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Another SPT crank for the number of smallest parts in overpartitions with even smallest part

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By using the M_2-rank of an overpartition as well as a residual crank, we give another combinatorial refinement of the congruences $\overline{\mathrm{spt}}_2(3n) \equiv \overline{\mathrm{spt}}_2(3n+1) \equiv 0 \pmod 3$. Here $\overline{\mathrm{spt}}_2(n)$ is the total number of appearances of the smallest parts among the overpartitions of n where the smallest part is even and not overlined. Our proof depends on Bailey's Lemma and the rank difference formulas of Lovejoy and Osburn for the M_2-rank of an overpartition. This congruence was previously refined using the rank of an overpartition.

1. Introduction and statement of results

We recall an overpartition of a positive integer n is a partition of n where the first occurrence of a part may (or may not) be overlined. For example, the overpartitions of 4 are 4, $\bar{4}$, $3+1$, $3+\bar{1}$, $\bar{3}+1$, $\bar{3}+\bar{1}$, $2+2$, $\bar{2}+2$, $2+1+1$, $2+\bar{1}+1$, $\bar{2}+1+1$, $\bar{2}+\bar{1}+1$, $1+1+1+1$, and $\bar{1}+1+1+1$.

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We have a weighted count on overpartitions given by counting an overpartition by the number of times the smallest appears. We use the convention of not including the overpartitions where the smallest part is overlined. We let $\overline{\mathrm{spt}}(n)$ denote the total number of occurrences of the smallest parts among the overpartitions of n without smallest part overlined. The function $\overline{\mathrm{spt}}(n)$ was introduced by Bringmann, Lovejoy, and Osburn in [2] after Andrews introduced the spt function for partitions in [1]. Two restrictions of $\overline{\mathrm{spt}}(n)$ are $\overline{\mathrm{spt}}_1(n)$ and $\overline{\mathrm{spt}}_2(n)$, where we restrict to overpartitions where the smallest part is odd and even respectively. We see $\overline{\mathrm{spt}}(4) = 13$, $\overline{\mathrm{spt}}_1(4) = 10$, and $\overline{\mathrm{spt}}_2(4) = 3$.

In [3] Garvan and the author gave combinatorial refinements of congruences satisfied by $\overline{\mathrm{spt}}(n)$, $\overline{\mathrm{spt}}_1(n)$, and $\overline{\mathrm{spt}}_2(n)$. The idea is to introduce an extra variable into the generating function of each spt-function to get a crank type statistic. This statistic can then be shown in certain cases to equally split up the numbers $\overline{\mathrm{spt}}(n)$, $\overline{\mathrm{spt}}_1(n)$, and $\overline{\mathrm{spt}}_2(n)$ based on the residue class of the statistic.

For $\overline{\mathrm{spt}}_2(n)$ we have the congruences

$$\overline{\mathrm{spt}}_2(3n) \equiv 0 \pmod 3, \tag{1.1}$$

$$\overline{\mathrm{spt}}_2(3n+1) \equiv 0 \pmod 3, \tag{1.2}$$

$$\overline{\mathrm{spt}}_2(5n+3) \equiv 0 \pmod 5. \tag{1.3}$$

In this paper we give another proof of the modulo 3 congruences.

To start, by summing according to the smallest part, we find a generating function for $\overline{\mathrm{spt}}_2(n)$ to be given by

$$\sum_{n=1}^{\infty} \overline{\mathrm{spt}}_2(n) q^n = \sum_{n=1}^{\infty} \frac{q^{2n}(-q^{2n+1};q)_\infty}{(1-q^{2n})^2 (q^{2n+1};q)_\infty}.$$

Here we use the standard product notation,

$$(a;q)_n = \prod_{j=0}^{n-1} \left(1 - aq^j\right),$$

$$(a;q)_\infty = \prod_{j=0}^{\infty} \left(1 - aq^j\right),$$

$$(a_1, a_2, \ldots, a_k; q)_n = (a_1;q)_n (a_2;q)_n \ldots (a_k;q)_n,$$

$$(a_1, a_2, \ldots, a_k; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \ldots (a_k;q)_\infty.$$

In [3] we considered the two variable generalization given by

$$\overline{\mathrm{S}}_2(z,q) = \sum_{n=1}^{\infty} \frac{q^{2n}(-q^{2n+1};q)_\infty (q^{2n+1};q)_\infty}{(zq^{2n};q)_\infty (z^{-1}q^{2n};q)_\infty}.$$

We note setting $z = 1$ gives the generating function for $\overline{\mathrm{spt}}_2(n)$. It turns out $\bar{\mathrm{S}}_2(z,q)$ can be expressed in terms of the Dyson rank of an overpartition and a residual crank from [2]. In the same paper, we also used the M_2-rank of a partition without repeated odd parts, but not the M_2-rank of an overpartition. Difference formulas for all three of these ranks were determined by Lovejoy and Osburn in [6–8].

In [4] the author gave higher order generalizations of $\overline{\mathrm{spt}}(n)$ and $\overline{\mathrm{spt}}_2(n)$ and noted that one could use the M_2-rank and another residual crank from [2] to explain the modulo 3 congruences for $\overline{\mathrm{spt}}_2(n)$. In this paper we instead use

$$\mathrm{S}(z,q) = \sum_{n=1}^{\infty} \frac{q^{2n}(-q^{2n+1};q)_\infty (q^{2n+1};q)_\infty}{(zq^{2n}, z^{-1}q^{2n};q^2)_\infty (q^{2n+1};q^2)_\infty^2} = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N_{\mathrm{S}}(m,n) z^m q^n.$$

Again setting $z = 1$ gives the generating function for $\overline{\mathrm{spt}}_2(n)$. This is not the same series $S(z,q)$ used in [3], however we do not want to overly complicate matters with additional notation.

For a positive integer t we let

$$N_{\mathrm{S}}(k,t,n) = \sum_{m \equiv k \pmod{t}} N_{\mathrm{S}}(m,n).$$

We then have

$$\overline{\mathrm{spt}}_2(n) = \sum_{m=-\infty}^{\infty} N_{\mathrm{S}}(m,n) = \sum_{k=0}^{t-1} N_{\mathrm{S}}(k,t,n).$$

Additionally we see if ζ_3 is a primitive third root of unity, then

$$\mathrm{S}(\zeta_3,q) = \sum_{n=1}^{\infty} \left(\sum_{k=0}^{2} N_{\mathrm{S}}(k,3,n)\zeta_3^k \right) q^n.$$

The minimal polynomial for ζ_3 is $x^2 + x + 1$. If $N_{\mathrm{S}}(0,3,N) + N_{\mathrm{S}}(1,3,N)\zeta_3 + N_{\mathrm{S}}(2,3,N)\zeta_3^2 = 0$ then we must in fact have $N_{\mathrm{S}}(0,3,N) = N_{\mathrm{S}}(1,3,N) = N_{\mathrm{S}}(2,3,N)$. That is to say, if the coefficient of q^N in $\mathrm{S}(\zeta_3,q)$ is zero, then $\overline{\mathrm{spt}}_2(N) = 3 \cdot N_{\mathrm{S}}(0,3,N)$ and so $\overline{\mathrm{spt}}_2(N) \equiv 0 \pmod{3}$.

Our proof of $\overline{\mathrm{spt}}_2(3n) \equiv \overline{\mathrm{spt}}_2(3n+1) \equiv 0$ is to find the 3-dissection of $\mathrm{S}(\zeta_3,q)$ with the q^{3n} and q^{3n+1} terms being all zero. This is the same idea that was used in [3], we are just using $\mathrm{S}(z,q)$ rather than $\bar{\mathrm{S}}_2(z,q)$.

We cannot use $\mathrm{S}(\zeta_5,q)$ to prove $\overline{\mathrm{spt}}_2(5n+3) \equiv 0 \pmod{5}$. In particular we find the coefficient of q^8 in $\mathrm{S}(\zeta_5,q)$ to be $z^3 + z^2 + 3z + 5 + 3z^{-1} + z^{-2} + z^{-3}$. That is to say, $N_{\mathrm{S}}(0,5,8) = 5$, $N_{\mathrm{S}}(1,5,8) = 3$, $N_{\mathrm{S}}(2,5,8) = 2$, $N_{\mathrm{S}}(3,5,8) = 2$, and $N_{\mathrm{S}}(4,5,8) = 3$.

However $\overline{\mathrm{spt}}_2(5n+3) \equiv 0 \pmod{5}$ does follow by considering $\bar{\mathrm{S}}_2(\zeta_5,q)$. This can be compared with the rank of a partition explaining the congruences for $p(5n+4)$ and $p(7n+5)$ but not $p(11n+6)$, whereas the crank of a partition does explain all three.

Theorem 1.1.

$$\mathrm{S}(\zeta_3,q)=q^2\frac{(q^{18};q^{18})_\infty^4}{(q^6;q^6)_\infty(q^9;q^9)_\infty^2}+\frac{2q^5(-q^9;q^9)_\infty}{(q^9;q^9)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n^2+18n}}{1-q^{18n+6}}.$$

We prove this theorem by relating $\mathrm{S}(z,q)$ to a certain rank and crank and using dissections of these related functions. We recall the M_2-rank of an overpartition π is given by

$$M_2\text{-rank}=\left\lceil\frac{l(\pi)}{2}\right\rceil-\#(\pi)+\#(\pi_o)-\chi(\pi),$$

where $l(\pi)$ is the largest part of π, $\#(\pi)$ is the number of parts of π, $\#(\pi_o)$ is the number of non-overlined odd parts, and $\chi(\pi)=1$ if the largest part of π is odd and non-overlined and otherwise $\chi(\pi)=0$. We let $\overline{N2}(m,n)$ denote the number of overpartitions of n with M_2-rank m. The M_2-rank for overpartitions was introduced by Lovejoy in [5], in the same paper Lovejoy found the generating function for $\overline{N2}$ is given by

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{N2}(m,n)z^mq^n=\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+2\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^nq^{n^2+2n}}{(1-zq^{2n})(1-z^{-1}q^{2n})}\right).\tag{1.4}$$

We also use a residual crank from [2]. We let

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{M2}(m,n)z^mq^n=\frac{(-q;q)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(zq^2;q^2)_\infty(z^{-1}q^2;q^2)_\infty}.\tag{1.5}$$

Theorem 1.2.

$$\mathrm{S}(z,q)=\frac{1}{(1-z)(1-z^{-1})}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\left(\overline{N2}(m,n)-\overline{M2}(m,n)\right)z^mq^n.$$

Theorem 1.3.

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{N2}(m,n)\zeta_3^mq^n\\&\quad=\frac{(-q^3;q^3)_\infty(q^9;q^9)_\infty^2}{(q^3;q^3)_\infty(-q^9;q^9)_\infty^2}+\frac{2q(q^9;q^9)_\infty(q^{18};q^{18})_\infty}{(q^3;q^3)_\infty}+\frac{4q^2(q^{18};q^{18})_\infty^4}{(q^6;q^6)_\infty(q^9;q^9)_\infty^2}\\&\qquad+\frac{6q^5(-q^9;q^9)_\infty}{(q^9;q^9)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n^2+18n}}{1-q^{9n+6}}.\end{aligned}$$

Theorem 1.3 follows from the rank difference formulas derived by Lovejoy and Osburn in [8].

Theorem 1.4.

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{M2}(m,n)\zeta_3^m q^n = \frac{(-q^3;q^3)_\infty(q^9;q^9)_\infty^2}{(q^3;q^3)_\infty(-q^9;q^9)_\infty^2} + 2q\frac{(q^9;q^9)_\infty(q^{18};q^{18})_\infty}{(q^3;q^3)_\infty} + q^2\frac{(q^{18};q^{18})_\infty^4}{(q^6;q^6)_\infty(q^9;q^9)_\infty^2}.$$

We see Theorem 1.1 follows from Theorems 1.2, 1.3, and 1.4. We give the proofs of Theorems 1.2 and 1.4 in the next section. In Section 3 we give brief combinatorial interpretations of the coefficients $N_{\mathrm{S}}(m,n)$, in particular they are non-negative.

2. The proofs

Proof of Theorem 1.2. We recall a pair of sequences (α_n,β_n) is a Bailey pair relative to (a,q) if

$$\beta_n = \sum_{r=0}^{n}\frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}.$$

A limiting case of Bailey's Lemma gives for a Bailey pair (α_n,β_n) that

$$\sum_{n=0}^{\infty}(\rho_1,\rho_2;q)_n\left(\frac{aq}{\rho_1\rho_2}\right)^n\beta_n = \frac{(aq/\rho_1,aq/\rho_2;q)_\infty}{(aq,aq/\rho_1\rho_2;q)_\infty}\sum_{n=0}^{\infty}\frac{(\rho_1,\rho_2;q)_n(\frac{aq}{\rho_1\rho_2})^n\alpha_n}{(aq/\rho_1,aq/\rho_2;q)_n}.$$

As in [4] the Bailey pair connecting the M_2-rank of an overpartition and the residual crank is

$$\alpha_n = \begin{cases} 1 & n=0 \\ (-1)^n 2q^{n^2} & n\geq 1, \end{cases} \qquad \beta_n = \frac{(q;q^2)_n^2}{(q^2;q^2)_{2n}}.$$

This is a Bailey pair with respect to $(1,q^2)$.

We note that

$$\frac{(q^2;q^2)_\infty}{(z,z^{-1};q^2)_\infty(q;q^2)_\infty^2}\cdot\frac{(zq^2,z^{-1}q^2;q^2)_\infty}{(q^2;q^2)_\infty^2} = \frac{(-q;q)_\infty}{(1-z)(1-z^{-1})(q;q)_\infty}.$$

With this Bailey pair we have

$$\begin{aligned}\mathrm{S}(z,q) &= \sum_{n=1}^{\infty}\frac{q^{2n}(-q^{2n+1};q)_\infty(q^{2n+1};q)_\infty}{(zq^{2n},z^{-1}q^{2n};q^2)_\infty(q^{2n+1};q^2)_\infty^2} \\ &= \frac{(q^2;q^2)_\infty}{(z,z^{-1};q^2)_\infty(q;q^2)_\infty^2}\sum_{n=0}^{\infty}q^{2n}\left(z,z^{-1};q^2\right)_n\beta_n - \frac{(q^2;q^2)_\infty}{(z,z^{-1};q^2)_\infty(q;q^2)_\infty^2}\end{aligned}$$

$$= \frac{(-q;q)_\infty}{(1-z)(1-z^{-1})(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n}(1-z)(1-z^{-1})\alpha_n}{(1-zq^{2n})(1-z^{-1}q^{2n})} - \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty (z,z^{-1};q^2)_\infty}$$

$$= \frac{(-q;q)_\infty}{(1-z)(1-z^{-1})(q;q)_\infty} \left(1 + 2\sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+2n}}{(1-zq^{2n})(1-z^{-1}q^{2n})}\right) - \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty (z,z^{-1};q^2)_\infty}.$$

By Eqs. (1.4) and (1.5) we then have

$$\mathrm{S}(z,q) = \frac{1}{(1-z)(1-z^{-1})} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \left(\overline{N2}(m,n) - \overline{M2}(m,n)\right) z^m q^n. \qquad \square$$

Proof of Theorem 1.4. We begin by noting

$$\frac{(-q;q)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty (\zeta_3 q^2, \zeta_3^{-1} q^2; q^2)_\infty} = \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2 (q^6;q^6)_\infty}.$$

By Gauss and the Jacobi Triple Product Identity we have

$$\begin{aligned}
\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} &= \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{1}{2} \sum_{n=-\infty}^{\infty} q^{n(n+1)/2} \\
&= \frac{1}{2} \sum_{k=0}^{2} \sum_{n=-\infty}^{\infty} q^{(3n+k)(3n+k+1)/2} \\
&= \left(-q^6, -q^3, q^9; q^9\right)_\infty + q\left(-q^9, -q^9, q^9; q^9\right)_\infty.
\end{aligned}$$

Using the above to expand $\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}$, dividing by $(q^6;q^6)_\infty$, and reducing the products finishes the proof. $\square$

3. Combinatorial interpretations

By viewing the summands $S(z,q)$ as the product of various types of partitions, we see $N_{\mathrm{S}}(m,n)$ could be interpreted in terms of vector partitions. However, the $(q^{2n+1};q)_\infty$ in the numerator would require us to count the vector partitions with a weight of $+1$ or -1. This interpretation would hide the fact that each $N_{\mathrm{S}}(m,n)$ is non-negative. We instead interpret $N_{\mathrm{S}}(m,n)$ in terms of partition pairs. This interpretation makes the non-negativity clear. Using the q-binomial theorem we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{q^{2n}(q^{4n+2};q^2)_\infty}{(zq^{2n}, z^{-1}q^{2n}; q^2)_\infty (q^{2n+1};q^2)_\infty^2} \\
&\quad = \sum_{n=1}^{\infty} \frac{q^{2n}}{(zq^{2n};q^2)_\infty (q^{2n+1};q^2)_\infty^2} \sum_{k=0}^{\infty} \frac{(zq^{2n+2};q^2)_k z^{-k} q^{2nk}}{(q^2;q^2)_k}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{q^{2n}}{(zq^{2n};q^2)_\infty (q^{2n+1};q^2)_\infty^2} \\
&\quad + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{z^{-k}q^{2n+2nk}}{(1-zq^{2n})(zq^{2n+2k+2};q^2)_\infty (q^2;q^2)_k (q^{2n+1};q^2)_\infty^2} \\
&= \sum_{n=1}^{\infty} \frac{q^{2n}}{(zq^{2n};q^2)_\infty (q^{2n+1};q^2)_\infty^2} \\
&\quad + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{q^{2n}}{(1-zq^{2n})(q^{2n+2};q^2)_k (zq^{2n+2k+2};q^2)_\infty (q^{2n+1};q^2)_\infty} \\
&\quad \times \frac{z^{-k}q^{2nk}(q^2;q^2)_{n+k}}{(q^2;q^2)_k (q^2;q^2)_n (q^{2n+1};q^2)_\infty}.
\end{aligned} \tag{3.1}
$$

For a partition π we let $s(\pi)$ denote the smallest part of π (with the convention that the empty partition has smallest part ∞), $\#(\pi)$ the number of parts in π, $\#(\pi_e)$ the number of even parts in π, and $|\pi|$ the sum of the parts of π. We say a pair of partitions (π^1, π^2) is a partition pair of n if $|\pi^1| + |\pi^2| = n$. We let PP denote the set of partition pairs (π^1, π^2) such that π^1 is non-empty, $s(\pi^1)$ is even, $s(\pi^1) \leq s(\pi^2)$, and the even parts of π^2 are at most $2s(\pi^1)$. For such a partition pair we let $k(\pi^1, \pi^2)$ denote the number of even parts of π^1 that are either the smallest part or are larger than $s(\pi^1) + 2\#(\pi_e^2)$. We note when π^2 contains no even parts that $k(\pi^1, \pi^2)$ reduces to $\#(\pi_e^1)$. We define a crank on the elements of PP by

$$
c(\pi^1, \pi^2) = k(\pi^1, \pi^2) - \#(\pi_e^2) - 1.
$$

We claim $N_{\mathrm{S}}(m, n)$ is also the number of partitions pairs of n from PP with $c(\pi^1, \pi^2) = m$.

For this we note the first series in (3.1) gives the cases when π^2 has no even parts. The second series in (3.1) gives the cases when π^2 has even parts, since $\frac{q^{2nk}(q^2;q^2)_{n+k}}{(q^2;q^2)_k (q^2;q^2)_n}$ is the generating function for partitions into even parts with exactly k parts and each part between $2n$ and $4n$ (inclusive).

It may be possible to define a bijection from these partition pairs to marked overpartitions with smallest part even, and through that determine a crank defined on marked overpartitions, similar to what was done in [3]. However, we do not pursue that here.

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