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General Section

Depth-graded motivic Lie algebra ☆

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A R T I C L E I N F O

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A B S T R A C T

In this paper we suggest a way to understand the structure of depth-graded motivic Lie subalgebra generated by the depth one part for the neutral Tannakian category mixed Tate motives over $\mathbb{Z}$. We will show that from an isomorphism conjecture proposed by K. Tasaka we can deduce the F. Brown's matrix conjecture and the nondegeneracy conjecture about depth-graded motivic Lie subalgebra generated by the depth one part.

1. Introduction

Denote by ζ_N an N-th primitive root of unity. Let $\mathcal{MT}(\mathbb{Z}[\zeta_N][1/N])$ denote the category of mixed Tate motives unramified over $\mathbb{Z}[\zeta_N][1/N]$. By the main result of [7], the motivic fundamental groupoid of $\mathbb{P}^1 - \{0, \mu_N, \infty\}$ can be realized in the category of $\mathcal{MT}(\mathbb{Z}[\zeta_N][1/N])$.

We call the Lie algebra of the maximal pro-unipotent subgroup of motivic fundamental group of $\mathcal{MT}(\mathbb{Z}[\zeta_N][1/N])$ the motivic Lie algebra of $\mathcal{MT}(\mathbb{Z}[\zeta_N][1/N])$. From

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Proposition 2.3 in [7], the motivic Lie algebra of $\mathcal{MT}(\mathbb{Z}[\zeta_N][1/N])$ is a graded free Lie algebra.

Since the Tannakian sub-category generated by the function ring of the motivic fundamental groupoid of $\mathbb{P}^1 - \{0, \mu_N, \infty\}$ is $\mathcal{MT}(\mathbb{Z}[\zeta_N][1/N]$ for $N = 1$ by [3] and for $N = 2, 3, 4, 6, 8$ by [6], from [7] we know that the motivic Lie algebra of $\mathcal{MT}(\mathbb{Z}[\zeta_N][1/N])$ has an induced depth filtration for $N = 1, 2, 3, 4, 6, 8$.

In [6] Deligne proved that the depth-graded motivic Lie algebra of $\mathcal{MT}(\mathbb{Z}[\zeta_N][1/N])$ is a free Lie algebra bi-graded by weight and depth for $N = 2, 3, 4, 6, 8$. While the structure of depth-graded motivic Lie algebra of $\mathcal{MT}(\mathbb{Z})$ is not fully understood up to now.

Schneps gave the structure of depth-graded motivic Lie algebra of $\mathcal{MT}(\mathbb{Z})$ in depth two [12]. Goncharov's work [10] gave the structure of the depth-graded motivic Lie algebra of $\mathcal{MT}(\mathbb{Z})$ in depth three. Brown gave some conjectural description of the structure of the depth-graded motivic Lie algebra of $\mathcal{MT}(\mathbb{Z})$ in all depths in [4].

It is widely believed that the Lie subalgebra of the depth-graded motivic Lie algebra generated by the depth one part only has the period polynomial relations in depth two among the generators (in [11] we call this statement the nondegeneracy conjecture). In fact, this statement follows from Brown's homological conjecture (see [9]). In this paper, we will show that from an isomorphism conjecture of K. Tasaka [13] we can deduce Brown's matrix conjecture and the nondegeneracy conjecture. Thus we reduce the well-known nondegeneracy conjecture to a purely linear algebra problem which probably is easier to handle.

The problems we tackle are purely combinatorial and algebraic. Some standard techniques in Lie algebra also play key roles in establishing our main results.

From the analysis of Section 4 in [9], our results give partial evidence to Brown's homological conjecture about depth-graded motivic Lie algebra in [4]. Since Brown's homological conjecture implies Broadhurst-Kreimer conjecture for motivic multiple zeta values, we see that Tasaka's isomorphism conjecture plays a vital role to understand motivic Broadhurst-Kreimer conjecture.

2. Mixed Tate motives and multiple zeta values

2.1. Mixed Tate motives

Denote by $\mathcal{MT}(\mathbb{Z})$ the category of mixed Tate motives over $\mathbb{Z}$. The references about mixed Tate motives are [5], [7]. $\mathcal{MT}(\mathbb{Z})$ is a neutral Tannakian category over $\mathbb{Q}$. Denote by $\pi_1(\mathcal{MT}(\mathbb{Z}))$ the fundamental group of $\mathcal{MT}(\mathbb{Z})$, then we have

$$\pi_1(\mathcal{MT}(\mathbb{Z})) = \mathbb{G}_m \ltimes U,$$

where U is pro-unipotent algebraic group with free Lie algebra generated by the formal symbol σ_{2n+1} in weight $2n + 1$ for $n \geq 1$.

By [7], the motivic fundamental groupoid of $\mathbb{P}^1 - \{0, 1, \infty\}$ can be realized in the category $\mathcal{MT}(\mathbb{Z})$.

Denote by ${}_0\Pi_1$ the motivic fundamental groupoid of $\mathbb{P}^1 - \{0, 1, \infty\}$ from the tangential base point $\overrightarrow{1}_0$ at 0 to the tangential base point $\overrightarrow{-1}_1$ at 1. Its ring of regular functions over $\mathbb{Q}$ is

$$\mathcal{O}({}_0\Pi_1) = \mathbb{Q}\langle e^0, e^1\rangle,$$

where $\mathbb{Q}\langle e^0, e^1\rangle$ is equipped with the shuffle product.

Denote by ${}_x\Pi_y$ the de-Rham realization of motivic fundamental groupoid of $\mathbb{P}^1\backslash\{0, 1, \infty\}$ from x to y, where $x, y \in \{\overrightarrow{1}_0, \overrightarrow{-1}_1\}$. We write $\overrightarrow{1}_0, \overrightarrow{-1}_1$ as 0, 1 respectively for short. Denote by G the group of automorphisms of the groupoid ${}_x\Pi_y$ for $x, y \in 0, 1$ which respect to the following structures:

(1) (Groupoid structure) The composition maps

$${}_x\Pi_y \times {}_y\Pi_z \to {}_x\Pi_z$$

for all $x, y, z \in \{0, 1\}$.

(2) (Inertia) The automorphism fixes the elements

$$\exp(e_0) \in {}_0\Pi_0(\mathbb{Q}),\ \exp(e_1) \in {}_1\Pi_1(\mathbb{Q}),$$

where e_0, e_1 respectively denotes the differential $\frac{dz}{z}, \frac{dz}{1-z}$.

From Proposition 5.11 in [7], it follows that ${}_x\Pi_y$ is a G-torsor. We have a natural morphism

$$\varphi : \mathrm{U} \to G \simeq {}_0\Pi_1.$$

From [3] φ is injective. Denote by $\mathfrak{g}$ the corresponding Lie algebra of U^{dR}, we have an injective map

$$i : \mathfrak{g} \to \operatorname{Lie} G \simeq (\mathbb{L}(e_0, e_1), \{\ ,\ \}),$$

where $(\mathbb{L}(e_0, e_1), \{\ ,\ \})$ is the free Lie algebra generated by e_0, e_1 with the following Ihara Lie bracket

$$\{f, g\} = [f, g] + D_f(g) - D_g(f)$$

and D_f is a derivation on $\mathbb{L}(e_0, e_1)$ which satisfies $D_f(e_0) = 0,\ D_f(e_1) = [e_1, f]$ for $f \in \mathbb{L}(e_0, e_1)$.

We denote by $\mathfrak{h}$ the Lie algebra $(\mathbb{L}(e_0, e_1), \{\ ,\ \})$ for short. There is a natural decreasing depth filtration on $\mathfrak{h}$ defined by

$$\mathfrak{D}^r\mathfrak{h} = \{\xi \in \mathfrak{h} \mid \deg_{e_1} \xi \geq r\}.$$

Define the weight grading by the total degree of e_0, e_1 for the elements of $\mathfrak{h}$. From the injective map i, there is an induced depth filtration on $\mathfrak{g}$, define

$$\mathfrak{dg} = \oplus_{r\geq 1}\mathfrak{D}^r\mathfrak{g}/\mathfrak{D}^{r+1}\mathfrak{g}$$

with induced Lie bracket as depth graded motivic Lie algebra of $\mathcal{MT}(\mathbb{Z})$. Denote by $\mathfrak{dg}_r$ the depth r part of $\mathfrak{dg}$. By Théorème 6.8(i) in [7], we have $i(\sigma_{2n+1}) = (\mathrm{ad}\, e_0)^{2n}(e_1)+$ terms of degree ≥ 2 in e_1. So $\mathfrak{dg}_1$ is essentially the $\mathbb{Q}$-linear combinations of $\overline{\sigma}_{2n+1} = (\mathrm{ad}\, e_0)^{2n}(e_1)$, $n \geq 1$ in $\mathfrak{h}$.

Here we give the definition of restricted even period polynomial:

Definition 2.1. For $N \geq 3$, the restricted even period polynomial of weight N is the polynomial $p(x_1, x_2)$ of degree $N-2$ which satisfies

(i) $p(x_1, 0) = 0$, i.e. p is restricted;
(ii) $p(\pm x_1, \pm x_2) = p(x_1, x_2)$, i.e. p is even;
(iii) $p(x_1, x_2) + p(x_1 - x_2, x_1) - p(x_1 - x_2, x_2) = 0$.

Denote by $\mathbb{P}_N$ the set of even restricted period polynomials of weight N.

For a $\mathbb{Q}$-vector space, denote by $\mathrm{Lie}(V)$ the free Lie algebra generated by the vector space V. Denote by $\mathrm{Lie}_n(V)$ elements of $\mathrm{Lie}(V)$ with exactly n occurrences of the formal Lie bracket [,].

For $n \geq 2$, define

$$\alpha : \mathbb{P} \otimes \underbrace{\mathfrak{dg}_1 \otimes \cdots \otimes \mathfrak{dg}_1}_{n-2} \to \mathrm{Lie}_n(\mathfrak{dg}_1)$$

by

$$\alpha : \sum p_{r,s} x_1^{r-1} x_2^{s-1} \otimes \overline{\sigma}_{i_1} \otimes \cdots \otimes \overline{\sigma}_{i_{n-2}} \mapsto \sum p_{r,s}[\cdots[[\overline{\sigma}_r, \overline{\sigma}_s], \overline{\sigma}_{i_1}], \cdots, \overline{\sigma}_{i_{n-2}}],$$

where [,] is the formal lie bracket. Denote by $\beta : \mathrm{Lie}_n(\mathfrak{dg}_1) \to \mathfrak{dg}_n$ the map that replacing the formal Lie bracket by the induced Ihara bracket.

The following conjecture is well-known.

Conjecture 2.2. *(nondegeneracy conjecture) For $n \geq 2$, the following sequence*

$$\mathbb{P} \otimes \underbrace{\mathfrak{dg}_1 \otimes \cdots \otimes \mathfrak{dg}_1}_{n-2} \xrightarrow{\alpha} \mathrm{Lie}_n(\mathfrak{dg}_1) \xrightarrow{\beta} \mathfrak{dg}_n$$

is exact.

From the main results of [9], Brown's homological conjecture about $\mathfrak{dg}$ implies the nondegeneracy conjecture.

2.2. Broadhurst-Kreimer conjecture and motivic Lie algebra

In this subsection we will give a short introduction to multiple zeta values and Broadhurst-Kreimer conjecture. Then we will explain Brown's motivic approach to understand Broadhurst-Kreimer conjecture.

For $r \geq 1, k_1, \cdots, k_{r-1} \geq 1, k_r \geq 2$, multiple zeta value $\zeta(k_1, k_2, \cdots, k_r)$ is defined by

$$\zeta(k_1, k_2, \cdots, k_r) = \sum_{0<n_1<n_2<\cdots<n_r} \frac{1}{n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r}}.$$

From the above expression it is clear that products of multiple zeta values are still $\mathbb{Q}$-linear combinations of multiple zeta values. For $\zeta(k_1, k_2, \cdots, k_r)$, we call $K = k_1 + k_2 + \cdots + k_r$ and r its weight and depth respectively.

For $K > 0$, denote by $\mathcal{Z}_K$ the $\mathbb{Q}$-linear subspace of $\mathbb{R}$ which is generated by weight K multiple zeta values. For $r > 0$, denote by $\mathfrak{D}_r\mathcal{Z}_K$ the $\mathbb{Q}$-linear subspace of $\mathcal{Z}_K$ which is generated by depth $\leq r$, weight K multiple zeta values. The weight and depth structures of multiple zeta values are quite mysterious. In fact, Broadhurst and Kreimer [2] proposed the following conjecture.

Conjecture 2.3. *(Broadhurst, Kreimer) For $r > 0$, denote by $gr_r^{\mathfrak{D}}\mathcal{Z}_K = \mathfrak{D}_r\mathcal{Z}_K/\mathfrak{D}_{r-1}\mathcal{Z}_K$, then*

$$1 + \sum_{K,r>0} \left(\dim_{\mathbb{Q}} gr_r^{\mathfrak{D}}\mathcal{Z}_K\right) s^K t^r = \frac{1 + \mathbb{E}(s)t}{1 - \mathbb{O}(s)t + \mathbb{S}(s)t^2 - \mathbb{S}(s)t^4},$$

where

$$\mathbb{E}(s) = \frac{s^2}{1-s^2}, \mathbb{O}(s) = \frac{s^3}{1-s^2}, \mathbb{S}(s) = \frac{s^{12}}{(1-s^4)(1-s^6)}.$$

Note that $\mathbb{S}(s)$ is the generating series of cusp forms of $SL_2(\mathbb{Z})$.

Brown [3] defined motivic multiple zeta values. Brown's definition of motivic multiple zeta values is a refinement of Goncharov's definition (see [5] and [10]) such that the motivic version of $\zeta(2)$ is non-zero. We review Brown's definition here. There is a point $dch \in {}_0\Pi_1(\mathbb{R})$ which comes from comparison between de-Rham fundamental groupoid and Betti fundamental groupoid of $\mathbb{P}^1 - \{0, 1, \infty\}$. The point dch determines a ring homomorphism

$$dch^* : \mathbb{Q}\langle e^0, e^1\rangle \to \mathbb{R}$$

which satisfies

$$dch^* \left(e^1(e^0)^{k_1-1}e^1(e^0)^{k_2-1}\cdots e^1(e^0)^{k_r-1}\right) = \zeta(k_1, k_2, \cdots, k_r).$$

Moreover, dch^* respects the shuffle product on $\mathbb{Q}\langle e^0, e^1\rangle$.

Denote by $\mathcal{I}$ the largest graded sub-ideal of Ker dch^* which is stable under the action of U. Define $\mathcal{H} = \mathcal{O}({}_0\Pi_1)/\mathcal{I} = \mathbb{Q}\langle e^0, e^1\rangle/\mathcal{I}$. Then $\mathcal{H}$ is the algebra of motivic multiple zeta values. Denote by $I : \mathbb{Q}\langle e^0, e^1\rangle \to \mathcal{H}$ the natural quotient map. Define $per : \mathcal{H} \to \mathbb{R}$ to be the map which satisfies $per \circ I = dch^*$. Denote by

$$\zeta^{\mathfrak{m}}(k_1, k_2, \cdots, k_r) = I\left(e^1(e^0)^{k_1-1}e^1(e^0)^{k_2-1}\cdots e^1(e^0)^{k_r-1}\right).$$

We call $\zeta^{\mathfrak{m}}(k_1, k_2, \cdots, k_r)$ motivic multiple zeta value. Clearly it satisfies

$$per\left(\zeta^{\mathfrak{m}}(k_1, k_2, \cdots, k_r)\right) = \zeta(k_1, k_2, \cdots, k_r).$$

There are also weight and depth structures on $\mathbb{Q}\langle e^0, e^1\rangle$. Since the weight structure and the depth filtration structure on $\mathbb{Q}\langle e^0, e^1\rangle$ are both motivic, one can define depth-graded motivic multiple zeta values $gr_r^{\mathfrak{D}}\mathcal{H}_K$. In order to study Broadhurst-Kreimer conjecture for motivic multiple zeta values (motivic Broadhurst-Kreimer conjecture), Brown [4] formulated the following conjecture.

Conjecture 2.4. *(Brown) The Lie algebra homology group of* $\mathfrak{dg}$ *is*

$$H_1(\mathfrak{dg}, \mathbb{Q}) \cong \mathfrak{dg}_1 \oplus e(\mathbb{P}), \tag{i}$$

$$H_2(\mathfrak{dg}, \mathbb{Q}) \cong \mathbb{P}, \tag{ii}$$

$$H_i(\mathfrak{dg}, \mathbb{Q}) = 0, \forall i \geq 3, \tag{iii}$$

where $e(\mathbb{P}) \subseteq \mathfrak{dg}_4$ *and* $e(\mathbb{P}) \cong \mathbb{P}$ *as* $\mathbb{Q}$*-vector space.*

Based on standard technique in Lie algebra homology theory, Brown showed that Conjecture 2.4 implies motivic Broadhurst-Kreimer conjecture. (Actually Brown gave explicitly conjectural description about $e(\mathbb{P})$, but we do not focus on this direction in our paper.) From Theorem 4.3 in [9], (*i*) and (*ii*) in Conjecture 2.4 are equivalent to the statement that the only relations among the generators of $\mathfrak{dg}$ are the period polynomial relations among $\overline{\sigma}_{2n+1}, n \geq 1$ in depth 2. Thus Conjecture 2.4 implies the nondegeneracy conjecture. Beware that $\mathfrak{dg}$ also has generators in depth ≥ 4. As a result, Brown's homological conjecture is much stronger than the nondegeneracy conjecture. From Theorem 4.3 in [9], we know that the nondegeneracy conjecture is the first step to prove Brown's homological conjecture.

3. Universal enveloping algebra

Denote by $\mathcal{U}\mathfrak{h}$ the universal enveloping algebra of $\mathfrak{h}$ and denote by $\mathbb{Q}\langle e_0, e_1\rangle$ the non-commutative polynomial ring in symbol e_0, e_1. From Proposition 5.9 in [7] we know that $\mathcal{U}\mathfrak{h}$ is isomorphic to $\mathbb{Q}\langle e_0, e_1\rangle$ as a vector space. But the new multiplication structure $\circ$ on $\mathbb{Q}\langle e_0, e_1\rangle$ which is transformed from $\mathcal{U}\mathfrak{h}$ is rather subtle. It's not the usual concatenation product.

Denote by $gr^r_{\mathfrak{D}}\mathbb{Q}\langle e_0, e_1\rangle$ the elements of $\mathbb{Q}\langle e_0, e_1\rangle$ with exactly r occurrences of e_1. We have the following map:

$$\begin{aligned}\rho : gr^r_{\mathfrak{D}}\mathbb{Q}\langle e_0, e_1\rangle &\to \mathbb{Q}[y_0, y_1, \cdots, y_r],\\ e_0^{a_0}e_1e_0^{a_1}e_1\cdots e_1e_0^{a_r} &\mapsto y_0^{a_0}y_1^{a_1}\cdots y_r^{a_r}.\end{aligned}$$

The map ρ is the polynomial representation of $\mathbb{Q}\langle e_0, e_1\rangle$ defined by Brown. The polynomial representation of non-commutative series has been used by Brown, Racinet, Ecalle and so on for the study of multiple zeta values.

Brown [4] introduced a $\mathbb{Q}$-bilinear map $\underline{\circ} : \mathbb{Q}\langle e_0, e_1\rangle \otimes_{\mathbb{Q}} \mathbb{Q}\langle e_0, e_1\rangle \to \mathbb{Q}\langle e_0, e_1\rangle$ which in the polynomial representation can be written as

$$\begin{aligned}f\underline{\circ}g(y_0, \cdots, y_{r+s}) = &\sum_{i=0}^{s} f(y_i, y_{i+1}, \cdots, y_{i+r})g(y_0, \cdots, y_i, y_{i+r+1}, \cdots, y_{r+s})+\\ &(-1)^{degf+r}\sum_{i=1}^{s} f(y_{i+r}, \cdots, y_{i+1}, y_i)g(y_0, \cdots, y_{i-1}, y_{i+r}, \cdots, y_{r+s})\end{aligned}$$

for $f \in \mathbb{Q}[y_0, \cdots, y_r] = \rho(gr^r_{\mathfrak{D}}\mathbb{Q}\langle e_0, e_1\rangle)$, $g \in \mathbb{Q}[y_0, \cdots, y_s] = \rho(gr^s_{\mathfrak{D}}\mathbb{Q}\langle e_0, e_1\rangle)$.

Since by the general theory of Lie algebra, the natural action of $\mathfrak{h}$ on $\mathcal{U}\mathfrak{h}$ is the form $(a, b_1 \otimes b_2 \otimes \cdots b_r) \mapsto a \otimes b_1 \otimes \cdots \otimes b_r$ in $\mathcal{U}\mathfrak{h}$ for $a, b_1, \cdots, b_r \in \mathfrak{h}$. By Proposition 2.2 in [4], we have

$$a_1 \circ a_2 \circ \cdots \circ a_r = a_1\underline{\circ}(a_2\underline{\circ}(\cdots(a_{r-1}\underline{\circ}a_r)\cdots))$$

for $a_i \in \mathfrak{h} \subseteq \mathbb{Q}\langle e_0, e_1\rangle, 1 \le i \le r-1$, $a_r \in \mathbb{Q}\langle e_0, e_1\rangle$.

The above formula is still not enough to give a very clear picture of the new multiplication $\circ$ on $\mathbb{Q}\langle e_0, e_1\rangle$. But it is enough for our purpose.

We first introduce some notation from Tasaka [13]. Denote by

$$S_{N,r} = \{(n_1, ..., n_r) \in \mathbb{Z}^r \mid n_1 + ... + n_r = N, n_1, ..., n_r \ge 3 : odd\}.$$

We write $\overrightarrow{m} = (m_1, ..., m_r)$ for short, while

$$\mathbf{Vect}_{N,r} = \{(a_{n_1,...,n_r})_{\overrightarrow{n}\in S_{N,r}} \mid a_{n_1,...,n_r} \in \mathbb{Q}\}.$$

For a matrix $P = \left(p\begin{pmatrix} m_1, ..., m_r \\ n_1, ..., n_r \end{pmatrix}\right)_{\substack{\vec{m}\in S_{N,r} \\ \vec{n}\in S_{N,r}}}$, the action of P on $a = (a_{m_1,...,m_r})_{\vec{m}\in S_{N,r}}$ means

$$aP = \left(\sum_{\vec{m}\in S_{N,r}} a_{m_1,...,m_r} p\begin{pmatrix} m_1, ..., m_r \\ n_1, ..., n_r \end{pmatrix}\right)_{\vec{n}\in S_{N,r}}$$

Denote by $\mathbb{P}_{N,r}$ the $\mathbb{Q}$-vector space spanned by the set

$$\{x_1^{n_1-1}\cdots x_r^{n_r-1} \mid (n_1, ..., n_r) \in S_{N,r}\}.$$

Obviously there is an isomorphism

$$\pi : \mathbb{P}_{N,r} \longrightarrow \mathbf{Vect}_{N,r}$$
$$\sum_{\vec{n}\in S_{N,r}} a_{n_1,...,n_r} x_1^{n_1-1}\cdots x_r^{n_r-1} \longmapsto (a_{n_1,...,n_r})_{\vec{n}\in S_{N,r}}.$$

Denote by

$$\mathbf{W}_{N,r} = \{p \in \mathbb{P}_{N,r} \mid p(x_1, ..., x_r) = p(x_2 - x_1, x_2, x_3, ..., x_r) - p(x_2 - x_1, x_1, x_3, ..., x_r)\}.$$

Denote by

$$e\begin{pmatrix} m_1, ..., m_r \\ n_1, ..., n_r \end{pmatrix} = \delta\begin{pmatrix} m_1, ..., m_r \\ n_1, ..., n_r \end{pmatrix} + \sum_{i=1}^{r-1} \delta\begin{pmatrix} m_2, ..., m_i, m_{i+2}, ..., m_r \\ n_1, ..., n_{i-1}, n_{i+2}, ..., n_r \end{pmatrix} b^{m_1}_{n_i,n_{i+1}},$$

where the $b^m_{n,n'}$ are defined by

$$b^m_{n,n'} = (-1)^n \binom{m-1}{n-1} + (-1)^{n'-m}\binom{m-1}{n'-1}$$

and $\delta\begin{pmatrix} m_1, ..., m_r \\ n_1, ..., n_r \end{pmatrix} = 1$ if $\vec{m} = \vec{n}$, $\delta\begin{pmatrix} m_1, ..., m_r \\ n_1, ..., n_r \end{pmatrix} = 0$ if $\vec{m} \neq \vec{n}$.

The matrix $E^{(r-i)}_{N,r}$, $i = 0, 1, ..., r-2$ are defined by

$$E^{(r-i)}_{N,r} = \left(\delta\begin{pmatrix} m_1, ..., m_i \\ n_1, ..., n_i \end{pmatrix} e\begin{pmatrix} m_{i+1}, ..., m_r \\ n_{i+1}, ..., n_r \end{pmatrix}\right)_{\substack{\vec{m}\in S_{N,r} \\ \vec{n}\in S_{N,r}}}.$$

We write $E^{(r)}_{N,r}$ as $E_{N,r}$. Denote by

$$C_{N,r} = E^{(2)}_{N,r} \cdot E^{(3)}_{N,r} \cdots E^{(r-1)}_{N,r} \cdot E_{N,r}$$

for $r \geq 2$. Denote by $C_{N,r}$ the one row, one column matrix 1 for $N > 1, \text{odd}$, $r = 1$.

By [1], we know $\pi(\mathbf{W}_{N,2}) = \mathrm{Ker}\ E_{N,2}$. For $r \geq 3$, K. Tasaka proved that

$$(\pi(\mathbf{W}_{N,r}))(E_{N,r} - I_{N,r}) \subseteq \mathrm{Ker}\ E_{N,r},$$

where $I_{N,r}$ denotes the identity matrix

$$\left(\delta\binom{m_1, ..., m_r}{n_1, ..., n_r}\right)_{\substack{\vec{m}\in S_{N,r}\\ \vec{n}\in S_{N,r}}}.$$

Furthermore, K. Tasaka proposed the following conjecture.

Conjecture 3.1. *(Tasaka's conjecture) The linear map*

$$\eta : \pi(\mathbf{W}_{N,r}) \to \mathrm{Ker}\ E_{N,r}$$
$$a_{\vec{m}} \mapsto (a_{\vec{m}})(E_{N,r} - I_{N,r})$$

is an isomorphism.

In [13], K. Tasaka suggested a way to prove the injectivity in the above conjecture. But there is a gap in his proof. We prove the injectivity for $r = 3$ in [11].

In [4], F. Brown proposed the following conjecture

Conjecture 3.2. *(Brown's matrix conjecture) The rank of the matrices $C_{N,r}$ satisfy*

$$1 + \sum_{N,r>0} \mathrm{rank}\ C_{N,r} x^N y^r = \frac{1}{1 - \mathbb{O}(x)y + \mathbb{S}(x)y^2}.$$

Now we can state our main result.

Theorem 3.3. *Tasaka's conjecture implies Brown's matrix conjecture and Brown's matrix conjecture implies nondegeneracy conjecture.*

From the discussion in Section 2, we hope that Theorem 3.3 will shed light on motivic Broadhurst-Kreimer conjecture.

4. Calculation

In this section we will prove Theorem 3.3. In fact we will prove a little bit more.

The strategy is firstly we use Poincaré-Birkhoff-Witt theorem to give the nondegeneracy conjecture a reformulation in the context of universal enveloping algebra. Then by some dimension counting trick we show that Brown's matrix conjecture implies the nondegeneracy conjecture. At last by explicit calculation of the polynomial representation of the motivic action on universal enveloping algebra, it will be clear that Tasaka's isomorphism conjecture implies Brown's matrix conjecture.

We will need the following result about Lie algebra.

Proposition 4.1. *Let $\mathfrak{L}$ be a Lie algebra over $\mathbb{Q}$, denote by $\mathcal{U}\mathfrak{L}$ its universal envelope algebra. $\mathfrak{M}$ is a Lie ideal in $\mathfrak{L}$, denote by $\mathcal{U}\mathfrak{L}(\mathfrak{M})$ the two-sided ideal generated by $\mathfrak{M}$ in $\mathcal{U}\mathfrak{L}$. Then we have*

$$\mathfrak{L} \cap (\mathcal{U}\mathfrak{L}(\mathfrak{M})) = \mathfrak{M}.$$

Proof. We have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{M} & \longrightarrow & \mathfrak{L} & \longrightarrow & \mathfrak{L}/\mathfrak{M} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{U}\mathfrak{L}(\mathfrak{M}) & \longrightarrow & \mathcal{U}\mathfrak{L} & \longrightarrow & \mathcal{U}(\mathfrak{L}/\mathfrak{M}) & \longrightarrow & 0
\end{array}$$

The first row and second row are short exact sequences. By Poincaré-Birkhoff-Witt theorem in Lie algebra, we know that the three vertical maps are all injective. $\mathfrak{L} \cap (\mathcal{U}\mathfrak{L}(\mathfrak{M})) = \mathfrak{M}$ follows by diagram chasing. □

Recall the injective Lie algebra homomorphism in Section 2

$$i : \mathfrak{g} \to \mathfrak{h}.$$

Since the Lie algebra $\mathfrak{h}$ is bigraded by weight and depth, the map i induces a natural injective Lie algebra homomorphism

$$\bar{i} : \mathfrak{dg} \to \mathfrak{h}.$$

The maps i and $\bar{i}$ induce the natural injective algebra homomorphisms on enveloping algebra

$$\mathcal{U}i : \mathcal{U}\mathfrak{g} \to \mathcal{U}\mathfrak{h} = (\mathbb{Q}\langle e_0, e_1\rangle, \circ)$$

and

$$\mathcal{U}\bar{i} : \mathcal{U}\mathfrak{dg} \to \mathcal{U}\mathfrak{h} = (\mathbb{Q}\langle e_0, e_1\rangle, \circ).$$

As $\mathfrak{g}$ is a free Lie algebra generated by elements σ_{2n+1} for $n \geq 1$ in weight $2n+1$. We have $\mathcal{U}\mathfrak{g} = \mathbb{Q}\langle \sigma_3, \sigma_5, \cdots, \sigma_{2n+1}, \cdots \rangle$ (the non-commutative polynomial ring generated by the symbol σ_{2n+1} for $n \geq 1$) with the usual concatenation product.

For $r = 0$, denote by L_r the rational field $\mathbb{Q}$. For $r \geq 1$, denote by L_r the $\mathbb{Q}$-linear space generated by elements

$$\overline{\sigma}_{n_1} \circ \overline{\sigma}_{n_2} \circ \cdots \circ \overline{\sigma}_{n_r} = (\mathrm{ad}\ e_0)^{n_1-1} e_1 \circ (\mathrm{ad}\ e_0)^{n_2-1} e_1 \circ \cdots \circ (\mathrm{ad}\ e_0)^{n_r-1} e_1$$

in $\mathcal{U}\mathfrak{h}$ for $n_i \geq 3$, odd, $1 \leq i \leq r$.

Define the map $B_r : L_1 \otimes_{\mathbb{Q}} L_{r-1} \to L_r$ by

$$B_r(\overline{\sigma}_{n_1} \otimes (\overline{\sigma}_{n_2} \circ \cdots \circ \overline{\sigma}_{n_r})) = \overline{\sigma}_{n_1} \circ \overline{\sigma}_{n_2} \circ \cdots \circ \overline{\sigma}_{n_r}$$

and define the map $A_r : \mathbb{P} \otimes_{\mathbb{Q}} L_{r-2} \to L_1 \otimes_{\mathbb{Q}} L_{r-1}$ by

$$\begin{aligned} A_r : ((\sum_{n_1,n_2\geq 3,\text{odd}} p_{n_1,n_2} x_1^{n_1-1} x_2^{n_2-1}) \otimes (\overline{\sigma}_{n_3} \circ \cdots\cdots \circ \overline{\sigma}_{n_r})) \\ = \sum_{n_1,n_2\geq 3,\text{odd}} p_{n_1,n_2} \overline{\sigma}_{n_1} \otimes (\overline{\sigma}_{n_2} \circ \cdots \circ \overline{\sigma}_{n_r}). \end{aligned}$$

We have the following lemma

Lemma 4.2. *The nondegeneracy conjecture for all $r \geq 2$ is equivalent to that the following sequence is exact*

$$0 \to \mathbb{P} \otimes_{\mathbb{Q}} L_{r-2} \xrightarrow{A_r} L_1 \otimes_{\mathbb{Q}} L_{r-1} \xrightarrow{B_r} L_r \to 0$$

for all $r \geq 2$.

Proof. If

$$\overline{x} = \sum_{\overrightarrow{n} \in S_{N,r}} a_{n_1,\cdots,n_r} \overline{\sigma}_{n_1} \otimes (\overline{\sigma}_{n_2} \circ \cdots \circ \overline{\sigma}_{n_r}) \in \text{Ker } B_r, \tag{1}$$

then by definition we will have

$$\sum_{\overrightarrow{n} \in S_{N,r}} a_{n_1,\cdots,n_r} \sigma_{n_1} \sigma_{n_2} \cdots \sigma_{n_r} \in \mathfrak{D}^{r+1}\mathcal{U}\mathfrak{g} = \mathfrak{D}^{r+1}\mathbb{Q}\langle \sigma_3, \cdots, \sigma_{2n+1}, \cdots \rangle. \tag{2}$$

Since $\mathcal{U}\mathfrak{g}$ is a non-commutative polynomial ring, from formula (1) and (2) we have

$$\begin{aligned} &\sum_{\overrightarrow{n} \in S_{N,r}} a_{n_1,\cdots,n_r} \sigma_{n_1} \sigma_{n_2} \cdots \sigma_{n_r} \\ &\qquad = \sum_{i=2}^{r} \sum_{\overrightarrow{m} \in S_{N,r}} b^i_{m_1,\cdots,m_i,m_{i+1},\cdots,m_r} [[\cdots[\sigma_{m_1}, \sigma_{m_2}], \cdots], \sigma_{m_i}] \sigma_{m_{i+1}} \cdots \sigma_{m_r} \end{aligned}$$

for some $b^i_{\overrightarrow{m}}, \overrightarrow{m} \in S_{N,r}$ and

$$\sum_{(m_1,\cdots,m_i) \in S_{N-m_{i+1}-\cdots-m_r}} b^i_{m_1,\cdots,m_i,m_{i+1},\cdots,m_r} [[\cdots[\sigma_{m_1}, \sigma_{m_2}], \cdots], \sigma_{m_i}] \subseteq \mathfrak{D}^{i+1}\mathfrak{g}$$

On one hand, if the nondegeneracy conjecture is true for all depth, we will have

$$\sum_{\vec{n}\in S_{N,r}} a_{n_1,\cdots,n_r}\overline{\sigma}_{n_1}\otimes\overline{\sigma}_{n_2}\otimes\cdots\otimes\overline{\sigma}_{n_r}$$
$$\subseteq \mathbb{P}\otimes(L_1)^{\otimes r-2}+L_1\otimes\mathbb{P}\otimes L_1^{\otimes r-3}+\cdots+L_1^{\otimes r-3}\otimes\mathbb{P}\otimes L_1+(L_1)^{\otimes r-2}\otimes\mathbb{P}.$$

From the definition of L_r and period polynomial relations of $\mathfrak{dg}$ in depth two, we have $\overline{x}\subseteq \text{Im } A_r$, i.e. $\text{Im } A_r=\text{Ker } B_r$.

Since it's obvious that B_r is surjective. While A_r is injective follows from $\text{Im } A_r = \text{Ker } B_r$ in depth $r-1$ and the fact that $\mathbb{P}\otimes L_1\cap L_1\otimes\mathbb{P}=\{0\}$. We deduce that from the nondegeneracy conjecture for all depth we will have the short exact sequence for all $r\geq 2$.

On the other hand, if the sequence is exact for all $r\geq 2$, let

$$\overline{x}=\sum_{\vec{m}\in S_{N,r}} b_{m_1,\cdots,m_r}\{\{\cdots\{\overline{\sigma}_{m_1},\overline{\sigma}_{m_2}\},\cdots\},\overline{\sigma}_{m_r}\}=0 \tag{3}$$

in $\mathfrak{dg}_r$, where $\{\,,\,\}$ denotes the induced Ihara Lie bracket on $\mathfrak{dg}$.

Then

$$x=\sum_{\vec{m}\in S_{N,r}} b_{m_1,m_2,\cdots,m_r}[[\cdots[\sigma_{m_1},\sigma_{m_2}],\cdots],\sigma_{m_r}]\in gr_{\mathfrak{D}}^{r+1}\mathcal{U}\mathfrak{g}, \tag{4}$$

where $[\,,\,]$ denotes the formal Lie bracket on the non-commutative polynomial ring $\mathcal{U}\mathfrak{g}$. Rewrite x as

$$x=\sum_{\vec{m}\in S_{N,r}} a_{m_1,m_2,\cdots,m_r}\sigma_{m_1}\sigma_{m_2}\cdots\sigma_{m_r}. \tag{5}$$

Denote

$$x_{\mathfrak{d}}=\sum_{\vec{m}\in S_{N,r}} a_{m_1,m_2,\cdots,m_r}\overline{\sigma}_{m_1}\otimes(\overline{\sigma}_{m_2}\circ\overline{\sigma}_{m_3}\circ\cdots\circ\overline{\sigma}_{m_r})\in L_1\otimes_{\mathbb{Q}} L_{r-1},$$

then from (3), (4) and (5), we have $x_{\mathfrak{d}}\in \text{Ker } B_r$. Since $\text{Im } A_r=\text{Ker } B_r$, we have

$$x_{\mathfrak{d}}=\sum_{\vec{m}\in S_{N,r}} c_{m_1,m_2,m_3,\cdots,m_r}\overline{\sigma}_{m_1}\otimes(\overline{\sigma}_{m_2}\circ\overline{\sigma}_{m_3}\circ\cdots\circ\overline{\sigma}_{m_r})$$

and $p=\sum_{m_1,m_2\geq 3,\text{odd}} c_{m_1,m_2,m_3,\cdots,m_r}x_1^{m_1-1}x_2^{m_2-1}\in\mathbb{P}$. Let $\iota:\mathbb{P}\to\mathfrak{g}$ be the map

$$\iota:\sum_{r,s\geq 3,\text{ odd}} p_{r,s}x_1^{r-1}x_2^{s-1}\mapsto\sum_{r,s\geq 3,\text{ odd}} p_{r,s}[\sigma_r,\sigma_s].$$

From Im A_i = Ker B_i for $i = 2, \cdots, r$, we deduce inductively that

$$x \in \mathcal{U}\mathfrak{g}(\iota(\mathbb{P})),$$

where $\mathcal{U}\mathfrak{g}(\iota(\mathbb{P}))$ means the two-sided ideal generated by $\iota(\mathbb{P})$ in $\mathcal{U}\mathfrak{g}$. By Proposition 4.1, x belongs to the Lie ideal generated by $\iota(\mathbb{P})$ in $\mathfrak{g}$. So from the short exact sequence for all depth we can deduce the nondegeneracy conjecture in all depth. $\square$

The following lemma reduces the nondegeneracy conjecture to a dimension conjecture of L_r in each weight N for all r.

Lemma 4.3. *Denote by $L_{N,r}$ the weight N part of L_r, then the formula*

$$1 + \sum_{N,r>0} \dim_{\mathbb{Q}} L_{N,r} x^N y^r = \frac{1}{1 - \mathbb{O}(x)y + \mathbb{S}(x)y^2}$$

is equivalent to that the following sequence is exact

$$0 \to \mathbb{P} \otimes_{\mathbb{Q}} L_{r-2} \xrightarrow{A_r} L_1 \otimes_{\mathbb{Q}} L_{r-1} \xrightarrow{B_r} L_r \to 0$$

for all $r \geq 2$, where $\mathbb{O}(x) = \frac{x^3}{1-x^2}$, $\mathbb{S}(x) = \frac{x^{12}}{(1-x^4)(1-x^6)}$.

Proof. $' \Rightarrow'$ It's clear that B_r is surjective and Im $A_r \subseteq$ Ker B_r for all $r \geq 2$. Since

$$\sum_{N>0} \dim_{\mathbb{Q}} L_{N,1} x^N = \mathbb{O}(x),$$

from the dimension formula we have

$$\dim_{\mathbb{Q}} L_{N,r} x^N - \sum_{N>0} \dim_{\mathbb{Q}} L_{N,r-1} \cdot \mathbb{O}(x) + \sum_{N>0} \dim_{\mathbb{Q}} L_{N,r-2} \cdot \mathbb{S}(x) = 0 \tag{6}$$

for all $r \geq 2$.

It's obvious that A_2 is injective, B_2 is surjective and Im $A_2 \subseteq$ Ker B_2. So from formula (6) in $r = 2$, we have Im A_2 = Ker B_2.

For $r \geq 3$, denote by

$$I : L_1 \otimes L_1 \otimes L_1 \otimes L_{r-3} \to L_1 \otimes L_1 \otimes L_{r-2},$$
$$\overline{\sigma}_{r_1} \otimes \overline{\sigma}_{r_2} \otimes \overline{\sigma}_{r_3} \otimes a \to \overline{\sigma}_{r_1} \otimes \overline{\sigma}_{r_2} \otimes (\overline{\sigma}_{r_3} \circ a).$$

Inductively, from Im A_{r-1} = Ker B_{r-1}, we have

$$\begin{aligned}\text{Ker } A_r &\subseteq \mathbb{P} \otimes \cap I(L_1 \otimes \mathbb{P} \otimes L_{r-3}) \\ &\subseteq I\left((\mathbb{P} \otimes L_1 \cap L_1 \otimes \mathbb{P}) \otimes L_{r-3}\right).\end{aligned}$$

From Goncharov's result $\mathbb{P} \otimes_{\mathbb{Q}} L_1 \cap L_1 \otimes_{\mathbb{Q}} \mathbb{P} = \{0\}$, we can deduce that A_r is injective. Then Im A_r = Ker B_r follows from formula (6) and the fact that A_r is injective, B_r is surjective and Im $A_r \subseteq$ Ker B_r.

$' \Leftarrow'$ It's clear that

$$\sum_{N>0} \dim_{\mathbb{Q}} L_{N,1} x^N = \mathbb{O}(x).$$

Then from the short exact sequence we have

$$\dim_{\mathbb{Q}} L_{N,r} x^N - \sum_{N>0} \dim_{\mathbb{Q}} L_{N,r-1} \cdot \mathbb{O}(x) + \sum_{N>0} \dim_{\mathbb{Q}} L_{N,r-2} \cdot \mathbb{S}(x) = 0$$

for all $r \geq 2$. So

$$1 + \sum_{N,r>0} \dim_{\mathbb{Q}} L_{N,r} x^N y^r = \frac{1}{1 - \mathbb{O}(x)y + \mathbb{S}(x)y^2}. \quad \square$$

Remark 4.4. In fact, if we only know that

$$1 + \sum_{N,r>0} \dim_{\mathbb{Q}} L_{N,r} x^N y^r \geq \frac{1}{1 - \mathbb{O}(x)y + \mathbb{S}(x)y^2},$$

where $\geq$ means the coefficient of the term $x^N y^r$ in the left side is bigger than the corresponding coefficient in the right side for all $N, r > 0$, then we can still deduce the short exact sequence exactly the same way as in the proof of Lemma 4.3.

Now we investigate the polynomial representation of $L_{N,r}$. From the main result of Section 3, we have

$$\begin{aligned}
&\rho(\overline{\sigma}_{m_1} \circ \overline{\sigma}_{m_2} \circ \cdots \circ \overline{\sigma}_{m_r}) \\
&\quad = (y_1 - y_0)^{m_1 - 1} \underline{\circ} ((y_1 - y_0)^{m_2 - 1} \underline{\circ} (\cdots ((y_1 - y_0)^{m_{r-1} - 1} \underline{\circ} (y_1 - y_0)^{m_r - 1})))
\end{aligned}$$

for $\overrightarrow{m} = (m_1, m_2, \cdots, m_r) \in S_{N,r}$.

For $\overrightarrow{n} = (n_1, n_2, \cdots, n_r) \in S_{N,r}$, the coefficient of $y_1^{n_1-1} y_2^{n_2-1} \cdots y_r^{n_r-1}$ in $\rho(\overline{\sigma}_{m_1} \circ \overline{\sigma}_{m_2} \circ \cdots \circ \overline{\sigma}_{m_r})$ is

$$c\binom{m_1, m_2, \cdots, m_r}{n_1, n_2, \cdots, n_r},$$

where $c\binom{m_1, m_2, \cdots, m_r}{n_1, n_2, \cdots, n_r}$ is the $(m_1, m_2, \cdots, m_r)$-th row, $(n_1, n_2, \cdots, n_r)$-th column term of the matrix $C_{N,r}$.

Now we have

Proposition 4.5. *The following map*

$$\widetilde{\eta} : W_{N,r} \to \mathbf{Vect}_{N,r}$$

$$(a_{m_1,\cdots,m_r})_{\vec{m}\in S_{N,r}} \mapsto \left(\sum_{\vec{m}\in S_{N,r}} a_{m_1,\cdots,m_r}\delta\binom{m_1}{n_1}e\binom{m_2,\cdots,m_r}{n_2,\cdots,n_r}\right)_{\vec{n}\in S_{N,r}}$$

satisfies $\widetilde{\eta}(W_{N,r}) \subseteq \mathrm{Ker}\ E_{N,r}$. *Furthermore,* $\widetilde{\eta}(a) + \eta(a) = 0$ *for*

$$a = (a_{m_1,\cdots,m_r})_{\vec{m}\in S_{N,r}} \in W_{N,r}.$$

Before we go to the long proof of Proposition 4.5, we explain the key observation firstly. Roughly speaking $\widetilde{\eta}(W_{N,r}) \subseteq \mathrm{Ker}\ E_{N,r}$ follows from the fact that generators in $\mathfrak{dg}_1$ have period polynomial relations among them. Thus the linear space $\widetilde{\eta}(W_{N,r})$, if viewed as sub-space of $\mathfrak{dg}_1 \otimes \mathcal{U}\mathfrak{dg}$, has trivial images in $\mathcal{U}\mathfrak{h}$, i.e. $\widetilde{\eta}(W_{N,r}) \subseteq \mathrm{Ker}\ E_{N,r}$. While $\widetilde{\eta}(a) + \eta(a) = 0$ essentially comes from anti-symmetry of even restricted period polynomials.

Proof of Proposition 4.5. Consider the natural action of $\mathfrak{dg}$ on $\mathcal{U}\mathfrak{h} = (\mathbb{Q}\langle e_0, e_1\rangle, \circ)$. By the main results of Section 3, we know that

$$\sum_{(m_2,\cdots,m_r)\in S_{N-n_1,r-1}} a_{n_1,m_2,\cdots,m_r} e\binom{m_2,\cdots,m_r}{n_2,\cdots,n_r}$$

is the coefficient of $y_2^{n_2-1}\cdots y_r^{n_r-1}$ in the polynomial representation of

$$\sum_{(m_2,\cdots,m_r)\in S_{N-n_1,r-1}} a_{n_1,m_2,\cdots,m_r}\overline{\sigma}_{m_2}\circ(e_1e_0^{m_3-1}e_1\cdots e_1e_0^{m_r-1}),$$

for $(n_2,\cdots,n_r) \in S_{N-n_1,r-1}$. Furthermore,

$$\sum_{\vec{m},\vec{n}\in S_{N,r}} a_{m_1,m_2,\cdots,m_r}\delta\binom{m_1}{n_1}e\binom{m_2,\cdots,m_r}{n_2,\cdots,n_r}e\binom{n_1,n_2,\cdots,n_r}{k_1,k_2,\cdots,k_r}$$

is the coefficient of $y_1^{k_1-1}y_2^{k_2-1}\cdots y_r^{k_r-1}$ in the polynomial representation of

$$\sum_{\vec{m}\in S_{N,r}} a_{m_1,m_2,\cdots,m_r}\overline{\sigma}_{m_1}\circ\overline{\sigma}_{m_2}\circ(e_1e_0^{m_3-1}e_1\cdots e_1e_0^{m_r-1}).$$

If $a = (a_{m_1,\cdots,m_r})_{\overrightarrow{m}\in S_{N,r}} \in W_{N,r}$, then

$$\begin{aligned}
&\sum_{\overrightarrow{m}\in S_{N,r}} a_{m_1,m_2,\cdots,m_r}\overline{\sigma}_{m_1}\circ\overline{\sigma}_{m_2}\circ(e_1e_0^{m_3-1}e_1\cdots e_1e_0^{m_r-1})\\
&=\frac{1}{2}\sum_{\overrightarrow{m}\in S_{N,r}} a_{m_1,m_2,\cdots,m_r}(\overline{\sigma}_{m_1}\circ\overline{\sigma}_{m_2}-\overline{\sigma}_{m_2}\circ\overline{\sigma}_{m_1})\circ(e_1e_0^{m_3-1}e_1\cdots e_1e_0^{m_r-1})\\
&=\frac{1}{2}\sum_{\overrightarrow{m}\in S_{N,r}} a_{m_1,m_2,\cdots,m_r}\{\overline{\sigma}_{m_1},\overline{\sigma}_{m_2}\}\circ(e_1e_0^{m_3-1}e_1\cdots e_1e_0^{m_r-1})\\
&=0.
\end{aligned}$$

So we have

$$\sum_{\overrightarrow{m},\overrightarrow{n}\in S_{N,r}} a_{m_1,m_2,\cdots,m_r}\delta\binom{m_1}{n_1}e\binom{m_2,\cdots,m_r}{n_2,\cdots,n_r}e\binom{n_1,n_2,\cdots,n_r}{k_1,k_2,\cdots,k_r}=0,$$

i.e. $\widetilde{\eta}(W_{N,r})\subseteq \text{Ker } E_{N,r}$.

Similarly, in order to prove $\widetilde{\eta}(a)+\eta(a)=0$ for $a=(a_{m_1,\cdots,m_r})_{\overrightarrow{m}\in S_{N,r}}\in W_{N,r}$, it suffices to show that the coefficient of the term $y_1^{n_1-1}y_2^{n_2-1}\cdots y_r^{n_r-1}$ in the polynomial representation of

$$-\sum_{\overrightarrow{m}\in S_{N,r}} a_{m_1,m_2,\cdots,m_r}[\overline{\sigma}_{m_1}\circ(e_1e_0^{m_2-1}\cdots e_1e_0^{m_r-1})-e_1e_0^{m_1-1}e_1e_0^{m_2-1}\cdots e_1e_0^{m_r-1}]$$

is equal to the coefficient of the term $y_1^{n_2-1}y_2^{n_3-1}\cdots y_{r-1}^{n_r-1}$ in the polynomial representation of

$$\sum_{(m_2,m_3,\cdots,m_r)\in S_{N-n_1,r-1}} a_{n_1,m_2,\cdots,m_r}\overline{\sigma}_{m_2}\circ(e_1e_0^{m_3-1}\cdots e_1e_0^{m_r-1}) \tag{7}$$

for all $\overrightarrow{n}\in S_{N,r}$. This follows from direct calculation of motivic Galois action and standard properties of period polynomial. □

Remark 4.6. From Proposition 4.5 we obtain Tasaka's result [13]

$$\eta(W_{N,r})\subseteq \text{Ker }(E_{N,r}-I_{N,r})$$

immediately. See the proof Proposition 5.5 in [11] for a proof of the fact

$$\widetilde{\eta}(a)+\eta(a)=0$$

based on an explicit matrix calculation. Also see the proof of Lemma 4.9 in [8] for a proof based on polynomial representation.

Now we can prove our main results.

Proof of Theorem 3.3. From linear algebra, we have

$$\mathrm{Ker}\ C_{N,r} \cong \mathrm{Ker}\ (E^{(2)}_{N,r}E^{(3)}_{N,r}\cdots E^{(r-1)}_{N,r}) \oplus \mathrm{Im}\ (E^{(2)}_{N,r}E^{(3)}_{N,r}\cdots E^{(r-1)}_{N,r}) \cap \mathrm{Ker}\ E_{N,r}. \tag{10}$$

By the definition of $C_{N,r}$, view $C_{N,r}$ as linear transformation on the vector space $\mathbf{Vect}_{N,r}$, then we have

$$\mathrm{Ker}(E^{(2)}_{N,r}E^{(3)}_{N,r}\cdots E^{(r-1)}_{N,r}) \cong \bigoplus_{m>1,\mathrm{odd}} \mathrm{Ker}\ C_{N-m,r-1}. \tag{11}$$

If Conjecture 3.1 (Tasaka conjecture) is true, then from Proposition 4.5 and the fact

$$\mathrm{Im}\ (E^{(2)}_{N,r}E^{(3)}_{N,r}\cdots E^{(r-2)}_{N,r}) \cap W_{N,r} \cong \bigoplus_{m>0,\mathrm{even}} \mathbb{P}_m \otimes_{\mathbb{Q}} \mathrm{Im}\ C_{N-m.r-2}$$

we have

$$\mathrm{Im}\ (E^{(2)}_{N,r}E^{(3)}_{N,r}\cdots E^{(r-1)}_{N,r}) \cap \mathrm{Ker}\ E_{N,r} \cong \bigoplus_{m>0,\mathrm{even}} \mathbb{P}_m \otimes_{\mathbb{Q}} \mathrm{Im}\ C_{N-m,r-2}. \tag{12}$$

From formula (10), (11) and (12), we have

$$\begin{aligned}\sum_{N,r>0} \dim_{\mathbb{Q}}\mathrm{Ker}\ C_{N,r}x^N y^r = &\sum_{N,r>0} \dim_{\mathbb{Q}}\mathrm{Ker}\ C_{N,r}x^N y^r \cdot \mathbb{O}(x)y \\ &+ \mathbb{S}(x)y^2 \cdot (1 + \sum_{N,r>0} \dim_{\mathbb{Q}}\mathrm{Im}\ C_{N,r}x^N y^r).\end{aligned} \tag{13}$$

From formula (13), we have

$$1 + \sum_{N,r>0} \mathrm{rank}\ C_{N,r}x^N y^r = \frac{1}{1 - \mathbb{O}(x)y + \mathbb{S}(x)y^2}. \tag{14}$$

The polynomial representation of element $\overline{\sigma}_{m_1} \circ \overline{\sigma}_{m_2} \circ \cdots \circ \overline{\sigma}_{m_r}$ in $L_{N,r}$ for $\overrightarrow{m} \in S_{N,r}$ is

$$(y_1 - y_0)^{m_1-1} \underline{\circ} ((y_1 - y_0)^{m_2-1} \underline{\circ} (\cdots \underline{\circ} (y_1 - y_0)^{m_r-1}) \cdots)). \tag{15}$$

The coefficient of the term $y_1^{n_1-1}y_2^{n_2-1}\cdots y_r^{n_r-1}$ in the formula (15) is the $(m_1, m_2, \cdots, m_r)$-th row, the $(n_1, n_2, \cdots, n_r)$-th column element of the matrix $C_{N,r}$. So from formula (14) we have

$$1 + \sum_{N,r>0} \dim_{\mathbb{Q}} L_{N,r}x^N y^r \geq \frac{1}{1 - \mathbb{O}(x)y + \mathbb{S}(x)y^2}.$$

From Remark 4.4 and Lemma 4.2, we have the nondegeneracy conjecture. □

Remark 4.7. Formula (12) is essentially the Conjecture 4.12 in [8], in the above proof we actually show that formula (12) is a corollary of Tasaka's isomorphism conjecture. See [11] for the application of nondegeneracy conjecture to motivic multiple zeta values.

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