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Explicit estimates for the number of rational points of singular complete intersections over a finite field ☆

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Let $V \subset \mathbb{P}^n(\overline{\mathbb{F}}_q)$ be a complete intersection defined over a finite field $\mathbb{F}_q$ of dimension r and singular locus of dimension at most $0 \leq s \leq r-2$. We obtain an explicit version of the Hooley–Katz estimate $||V(\mathbb{F}_q)| - p_r| = \mathcal{O}(q^{(r+s+1)/2})$, where $|V(\mathbb{F}_q)|$ denotes the number of $\mathbb{F}_q$-rational points of V and $p_r := |\mathbb{P}^r(\mathbb{F}_q)|$. Our estimate improves all the previous estimates in several important cases. Our approach relies on tools of classical algebraic geometry. A crucial ingredient is a new effective version of the Bertini smoothness theorem, namely an explicit upper bound of the degree of a proper Zariski closed subset of $(\mathbb{P}^n)^{s+1}(\overline{\mathbb{F}}_q)$ which contains all the singular linear sections of V of codimension $s+1$.

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Bertini's smoothness theorem
Hooley–Katz estimate

1. Introduction

Let $\mathbb{F}_q$ be the finite field of q elements and let $\overline{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$. By $\mathbb{P}^n := \mathbb{P}^n(\overline{\mathbb{F}}_q)$ and $\mathbb{A}^n := \mathbb{A}^n(\overline{\mathbb{F}}_q)$ we denote the n-dimensional projective and affine spaces defined over $\overline{\mathbb{F}}_q$ respectively. For any affine or projective variety V, we denote by $V(\mathbb{F}_q)$ the set of $\mathbb{F}_q$-rational points of V, that is, the set of points of V with coordinates in $\mathbb{F}_q$, and by $|V(\mathbb{F}_q)|$ its cardinality. In particular, it is well-known that, for $r \geq 0$,

$$p_r := |\mathbb{P}^r(\mathbb{F}_q)| = q^r + \cdots + q + 1.$$

Let $V \subset \mathbb{P}^n$ be an ideal-theoretic complete intersection defined over $\mathbb{F}_q$, of dimension r and multidegree $\boldsymbol{d} := (d_1, \ldots, d_{n-r})$. In a fundamental work [9], P. Deligne showed that, if V is nonsingular, then

$$\big||V(\mathbb{F}_q)| - p_r\big| \leq b'_r(n, \boldsymbol{d})\, q^{\frac{r}{2}}, \tag{1}$$

where $b'_r(n, \boldsymbol{d})$ is the rth primitive Betti number of any nonsingular complete intersection of $\mathbb{P}^n$ of dimension r and multidegree $\boldsymbol{d}$ (see [11, Theorem 4.1] for an explicit expression of $b'_r(n, \boldsymbol{d})$ in terms of n, r and $\boldsymbol{d}$).

This result was extended by C. Hooley and N. Katz to singular complete intersections. More precisely, in [15] it is proved that, if the singular locus of V has dimension at most s, then

$$|V(\mathbb{F}_q)| = p_r + \mathcal{O}(q^{\frac{r+s+1}{2}}), \tag{2}$$

where the constant implied by the $\mathcal{O}$-notation depends only on n, r and $\boldsymbol{d}$, and it is not explicitly given. Finally, S. Ghorpade and G. Lachaud obtained the following explicit version of the Hooley–Katz bound (2) in [11] (see also [12]):

$$\big||V(\mathbb{F}_q)| - p_r\big| \leq b'_{r-s-1}(n - s - 1, \boldsymbol{d})\, q^{\frac{r+s+1}{2}} + C(n, r, \boldsymbol{d})\, q^{\frac{r+s}{2}}, \tag{3}$$

where $C(n, r, \boldsymbol{d}) := 9 \cdot 2^{n-r}\big((n-r)d + 3\big)^{n+1}$ and $d := \max_{1 \leq i \leq n-r} d_i$.

For the potential applications of (3), the fact that the constant $C(n, r, \boldsymbol{d})$ depends exponentially on the dimension n of the ambient space $\mathbb{P}^n$ may be inconvenient. This can be seen for example in [7,17,6], where we use estimates on the number of $\mathbb{F}_q$-rational points of singular complete intersections to determine the asymptotic behavior of the average cardinality of value sets and the distribution of factorization patterns of families of univariate polynomials defined over $\mathbb{F}_q$ having certain coefficients with prescribed values. For this reason, in this paper we obtain another explicit estimate on $|V(\mathbb{F}_q)|$ where this exponential dependency on n is avoided.

From a methodological point of view, the estimates in [11] are based on the Grothendieck–Lefschetz Trace Formula, together with estimates for the dimension of certain spaces of étale ℓ-adic cohomology associated with the complete intersection $V \subset \mathbb{P}^n$ under consideration.

Our approach is rather different and relies on tools of classical projective algebraic geometry, combined with Deligne's estimate (1). The crucial geometric ingredient is the following effective version of the Bertini smoothness theorem, which provides quantitative information on the set of linear sections $\mathcal{L} \subset \mathbb{P}^n$ defined over $\mathbb{F}_q$ such that $V \cap \mathcal{L}$ has codimension $s+1$ and is nonsingular.

Theorem 1.1. *Let $V \subset \mathbb{P}^n$ be a complete intersection defined over $\mathbb{F}_q$, of dimension r, multidegree $\boldsymbol{d} := (d_1, \dots, d_{n-r})$, and singular locus of dimension at most $0 \leq s \leq r-2$. Let $\delta := \prod_{i=1}^{n-r} d_i$ and $D := \sum_{i=1}^{n-r}(d_i - 1)$. There exists a hypersurface $\mathcal{H} \subset (\mathbb{P}^n)^{s+1}$, defined by a multihomogeneous polynomial of degree at most $D^{r-s-1}(D+r-s)\delta$ in each group of variables, with the following property: if $\boldsymbol{\gamma} \in (\mathbb{P}^n)^{s+1} \setminus \mathcal{H}$ and $\mathcal{L} := \{\boldsymbol{\gamma} \cdot x = 0\}$, then $V \cap \mathcal{L}$ is nonsingular of pure dimension $r-s-1$.*

Refs. [1] and [4] provide effective versions of the Bertini smoothness theorem for hypersurfaces and normal complete intersections respectively. Theorem 1.1 significantly improves and generalizes both results. We also remark that a different variant of an effective Bertini smoothness theorem is obtained in [5].

Combining Theorem 1.1 with upper bounds on the number of $\mathbb{F}_q$-rational zeros of multihomogeneous polynomials we obtain rather precise estimates on the number of nonsingular $\mathbb{F}_q$-definable linear sections of codimension $s+1$ of V. Then the analysis of the second moment of the number of $\mathbb{F}_q$-rational points of V in linear sections of codimension $s+1$ yields an estimate on the number of $\mathbb{F}_q$-rational points of V. More precisely, we obtain the following result.

Theorem 1.2. *Let $V \subset \mathbb{P}^n$ be a complete intersection defined over $\mathbb{F}_q$, of dimension r, multidegree $\boldsymbol{d} := (d_1, \dots, d_{n-r})$, and singular locus of dimension at most $0 \leq s \leq r-2$. Let $\delta := \prod_{i=1}^{n-r} d_i$ and $D := \sum_{i=1}^{n-r}(d_i - 1)$. If $q > 2(s+1)D^{r-s-1}(D+r-s)\delta$, then*

$$\big||V(\mathbb{F}_q)| - p_r\big| \leq \big(b'_{r-s-1}(n-s-1, \boldsymbol{d}) + 2\sqrt{\delta} + 1\big)\, q^{\frac{r+s+1}{2}}. \tag{4}$$

According to [11, Proposition 4.2], the Betti number $b'_{r-s-1}(n-s-1, \boldsymbol{d})$ can be bounded from above by a quantity which is roughly of order $D^{r-s}\delta$. As D is in general much smaller than δ, we may say that the error term of (4) grows linearly with δ. In this sense, (4) significantly improves (3), whose error term may include an exponential term δ^{n+1} when V is a hypersurface (although in this case our error term grows with rate proportional to δ^{r-s+1}). On the other hand, (3) is valid without restrictions on q, while (4) only holds for $q > 2(s+1)D^{r-s-1}(D+r-s)\delta$.

The paper is organized as follows. In Section 2 we include a brief review of the notions of classical algebraic geometry which we use. We also obtain an upper bound on the number of $\mathbb{F}_q$-rational zeros of a multihomogeneous polynomial. Section 3 is devoted to the proof of Theorem 1.1. Finally, in Section 4 we combine the upper bound of Section 2 with Theorem 1.1 and the analysis of the second moment mentioned before to prove Theorem 1.2. Taking into account that the condition on q of the statement of Theorem 1.2 may restrict its applicability, we obtain a further estimate for normal complete intersections which is valid without restrictions on q (Corollary 4.6).

2. Notions, notations and preliminary results

We use standard notions and notations of commutative algebra and algebraic geometry as can be found in, e.g., [13,16,18,19].

Let K be any of the fields $\mathbb{F}_q$ or $\overline{\mathbb{F}}_q$. We denote by $\mathbb{A}^n$ the affine n-dimensional space $\overline{\mathbb{F}}_q^n$ and by $\mathbb{P}^n$ the projective n-dimensional space over $\overline{\mathbb{F}}_q^{n+1}$. Both spaces are endowed with their respective Zariski topologies over K, for which a closed set is the zero locus of a set of polynomials of $\mathsf{K}[X_1, \ldots, X_n]$, or of a set of homogeneous polynomials of $\mathsf{K}[X_0, \ldots, X_n]$.

A subset $V \subset \mathbb{P}^n$ is a *projective variety defined over* K (or a projective K-variety for short) if it is the set of common zeros in $\mathbb{P}^n$ of homogeneous polynomials $F_1, \ldots, F_m \in \mathsf{K}[X_0, \ldots, X_n]$. Correspondingly, an *affine variety of* $\mathbb{A}^n$ *defined over* K (or an affine K-variety for short) is the set of common zeros in $\mathbb{A}^n$ of polynomials $F_1, \ldots, F_m \in \mathsf{K}[X_1, \ldots, X_n]$. We think a projective or affine K-variety to be equipped with the induced Zariski topology. We shall frequently denote by $V(F_1, \ldots, F_m)$ or $\{F_1 = 0, \ldots, F_m = 0\}$ the affine or projective K-variety consisting of the common zeros of the polynomials $F_1, \ldots, F_m$.

In the remaining part of this section, unless otherwise stated, all results referring to varieties in general should be understood as valid for both projective and affine varieties.

A K-variety V is K-*irreducible* if it cannot be expressed as a finite union of proper K-subvarieties of V. Further, V is *absolutely irreducible* if it is $\overline{\mathbb{F}}_q$-irreducible as a $\overline{\mathbb{F}}_q$-variety. Any K-variety V can be expressed as an irredundant union $V = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_s$ of irreducible (absolutely irreducible) K-varieties, unique up to reordering, which are called the *irreducible* (*absolutely irreducible*) K-*components* of V.

For a K-variety V contained in $\mathbb{P}^n$ or $\mathbb{A}^n$, we denote by $I(V)$ its *defining ideal*, namely the set of polynomials of $\mathsf{K}[X_0, \ldots, X_n]$, or of $\mathsf{K}[X_1, \ldots, X_n]$, vanishing on V. The *coordinate ring* $\mathsf{K}[V]$ of V is defined as the quotient ring $\mathsf{K}[X_0, \ldots, X_n]/I(V)$ or $\mathsf{K}[X_1, \ldots, X_n]/I(V)$. The *dimension* $\dim V$ of V is the length r of the longest chain $V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r$ of nonempty irreducible K-varieties contained in V. We call V *equidimensional* if all its irreducible K-components are of the same dimension. We say that V has *pure dimension* r if it is equidimensional of dimension r.

A K-variety of $\mathbb{P}^n$ or $\mathbb{A}^n$ of pure dimension $n-1$ is called a K-*hypersurface*. A K-hypersurface of $\mathbb{P}^n$ (or $\mathbb{A}^n$) is the set of zeros of a single nonzero polynomial of $\mathsf{K}[X_0, \ldots, X_n]$ (or of $\mathsf{K}[X_1, \ldots, X_n]$).

The *degree* $\deg V$ of an irreducible K-variety V is the maximum number of points lying in the intersection of V with a linear space L of codimension $\dim V$, for which $V \cap L$

is a finite set. More generally, following [14] (see also [10]), if $V = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_s$ is the decomposition of V into irreducible K-components, we define its degree as

$$\deg V := \sum_{i=1}^{s} \deg \mathcal{C}_i.$$

According to this definition, the degree of a K-hypersurface V is the degree of a polynomial of minimal degree defining V.

Let $V \subset \mathbb{A}^n$ be a K-variety and let $I(V) \subset \mathsf{K}[X_1, \ldots, X_n]$ be the defining ideal of V. Let x be a point of V. The *dimension* $\dim_x V$ *of* V *at* x is the maximum of the dimensions of the irreducible K-components of V that contain x. If $I(V) = (F_1, \ldots, F_m)$, the *tangent space* $\mathcal{T}_x V$ to V at x is the kernel of the Jacobian matrix $(\partial F_i/\partial X_j)_{1 \le i \le m, 1 \le j \le n}(x)$ of the polynomials $F_1, \ldots, F_m$ with respect to $X_1, \ldots, X_n$ at x. We have the following inequality (see, e.g., [18, p. 94]):

$$\dim \mathcal{T}_x V \ge \dim_x V.$$

The point x is *regular* if $\dim \mathcal{T}_x V = \dim_x V$. Otherwise, the point x is called *singular*. The set of singular points of V is the *singular locus* $\mathrm{Sing}(V)$ of V; it is a closed K-subvariety of V. A variety is called *nonsingular* if its singular locus is empty. For a projective variety, the concepts of tangent space, regular and singular point can be defined by considering an affine neighborhood of the point under consideration.

2.1. Complete intersections

A K-variety V of dimension r in the n-dimensional space is an (ideal-theoretic) *complete intersection* if its ideal $I(V)$ over K can be generated by $n - r$ polynomials. If $V \subset \mathbb{P}^n$ is a complete intersection defined over K, of dimension r and degree δ, and $F_1, \ldots, F_{n-r}$ is a system of homogeneous generators of $I(V)$, the degrees $d_1, \ldots, d_{n-r}$ depend only on V and not on the system of homogeneous generators. Arranging the d_i in such a way that $d_1 \ge d_2 \ge \cdots \ge d_{n-r}$, we call $\boldsymbol{d} := (d_1, \ldots, d_{n-r})$ the *multidegree* of V.

If $V \subset \mathbb{P}^n$ is a complete intersection of multidegree $\boldsymbol{d} := (d_1, \ldots, d_{n-r})$, then the *Bézout theorem* (see, e.g., [13, Theorem 18.3] or [19, §5.5, p. 80]) asserts that

$$\deg V = d_1 \cdots d_{n-r}.$$

We shall consider a particular class of complete intersections, which we now define. A K-variety is *regular in codimension* m if the singular locus $\mathrm{Sing}(V)$ of V has codimension at least $m+1$ in V, namely if $\dim V - \dim \mathrm{Sing}(V) \ge m+1$. A complete intersection V which is regular in codimension 1 is called *normal* (actually, normality is a general notion that agrees on complete intersections with the one we define here). A fundamental result for projective complete intersections is the Hartshorne connectedness theorem (see, e.g., [16, Theorem VI.4.2]), which we now state. If $V \subset \mathbb{P}^n$ is a complete intersection defined over K and $W \subset V$ is any K-subvariety of codimension at least 2, then $V \setminus W$ is connected in the Zariski topology of $\mathbb{P}^n$ over K. Applying the Hartshorne connectedness theorem with $W := \mathrm{Sing}(V)$, one deduces the following result.

Theorem 2.1. *If $V \subset \mathbb{P}^n$ is a normal complete intersection, then V is absolutely irreducible.*

2.2. Rational points

Let $\mathbb{P}^n(\mathbb{F}_q)$ be the n-dimensional projective space over $\mathbb{F}_q$ and let $\mathbb{A}^n(\mathbb{F}_q)$ be the n-dimensional $\mathbb{F}_q$-vector space $\mathbb{F}_q^n$. For a projective variety $V \subset \mathbb{P}^n$ or an affine variety $V \subset \mathbb{A}^n$, we denote by $V(\mathbb{F}_q)$ the set of $\mathbb{F}_q$-rational points of V, namely $V(\mathbb{F}_q) := V \cap \mathbb{P}^n(\mathbb{F}_q)$ or $V(\mathbb{F}_q) := V \cap \mathbb{A}^n(\mathbb{F}_q)$ respectively.

For a projective variety V of dimension r and degree δ, we have (see [11, Proposition 12.1] or [4, Proposition 3.1]):

$$|V(\mathbb{F}_q)| \leq \delta p_r. \tag{5}$$

On the other hand, if V is an affine variety of dimension r and degree δ, then (see, e.g., [3, Lemma 2.1])

$$|V(\mathbb{F}_q)| \leq \delta q^r. \tag{6}$$

2.3. Multiprojective space

Let $\mathbb{N} := \mathbb{Z}_{\geq 0}$ be the set of nonnegative integers. For $\boldsymbol{n} := (n_1, \ldots, n_m) \in \mathbb{N}^m$, we define $|\boldsymbol{n}| := n_1 + \cdots + n_m$. Denote by $\mathbb{P}^{\boldsymbol{n}} := \mathbb{P}^{\boldsymbol{n}}(\overline{\mathbb{F}}_q)$ the multiprojective space $\mathbb{P}^{\boldsymbol{n}} := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$. For $1 \leq i \leq m$, let $\Gamma_i := \{\Gamma_{i,0}, \ldots, \Gamma_{i,n_i}\}$ be group of n_i+1 variables and let $\boldsymbol{\Gamma} := \{\Gamma_1, \ldots, \Gamma_m\}$. For $\mathsf{K} := \overline{\mathbb{F}}_q$ or $\mathsf{K} := \mathbb{F}_q$, a *multihomogeneous* polynomial of $\mathsf{K}[\boldsymbol{\Gamma}]$ of multidegree $\boldsymbol{d} := (d_1, \ldots, d_m)$ is an element which is homogeneous of degree d_i in Γ_i for $1 \leq i \leq m$. An ideal $I \subset \mathsf{K}[\boldsymbol{\Gamma}]$ is *multihomogeneous* if it is generated by a family of multihomogeneous polynomials. For any such ideal, we denote by $V(I) \subset \mathbb{P}^{\boldsymbol{n}}$ the variety defined as its set of common zeros. In particular, a hypersurface in $\mathbb{P}^{\boldsymbol{n}}$ defined over K is the set of zeros of a multihomogeneous polynomial of $\mathsf{K}[\boldsymbol{\Gamma}]$. The notions of irreducible variety and dimension of a subvariety of $\mathbb{P}^{\boldsymbol{n}}$ are defined as in the projective space.

2.3.1. Number of zeros of multihomogeneous hypersurfaces

With notations as above, let $\mathbb{F}_q^{\boldsymbol{n}+\boldsymbol{1}} := \mathbb{F}_q^{n_1+1} \times \cdots \times \mathbb{F}_q^{n_m+1}$. Let $F \in \overline{\mathbb{F}}_q[\boldsymbol{\Gamma}]$ be a multihomogeneous polynomial of multidegree $\boldsymbol{d} := (d_1, \ldots, d_m)$. In this section we establish two basic results concerning the number of $\mathbb{F}_q$-rational zeros of F. The first result is a nontrivial upper bound on the number of zeros of F in $\mathbb{F}_q^{\boldsymbol{n}+\boldsymbol{1}}$, which improves (6) for multiprojective hypersurfaces.

For $\boldsymbol{\alpha} \in \mathbb{N}^m$, we denote $\boldsymbol{d}^{\boldsymbol{\alpha}} := d_1^{\alpha_1} \cdots d_m^{\alpha_m}$. Further, let

$$\eta_m(\boldsymbol{d}, \boldsymbol{n}) := \sum_{\boldsymbol{\varepsilon} \in \{0,1\}^m \setminus \{\boldsymbol{0}\}} (-1)^{|\boldsymbol{\varepsilon}|+1} \boldsymbol{d}^{\boldsymbol{\varepsilon}} q^{|\boldsymbol{n}|+m-|\boldsymbol{\varepsilon}|}.$$

Observe that $\eta_m(\boldsymbol{d}, \boldsymbol{n}) < q^{|\boldsymbol{n}|+m} = |\mathbb{F}_q^{\boldsymbol{n}+\boldsymbol{1}}|$ if $q > \max_{1 \leq i \leq m} d_i$, while this inequality may not hold for $q \leq \max_{1 \leq i \leq m} d_i$. We have the following result.

Proposition 2.2. *Let $F \in \overline{\mathbb{F}}_q[\boldsymbol{\Gamma}]$ be a multihomogeneous polynomial of multidegree $\boldsymbol{d}$ with $\max_{1 \leq i \leq m} d_i < q$ and let N be the number of zeros of F in $\mathbb{F}_q^{\boldsymbol{n}+\boldsymbol{1}}$. Then*

$$N \leq \eta_m(\boldsymbol{d}, \boldsymbol{n}).$$

Proof. We argue by induction on m. The case $m = 1$ is (6).

Suppose that the statement holds for $m - 1$ and let $F \in \overline{\mathbb{F}}_q[\boldsymbol{\Gamma}]$ be an m-homogeneous polynomial of multidegree $\boldsymbol{d} := (d_1, \ldots, d_m)$. Let N be the number of zeros of F in $\mathbb{F}_q^{\boldsymbol{n}+\boldsymbol{1}}$, and let Z_m be the set of γ_m in $\mathbb{F}_q^{n_m+1}$ such that the substitution $F(\Gamma_1, \ldots, \Gamma_{m-1}, \gamma_m)$ of γ_m for Γ_m in F yields the zero polynomial of $\overline{\mathbb{F}}_q[\Gamma_1, \ldots, \Gamma_{m-1}]$. Consider F as an element of $\overline{\mathbb{F}}_q[\Gamma_m][\Gamma_1, \ldots, \Gamma_{m-1}]$ and let $A \in \overline{\mathbb{F}}_q[\Gamma_m]$ be a nonzero homogeneous polynomial of degree d_m which occurs as the coefficient of a monomial $\Gamma_1^{\boldsymbol{\alpha}_1} \cdots \Gamma_{m-1}^{\boldsymbol{\alpha}_{m-1}}$ in the dense representation of F. As Z_m is contained in the set of zeros in $\mathbb{F}_q^{n_m+1}$ of A, by (6) we have $|Z_m| \leq d_m q^{n_m}$.

Since $d_m < q$ by hypothesis, it follows that $|Z_m| \leq d_m q^{n_m} < q^{n_m+1} = |\mathbb{F}_q^{n_m+1}|$, which implies that $\mathbb{F}_q^{n_m+1} \setminus Z_m$ is nonempty. For $\gamma_m \in \mathbb{F}_q^{n_m+1} \setminus Z_m$, denote by $N_{m-1} := N_{m-1}(\gamma_m)$ the number of zeros of $F(\Gamma_1, \ldots, \Gamma_{m-1}, \gamma_m)$ in $\mathbb{F}_q^{n_1+1} \times \cdots \times \mathbb{F}_q^{n_{m-1}+1}$. By the inductive hypothesis and the fact that $\max_{1 \leq i \leq m-1} d_i < q$, we see that

$$N_{m-1} \leq \eta_{m-1}(\boldsymbol{d}^*, \boldsymbol{n}^*) < q^{|\boldsymbol{n}^*|+m-1},$$

where $\boldsymbol{d}^* := (d_1, \ldots, d_{m-1})$ and $\boldsymbol{n}^* := (n_1, \ldots, n_{m-1})$. As a consequence,

$$\begin{aligned} N &\leq |Z_m| q^{|\boldsymbol{n}^*|+m-1} + (q^{n_m+1} - |Z_m|)\eta_{m-1}(\boldsymbol{d}^*, \boldsymbol{n}^*) \\ &= |Z_m| \left(q^{|\boldsymbol{n}^*|+m-1} - \eta_{m-1}(\boldsymbol{d}^*, \boldsymbol{n}^*)\right) + \eta_{m-1}(\boldsymbol{d}^*, \boldsymbol{n}^*) q^{n_m+1} \leq \eta_m(\boldsymbol{d}, \boldsymbol{n}). \end{aligned}$$

This completes the proof of the proposition. □

The second result is concerned with conditions of existence of a point of $\mathbb{P}^{\boldsymbol{n}}(\mathbb{F}_q)$ which does not annihilate F and will be used to obtain an effective version of the Bertini smoothness theorem (Theorem 3.5).

Corollary 2.3. *Let $F \in \overline{\mathbb{F}}_q[\boldsymbol{\Gamma}]$ be a multihomogeneous polynomial of multidegree $\boldsymbol{d}$ and let $d := \max_{1 \leq i \leq m} d_i$. If $q > d$, then there exists $\boldsymbol{\gamma} \in \mathbb{P}^{\boldsymbol{n}}(\mathbb{F}_q)$ with $F(\boldsymbol{\gamma}) \neq 0$.*

Proof. It suffices to show that there exists $\boldsymbol{\gamma}' \in \mathbb{F}_q^{\boldsymbol{n}+\mathbf{1}}$ with $F(\boldsymbol{\gamma}') \neq 0$. Let N be the number of zeros of F in $\mathbb{F}_q^{\boldsymbol{n}+\mathbf{1}}$. According to Proposition 2.2, the number $N_{\neq 0}$ of elements in $\mathbb{F}_q^{\boldsymbol{n}+\mathbf{1}}$ not annihilating F is bounded as follows:

$$N_{\neq 0} \geq q^{|\boldsymbol{n}|+m} - \eta_m(\boldsymbol{d}, \boldsymbol{n}) = \sum_{\boldsymbol{\varepsilon} \in \{0,1\}^m} (-1)^{|\boldsymbol{\varepsilon}|} \boldsymbol{d}^{\boldsymbol{\varepsilon}} q^{|\boldsymbol{n}|+m-|\boldsymbol{\varepsilon}|} = \prod_{i=1}^{m} (q^{n_i+1} - d_i q^{n_i}).$$

Since $q > d$, we have $q^{n_i+1} > d_i q^{n_i}$ for $1 \leq i \leq m$, which yields the corollary. $\square$

3. On the existence of nonsingular linear sections

In this section we establish a Bertini-type theorem, namely we show the existence of nonsingular linear sections of a singular complete intersection. The Bertini smoothness theorem asserts that a generic hyperplane section of a nonsingular variety V is nonsingular. A more precise variant of this result asserts that, if $V \subset \mathbb{P}^n$ is a projective variety with singular locus of dimension at most s, then the section of V defined by a generic linear space of $\mathbb{P}^n$ of codimension at least $s + 1$ is nonsingular (see, e.g., [11, Proposition 1.3]). Identifying each section of this type with a point in the multiprojective space $(\mathbb{P}^n)^{s+1}$, we show the existence of a hypersurface $\mathcal{H} \subset (\mathbb{P}^n)^{s+1}$ containing all the linear subvarieties of codimension $s + 1$ of $(\mathbb{P}^n)^{s+1}$ which yield singular sections of V. We also estimate the multidegree of this hypersurface.

Let $V \subset \mathbb{P}^n$ be a complete intersection defined by homogeneous polynomials $F_1, \dots, F_{n-r} \in \mathbb{F}_q[X_0, \dots, X_n]$ of degrees $d_1 \geq \cdots \geq d_{n-r} \geq 2$ respectively. Let $\Sigma := \operatorname{Sing} V$ and suppose that there exists s with $0 \leq s \leq r - 2$ such that $\dim \Sigma \leq s$. In particular, V is a normal complete intersection, and therefore absolutely irreducible (Theorem 2.1). We denote by $V_{\mathrm{sm}} := V \setminus \Sigma$ the smooth locus of V. Finally, set $\delta := \deg V = d_1 \cdots d_{n-r}$ and $D := \sum_{i=1}^{n-r} (d_i - 1)$.

Set $X := (X_0, \dots, X_n)$. For $\mu := (\mu_0 : \cdots : \mu_n) \in \mathbb{P}^n$, we shall use the notation $\mu \cdot X := \mu_0 X_0 + \cdots + \mu_n X_n$. Let $\boldsymbol{\gamma} := (\gamma_0, \dots, \gamma_s) \in (\mathbb{P}^n)^{s+1}$, where $\gamma_0, \dots, \gamma_s$ are $\overline{\mathbb{F}}_q$-linearly independent, and consider the linear variety $\mathcal{L} \subset \mathbb{P}^n$ defined by

$$\mathcal{L} := \{\boldsymbol{\gamma} \cdot x = 0\} := \{x \in \mathbb{P}^n : \gamma_0 \cdot x = \cdots = \gamma_s \cdot x = 0\}.$$

Our goal is to prove the existence of a hypersurface $\mathcal{H} \subset (\mathbb{P}^n)^{s+1}$ with the following property: if $\boldsymbol{\gamma} \in (\mathbb{P}^n)^{s+1} \setminus \mathcal{H}$ and $\mathcal{L} := \{\boldsymbol{\gamma} \cdot x = 0\}$, then $V \cap \mathcal{L}$ is nonsingular of pure dimension $r - s - 1$.

Let $\Gamma_i := (\Gamma_{i,0}, \dots, \Gamma_{i,n})$ be a group of $n + 1$ variables for $0 \leq i \leq s$ and denote $\boldsymbol{\Gamma} := (\Gamma_0, \dots, \Gamma_s)$. We consider the incidence variety

$$\mathcal{W} := (V_{\mathrm{sm}} \times \mathcal{U}) \cap \{\Gamma_0 \cdot X = 0, \dots, \Gamma_s \cdot X = 0, \Delta_1(\boldsymbol{\Gamma}, X) = 0, \dots, \Delta_m(\boldsymbol{\Gamma}, X) = 0\},$$

where $\mathcal{U} \subset (\mathbb{P}^n)^{s+1}$ is the Zariski open subset of $(s + 1) \times (n + 1)$-matrices of maximal rank and $\Delta_1, \dots, \Delta_m$ are the maximal minors of the matrix

$$\mathcal{M}(X,\mathbf{\Gamma}) := \begin{pmatrix} \frac{\partial F_1}{\partial X_0} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & & \vdots \\ \frac{\partial F_{n-r}}{\partial X_0} & \cdots & \frac{\partial F_{n-r}}{\partial X_n} \\ \Gamma_{0,0} & \cdots & \Gamma_{0,n} \\ \vdots & & \vdots \\ \Gamma_{s,0} & \cdots & \Gamma_{s,n} \end{pmatrix}. \tag{7}$$

Let $l := n(s+1)$. The following result states the main property of the incidence variety $\mathcal{W}$ we shall use.

Proposition 3.1. *$\mathcal{W}$ is a subvariety of $V_{\mathrm{sm}} \times \mathcal{U}$ of dimension $l-1$.*

Proof. Let $\pi_1 : \mathcal{W} \to V_{\mathrm{sm}}$ be the mapping $\pi_1(x, \boldsymbol{\gamma}) := x$. Fix $x \in V_{\mathrm{sm}}$ and consider the fiber $\pi_1^{-1}(x)$. We have $\pi_1^{-1}(x) = \{x\} \times \Omega$, where $\Omega \subset \mathcal{U}$ is the set of $\boldsymbol{\gamma} := (\gamma_0, \ldots, \gamma_s)$ such that $\gamma_0 \cdot x = \cdots = \gamma_s \cdot x = 0$ and the matrix $\mathcal{M}(x, \boldsymbol{\gamma})$ is not of full rank. The latter condition is equivalent to

$$\langle \gamma_0, \ldots, \gamma_s \rangle \cap \big\langle \nabla F_1(x), \ldots, \nabla F_{n-r}(x) \big\rangle \neq \{\mathbf{0}\}, \tag{8}$$

where $\langle v_0, \ldots, v_m \rangle \subset \mathbb{A}^{n+1}$ denotes the linear variety spanned by $v_0, \ldots, v_m$ in $\mathbb{A}^{n+1}$. Let $\mathbb{V} := \{v \in \mathbb{A}^{n+1} : v \cdot x = 0\}$. Observe that $\nabla F_j(x) \in \mathbb{V}$ for $1 \le j \le n-r$. Then (8) holds if and only if $\gamma_0, \ldots, \gamma_s$ are not linearly independent in the quotient $\overline{\mathbb{F}}_q$-vector space

$$\mathbb{W} := \mathbb{V} / \langle \nabla F_1(x), \ldots, \nabla F_{n-r}(x) \rangle.$$

As Ω and $\mathcal{U}$ are subsets of the multiprojective space $(\mathbb{P}^n)^{s+1}$, we may consider their multi-affine cones Ω_{aff} and $\mathcal{U}_{\mathrm{aff}}$ in $(\mathbb{A}^{n+1})^{s+1}$. Since $\Omega_{\mathrm{aff}} \subset \mathbb{V}^{s+1}$ and $\mathbb{V}^{s+1}$ is isomorphic to $\mathrm{Hom}_{\overline{\mathbb{F}}_q}(\mathbb{A}^{s+1}, \mathbb{V})$, the multi-affine cone Ω_{aff} may be identified with a subset of the latter.

Now we consider the situation modulo $\mathbb{S} := \langle \nabla F_1(x), \ldots, \nabla F_{n-r}(x) \rangle$, that is, we consider the surjective mapping $\Phi : \mathrm{Hom}_{\overline{\mathbb{F}}_q}(\mathbb{A}^{s+1}, \mathbb{V}) \to \mathrm{Hom}_{\overline{\mathbb{F}}_q}(\mathbb{A}^{s+1}, \mathbb{W})$ induced by the quotient mapping $\mathbb{V} \to \mathbb{W}$. With a slight abuse of notation, we shall extend Φ to a surjective mapping from $\mathrm{Hom}_{\overline{\mathbb{F}}_q}(\mathbb{A}^{s+1}, \mathbb{A}^{n+1})$ to $\mathrm{Hom}_{\overline{\mathbb{F}}_q}(\mathbb{A}^{s+1}, \mathbb{A}^{n+1}/\mathbb{S})$ and denote this extension by Φ. From (8) it follows that Ω_{aff} modulo $\mathbb{S}$ is isomorphic to the Zariski open subset $L_s'(\mathbb{A}^{s+1}, \mathbb{W}) \cap \Phi(\mathcal{U}_{\mathrm{aff}})$ of $L_s'(\mathbb{A}^{s+1}, \mathbb{W})$, where

$$L_s'(\mathbb{A}^{s+1}, \mathbb{W}) := \{f \in \mathrm{Hom}_{\overline{\mathbb{F}}_q}(\mathbb{A}^{s+1}, \mathbb{W}) : \mathrm{rank}(f) \le s\}.$$

According to [2, Proposition 1.1], $L_s'(\mathbb{A}^{s+1}, \mathbb{W})$ is an absolutely irreducible variety of dimension $s(r+1)$. Since we are considering elements of $\mathrm{Hom}_{\overline{\mathbb{F}}_q}(\mathbb{A}^{s+1}, \mathbb{V}) \cong \mathbb{V}^{s+1}$ modulo $\mathbb{S}^{s+1}$, and $\mathbb{S} := \langle \nabla F_1(x), \ldots, \nabla F_{n-r}(x) \rangle$ has dimension $n-r$ because $x \in V_{\mathrm{sm}}$, it follows that the multi-affine cone of $\pi_1^{-1}(x) = \{x\} \times \Omega$ is an open dense subset of an irreducible variety of $V_{\mathrm{sm}} \times \mathcal{U}_{\mathrm{aff}}$ of dimension $s(r+1) + (n-r)(s+1) = l + s - r$.

This implies that $\pi_1^{-1}(x) = \{x\} \times \Omega$ is an irreducible subvariety of $V_{\mathrm{sm}} \times \mathcal{U}$ of dimension $l + s - r - (s+1) = l - r - 1$.

Let $\mathcal{W} = \bigcup_j \mathcal{C}_j$ be the decomposition of $\mathcal{W}$ into irreducible components. Our previous arguments show that $\pi_1 : \mathcal{W} \to V_{\mathrm{sm}}$ is surjective. Then $\pi_1(\mathcal{W}) = V_{\mathrm{sm}} = \bigcup_j \pi_1(\mathcal{C}_j)$. As a consequence, there exists i with $\dim \pi_1(\mathcal{C}_i) = r$. The restriction $\pi_1|_{\mathcal{C}_i} : \mathcal{C}_i \to \overline{\pi_1(\mathcal{C}_i)}$ is dominant. For any $x \in \pi_1(\mathcal{C}_i)$, the fiber $\pi_1^{-1}(x)$ is an irreducible subvariety of $\mathcal{C}_i$. Hence, the theorem on the dimension of fibers (see, e.g., [18, §I.6.3, Theorem 7]) shows that, for any $x \in \pi_1(\mathcal{C}_i)$,

$$l - r - 1 = \dim \pi_1^{-1}(x) = \dim \mathcal{C}_i - \dim \overline{\pi_1(\mathcal{C}_i)} = \dim \mathcal{C}_i - r.$$

This shows that $\mathcal{C}_i$ has dimension $l - 1$. On the other hand, for any component $\mathcal{C}_j$ of $\mathcal{W}$ we have that $\pi_1|_{\mathcal{C}_j} : \mathcal{C}_j \to \overline{\pi_1(\mathcal{C}_j)}$ is dominant and the theorem on the dimension of fibers asserts that, for any $x \in \pi_1(\mathcal{C}_j)$,

$$l - r - 1 = \dim \pi_1^{-1}(x) = \dim \mathcal{C}_j - \dim \overline{\pi_1(\mathcal{C}_j)} \geq \dim \mathcal{C}_j - r.$$

We conclude that $\dim \mathcal{C}_j \leq l - 1$, which finishes the proof of the proposition. $\square$

An immediate consequence of Proposition 3.1 is that the Zariski closure of the image of the projection $\pi_2 : \mathcal{W} \to \mathcal{U}$ on the second argument is a variety of dimension at most $l - 1$. Our interest in the set $\pi_2(\mathcal{W})$ is based on the following lemma.

Lemma 3.2. *If $\boldsymbol{\gamma} \in \mathcal{U} \setminus \pi_2(\mathcal{W})$, then the linear section $V_{\mathrm{sm}} \cap \mathcal{L}$ defined by $\mathcal{L} := \{\boldsymbol{\gamma} \cdot x = 0\}$ is nonsingular of pure dimension $r - s - 1$.*

Proof. Fix $\boldsymbol{\gamma} \in \mathcal{U} \setminus \pi_2(\mathcal{W})$ and denote $\mathcal{L} := \{\boldsymbol{\gamma} \cdot x = 0\}$. According to [11, Lemma 1.1], $\mathrm{Sing}(V_{\mathrm{sm}} \cap \mathcal{L}) = N(V_{\mathrm{sm}}, \mathcal{L})$, where $N(V_{\mathrm{sm}}, \mathcal{L})$ is the set of points $x \in V_{\mathrm{sm}}$ where V and $\mathcal{L}$ do not meet transversely, that is, $\dim \mathcal{T}_x V \cap \mathcal{L} > \dim \mathcal{T}_x V - \mathrm{codim} \mathcal{L} = r - s - 1$. For $x \in V_{\mathrm{sm}} \cap \mathcal{L}$, we have $(x, \boldsymbol{\gamma}) \notin \mathcal{W}$ and then $\mathcal{M}(x, \boldsymbol{\gamma})$ has maximal rank, where $\mathcal{M}(X, \boldsymbol{\Gamma})$ is the matrix of (7). As a consequence, $\dim \mathcal{T}_x V \cap \mathcal{L} = r - s - 1$. This implies that V and $\mathcal{L}$ meet transversely at x, and hence x is a nonsingular point of $V_{\mathrm{sm}} \cap \mathcal{L}$. This shows that $V_{\mathrm{sm}} \cap \mathcal{L}$ is nonsingular.

By [18, §I.6.2, Corollary 5], each irreducible component of $V_{\mathrm{sm}} \cap \mathcal{L}$ has dimension at least $r - s - 1$. For $x \in V_{\mathrm{sm}} \cap \mathcal{L}$, the matrix $\mathcal{M}(x, \boldsymbol{\gamma})$ has maximal rank. Hence, $\dim \mathcal{T}_x(V \cap \mathcal{L}) \leq r - s - 1$, which implies that each irreducible component of $V_{\mathrm{sm}} \cap \mathcal{L}$ containing x has dimension at most $r - s - 1$. We conclude that $V_{\mathrm{sm}} \cap \mathcal{L}$ is of pure dimension $r - s - 1$. $\square$

We shall show that the set $\pi_2(\mathcal{W})$ is contained in a hypersurface of $(\mathbb{P}^n)^{s+1}$ of "low" degree. Denote

$$\begin{aligned} L_\Gamma &:= \{\Gamma_0 \cdot X = 0, \ldots, \Gamma_s \cdot X = 0\} \\ &:= \{(x, \boldsymbol{\gamma}) \in (\mathbb{P}^n)^{s+2} : \gamma_0 \cdot x = 0, \ldots, \gamma_s \cdot x = 0\}, \end{aligned}$$

and let $\mathcal{W}'' \subset (\mathbb{P}^n)^{s+2}$ be the following variety:

$$\mathcal{W}'' := \mathcal{W} \cup \big((\Sigma \times (\mathbb{P}^n)^{s+1}) \cap L_\Gamma\big) \cup \big((V \times ((\mathbb{P}^n)^{s+1} \setminus \mathcal{U})) \cap L_\Gamma\big). \tag{9}$$

We have the following result.

Lemma 3.3. *The variety $\mathcal{W}''$ has dimension $l-1$ and the following identity holds:*

$$\mathcal{W}'' = \big((V \times (\mathbb{P}^n)^{s+1}) \cap L_\Gamma\big) \cap \{\Delta_1(\mathbf{\Gamma}, X) = 0, \ldots, \Delta_m(\mathbf{\Gamma}, X) = 0\}. \tag{10}$$

Proof. First we prove (10). It is easy to see that the left-hand side is contained in the right-hand side. On the other hand, for $(x, \boldsymbol{\gamma}) \in \big(V \times (\mathbb{P}^n)^{s+1}\big) \cap L_\Gamma$, either $x \in \Sigma$, or $\boldsymbol{\gamma} \in (\mathbb{P}^n)^{s+1} \setminus \mathcal{U}$, or $(x, \boldsymbol{\gamma}) \in V_{\mathrm{sm}} \times \mathcal{U}$. In the first two cases, $(x, \boldsymbol{\gamma}) \in \mathcal{W}''$ and the identity $\Delta_j(x, \boldsymbol{\gamma}) = 0$ is satisfied for $1 \le j \le m$. In the third case we have $(x, \boldsymbol{\gamma}) \in \mathcal{W}''$ if and only if $\Delta_j(x, \boldsymbol{\gamma}) = 0$ for $1 \le j \le m$. This shows the claim.

Next we determine the dimension of $\mathcal{W}''$. Observe that $\Sigma \times (\mathbb{P}^n)^{s+1}$ is a cylinder whose intersection with the equations $\Gamma_0 \cdot X = 0, \ldots, \Gamma_s \cdot X = 0$ has codimension $s+1$. Hence, $(\Sigma \times (\mathbb{P}^n)^{s+1}) \cap L_\Gamma$ has dimension at most $s + l - (s+1) = l - 1$. On the other hand, the affine cone of $(\mathbb{P}^n)^{s+1} \setminus \mathcal{U}$ is the closed set $L_s(\mathbb{A}^{s+1}, \mathbb{A}^{n+1})$ of matrices of rank at most s. By [2, Proposition 1.1], $\dim L_s(\mathbb{A}^{s+1}, \mathbb{A}^{n+1}) = s(n+2)$; thus, $(\mathbb{P}^n)^{s+1} \setminus \mathcal{U}$ has dimension $s(n+2) - (s+1) = l + s - n - 1$. Then $V \times \big((\mathbb{P}^n)^{s+1} \setminus \mathcal{U}\big)$ has dimension $r + l + s - n - 1$. Consider the projection $\pi_2 : \big(V \times ((\mathbb{P}^n)^{s+1} \setminus \mathcal{U})\big) \cap L_\Gamma \to (\mathbb{P}^n)^{s+1} \setminus \mathcal{U}$ on the second argument. The intersection of V with a generic linear variety of $\mathbb{P}^n$ of codimension s is of pure dimension $r - s$. Let $\boldsymbol{\gamma} := (\gamma_0, \ldots, \gamma_s)$ be a point of $(\mathbb{P}^n)^{s+1} \setminus \mathcal{U}$ with $\{\gamma_0 \cdot x = 0, \ldots, \gamma_{s-1} \cdot x = 0\} \subset \mathbb{P}^n$ generic in the sense above. Then the fiber $\pi_2^{-1}(\boldsymbol{\gamma})$ has dimension $r - s$ and the theorem on the dimension of fibers implies

$$r - s = \dim \pi_2^{-1}(\boldsymbol{\gamma}) \ge \dim\big(V \times ((\mathbb{P}^n)^{s+1} \setminus \mathcal{U})\big) \cap L_\Gamma - (l + s - n - 1).$$

We deduce that $\dim\big(V \times ((\mathbb{P}^n)^{s+1} \setminus \mathcal{U})\big) \cap L_\Gamma \le l - n + r - 1 < l - 1$. Combining these facts with Proposition 3.1 we conclude that $\mathcal{W}''$ has dimension $l - 1$. $\square$

As an immediate consequence of Lemma 3.3, we obtain the following result.

Corollary 3.4. *There exist linear combinations $\Delta^1, \ldots, \Delta^{r-s}$ of the maximal minors $\Delta_1(\mathbf{\Gamma}, X), \ldots, \Delta_m(\mathbf{\Gamma}, X)$ of the matrix $\mathcal{M}(X, \mathbf{\Gamma})$ of (7) such that the variety $\mathcal{W}' \subset (\mathbb{P}^n)^{s+2}$ defined as the set of common solutions of*

$$F_1 = 0, \ldots, F_{n-r} = 0, \Gamma_0 \cdot X = 0, \ldots, \Gamma_s \cdot X = 0, \Delta^1 = 0, \ldots, \Delta^{r-s} = 0, \tag{11}$$

is of pure dimension $l - 1$ and contains $\mathcal{W}''$.

Proof. Observe that the intersection of $V \times (\mathbb{P}^n)^{s+1}$ with L_Γ has codimension $s+1$, that is,

$$\dim\big(V \times (\mathbb{P}^n)^{s+1}\big) \cap L_\Gamma = r + (n-1)(s+1) = r - s + l - 1.$$

Then the result is an easy consequence of the fact that $\mathcal{W}''$ has codimension at least $r-s$ in $\big(V \times (\mathbb{P}^n)^{s+1}\big) \cap L_\Gamma$. Indeed, applying, e.g., [5, Lemma 4.4] to $\big(V \times (\mathbb{P}^n)^{s+1}\big) \cap L_\Gamma$ and $\mathcal{W}''$, we readily deduce the corollary. □

Now we are in a position to prove that the main result of this section, namely that the set of $\boldsymbol{\gamma} \in (\mathbb{P}^n)^{s+1}$ for which the linear section $V \cap \{\boldsymbol{\gamma} \cdot x = 0\}$ is not smooth of codimension $s+1$, is contained in a hypersurface of $(\mathbb{P}^n)^{s+1}$ of "low" degree.

Theorem 3.5. *Let $V \subset \mathbb{P}^n$ be a complete intersection defined over $\mathbb{F}_q$, of dimension r, multidegree $\boldsymbol{d} := (d_1, \ldots, d_{n-r})$, and singular locus of dimension at most $0 \le s \le r-2$. Let $\delta := \prod_{i=1}^{n-r} d_i$ and $D := \sum_{i=1}^{n-r}(d_i - 1)$. There exists a hypersurface $\mathcal{H} \subset (\mathbb{P}^n)^{s+1}$, defined by a multihomogeneous polynomial of degree at most $D^{r-s-1}(D+r-s)\delta$ in each group of variables Γ_i, with the following property: if $\boldsymbol{\gamma} \in (\mathbb{P}^n)^{s+1} \setminus \mathcal{H}$ and $\mathcal{L} := \{\boldsymbol{\gamma} \cdot x = 0\}$, then $V \cap \mathcal{L}$ is nonsingular of pure dimension $r-s-1$.*

Proof. By the version of the Bertini smoothness theorem of, e.g., [11, Proposition 1.3], for generic $\boldsymbol{\gamma} \in (\mathbb{P}^n)^{s+1}$ and $\mathcal{L} := \{\boldsymbol{\gamma} \cdot x = 0\}$, the linear section $V \cap \mathcal{L}$ is nonsingular of pure codimension $s+1$. Furthermore, as the polynomials $\Delta^1, \ldots, \Delta^{r-s}$ of (11) are generic linear combinations of $\Delta_1, \ldots, \Delta_m$, we may assume without loss of generality that the equations

$$F_1 = 0, \ldots, F_{n-r} = 0, \boldsymbol{\gamma} \cdot X = 0, \Delta^1(\boldsymbol{\gamma}, X) = 0, \ldots, \Delta^{r-s}(\boldsymbol{\gamma}, X) = 0,$$

do not have common solutions in $\mathbb{P}^n$. Denote $\mathsf{K} := \overline{\mathbb{F}_q(\Gamma)}$. Then the equations

$$F_1 = 0, \ldots, F_{n-r} = 0, \boldsymbol{\Gamma} \cdot X = 0, \Delta^1(\boldsymbol{\Gamma}, X) = 0, \ldots, \Delta^{r-s}(\boldsymbol{\Gamma}, X) = 0 \tag{12}$$

do not have common solutions in the n-dimensional projective space $\mathbb{P}^n_{\mathsf{K}}$ over K. As a consequence, the multidimensional resultant of the corresponding polynomials is a nonzero element of $\mathbb{F}_q[\boldsymbol{\Gamma}]$ which vanishes on $\boldsymbol{\gamma} \in (\mathbb{P}^n)^{s+1}$ if and only if the substitution of $\boldsymbol{\gamma}$ for $\boldsymbol{\Gamma}$ in (12) yields a nonempty variety of $\mathbb{P}^n$.

Define $d_i := 1$ for $n-r+1 \le i \le n-r+s+1$ and $d_i := D$ for $n-r+s+2 \le i \le n+1$ so that the polynomials in (12) have degree $d_1, \ldots, d_{n+1}$ in X respectively. Set $D_i := \binom{d_i+n}{n} - 1$ for $1 \le i \le n+1$ and denote $\boldsymbol{D} := (D_1, \ldots, D_{n+1})$ and $\mathbb{P}^{\boldsymbol{D}} = \mathbb{P}^{D_1} \times \cdots \times \mathbb{P}^{D_{n+1}}$. Let $\boldsymbol{\Lambda}_i$ be a group of $D_i + 1$ indeterminates over $\overline{\mathbb{F}}_q$ for $1 \le i \le n+1$, $\overline{\mathbb{F}}_q[\boldsymbol{\Lambda}] := \overline{\mathbb{F}}_q[\boldsymbol{\Lambda}_1, \ldots, \boldsymbol{\Lambda}_{n+1}]$ and let $P \in \overline{\mathbb{F}}_q[\boldsymbol{\Lambda}]$ be the multivariate resultant of generic polynomials of $\overline{\mathbb{F}}_q[\boldsymbol{\Lambda}_1][X], \ldots, \overline{\mathbb{F}}_q[\boldsymbol{\Lambda}_{n+1}][X]$ of degrees $d_1, \ldots, d_{n+1}$ respectively. Denote by $\mathcal{H}_{gen} \subset \mathbb{P}^{\boldsymbol{D}}$ the hypersurface defined by P. For $\boldsymbol{\gamma} \in (\mathbb{P}^n)^{s+1}$, the substitution of $\boldsymbol{\gamma}$ for

$\boldsymbol{\Gamma}$ in (12) yields a nonempty variety of $\mathbb{P}^n$ if and only if the corresponding $(n+1)$-tuple of polynomials $\big(F_1,\ldots,F_{n-r},\boldsymbol{\gamma}\cdot X,\Delta^1(\boldsymbol{\gamma},X),\ldots,\Delta^{r-s}(\boldsymbol{\gamma},X)\big)$ belongs to $\mathcal{H}_{gen}$. Let $\phi:(\mathbb{P}^n)^{s+1}\to\mathbb{P}^{\boldsymbol{D}}$ be the regular mapping defined as

$$\phi(\boldsymbol{\gamma}):=\big(F_1,\ldots,F_{n-r},\boldsymbol{\gamma}\cdot X,\Delta^1(\boldsymbol{\gamma},X),\ldots,\Delta^{r-s}(\boldsymbol{\gamma},X)\big).$$

Finally, let $\mathcal{H}$ be the hypersurface of $(\mathbb{P}^n)^{s+1}$ defined by the (nonzero) polynomial $\phi^*(P)$, where $\phi^*:\overline{\mathbb{F}}_q[\boldsymbol{\Lambda}]\to\overline{\mathbb{F}}_q[\boldsymbol{\Gamma}]$ is the $\overline{\mathbb{F}}_q$-algebra homomorphism induced by ϕ. We claim that $\mathcal{H}$ satisfies the requirements in the statement of the theorem.

Indeed, let $\boldsymbol{\gamma}\notin\mathcal{H}$. Then the substitution of $\boldsymbol{\gamma}$ for $\boldsymbol{\Gamma}$ in (12) yields the empty variety of $\mathbb{P}^n$. In particular, $\boldsymbol{\gamma}\notin\pi_2(\mathcal{W}')$, where $\mathcal{W}'$ is the variety of Corollary 3.4. By the definition of $\mathcal{W}'$ we have $\boldsymbol{\gamma}\notin\pi_2(\mathcal{W}'')$, where $\mathcal{W}''$ is the variety of (9). This implies that $\boldsymbol{\gamma}\in\mathcal{U}\setminus\pi_2(\mathcal{W})$, and Lemma 3.2 shows that $V_{\mathrm{sm}}\cap\mathcal{L}$ is smooth of pure codimension $s+1$. Furthermore, from the definition of $\mathcal{W}''$ it follows that $\boldsymbol{\gamma}\notin\pi_2\big((\Sigma\times(\mathbb{P}^n)^{s+1})\cap L_\Gamma\big)$, which implies that $\Sigma\cap\mathcal{L}=\emptyset$. We conclude that $V\cap\mathcal{L}=V_{\mathrm{sm}}\cap\mathcal{L}$ is smooth of pure codimension $s+1$.

Finally we prove the bound on the multidegree of $\mathcal{H}$ of the statement of the theorem. According to, e.g., [8, Chapter 3, Theorem 3.1], the multivariate resultant $P\in\overline{\mathbb{F}}_q[\boldsymbol{\Lambda}]$ is a multihomogeneous polynomial with

$$\deg P_{\Lambda_i}=\begin{cases} D^{r-s}\delta & \text{for } n-r+1\le i\le n-r+s+1,\\ D^{r-s-1}\delta & \text{for } n-r+s+2\le i\le n+1.\end{cases}$$

The homomorphism $\phi^*:\overline{\mathbb{F}}_q[\boldsymbol{\Lambda}]\to\overline{\mathbb{F}}_q[\boldsymbol{\Gamma}]$ maps $\Lambda_{n-r+1+i}$ to Γ_i for $0\le i\le s$ and $\Lambda_{n-r+s+1+i}$ on the vector of coefficients of $\Delta^i\in\mathbb{F}_q[\boldsymbol{\Gamma}][X]$ for $1\le i\le r-s$. Since each coefficient of $\Delta^i\in\mathbb{F}_q[\boldsymbol{\Gamma}]$ is homogeneous of degree 1 in Γ_j for $0\le j\le s$, we see that

$$\deg_{\Gamma_i}\phi^*(P)=\deg_{\Lambda_{n-r+1+i}}P+\sum_{j=n-r+s+2}^{n+1}\deg_{\Lambda_j}P=D^{r-s}\delta+(r-s)D^{r-s-1}\delta.$$

This finishes the proof of the theorem. □

According to Theorem 3.5, for "most" elements $\boldsymbol{\gamma}\in(\mathbb{P}^n)^{s+1}$ the linear section $V\cap\mathcal{L}:=V\cap\{\boldsymbol{\gamma}\cdot x=0\}$ is nonsingular of codimension $s+1$. Furthermore, combining Theorem 3.5 with the results of Section 2.3.1 we are able to estimate the number of "good" linear sections $V\cap\mathcal{L}$ which are defined over $\mathbb{F}_q$, which is essential for the results of Section 4. In particular, for $q>D^{r-s-1}(D+r-s)\delta$, Corollary 2.3 proves that there exists $\boldsymbol{\gamma}\in(\mathbb{P}^n(\mathbb{F}_q))^{s+1}\setminus\mathcal{H}$. This yields an effective version of the Bertini smoothness theorem, which may be of independent interest. We remark that this result will not be used in the sequel.

Theorem 3.6. *Let $V\subset\mathbb{P}^n$ be a complete intersection defined over $\mathbb{F}_q$, of dimension r, multidegree $\boldsymbol{d}:=(d_1,\ldots,d_{n-r})$, and singular locus of dimension at most $0\le s\le r-2$.*

Let $\delta := \prod_{i=1}^{n-r} d_i$ and $D := \sum_{i=1}^{n-r}(d_i - 1)$. If $q > D^{r-s-1}(D+r-s)\delta$, then there exists a linear variety $\mathcal{L} \subset \mathbb{P}^n$ defined over $\mathbb{F}_q$ of dimension $n-s-1$ such that the linear section $V \cap \mathcal{L}$ is nonsingular of pure codimension $s+1$.

An effective version of a weak form of a Bertini smoothness theorem for hypersurfaces is obtained in [1]. Nevertheless, the bound given in [1] is exponentially higher than ours and therefore not suitable for our purposes, even in the hypersurface case. On the other hand, in [4] a version of the Bertini smoothness theorem for normal complete intersections is established, which is significantly generalized and improved by Theorem 3.6. Finally, the result of Theorem 3.6 is similar both quantitatively and qualitatively to [5, Corollary 6.6], the main contribution over the latter being the simplicity of the approach. Nevertheless, neither Theorem 3.6 nor [5, Corollary 6.6] provides enough information on the nonsingular linear sections of V of codimension $s+1$ defined over $\mathbb{F}_q$ for the purposes of Section 4.

4. Estimates on the number of rational points

Let $V \subset \mathbb{P}^n$ be an ideal-theoretic complete intersection defined over $\mathbb{F}_q$, of dimension r, multidegree $\boldsymbol{d} := (d_1, \dots, d_{n-r})$ and singular locus of dimension at most $0 \le s \le r-2$. As before, we denote $\delta := \deg V = d_1 \cdots d_{n-r}$ and $D := \sum_{i=1}^{n-r}(d_i - 1)$. In this section we obtain an explicit version of the Hooley–Katz estimate (2) for V.

The proof of (2) in [15] proceeds in $s+1$ steps, considering successive hyperplane sections of V until nonsingular sections are obtained. The number of $\mathbb{F}_q$-rational points of each of these nonsingular sections is estimated using Deligne's estimate. A key ingredient in [15] is an upper bound for the second moment

$$M_1 := \sum_{\boldsymbol{m} \in \mathbb{F}_q^{n+1}} \big(N - q\,N(\boldsymbol{m})\big)^2,$$

where N and $N(\boldsymbol{m})$ are the number of $\mathbb{F}_q$-rational points of V and of the linear section of V determined by the hyperplane defined by $\boldsymbol{m}$. In this section we introduce a variant of the second moment M_1: the second moment M_{s+1} obtained by considering the linear sections of V determined by all the linear varieties of codimension $s+1$ of $\mathbb{P}^n$ defined over $\mathbb{F}_q$.

First we estimate the number of nonsingular linear sections of V defined over $\mathbb{F}_q$ of pure codimension $s+1$.

Lemma 4.1. *Assume that $q > d := D^{r-s-1}(D+r-s)\delta$. Let N_{ns} be the number of $\boldsymbol{\gamma} \in (\mathbb{F}_q^{n+1})^{s+1}$ for which $V \cap \mathcal{L}$ is nonsingular of pure codimension $s+1$, where $\mathcal{L} := \{\boldsymbol{\gamma} \cdot x = 0\} \subset \mathbb{P}^n$. Then*

$$N_{ns} \ge (q-d)^{s+1} q^{n(s+1)}.$$

Proof. Let $\mathcal{H} \subset (\mathbb{P}^n)^{s+1}$ be the hypersurface of the statement of Theorem 3.5. The hypersurface $\mathcal{H}$ is defined by a multihomogeneous polynomial $F \in \overline{\mathbb{F}}_q[\mathbf{\Gamma}]$ of degree at most d in each group of variables Γ_i. For any $\boldsymbol{\gamma} \in (\mathbb{F}_q^{n+1})^{s+1}$ with $F(\boldsymbol{\gamma}) \neq 0$, the corresponding linear section $V \cap \{\boldsymbol{\gamma} \cdot x = 0\}$ is nonsingular of pure codimension $s+1$. As a consequence, from Proposition 2.2 we obtain

$$\begin{aligned} N_{ns} &\geq q^{(n+1)(s+1)} - \sum_{\boldsymbol{\varepsilon} \in \{0,1\}^{s+1} \setminus \{\mathbf{0}\}} (-1)^{|\boldsymbol{\varepsilon}|+1} d^{|\boldsymbol{\varepsilon}|} q^{(n+1)(s+1)-|\boldsymbol{\varepsilon}|} \\ &= \sum_{\boldsymbol{\varepsilon} \in \{0,1\}^{s+1}} (-d)^{|\boldsymbol{\varepsilon}|} q^{(n+1)(s+1)-|\boldsymbol{\varepsilon}|} = \sum_{i=0}^{s+1} \sum_{\boldsymbol{\varepsilon}:\, |\boldsymbol{\varepsilon}|=i} (-d)^i q^{(n+1)(s+1)-i}. \end{aligned}$$

This implies

$$N_{ns} \geq q^{n(s+1)} \left(\sum_{i=0}^{s+1} \binom{s+1}{i} (-d)^i q^{s+1-i} \right) = q^{n(s+1)} (q-d)^{s+1},$$

which proves the statement of the lemma. □

Now we consider the second moment defined as

$$M_{s+1} := \sum_{\boldsymbol{\gamma} \in \mathbb{F}_q^{(n+1)(s+1)}} \big(N - q^{s+1} N(\boldsymbol{\gamma}) \big)^2, \tag{13}$$

where $N := |V(\mathbb{F}_q)|$, $N(\boldsymbol{\gamma}) := |V \cap \mathcal{L}(\mathbb{F}_q)|$ and $\mathcal{L} := \{\boldsymbol{\gamma} \cdot x = 0\}$.

Lemma 4.2. *We have $M_{s+1} = N q^{(n+1)(s+1)} (q^{s+1} - 1)$.*

Proof. Set $t := (n+1)(s+1)$ and observe that

$$M_{s+1} = \sum_{\boldsymbol{\gamma} \in \mathbb{F}_q^t} N^2 - 2q^{s+1} N \sum_{\boldsymbol{\gamma} \in \mathbb{F}_q^t} N(\boldsymbol{\gamma}) + q^{2(s+1)} \sum_{\boldsymbol{\gamma} \in \mathbb{F}_q^t} N(\boldsymbol{\gamma})^2. \tag{14}$$

First we consider the second term in the right-hand side of (14):

$$\sum_{\boldsymbol{\gamma} \in \mathbb{F}_q^t} N(\boldsymbol{\gamma}) = \sum_{\boldsymbol{\gamma} \in \mathbb{F}_q^t} \sum_{\substack{x \in V(\mathbb{F}_q) \\ \boldsymbol{\gamma} \cdot x = 0}} 1 = \sum_{x \in V(\mathbb{F}_q)} \sum_{\substack{\boldsymbol{\gamma} \in \mathbb{F}_q^t \\ \boldsymbol{\gamma} \cdot x = 0}} 1 = q^{t-s-1} N. \tag{15}$$

On the other hand, concerning the third term of the right-hand side of (14),

$$\sum_{\boldsymbol{\gamma} \in \mathbb{F}_q^t} N(\boldsymbol{\gamma})^2 = \sum_{\boldsymbol{\gamma} \in \mathbb{F}_q^t} \Bigg(\sum_{\substack{x \in V(\mathbb{F}_q) \\ \boldsymbol{\gamma} \cdot x = 0}} 1 \Bigg) \Bigg(\sum_{\substack{x' \in V(\mathbb{F}_q) \\ \boldsymbol{\gamma} \cdot x' = 0}} 1 \Bigg) = \sum_{\boldsymbol{\gamma} \in \mathbb{F}_q^t} \Bigg(\sum_{\substack{x \in V(\mathbb{F}_q) \\ \boldsymbol{\gamma} \cdot x = 0}} 1 + \sum_{\substack{x, x' \in V(\mathbb{F}_q),\, x \neq x' \\ \boldsymbol{\gamma} \cdot x = \boldsymbol{\gamma} \cdot x' = 0}} 1 \Bigg).$$

Further, we have

$$\sum_{\boldsymbol{\gamma}\in\mathbb{F}_q^t}\ \sum_{\substack{x,x'\in V(\mathbb{F}_q),\,x\neq x'\\ \boldsymbol{\gamma}\cdot x=\,\boldsymbol{\gamma}\cdot x'=0}} 1=\sum_{\substack{x,x'\in V(\mathbb{F}_q)\\ x\neq x'}}\ \sum_{\substack{\boldsymbol{\gamma}\in\mathbb{F}_q^t\\ \boldsymbol{\gamma}\cdot x=\,\boldsymbol{\gamma}\cdot x'=0}} 1=\sum_{\substack{x,x'\in V(\mathbb{F}_q)\\ x\neq x'}} q^{t-2(s+1)}$$
$$=q^{t-2(s+1)}N(N-1).$$

We conclude that

$$\sum_{\boldsymbol{\gamma}\in\mathbb{F}_q^t} N(\boldsymbol{\gamma})^2=q^{t-s-1}N+q^{t-2(s+1)}N(N-1). \tag{16}$$

Combining (14), (15) and (16) we easily deduce the statement of the lemma. □

From Lemma 4.2 we deduce that there are at least $\frac{1}{2}q^{(n+1)(s+1)}$ elements $\boldsymbol{\gamma}\in\mathbb{F}_q^{(n+1)(s+1)}$ such that the linear variety $\mathcal{L}:=\{\boldsymbol{\gamma}\cdot x=0\}$ satisfies the condition

$$\big||V(\mathbb{F}_q)|-q^{s+1}|(V\cap\mathcal{L})(\mathbb{F}_q)|\big|\leq\sqrt{2N(q^{s+1}-1)}.$$

Otherwise, $\big||V(\mathbb{F}_q)|-q^{s+1}|(V\cap\mathcal{L})(\mathbb{F}_q)|\big|>\sqrt{2N(q^{s+1}-1)}$ for at least $\frac{1}{2}q^{(n+1)(s+1)}$ linear varieties $\mathcal{L}$ defined over $\mathbb{F}_q$, and then

$$M_{s+1}>N(q^{s+1}-1)\,q^{(n+1)(s+1)},$$

which contradicts Lemma 4.2. In other words, we have the following result.

Corollary 4.3. *There exist at least* $\frac{1}{2}q^{(n+1)(s+1)}$ *elements* $\boldsymbol{\gamma}\in\mathbb{F}_q^{(n+1)(s+1)}$ *such that the linear variety* $\mathcal{L}:=\{\boldsymbol{\gamma}\cdot x=0\}$ *satisfies the condition*

$$\big||V(\mathbb{F}_q)|-q^{s+1}|(V\cap\mathcal{L})(\mathbb{F}_q)|\big|\leq\sqrt{2N(q^{s+1}-1)}. \tag{17}$$

Assume that $q>d:=D^{r-s-1}(D+r-s)\delta$. According to Lemma 4.1, there exist at least $(q-d)^{s+1}q^{n(s+1)}$ elements $\boldsymbol{\gamma}\in(\mathbb{F}_q^{n+1})^{s+1}$ such that the linear section $V\cap\mathcal{L}$ defined by $\mathcal{L}:=\{\boldsymbol{\gamma}\cdot x=0\}$ is nonsingular of codimension $s+1$. In particular, for

$$(q-d)^{s+1}q^{n(s+1)}>\frac{1}{2}q^{(n+1)(s+1)}, \tag{18}$$

there exists a nonsingular $\mathbb{F}_q$-definable linear section $V\cap\mathcal{L}$ of codimension $s+1$ satisfying (17).

Observe that (18) is equivalent to the inequality $(1-\frac{d}{q})^{s+1}>\frac{1}{2}$. By the Bernoulli inequality, $(1-\frac{d}{q})^{s+1}\geq 1-(s+1)\frac{d}{q}$. Therefore, the condition $1-(s+1)\frac{d}{q}>\frac{1}{2}$ implies (18). As a consequence, we obtain the following result.

Corollary 4.4. *For $q > 2(s+1)D^{r-s-1}(D+r-s)\delta$, there exists a nonsingular $\mathbb{F}_q$-definable linear section of V of codimension $s+1$ which satisfies (17).*

Finally, we are ready to state our estimate on the number of $\mathbb{F}_q$-rational points of a singular complete intersection.

Theorem 4.5. *Let $V \subset \mathbb{P}^n$ be a complete intersection defined over $\mathbb{F}_q$, of dimension r, multidegree $\boldsymbol{d} := (d_1, \ldots, d_{n-r})$, and singular locus of dimension at most $0 \leq s \leq r-2$. Let $\delta := \prod_{i=1}^{n-r} d_i$ and $D := \sum_{i=1}^{n-r}(d_i - 1)$. If $q > 2(s+1)D^{r-s-1}(D+r-s)\delta$, then*

$$\big||V(\mathbb{F}_q)| - p_r\big| \leq \big(b'_{r-s-1} + 2\sqrt{\delta} + 1\big)\, q^{\frac{r+s+1}{2}}, \tag{19}$$

where $b'_{r-s-1} := b'_{r-s-1}(n-s-1, \boldsymbol{d})$ is the $(r-s-1)$th primitive Betti number of any nonsingular complete intersection of $\mathbb{P}^{n-s-1}$ of dimension r and multidegree $\boldsymbol{d}$.

Proof. Since $q > 2(s+1)D^{r-s-1}(D+r-s)\delta$, by Corollary 4.4 there exists $\boldsymbol{\gamma} \in \mathbb{F}_q^{(n+1)(s+1)}$ such that the linear section $V \cap \mathcal{L}$ defined by $\mathcal{L} := \{\boldsymbol{\gamma} \cdot x = 0\}$ is nonsingular of dimension $r-s-1$ and satisfies

$$\big||V(\mathbb{F}_q)| - q^{s+1}|(V \cap \mathcal{L})(\mathbb{F}_q)|\big| \leq \sqrt{2N(q^{s+1}-1)}.$$

Fix such an element $\boldsymbol{\gamma} \in \mathbb{F}_q^{(n+1)(s+1)}$. We have

$$\big||V(\mathbb{F}_q)| - p_r\big| \leq \big||V(\mathbb{F}_q)| - q^{s+1}|V \cap \mathcal{L}(\mathbb{F}_q)|\big| + \big|q^{s+1}|V \cap \mathcal{L}(\mathbb{F}_q)| - p_r\big|.$$

By the definition of $\boldsymbol{\gamma}$ and the identity $p_r = q^{s+1}p_{r-s-1} + p_s$, it follows that

$$\big||V(\mathbb{F}_q)| - p_r\big| \leq \sqrt{2N(q^{s+1}-1)} + q^{s+1}\big||V \cap \mathcal{L}(\mathbb{F}_q)| - p_{r-s-1}\big| + p_s.$$

Since $V \cap \mathcal{L}$ is a nonsingular complete intersection of $\mathcal{L}$ of dimension $r-s-1$ and multidegree $\boldsymbol{d}$, applying (1) we obtain

$$\big||V(\mathbb{F}_q)| - p_r\big| \leq \sqrt{2N(q^{s+1}-1)} + b'_{r-s-1}(n-s-1, \boldsymbol{d})\, q^{\frac{r+s+1}{2}} + p_s.$$

By the bound $N \leq \delta p_r$ and elementary calculations, the theorem follows. □

Let $V \subset \mathbb{P}^n$ be a singular complete intersection as in the statement of Theorem 4.5. In [11, Theorem 6.1], the following estimate is obtained:

$$\big||V(\mathbb{F}_q)| - p_r\big| \leq b'_{r-s-1}\, q^{\frac{r+s+1}{2}} + 9 \cdot 2^{n-r}\big((n-r)d + 3\big)^{n+1} q^{\frac{r+s}{2}}, \tag{20}$$

where $d := \max_{1 \leq i \leq n-r} d_i$. We observe that the error term in (19) avoids the exponential dependency on n present in (20). On the other hand, (20) holds without any condition on q, while (19) is valid for $q > 2(s+1)D^{r-s-1}(D+r-s)\delta$.

4.1. Normal complete intersections

Let $V \subset \mathbb{P}^n$ be a complete intersection defined over $\mathbb{F}_q$, of dimension r, multidegree $\boldsymbol{d}$ and singular locus of codimension at least 2. By the case $s = r - 2$ of Theorem 4.5 we conclude that, if $q > 2(r-1)D(D+2)\delta$, then

$$\big||V(\mathbb{F}_q)| - p_r\big| \leq \big(b_1'(n-r+1, \boldsymbol{d}) + 2\sqrt{\delta} + 1\big)\, q^{r-\frac{1}{2}}.$$

Nevertheless, the condition on q may restrict the range of applicability of this estimate. For this reason, the next result provides a further estimate which holds without restrictions on q.

Corollary 4.6. *Let $V \subset \mathbb{P}^n$ be a normal complete intersection defined over $\mathbb{F}_q$, of dimension $r \geq 2$ and multidegree $\boldsymbol{d}$. Let $\delta := \prod_{i=1}^{n-r} d_i$ and $D := \sum_{i=1}^{n-r}(d_i - 1)$. Then*

$$\big||V(\mathbb{F}_q)| - p_r\big| \leq 3\, r^{1/2}(D+1)\delta^{3/2}\, q^{r-\frac{1}{2}}. \tag{21}$$

Proof. Suppose first that $q > 2(r-1)D(D+2)\delta$. Since $b_1'(n-r+1, \boldsymbol{d}) = (D-2)\delta + 2$ (see, e.g., [11, Theorem 4.1]), Theorem 4.5 readily implies the corollary. As a consequence, we may assume $q \leq 2(r-1)D(D+2)\delta$. By (5), it follows that $|V(\mathbb{F}_q)| \leq \delta p_r$. Therefore,

$$\big||V(\mathbb{F}_q)| - p_r\big| \leq (\delta - 1)p_r \leq 2\delta q^r \leq 3\, r^{1/2}(D+1)\delta^{3/2} q^{r-1/2}.$$

This finishes the proof of the corollary. □

Let $V \subset \mathbb{P}^n$ be a normal complete intersection as in Corollary 4.6. According to [11, Corollary 6.2],

$$\big||V(\mathbb{F}_q)| - p_r\big| \leq (\delta(D-2) + 2)q^{r-1/2} + 9 \cdot 2^{n-r}((n-r)d + 3)^{n+1} q^{r-1}, \tag{22}$$

where $d := \max_{1 \leq i \leq n-r} d_i$. On the other hand, [5, Corollary 8.3] shows that

$$\big||V(\mathbb{F}_q)| - p_r\big| \leq (\delta(D-2) + 2)q^{r-1/2} + 14D^2\delta^2 q^{r-1}. \tag{23}$$

These are the most accurate estimates to the best of our knowledge.

For the sake of comparison, it can be seen that

$$2^{n-r}((n-r)d + 3)^{n+1} \geq \big(2(n-r)\big)^{n-r} D^{r+1}\delta.$$

This shows that for varieties of high dimension, say $r \geq (n+1)/2$, (21) and (23) are clearly preferable to (22). In particular, for hypersurfaces the error term in both (21) and (23) is at most quartic in δ, while that of (22) contains an exponential term δ^{n+1}. On the other hand, for varieties of low dimension (22) might be more accurate than both (21) and (23). In this sense, we may say that (21) and (23) somewhat complement (22). Finally, the right-hand side of (21) depends on a lower power of δ than that of (23), which may yield a significant improvement in estimates for varieties of large degree.

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