geq c_1 e^{-\sigma_2 x} \quad (c_1 > 0) \qquad \text{and} \qquad \operatorname{Re} e(b\theta_{p_n})G_2(\log p_n) \geq 0.$$

Hence

$$\sum_{\substack{\log p_n \in I \\ n \equiv k \bmod l}} \langle u^*_{p_n}, \phi\rangle \geq c_1' \frac{e^{(1-\sigma_2)x}}{x^{N+2}}$$

for some positive constant c_1'.

In order to prove the existence of such k, notice that for every θ with $|2\pi a\theta+\omega_1| \le \pi/2-2\eta$ we have

$$\arg e(a\theta)G_1(t) \in \left[-\frac{\pi}{2}+\eta, \frac{\pi}{2}-\eta\right], \qquad t \in I,$$

and

$$\left|2\pi b\theta+\frac{b}{a}\omega_1\right| = \frac{|b|}{|a|}|2\pi a\theta+\omega_1| \le \frac{|b|}{|a|}\frac{\pi}{2}-\frac{|b|}{|a|}2\eta.$$

From the assumption $|b| > |a|$ it is easy to observe that for sufficiently small $\eta := \eta(a,b) > 0$ the right hand side is at least $\frac{\pi}{2}+\frac{\pi}{4|a|}$. Hence the set

$$\mathcal{A} = \{2\pi b\theta+\omega_2 : |2\pi a\theta+\omega_1| \le \pi/2-2\eta\}$$

covers all values in the interval

$$\left[-\frac{|b|}{|a|}\frac{\pi}{2}+\frac{|b|}{|a|}2\eta+\omega_2-\frac{b}{a}\omega_1, \frac{|b|}{|a|}\frac{\pi}{2}-\frac{|b|}{|a|}2\eta+\omega_2-\frac{b}{a}\omega_1\right]$$

of length $\ge \pi+\frac{\pi}{2|a|}$. Then, for sufficiently small η, the set $\mathcal{A}$ contains an interval of size $\ge \frac{\pi}{4|a|}$ for which $\arg e(b\theta)G_2(t) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $t \in I$. Therefore, the set of θ satisfying

$$\arg e(a\theta)G_1(t) \in \left[-\frac{\pi}{2}+\eta, \frac{\pi}{2}-\eta\right] \qquad \text{and} \qquad \arg e(b\theta)G_2(t) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

has the measure $\ge \frac{1}{8|b||a|}$, so for sufficiently small l, depending only on a and b, this set contains a rational number of the form $\frac{k^*}{l}$, and (6) holds by taking $k \equiv k^* \bmod l$ with $0 \le k < l$. $\square$

3. Proof of Theorem 1.1

In order to prove Theorem 1.1 we shall use some results from the theory of diophantine approximation.

Let us recall that a vector $\mathbf{x} \in \mathbb{R}^n$ belongs to $\gamma \subset \mathbb{R}^n$ mod 1 if there exists a vector $\mathbf{y} \in \mathbb{Z}^n$ such that $\mathbf{x}-\mathbf{y} \in \gamma$.

Theorem B *(Kronecker). Let $\alpha_1, \ldots, \alpha_n$ be real numbers linearly independent over $\mathbb{Q}$ and γ be a subregion of the n-dimensional unit cube with Jordan measure $m(\gamma)$. Then*

$$\lim_{T\to\infty}\frac{1}{T}\operatorname{meas}\{\tau \in (0,T] : (\alpha_1\tau, \ldots, \alpha_n\tau) \in \gamma \bmod 1\} = m(\gamma).$$

Proof. This is [8, Theorem A.8.1]. □

We say that the curve $\gamma(\tau) : [0, \infty] \to \mathbb{R}^n$ is *uniformly distributed mod* 1 in $\mathbb{R}^n$ if for every α_j, β_j, $j = 1, 2, \ldots, n$, with $0 \le \alpha_j < \beta_j \le 1$ we have

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas}\{\tau \in (0, T] : \gamma(\tau) \in [\alpha_1, \beta_1] \times \cdots \times [\alpha_n, \beta_n] \bmod 1\} = \prod_{j=1}^{n} (\beta_j - \alpha_j).$$

Lemma 3.1. *Let $\gamma(\tau)$ be uniformly distributed mod 1 in $\mathbb{R}^n$ and X be a closed and Jordan measurable subregion of the unit cube in $\mathbb{R}^n$. Suppose that Ω is a family of complex-valued continuous functions defined on X. If Ω is uniformly bounded and equivcontinuous, then, uniformly on Ω, we have*

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_{A_T} f(\{\gamma(\tau)\}) d\tau = \int \cdots \int\limits_{X} f(x_1, \ldots, x_n) \prod_{j=1}^{n} dx_j,$$

where A_T denotes the set of $\tau \in (0, T]$ such that $\gamma(\tau) \in X$ mod 1 and $\{\gamma(\tau)\}$ denotes the fractional part of $\gamma(\tau)$.

Proof. This is [8, Theorem A.8.3]. □

Moreover, we shall need the following result due to Mergelyan.

Theorem C *(Mergelyan). Let K be a compact set with connected complement and $f(s)$ be continuous on K and analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $P(s)$ such that*

$$\max_{s\in K} |f(s) - P(s)| < \varepsilon.$$

Proof. See [4, Chapter III]. □

Proof of Theorem 1.1. By Mergelyan's theorem it suffices to assume that f_a, f_b are polynomials without zeros on K. Then we can find an admissible set U such that f_a, f_b have no zeros on the closure of U and $K \subset U \subset \overline{U} \subset \{s \in \mathbb{C} : \frac{1}{2} < \operatorname{Re}(s) < 1\}$.

Let us fix $\varepsilon > 0$. By Lemma 2.4, for any $y > 0$, there exists a finite set of primes M containing all primes $p \le y$ and the sequence θ_p, $p \in M$, such that

$$\max_{c\in\{a,b\}} \max_{s\in\overline{U}} |\zeta_M(s; c\theta_p) - f_c(s)| < \frac{\varepsilon}{2}.$$

Moreover, if

$$\max_{p\in M} \left\| \tau \frac{\log p}{2\pi} - \theta_p \right\| < \delta \qquad \text{for sufficiently small } \delta, \tag{7}$$

then

$$\max_{c\in\{a,b\}}\max_{p\in M}\left\|\tau\frac{c\log p}{2\pi}-c\theta_p\right\|<c\delta,$$

and, by continuity,

$$\max_{c\in\{a,b\}}\max_{s\in\overline{U}}|\zeta_M(s+ic\tau;\mathbf{0})-f_c(s)|<\varepsilon; \tag{8}$$

here $\mathbf{0}$ is the sequence of zeros and $||\cdot||$ denotes the distance to the nearest integer.

Put $Q:=\{p:p\le z\}$ for $z>y$ such that $M\subset Q$ and define the set

$$\mathcal{D}:=\{(\omega_p)_{p\in Q}:\max_{p\in M}\|\omega_p-\theta_p\|<\delta\}.$$

Next let us consider

$$S=\sum_{c\in\{a,b\}}\frac{1}{T}\int\limits_{A_T}\left(\iint\limits_{U}|\zeta(s+ic\tau)-\zeta_M(s+ic\tau;\mathbf{0})|^2\,d\sigma dt\right)d\tau,$$

where $T>1$ and A_T is the set of $\tau\in[0,T]$ satisfying (7).

By Cauchy–Schwarz inequality, we get $S\le 2S_1+2S_2$, where

$$S_1=\sum_{c\in\{a,b\}}\frac{1}{T}\int\limits_{A_T}\left(\iint\limits_{U}|\zeta_Q(s+ic\tau,\mathbf{0})-\zeta_M(s+ic\tau;\mathbf{0})|^2\,d\sigma dt\right)d\tau$$

and

$$S_2=\sum_{c\in\{a,b\}}\frac{1}{T}\int\limits_{A_T}\left(\iint\limits_{U}|\zeta(s+ic\tau)-\zeta_Q(s+ic\tau;\mathbf{0})|^2\,d\sigma dt\right)d\tau.$$

By the unique factorization of integers, the curve $\gamma(\tau)=(\tau\frac{\log p}{2\pi})_{p\in Q}$ is uniformly distributed modulo 1 on $\mathbb{R}^{\pi(z)}$. Hence, by Lemma 3.1 and the fact that there is only restriction on ω_p with $p\in M$ in the definition of the set $\mathcal{D}$, we have

$$\begin{aligned}&\lim_{T\to\infty}\frac{1}{T}\int\limits_{A_T}\left|\zeta_Q\left(s,\left(\tau\frac{c\log p}{2\pi}\right)\right)-\zeta_M\left(s;\left(\tau\frac{c\log p}{2\pi}\right)\right)\right|^2d\tau\\&=\int\cdots\int\limits_{\mathcal{D}}|\zeta_M(s;(c\omega_p))|^2\left|\zeta_{Q\setminus M}(s;(c\omega_p))-1\right|^2\prod_{p\in Q}d\omega_p\end{aligned}$$

$$\le \left(\max_{s\in\overline{U}} |f_c(s)| + \varepsilon\right)^2 \int_{\mathcal{D}} \cdots \int \left|\zeta_{Q\setminus M}(s;(c\omega_p)) - 1\right|^2 \prod_{p\in Q} d\omega_p$$

$$\ll m(\mathcal{D}) \int_0^1 \ldots \int_0^1 \left|\zeta_{Q\setminus M}(s;(c\omega_p)) - 1\right|^2 \prod_{p\in Q\setminus M} d\omega_p.$$

Moreover, by easy calculation and the fact that $Q \setminus M$ contains only primes greater than y, one can show that

$$\int_0^1 \cdots \int_0^1 \left|\zeta_{Q\setminus M}(s;(c\omega_p)) - 1\right|^2 \prod_{p\in Q\setminus M} d\omega_p \le \sum_{n>y} \frac{1}{n^{2\sigma}}, \qquad s = \sigma + it \in \overline{U}.$$

Therefore, since $\sigma > \frac{1}{2}$ for all $s \in \overline{U}$, we have

$$S_1 \le \frac{1}{4} m(\mathcal{D})\varepsilon^2$$

for sufficiently large $y > 0$.

Now, using the well-known estimate for the mean-square of the Riemann zeta function and Carlson's theorem (see [8, Theorem A.2.10]) gives that

$$S_2 \le \frac{1}{4} m(\mathcal{D})\varepsilon^2$$

for sufficiently large $z > 0$, and we have

$$S \le m(\mathcal{D})\varepsilon^2.$$

On the other hand, we know that the sequence $\frac{\log p}{2\pi}$, $p \in Q$, is linearly independent over $\mathbb{Q}$, so by the Kronecker approximation theorem, we get

$$\lim_{T\to\infty} \frac{1}{T} \int_{A_T} d\tau = m(\mathcal{D}).$$

Thus, by (8), one can show that the set of $\tau \in A_T$ satisfying

$$\max_{c\in\{a,b\}} \iint_U |\zeta(s + ic\tau) - f_c(s)|^2 d\sigma dt \ll \varepsilon^2$$

has measure $\gg T$. Therefore, since approximation with respect to $L^2(U)$-norm implies uniform approximation on $K \subset U$ (see e.g. [4, Chapter I, Section 1, Lemma 1]), the proof is complete. □

References

[1] B. Bagchi, The statistical behavior and universality properties of the Riemann Zeta-Function and other allied Dirichlet series, PhD thesis, Indian Statistical Institute, Calcutta, 1981.
[2] B. Bagchi, A joint universality theorem for Dirichlet L-functions, Math. Z. 181 (1982) 319–334.
[3] B. Bagchi, Recurrence in topological dynamics and the Riemann hypothesis, Acta Math. Hungar. 50 (1987) 227–240.
[4] D. Gaier, Vorlesungen über Approximation im Komplexen, Birkhäuser, Basel, 1980.
[5] R. Garunkštis, Self-approximation of Dirichlet L-functions, J. Number Theory 131 (2011) 1286–1295.
[6] W.H. Gottschalk, G.A. Hedlund, Recursive properties of topological transformation groups, Bull. Amer. Math. Soc. 52 (1946) 488–489.
[7] J. Kaczorowski, M. Kulas, On the non-trivial zeros off the critical line for L-functions from extended Selberg class, Monatsh. Math. 150 (2007) 217–232.
[8] A.A. Karatsuba, S.M. Voronin, The Riemann Zeta Function, de Gruyter, Berlin, 1992.
[9] H. Mishou, Joint universality theorems for pairs of automorphic zeta functions, Math. Z. 277 (2014) 1113–1154.
[10] T. Nakamura, The joint universality and the generalized strong recurrence for Dirichlet L-functions, Acta Arith. 138 (2009) 357–362.
[11] T. Nakamura, The generalized strong recurrence for non-zero rational parameters, Arch. Math. 95 (2010) 549–555.
[12] T. Nakamura, Ł. Pańkowski, Erratum to: The generalized strong recurrence for non-zero rationals parameters, Arch. Math. 99 (2012) 43–47.
[13] Ł. Pańkowski, Some remarks on the generalized strong recurrence for L-functions, in: New Directions in Value Distribution Theory of Zeta and L-Functions, in: Ber. Math., Shaker Verlag, Aachen, 2009, pp. 305–315.
[14] D.V. Pecherskii, On rearrangements of terms in functional series, Sov. Math. Dokl. 14 (1973) 633–636.
[15] H. Queffélec, M. Queffélec, Diophantine Approximation and Dirichlet Series, HRI Lecture Notes Series, vol. 2, AMS, 2013.
[16] A.C. Schaeffer, Inequalities of A. Markoff and S. Bernstein for polynomials and related functions, Bull. Amer. Math. Soc. 47 (1941) 565–579.
[17] J. Steuding, Value-Distribution of L-functions, Springer, Berlin, 2007.
[18] S.M. Voronin, Theorem on the universality of the Riemann zeta function, Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975) 475–486 (in Russian); Math. USSR, Izv. 9 (1975) 443–453.