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An arithmetical approach to the convergence problem of series of dilated functions and its connection with the Riemann Zeta function

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A B S T R A C T

Given a periodic function f, we study the convergence almost everywhere and in norm of the series $\sum_k c_k f(kx)$. Let $f(x) = \sum_{m=1}^{\infty} a_m \sin 2\pi mx$ where $\sum_{m=1}^{\infty} a_m^2 d(m) < \infty$ and $d(m) = \sum_{d|m} 1$, and let $f_n(x) = f(nx)$. We show by using a new decomposition of squared sums that for any $K \subset \mathbb{N}$ finite, $\| \sum_{k\in K} c_k f_k\|_2^2 \le (\sum_{m=1}^{\infty} a_m^2 d(m)) \sum_{k\in K} c_k^2 d(k^2)$. If $f^{(s)}(x) = \sum_{j=1}^{\infty} \frac{\sin 2\pi jx}{j^s}$, $s > 1/2$, by only using elementary Dirichlet convolution calculus, we show that for $0 < \varepsilon \le 2s-1$, $\zeta(2s)^{-1}\|\sum_{k\in K} c_k f_k^{(s)}\|_2^2 \le \frac{1+\varepsilon}{\varepsilon}(\sum_{k\in K} |c_k|^2 \sigma_{1+\varepsilon-2s}(k))$, where $\sigma_h(n) = \sum_{d|n} d^h$. From this we deduce that if $f \in \mathrm{BV}(\mathbb{T})$, $\langle f, 1\rangle = 0$ and

$$\sum_k c_k^2 \frac{(\log\log k)^4}{(\log\log\log k)^2} < \infty,$$

then the series $\sum_k c_k f_k$ converges almost everywhere. This slightly improves a recent result, depending on a fine analysis on the polydisc [1, th. 3] ($n_k = k$), where it was assumed that $\sum_k c_k^2 (\log\log k)^{\gamma}$ converges for some $\gamma > 4$. We further show that the same conclusion holds under the arithmetical condition

$$\sum_k c_k^2 (\log\log k)^{2+b} \sigma_{-1+\frac{1}{(\log\log k)^{b/3}}}(k) < \infty,$$

for some $b > 0$, or if $\sum_k c_k^2 d(k^2)(\log\log k)^2 < \infty$. We also derive from a recent result of Hilberdink an Ω-result for the

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Riemann Zeta function involving factor closed sets. As an application we find that simple conditions on T and ν ensuring that for any $\sigma > 1/2$, $0 \leq \varepsilon < \sigma$, we have

$$\max_{1\leq t\leq T} |\zeta(\sigma + it)| \geq C(\sigma)\big(\frac{1}{\sigma_{-2\varepsilon}(\nu)} \sum_{n|\nu} \frac{\sigma_{-s+\varepsilon}(n)^2}{n^{2\varepsilon}}\big)^{1/2}.$$

We finally prove an important complementary result to Wintner's famous characterization of mean convergence of series $\sum_{k=0}^{\infty} c_k f_k$.

1. Introduction

Given a periodic function f and an increasing sequence $\mathcal{N} = \{n_k, k \geq 1\}$ of positive integers, one can formally define the series $\sum_{k=1}^{\infty} c_k f(n_k x)$ and ask under which conditions this series converges in norm or almost everywhere, for instance for any real coefficient sequence $\underline{c} = (c_k)_k \in \ell^2(\mathbb{N})$. This is one of the oldest and most central problems in the theory of systems of dilated sums. We only briefly outline the kind of results obtained. First studies were made at the beginning of the last century (see Jerosch and Weyl [24] where a.e. convergence is obtained under growth conditions on coefficients and Fourier coefficients of f), parallel to similar ones for the trigonometrical system. This partly explains why until Carleson's famous proof of Lusin's hypothesis, the results obtained essentially concerned functions with slowly growing modulus or integral modulus of continuity and/or sequences $\mathcal{N}$ verifying the classical Hadamard gap condition: $n_{k+1}/n_k \geq q > 1$ for all k. Carleson's result triggered a new interest, permitting substantial progresses in this direction, under the main impulse of Russian analysts, among them Gaposhkin and later by Berkes. We refer to [4] for more details and references. Then the attention to these problems declined until very recently where there is a renewed activity, notably concerning their connection with some questions (Ω-results) on the Riemann Zeta function.

In analogy with parallel questions concerning partial sums $\sum_{k=1}^{n} f(kx)$, $n = 1, 2, \ldots$, strong law of large numbers, studied by Gál, Koksma (see also [6]), and law of the iterated logarithm, central limit theorem, invariance principle, much explored by Erdös, Berkes and Philipp, and Gaposhkin notably, recent works show that the arithmetical nature of the support of the coefficient sequence, as well as the analytic nature of f, interact in a complex way in the study of the convergence almost everywhere and in norm of these series. The part of the theory devoted to individual results, namely the search of convergence conditions linking f, $\mathcal{N}$ and $\underline{c}$ is, to say the least, barely investigated. Our main concern in this work is precisely the search of individual conditions ensuring the almost everywhere convergence of the series $\sum_{k=1}^{\infty} c_k f(n_k x)$. We propose new approaches for treating these questions. Notice before continuing, that the problem under consideration is a natural continuation of the study of the trigonometrical system, since by Carleson's result, the series $\sum_k c_k f(n_k x)$ converges almost everywhere for any trigonometrical polynomial f. And this is in fact a convergence problem that can be put inside the study of the two-index trigonometrical system with $\{e_{jk}, j, k \geq 1\}$ where we denote $e(x) = e^{2i\pi x}$, $e_n(x) = e(nx)$, $n \geq 1$. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0,1[$. Let $f(x) \sim \sum_{j=1}^{\infty} a_j e_j(x)$. Let $f_n(x) = f(nx)$, $n \in \mathbb{N}$. We assume throughout that

$$f \in L^2(\mathbb{T}), \qquad \langle f, 1\rangle = 0.$$

A key preliminary step naturally consists in the search of bounds of $\| \sum_{k\in K} c_k f_k\|_2$ integrating in their formulation the arithmetical structure of K. That question has received a satisfactory answer only for specific cases. In this work we propose an approach based on elementary Dirichlet convolution calculus and on a new decomposition of squared sums. Although quite natural in regard of the posed problem it seems, at least to our knowledge, that this direction was not prospected before, except in the recent works [38,5].

We show that our approach is strong enough to recover and even slightly improve a recent a.e. convergence result [1, Theorem 3] in the case $\mathcal{N} = \mathbb{N}$ without using analysis on the polydisc, see Theorem 3.2. We obtain norm estimates of arithmetical type of $\sum_k c_k f(kx)$ and also a related Ω-result for the Riemann Zeta function involving factor closed sets. We begin with stating and commenting on them. Next we will state almost everywhere convergence results derived from these estimates. The remaining part of the paper is devoted to the proofs of the results.

Notation. We write $u \vee v = \max\{u, v\}$. We also write $\log\log x = \log\log(x \vee e^e)$, $\log\log\log x = \log\log\log(x \vee e^{e^e})$, $x > 0$. The notations (a, b), $[a, b]$ respectively stand for the greatest common divisor and the lest common multiple of the numbers a and b.

2. Arithmetical results

2.1. A general arithmetical norm estimate

Let $d(n)$ be the divisor function, namely the number of divisors of n.

Theorem 2.1. *Assume that $\sum_{m=1}^{\infty} a_m^2 d(m) < \infty$. Then, for any finite set K of positive integers,*

$$\| \sum_{k\in K} c_k f_k\|_2^2 \leq \Big(\sum_{m=1}^{\infty} a_m^2 d(m)\Big) \sum_{k\in K} c_k^2 d(k^2).$$

The presence of the arithmetical factor $d(k^2)$ comes from formula (4.1). In [38], we recently showed a similar estimate, however restricted to sets K such that $K \subset]e^r, e^{r+1}]$ for some integer r. Then,

$$\Big\| \sum_{k\in K} c_k f_k \Big\|_2^2 \le \Big(\sum_{\nu=1}^{\infty} a_\nu^2 \Delta(\nu) \Big) \sum_{k\in K} c_k^2 d(k), \tag{2.1}$$

where $\Delta(v)$ is Hooley's Delta function,

$$\Delta(v) = \sup_{u\in\mathbb{R}} \sum_{\substack{d|v \\ u<d\le eu}} 1.$$

This one can be used to prove that under the conditions

$$A = \sum_{\nu\ge 1} a_\nu^2 \Delta(\nu) < \infty, \qquad B = \sum_j c_j^2 d(j)(\log j)^2 < \infty$$

the series $\sum_{k=0}^{\infty} c_k f_k(x)$ converges for almost all x. A slightly weaker result was established in [38] (see Theorem 1.1). Condition $A < \infty$ is very weak. As by [33],

$$\frac{1}{x} \sum_{n\le x} \Delta(n) = \mathcal{O}\big(e^{c\sqrt{\log\log x \cdot \log\log\log x}} \big)$$

for a suitable constant $c > 0$ (whereas $\frac{1}{x}\sum_{v\le x} d(v) \sim \log x$), it reduces when the Fourier coefficients are monotonic to

$$\sum_{\nu\ge 1} a_\nu^2 e^{c\sqrt{\log\log \nu \cdot \log\log\log \nu}} < \infty. \tag{2.2}$$

Theorem 2.1 is proved in Section 4 and will be deduced from a more general but also more technical result.

2.2. Related classes of arithmetical quadratic forms

The question of the reduction of a quadratic form whose coefficients are a function of the greater common divisor of their indices

$$X = \sum_{i,j} x_i x_j F((i,j))$$

was considered long ago by Cesàro [10,11] in 1885–1886, after the works of Smith [32] and Mansion [28] in 1875–1876 who calculated their determinant (Cesàro also calculated other classes of arithmetical determinants). These quadratic forms are in turn directly related to an important class of functions (see (2.7)) whose associated systems of dilated functions was recently intensively studied. Other authors, among them Jacobstahl [22], Carlitz [9], investigated before this problem (see the survey on GCD matrices by Haukkanen, Wang and Sillanpää [18] for more references). In the present case, the reduction takes the following form:

$$\sum_{k,\ell=1}^{n} c_k c_\ell \frac{(k,\ell)^{2s}}{j^s \ell^s} = \sum_{i=1}^{n} J_{2s}(i)\Big(\sum_{k=1}^{n} c_k k^{-s}\mathbf{1}_{i|k}\Big)^2. \tag{2.3}$$

Here $J_{2s}(i) = i^{2s}\prod_{p|i}(1-p^{-2s})$ is the generalized Euler totient function (see Section 5). This formula, which is used in [5] (see Lemma 1.1), was already known by Cesàro. Obviously, (2.3) remains true when replacing $\{1,\dots,n\}$ by a factor closed set. A dual problem is Möbius inversion of a family of vectors (with Gram matrix $\{F((i,j))\}_{i,j}$). Recent related works are in Balazard and Saias [3], Brémont [8].

In matrix form, this can be condensed in the following proposition, which generalizes Proposition 2.2 in [8] based on Carlitz Lemma, see also Li's representation of GCD matrices [26] and [20]. As the proof is elementary and short, we shall give it right after some necessary complementary remarks.

Proposition 2.2. *Let $T = \big(t_{i,j}\big)_{n\times n}$ and $\check{T} = \big(\check{t}_{i,j}\big)_{n\times n}$ be matrices defined by*

$$t_{i,j} = \delta_i\theta_j\mathbf{1}_{i|j}, \qquad \check{t}_{i,j} = \frac{1}{\theta_i\delta_j}\mathbf{1}_{i|j}\mu\big(\frac{j}{i}\big), \qquad i=1,\dots,n, \tag{2.4}$$

where δ_i, θ_i, $i=1,\dots,n$ are real numbers satisfying $\delta_i>0, \theta_i\neq 0$ for all i. Then,

(a) *T is invertible and $T^{-1} = \check{T}$.*
(b) *Let $H_m, m=1,\dots,n$ be real numbers defined as follows:*

$$H_m = \sum_{k|m}\delta_k^2, \quad m=1,\dots,n. \tag{2.5}$$

Then ${}^tTT = A$ where $A = \big(\theta_i\theta_j H_{(i,j)}\big)_{n\times n}$. Further, A is positive definite.

(c) *Let $G = (g_i)_{1\le i\le n}$ be vectors in an inner product space such that* $\mathrm{Gram}(G) = A$. *Then $F = {}^t\check{T}G = (f_i)_{1\le i\le n}$ has orthonormal components.*

(d) *For any reals c_i, we have*

$$\sum_{k=1}^{n} c_k g_k = \sum_{i=1}^{n} b_i f_i, \quad \textit{where} \quad b_i = \delta_i\Big(\sum_{k=1}^{n} c_k\theta_k\mathbf{1}_{i|k}\Big),\ i=1,\dots,n. \tag{2.6}$$

In particular,

$$\Big\|\sum_{k=1}^{n} c_k g_k\Big\|^2 = \sum_{i=1}^{n} b_i^2.$$

Remark 2.3. We recall that positive semi-definite matrices are always Gram matrices (of vectors in an inner product space), hence the existence of G in (c). Further, a matrix B is positive definite if and only if there exists a non-singular lower triangular matrix L such that $A = L^tL$, see [21, Corollary 7.2.9]. Furthermore, by the Möbius inversion formula,

$$H_m = \sum_{k|m} \delta_k^2, \quad m = 1, \ldots, n \qquad \Longleftrightarrow \qquad \delta_k^2 = \sum_{\ell|k} \mu\Big(\frac{k}{\ell}\Big) H_\ell, \quad k = 1, \ldots, n.$$

Remark 2.4. Take $\theta_j = j^{-s}$, $H(x) = x^{2s}$. Then $\delta_k = \sqrt{J_{2s}(k)}$ and

$$a_{i,j} = \frac{(i,j)^{2s}}{i^s j^s}, \qquad b_i = \sqrt{J_{2s}(i)}\Big(\sum_{k=1}^n c_k k^{-s} \mathbf{1}_{i|k}\Big).$$

Hence (2.3). Next consider the class of functions introduced in [27],

$$f^{(s)}(x) = \sum_{j=1}^\infty \frac{\sin 2\pi j x}{j^s} \tag{2.7}$$

where $s > 1/2$. Recall that

$$\langle f_k^{(s)}, f_\ell^{(s)} \rangle = \zeta(2s) \frac{(k,\ell)^{2s}}{k^s \ell^s}.$$

And so

$$\Big\| \sum_{k=1}^n c_k f_k^{(s)} \Big\|_2^2 = \zeta(2s) \sum_{k,\ell=1}^n c_k c_\ell \frac{(k,\ell)^{2s}}{k^s \ell^s}. \tag{2.8}$$

This class of functions was recently much studied. We begin with examining.

Further, the system

$$h_i^{(s)} = \frac{1}{\sqrt{J_{2s}(i)}} \sum_{j|i} j^s \mu\Big(\frac{i}{j}\Big) f_j^{(s)} \qquad i = 1, \ldots \tag{2.9}$$

is orthonormal and

$$\sum_{k=1}^n c_k f_k^{(s)} = \sum_{i=1}^n b_i h_i^{(s)}, \quad \text{where} \quad b_i = \sqrt{J_{2s}(i)}\Big(\sum_{k=1}^n c_k k^{-s} \mathbf{1}_{i|k}\Big), \ i = 1, \ldots, n, \quad n \geq 1. \tag{2.10}$$

This is Brémont's result, who deduced the following characterization:

The series $\sum_k c_k f_k^{(s)}$ converges in $L^2(\mathbb{T})$ if and only if the following uniformity condition holds:

$$\lim_{n\to\infty} \sup_{N>n} \sum_{i=1}^\infty J_{2s}(i) \Big(\sum_{k=N+1}^\infty c_k k^{-s} \mathbf{1}_{i|k} \Big)^2 = 0. \tag{2.11}$$

Notice that by assumption and Cauchy–Schwarz's inequality, the series $\sum_{k\geq 1} |c_k| k^{-s}$ is convergent. See [8, Proposition 2.2 and Corollary 2.3]. Although satisfactory, (2.11) is however implicit, and it would be desirable to find a more concrete characterization, namely depending more directly on the coefficient sequence $(c_k)_k$. As a (non-trivial) application, it was shown in [8], that $L^2(\mathbb{T})$-convergence holds if $|c_k| \le \delta(k)$ where δ is multiplicative and $\sum_n \delta^2(n) < \infty$.

Proof of Proposition 2.2. Let I denote the $n \times n$ identity matrix.

(a) We have

$$\sum_{k=1}^{n} \check{t}_{i,k} t_{k,j} = \sum_{k=1}^{n} \frac{1}{\theta_i \delta_k} \mathbf{1}_{i|k} \mu\Big(\frac{k}{i}\Big) \delta_k \theta_j \mathbf{1}_{k|j} = \frac{\theta_j}{\theta_i} \sum_{k=1}^{n} \mathbf{1}_{i|k} \mu\Big(\frac{k}{i}\Big) \mathbf{1}_{k|j} = \frac{\theta_j}{\theta_i} \mathbf{1}_{i|j} \sum_{\ell | \frac{j}{i}} \mu(\ell) = 0,$$

if $i \neq j$, since $\sum_{\ell|m} \mu(\ell) = 0$, if $m \ge 2$. Hence $\check{T}T = I$, and similarly $T\check{T} = I$.

(b) We compute the (i,j)-th entry of ${}^tTT$.

$$\sum_{k=1}^{n} t_{k,i} t_{k,j} = \theta_i \theta_j \sum_{k|(i,j)} \delta_k^2 = \theta_i \theta_j H_{(i,j)}$$

by the Möbius inversion formula. We also have $\det A = (\det T)^2 = \prod_{i=1}^{n} \delta_i \theta_i \neq 0$. And if $X \neq 0$, then ${}^tXAX = {}^tYY > 0$ with $Y = TX \neq 0$.

(c)

$$\begin{aligned}\langle f_i, f_j \rangle &= \sum_{k=1}^{n} \sum_{\ell=1}^{n} \check{t}_{k,i} \check{t}_{\ell,j} \langle g_k, g_\ell \rangle = \sum_{k=1}^{n} \sum_{\ell=1}^{n} \check{t}_{k,i} \Big(\sum_{u=1}^{n} t_{k,u} t_{u,\ell} \Big) \check{t}_{\ell,j} \\ &= \sum_{k=1}^{n} \check{t}_{k,i} \sum_{u=1}^{n} t_{k,u} \mathbf{1}_{u=j} = \sum_{k=1}^{n} \check{t}_{k,i} t_{k,j} = \mathbf{1}_{i=j}.\end{aligned}$$

(d) As $G = {}^tTF$, we have $t_{i,j} = \delta_i \theta_j \mathbf{1}_{i|j}$

$$\begin{aligned}\sum_{k=1}^{n} c_k g_k &= \sum_{k=1}^{n} c_k \sum_{i=1}^{n} t_{i,k} f_i = \sum_{k=1}^{n} c_k \Big(\theta_k \sum_{i=1}^{n} \delta_i \mathbf{1}_{i|k} f_i \Big) \\ &= \sum_{i=1}^{n} f_i \delta_i \Big(\sum_{k=1}^{n} c_k \theta_k \mathbf{1}_{i|k} \Big) = \sum_{i=1}^{n} f_i b_i. \qquad \square\end{aligned}$$

Another approach was proposed by Hilberdink [19] who has estimated the sums

$$\sum_{k,\ell=1}^{N} \frac{(n_k, n_\ell)^{2s}}{n_k^s n_\ell^s}$$

when $n_k = k$ and obtained optimal bounds in this case. He showed that if $b_n = n^{-s} \sum_{d|n} d^s a_d$, then

$$\sum_{n=1}^{N}|b_n|^2=\sum_{m,n\le N}a_m\overline{a}_n\frac{(m,n)^{2s}}{m^sn^s}\sum_{k\le N/[m,n]}^{*}\frac{1}{k^{2s}}. \tag{2.12}$$

Here we introduce the symbol (*) to mean that the sum is 0 when the summation index is empty. And this requires some restriction with respect to the original statement, see Proposition 3.1 and after in [19]. More precisely, if $a_n\ge 0$, the right-term is less than $\zeta(2s)\sum_{m,n\le N}a_m\overline{a}_n\frac{(m,n)^{2s}}{m^sn^s}$. And a similar lower bound occurs when $a_n\ge 0$.

When the n_k's are arbitrary but distinct positive integers, the initial result is due to Gál [16], who showed for $s=1$ that

$$\sum_{k,\ell=1}^{N}\frac{(n_k,n_\ell)^2}{n_kn_\ell}\le CN(\log\log N)^2, \tag{2.13}$$

where C is an absolute constant, and moreover that estimate is optimal. It follows for this choice of values of n_k and by taking $c_k\equiv 1$ that in this case

$$\Big\|\sum_{k=1}^{N}c_kf_{n_k}^{(1)}\Big\|_2^2\ge CN(\log\log N)^2\gg\sum_{k=1}^{N}c_k^2. \tag{2.14}$$

This is a famous result and a few explanatory words concerning the proof are necessary. Gál's proof is based on the observation that the sum in (2.13) will be not maximal unless $\{n_1,\dots,n_k\}$ is factor closed, namely $d|n_j\Rightarrow d=n_i$ for some i. These sets are usually termed FC sets (see [13, §3.3], [18]). Hence it follows that if the sum is maximal, then the corresponding n_i are products of powers of at most $C\log N$ primes.

This result was recently extended in [1] to the case $0<s<1$ (see also [7] for recent improvements, in the case $s=1/2$ notably) by representing these sums as Poisson integrals on the polydisc and by suitably modifying Gál's combinatorial argument. When sieving the coefficients c_k according to their order of magnitude, that estimate can be implemented and then becomes a decisive tool when f has slowly decreasing Fourier coefficients, typically when $f=f^1$. That allowed the authors to establish quite sharp results for the a.e. convergence of series $\sum_k c_kf_{n_k}^1$, and in fact by a plain monotonicity argument on the Fourier coefficients, for any $f\in\mathrm{BV}(\mathbb{T})$. The authors further extended their result to any $f\in\mathrm{Lip}_{1/2}(\mathbb{T})$. These results are of relevance in the present work.

Representation using Cauchy measures. Notice before continuing that

$$\frac{(n_k,n_\ell)^{2s}}{n_k^sn_\ell^s}=\prod_p p^{-s|v_p(n_k)-v_p(n_\ell)|}$$

where $v_p(n)$ is the p-valuation of n (namely $p^{v_p(n)}\|n$ and $v_p(n)=0$ if $p\nmid n$). From the relation $e^{-|\vartheta|}=\int_{\mathbb{R}}e^{i\vartheta t}\frac{\mathrm{d}t}{\pi(t^2+1)}$, it follows that

$$e^{-s|v_p(n_k)-v_p(n_\ell)|\log p}=\int_{\mathbb{R}}\frac{1}{p^{iv_p(n_k)st}p^{-iv_p(n_\ell)st}}\frac{\mathrm{d}t}{\pi(t^2+1)}.$$

So that

$$\sum_{k,\ell=1}^{N} c_k\overline{c}_\ell \frac{(n_k,n_\ell)^{2s}}{n_k^s n_\ell^s} = \sum_{k,\ell=1}^{N} c_k\overline{c}_\ell \int_{\mathbb{R}^\infty} \prod_p \Big(\frac{1}{p^{iv_p(n_k)st_p}p^{-iv_p(n_\ell)st_p}}\frac{\mathrm{d}t_p}{\pi(t_p^2+1)}\Big)$$
$$= \int_{\mathbb{R}^\infty} \Big|\sum_{k=1}^{N} \frac{c_k}{\prod_p p^{iv_p(n_k)st_p}}\Big|^2 \prod_p \frac{\mathrm{d}t_p}{\pi(t_p^2+1)}, \tag{2.15}$$

namely, the sum directly expresses as a squared norm with respect to the infinite Cauchy measure.

2.3. A new arithmetical quadratic estimate

It turns up that even for this specific class of functions, another much simpler device can be used, based on Dirichlet convolution calculus, which also leads, at least when $n_k = k$, to slightly sharper convergence results. The basic tool, which we are going to state now, provides a new estimate of $\|\sum_{k\in K} c_k f_k^{(s)}\|$, K arbitrary. This estimate is of individual type, in the sense that it is expressed by means of the values taken on K by some elementary arithmetical functions. For $u\in\mathbb{R}$, let $\sigma_u(k)=\sum_{d|k} d^u$. In particular $\sigma_0 = d$, $\sigma_1=\sigma$ is the usual sum-divisor function and $\sigma_{-\alpha}(n) = n^{-\alpha}\sigma_\alpha(n)$.

Theorem 2.5. *Let $s>0$ and $0\le\tau\le 2s$. Also let $\psi_1(u)>0$ be non-decreasing. Then for any finite set K of integers,*

$$\sum_{k,\ell\in K} c_k c_\ell \frac{(k,\ell)^{2s}}{k^s\ell^s} \le \Big(\sum_{u\in F(K)} \frac{1}{\psi_1(u)\sigma_\tau(u)}\Big)\Big(\sum_{\nu\in K} c_\nu^2\psi_1(\nu)\sigma_{\tau-2s}(\nu)\Big).$$

In particular,

(i)

$$\sum_{k,\ell\in K} |c_k||c_\ell|\frac{(k,\ell)^{2s}}{k^s\ell^s} \le M(K)\Big(\sum_{k\in K} c_k^2\sigma_{\tau-2s}(k)\Big),$$

with

$$M(K) = \sum_{k\in F(K)} \frac{1}{\sigma_\tau(k)}.$$

(ii) *Further,*

$$\sum_{k,\ell\in K} |c_k||c_\ell|\frac{(k,\ell)^2}{k\ell} \le \frac{\pi^2}{6}\Big(\sum_{u\in F(K)} \frac{\varphi(u)}{u^2\log\log u}\Big)\Big(\sum_{\nu\in K} c_\nu^2\sigma_{-1}(\nu)\log\log\nu\Big),$$

As an immediate consequence, we get recalling (2.8),

Corollary 2.6. *Let $s > 1/2$. Then for any finite set K,*

$$\zeta(2s)^{-1}\Big\| \sum_{k\in K} c_k f_k^{(s)} \Big\|_2^2 \le \inf_{0<\varepsilon\le 2s-1} \frac{1+\varepsilon}{\varepsilon} \sum_{k\in K} c_k^2 \sigma_{1+\varepsilon-2s}(k).$$

Proof. Let indeed $0 < \varepsilon \le 2s-1$ and take $\tau = 1+\varepsilon$. From the obvious inequality $\sigma_\tau(k) \ge k^\tau$, it follows that

$$M(K) \le \sum_{k\in F(K)} \frac{1}{k^{1+\varepsilon}} \le \sum_{k\ge 1} \frac{1}{k^{1+\varepsilon}} \le \frac{1+\varepsilon}{\varepsilon}.$$

So Corollary 2.6 just follows from assertion (i) of Theorem 2.5. □

Remark 2.7. Letting for instance $s = 1$ and using monotonicity of the Fourier coefficients, Corollary 2.6 implies in particular:

For any $f \in \mathrm{BV}(\mathbb{T})$ with $\int_{\mathbb{T}} f \,\mathrm{d}\lambda = 0$,

$$\Big\| \sum_{k\in K} c_k f_k \Big\|_2^2 \le C(f) \inf_{0<\varepsilon\le 2s-1} \frac{1+\varepsilon}{\varepsilon} \sum_{k\in K} |c_k|^2 \sigma_{-1+\varepsilon}(k).$$

And $C(f)$ depends on f only.

We derive from Corollary 2.6 a sharp arithmetical sufficient condition for the in-norm convergence of the series $\sum_{k\ge 1} c_k f_k^{(s)}$.

Corollary 2.8. *Let $s > 1/2$. Assume that the following condition is fulfilled:*

$$\textit{For some } \varepsilon > 0, \qquad \sum_{k\ge 1} |c_k|^2 \sigma_{1+\varepsilon-2s}(k) < \infty.$$

Then the series $\sum_{k\ge 1} c_k f_k^{(s)}$ converges in $L^2(\mathbb{T})$.

Proof. This is now straightforward, since by Corollary 2.6

$$\sup_{n,m\ge N} \Big\| \sum_{n\le k\le m} c_k f_k^{(s)} \Big\|_2^2 \le C(s,\varepsilon) \sum_{k\ge N} |c_k|^2 \sigma_{1+\varepsilon-2s}(k) \to 0,$$

as N tends to infinity. Hence $\{\sum_{1\le k\le m} c_k f_k^{(s)}, m \ge 1\}$ is a Cauchy sequence in $L^2(\mathbb{T})$. □

See also the recent work [2] for a different proof.

Remark 2.9. By monotonicity of Fourier coefficients, Corollary 2.8 immediately extends with no change to functions $f \sim \sum_{j=1}^{\infty} a_j \sin 2\pi jx$ such that $a_j = \mathcal{O}(j^{-s})$, $s > 1/2$.

Remark 2.10. Estimates (i)–(iii) also provide sharp bounds to GCD sums indexed on FC sets. Estimate (i) with $s = \tau = 1$ further implies

$$\sum_{k,\ell\in K} c_k c_\ell \frac{(k,\ell)^2}{k\ell} \le \frac{\pi^2}{6}\Big(\sum_{k\in K}\frac{\varphi(k)}{k^2}\Big)\Big(\sum_{k\in K}|c_k|^2\sigma_{-1}(k)\Big),$$

where $\varphi(n)$ is Euler totient function. Indeed

$$\begin{aligned}\sum_{k,\ell\in K} c_k c_\ell \frac{(k,\ell)^2}{k\ell} &\le \Big(\sum_{k\in K}\frac{1}{\sigma(k)}\Big)\Big(\sum_{k\in K}|c_k|^2\sigma_{-1}(k)\Big)\\ &\le \frac{\pi^2}{6}\Big(\sum_{k\in K}\frac{\varphi(k)}{k^2}\Big)\Big(\sum_{k\in K}|c_k|^2\sigma_{-1}(k)\Big),\end{aligned}$$

since [12, p. 10] $\sigma(n)\varphi(n) > 6n^2/\pi^2$. Concerning the first factor, notice that

$$\sum_{k=1}^{\infty}\frac{\varphi(k)}{k^s} = \frac{\zeta(s-1)}{\zeta(s)} \qquad \text{for } s > 2.$$

Recall for later use that by Gronwall's estimates [17, pp. 119–122],

$$\limsup_{n\to\infty}\frac{\log\left(\frac{\sigma_\alpha(n)}{n^\alpha}\right)}{\frac{(\log n)^{1-\alpha}}{\log\log n}} = \frac{1}{1-\alpha} \qquad (0<\alpha<1) \tag{2.16}$$

Further,

$$\limsup_{n\to\infty}\frac{\sigma(n)}{n\log\log n} = e^\gamma, \tag{2.17}$$

where γ is Euler's constant.

2.4. Eigenvalues arithmetical estimates

The recent estimates of the eigenvalues of the arithmetical matrix

$$M(K,s) = \Big\{\frac{(k,\ell)^{2s}}{k^s\ell^s}\Big\}_{k,\ell\in K}$$

established in [19,1,7] are sharp but not of arithmetical type. An important and quite challenging question is precisely to know whether it is possible to provide bounds of this type, expressed in a simple way by arithmetical functions. In this direction, the following GCD sum estimate established in [5, Proposition 1.16] is relevant.

Proposition 2.11. *Let* $0 < s \le 1$. *For any* $k \in K$ *(letting* $K_- = \min K$, $K_+ = \max(K)$*),*

$$\sum_{\substack{\ell \in K \\ \ell \neq k}} \frac{(k,\ell)^{2s}}{k^s \ell^s} \le \begin{cases} 2\left(\log \frac{K_+}{K_-}\right)\sigma_{-1}(k) & \text{if } s = 1, \\ 2^s k^{s-1}\left(\int_{K_-}^{K_+} \frac{\mathrm{d}u}{u^s}\right)\sigma_{1-2s}(k) & \text{if } 0 < s < 1. \end{cases}$$

It is easy to derive eigenvalues estimates of $M(K,s)$ for K arbitrary.

Corollary 2.12. *Let* $s > \frac{1}{2}$. *Let* $\lambda(k,s), k \in K$ *be the eigenvalues of* $M(K,s)$. *Then for any* $k \in K$,

$$|\lambda(k,s) - \zeta(2s)| \le \begin{cases} 2\left(\log \frac{K_+}{K_-}\right) \sup\limits_{k \in K} \sigma_{-1}(k) & \text{if } s = 1, \\ 2^s \left(\frac{K_+}{K_-}\right)^{1-s} \sup\limits_{k \in K} \sigma_{1-2s}(k) & \text{if } s < 1. \end{cases}$$

Gronwall's estimates (2.16) provide further quantitative bounds.

Proof. We apply Geršgorin's theorem stating that the eigenvalues of an $n \times n$ matrix $(a_{i,j})$ with complex entries lie in the union of the closed disks (Geršgorin disks)

$$|z - a_{i,i}| \le \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{i,j}| \qquad (i = 1, 2, \ldots, n) \tag{2.18}$$

in the complex plane, see for instance [36]. Hence

$$|\lambda(k,s) - \zeta(2s)| \le \sup_{k \in K} \sum_{\substack{\ell \in K \\ \ell \neq k}} \frac{(k,\ell)^{2s}}{k^s \ell^s}.$$

Applying Proposition 2.11 and noticing that when $s < 1$,

$$\sum_{\substack{\ell \in K \\ \ell \neq k}} \frac{(k,\ell)^{2s}}{k^s \ell^s} \le 2^s \left(\frac{K_+}{K_-}\right)^{1-s} \sigma_{1-2s}(k),$$

allows us to conclude. □

When combined with the classical weighted estimate for quadratic forms:
For any system of complex numbers $\{x_i\}$ *and* $\{\alpha_{i,j}\}$,

$$\Big| \sum_{\substack{1 \le i,j \le n \\ i \neq j}} x_i x_j \alpha_{i,j} \Big| \le \frac{1}{2} \sum_{i=1}^{n} |x_i|^2 \Big(\sum_{\substack{\ell=1 \\ \ell \neq i}}^{n} (|\alpha_{i,\ell}| + |\alpha_{\ell,i}|) \Big), \tag{2.19}$$

Proposition 2.11 immediately implies that

$$\|\sum_{k\in K} c_k f_k^{(s)}\|_2^2 \le \begin{cases} 2^s\big(\frac{K_+}{K_-}\big)^{1-s}\sum_{k\in K}\sigma_{1-2s}(k)c_k^2 & \text{if } 1/2<s\le 1\\ 2\big(\log\frac{K_+}{K_-}\big)\sum_{k\in K} d(k)c_k^2 & \text{if } s=1/2,\end{cases} \tag{2.20}$$

as observed in [5]. These estimates turn up to be of crucial use in the last part of the proof of Theorem 3.2.

Remark 2.13. Let $p_1,\dots,p_N$ be distinct prime numbers and let $G=\langle p_1,\dots,p_N\rangle$ be the associated multiplicative semi-group. A simple consequence of (2.19) is also that

$$\sum_{i,j=1}^N x_i\overline{x}_j\frac{(n_i,n_j)^{2\alpha}}{n_i^\alpha n_j^\alpha} \le C\prod_{i=1}^N\Big(\frac{1}{1-p_i^{-\alpha}}\Big)\sum_{i=1}^N |x_i|^2,$$

for any $\alpha>0$, any $n_1,\dots,n_N\in G$ and any complex numbers $x_1,\dots,x_N$. And the constant C is absolute.

2.5. An arithmetical Ω-theorem for the Zeta function

We derive from a recent result of Hilberdink [19] (see Theorem 3.3) an arithmetical type Ω-result for the Riemann Zeta function which is, to our knowledge, the first of this kind in the theory.

Theorem 2.14. *Let $\sigma>1/2$. There exist a positive constant c_σ depending on σ only and a positive absolute constant C, such that for any FC set K such that*

$$K_+=\max\{k,k\in K\}\le T,\qquad \max_{k,\ell\in K}\frac{k\vee\ell}{(k,\ell)}\ge c_\sigma,$$

and any $0\le\varepsilon<\sigma$,

$$\begin{aligned}(1+C)\Big(\sum_{n\in K}\frac{1}{n^{2\varepsilon}}\Big)\max_{1\le t\le T}|\zeta(\sigma+it)|^2 \ge \frac{\zeta(2\sigma)}{2}\sum_{n\in K}\frac{1}{n^{2\varepsilon}}\sigma_{-\sigma+\varepsilon}(n)^2\\ -\frac{1}{T^{2\sigma-1}}\Big(\sum_{\ell\in K}\frac{1}{\ell^\varepsilon}\Big)\Big(\sum_{k\in K}k^{1-\sigma-\varepsilon}\log(kT)\Big\}.\end{aligned}$$

Theorem 2.15. *Let $\sigma>1/2$. For any integer $\nu\ge 2$ such that $\max_{[k,\ell]|\nu}\frac{(k\vee\ell)}{(k,\ell)}\ge c_\sigma$, and $0\le\varepsilon<\sigma$, we have*

$$\max_{1\le t\le T}|\zeta(\sigma+it)|\ge c\Big(\frac{1}{\sigma_{-2\varepsilon}(\nu)}\sum_{n|\nu}\frac{\sigma_{-\sigma+\varepsilon}(n)^2}{n^{2\varepsilon}}\Big)^{1/2},$$

whenever ν and T are such that

$$\frac{\sigma_{-\varepsilon}(\nu)\sigma_{1-\sigma-\varepsilon}(\nu)\log(\nu T)}{\sum_{n|\nu}\frac{\sigma_{-\sigma+\varepsilon}(n)^2}{n^{2\varepsilon}}} \le \frac{\zeta(2\sigma)^{1/2}}{4} T^{(2\sigma-1)}.$$

Here $c = \frac{\zeta(2\sigma)}{2\sqrt{1+C}}$ *and* c_σ, C *are as in Theorem 2.14.*

By taking ν a product of primes, it is easy to recover Hilberdink's result from Theorem 2.15.

Remark 2.16. The inequality linking ν and T clearly depends on the divisors of ν. Otherwise, a typical case where this is satisfied is provided by the simple condition

$$\nu = \mathcal{O}\big(T^{\frac{2\sigma-1}{1-\sigma}}/\log T\big).$$

Indeed, by using estimates Gronwall's estimates (2.16)

$$\sigma_{-\varepsilon}(\nu)\sigma_{1-\sigma-\varepsilon}(\nu) \le \nu^{1-\sigma-\varepsilon} e^{\frac{c(\sigma,\varepsilon)}{\log\log n}((\log n)^{1-\varepsilon}+(\log n)^{\sigma+\varepsilon})} = o(\nu^{1-\sigma}).$$

And trivially $\sum_{n|\nu}\frac{\sigma_{-\sigma+\varepsilon}(n)^2}{n^{2\varepsilon}} \ge 1$. This suffices to conclude. We don't know how large the right-hand side can be made. This will be considered in a separate study with further developments of the arithmetical approach used here, in the context of Dirichlet approximating polynomials.

3. Almost everywhere convergence results

We first apply Theorem 2.1 to almost everywhere convergence. We obtain new convergence conditions of mixed type, namely multipliers partly expressed by arithmetical functions. We will prove

Theorem 3.1. *Assume that* $a_m = \mathcal{O}(m^{-\alpha})$ *for some* $\alpha > 1/2$.

(i) Let $1/2 < \alpha < 1$. *Then the series* $\sum_{k\ge1} c_k f_k$ *converges almost everywhere whenever the following condition is satisfied:*

$$\sum_{k\ge3} c_k^2(\log k)^{4(1-\alpha)}(\log\log k)^{2(1-\alpha)}d(k^2) < \infty.$$

(ii) Let $\alpha = 1$. *Then the same conclusion holds true if the above condition is replaced by*

$$\sum_{k\ge3} c_k^2 d(k^2)(\log\log k)^2 < \infty.$$

(iii) Assume that $a_m = \mathcal{O}(m^{-1/2}(\log m)^{-(1+h)/2})$ *for some* $h > 1$. *Then the same conclusion holds true under the following condition:*

$$\sum_{k\ge3} c_k^2 d(k^2)(\log k)^2(\log\log k)^{1-h} < \infty.$$

These arithmetical conditions are meaningful for coefficient sequences supported by sets of integers k having few divisors. In [5, Theorem 2.8], we showed that the condition

$$\sum_{k\geq 1} c_k^2(\log k)^2\sigma_{1-2\alpha}(k) < \infty$$

also implies the convergence almost everywhere of the series $\sum_{k\geq 1} c_k f_k$. Although not exactly comparable with the condition given in (i), this one yields a better condition for coefficient sequences supported by integers with few divisors. A similar remark holds concerning the general condition given in [5] (see Corollary 2.6 and Remark 2.7). The condition given in (ii) has to be compared with the one in Theorems 3.2, 8.4.

As to (ii) and (iii), the non-arithmetical factors of the multipliers are significantly better than those in Theorem 3.2, and Theorem 1.1 in [38], respectively. Recall concerning (ii) that condition (see Theorems 3.7 in [1])

$$\sum_{k\geq 3} c_k^2(\log\log k)^\gamma < \infty,$$

for every $\gamma < 2$ is necessary for the convergence almost everywhere of the series $\sum_{k\geq 1} c_k f_k$.

We further prove the following almost everywhere convergence result concerning the Banach space $\mathrm{BV}(\mathbb{T})$ of functions with bounded variation.

Theorem 3.2. *Let $f\in \mathrm{BV}(\mathbb{T})$, $\langle f,1\rangle = 0$. Assume that*

$$\sum_{k\geq 3} c_k^2\frac{(\log\log k)^4}{(\log\log\log k)^2} < \infty. \tag{3.1}$$

Then the series $\sum_k c_k f_k$ converges almost everywhere.

This slightly improves Theorem 3 in [1] ($n_k = k$), where it was assumed that the series

$$\sum_{k=1}^{\infty} c_k^2(\log\log k)^\gamma \tag{3.2}$$

converges for some $\gamma > 4$.

Theorem 3.2 will be derived directly from Theorem 2.5, thus without using analysis on the polydisc as in [1]. By modifying the proof, we also obtain in Section 8 a delicate sufficient condition where multipliers have arithmetical factors (see Remark 8.3).

Remark 3.3. In spite of the regular decay of its Fourier coefficients, it is well known that a function $f \in \mathrm{BV}(\mathbb{T})$ may have very pathological behavior. Jordan [25] gave in 1881 a remarkably simple and elegant construction of a function with bounded variation, having positive jumps on each rational, and being continuous almost everywhere.

Remark 3.4. Theorem 3.2 applies to the case $s = 1$ in (2.7) which corresponds to the Fourier expansion of the function $\langle x\rangle = \frac{\pi}{2}(1-2x)$, $0 \le x \le 1$

$$\langle x\rangle = \sum_{1}^{\infty} \frac{\sin 2\pi n x}{n} \qquad (0 < x < 1) \tag{3.3}$$

the series being discontinuous at $x = 0$. It is quite interesting to notice by expanding $\langle x\rangle$ with respect to the system $\cos(n+\frac{1}{2})x$, $\sin(n+\frac{1}{2})x$, $n = 0, 1, \ldots$ (which is orthogonal and complete over any interval of length 2π), that one also gets [40, p. 71]

$$\langle x\rangle = \frac{4}{\pi} \sum_{0}^{\infty} \frac{\cos 2\pi(n+\frac{1}{2})x}{(2n+1)^2} \qquad (0 \le x \le 1), \tag{3.4}$$

where this time the series is absolutely and uniformly convergent. Let $\varsigma(x)$ denote the series in the right-hand side. Further, it is not a complicated task to prove that the series $\sum c_k \varsigma(n_k x)$ converges for almost every x under the minimal condition $\sum c_k^2 < \infty$. However $\varsigma(x)$ is 2-periodic whereas $\langle x\rangle$ is 1-periodic. The study of the system $\{\langle nx\rangle, n \in \mathbb{N}\}$ goes back to Riemann's work [31]. Davenport [14,15] much investigated its properties. It is known that this system possesses smoothness properties going at the opposite of those of the trigonometrical system (the series $\sum_k c_k\langle kx\rangle$ is never continuous unless the coefficients c_k all vanish). We refer to Jaffard [23]. However, the a.s. convergence properties of series attached to this system seem to remain relatively close to those of the trigonometrical system, namely to belong close to the domain of applicability of Carleson's theorem.

3.1. A complement to Wintner's Theorem

We finally also prove an important complementary result to Wintner's famous characterization of mean convergence of series $\sum_{k=0}^{\infty} c_k f_k$. Recall some necessary facts. Let $f \in L^2(\mathbb{T})$ with $\langle f, 1\rangle = 0$ and denote $\bar{f} = (f_n)_n$ where we recall that $f_n(x) = f(nx)$. We say that the system $\bar{f}$ is *mean convergent* if the series $\sum_{k=0}^{\infty} c_k f_k$ converges in $L^2(\mathbb{T})$ for any $(c_k)_k \in \ell^2$. This property is characterized by the following well-known theorem.

Theorem 3.5 *(Wintner [39]). Let $f \in L^2(\mathbb{T})$ with $\langle f, 1\rangle = 0$ and with Fourier series $f(x) \sim \sum_{j=1}^{\infty}(a_j \cos 2\pi jx + b_j \sin 2\pi jx)$. The following statements are equivalent:*

1. The series $\sum_{k=1}^{\infty} c_k f_k(x)$ converges in $L^2(\mathbb{T})$ for any coefficient sequence $(c_k)_k \in \ell^2$.
2. There exists a constant $c > 0$ such that for any $n \ge 1$ and any reals $\{c_k, 1 \le k \le n\}$ we have

$$\Big\| \sum_{k=1}^{n} c_k f_k \Big\|^2 \le c \sum_{k=1}^{n} c_k^2. \tag{3.5}$$

3. The infinite matrix $\big(\langle f_k, f_\ell\rangle\big)_{k,\ell}$ *defines a bounded operator on* ℓ^2.
4. The Dirichlet series $\sum_{n=1}^\infty a_n n^{-s}$ *and* $\sum_{n=1}^\infty b_n n^{-s}$ *are regular and bounded in the half-plane* $\Re(s) > 0$.

Suppose $\bar f$ is mean convergent. It is natural to ask whether there always exists a class of coefficients $(c_k)_k$ for which the series $\sum_{k=1}^\infty c_k f_k$ will converge almost everywhere. The theorem below answers this affirmatively by identifying a general class of coefficients.

Recall a useful notion. A sequence of coefficients $(c_k)_k$ is called *universal* if for any orthonormal system $\Phi = (\varphi_k)$ of functions defined on a bounded interval (and possibly extended periodically over the real line), the series $\sum_{k=1}^\infty c_k\varphi_k$ converges a.e.

Theorem 3.6. *Assume that* $\bar f$ *is mean convergent. Then the series* $\sum_{k=1}^\infty c_k f_k$ *converges a.e. for any universal coefficient sequence* $(c_k)_k$.

The paper is organized as follows: in Sections 4 to 9 we prove the results stated before.

4. Proof of Theorem 2.1

We will deduce Theorem 2.1 from the following more general but more technical result. Introduce the necessary notation. Let

$$A_k = \sum_{\nu=1}^\infty a_{\nu k}^2.$$

Let ζ_h denote the arithmetic function defined by $\zeta_h(n) = n^h$ for all positive n. In particular $\zeta_0(n) = 1$ for all n. Let $\theta(n)$ denote the number of square-free divisors of n. Then $\theta(n) = 2^{\omega(n)}$ where $\omega(n)$ is the prime divisor function, and by the Mertens estimate, $\sum_{k\le x} 2^{\omega(k)} = Cx\log x + \mathcal{O}(x)$, $x \ge 2$, where C is some positive constant [12, p. 70].

Given $K \subset \mathbb{N}$, we note $F(K) = \{d \ge 1; \exists k \in K : d|k\}$. If K is factor closed ($d|k \Rightarrow d \in K$ for all $k \in K$), then $F(K) = K$. Typical examples are $\{1, \dots, n\}$ or sets of mutually coprime integers. Recall that if ψ, ϕ are arithmetical functions, the Dirichlet convolution $\psi * \phi$ is defined by $\psi * \phi(n) = \sum_{d|n}\psi(d)\phi(n/d)$.

Proposition 4.1. *Let* ψ *be any arithmetical function taking only positive values.*

i) For any finite set K *of positive integers,*

$$\Big\|\sum_{k\in K} c_k f_k\Big\|_2^2 \le \sum_{d\in F(K)}\Big(\sum_{\substack{k\in K\\ d|k}} c_k^2\psi\big(\frac{k}{d}\big)\Big)\Big(\sum_{\substack{k\in K\\ d|k}}\frac{A_{\frac{k}{d}}}{\psi(\frac{k}{d})}\theta\big(\frac{k}{d}\big)\Big).$$

ii) In particular,

$$\Big\|\sum_{k\in K} c_k f_k\Big\|_2^2 \le B\sum_{k\in K} c_k^2\psi * \zeta_0(k),$$

where

$$B = \sup_{d\in F(K)} \Big(\sum_{\substack{k\in K\\ d|k}} \frac{A_{\frac{k}{d}}}{\psi(\frac{k}{d})}\theta(\frac{k}{d})\Big) < \infty.$$

Let us first indicate how Theorem 2.1 follows from Proposition 4.1. Choose for instance $\psi = \theta$ and note [29, formula (1.54)] that

$$\psi * \zeta_0(k) = \sum_{d|k} \theta(d) = d(k^2). \tag{4.1}$$

Let $d \in F(K)$, then

$$\begin{aligned}\sum_{\substack{k\in K\\ d|k}} \frac{A_{\frac{k}{d}}}{\psi(\frac{k}{d})}\theta(\frac{k}{d}) &= \sum_{\substack{k\in K\\ d|k}} \sum_{\nu=1}^{\infty} a^2_{\nu\frac{k}{d}} = \sum_{m=1}^{\infty} a_m^2\Big(\sum_{\substack{k\in K\\ d|k,\, \frac{k}{d}|m}} 1\Big)\\ &= \sum_{m=1}^{\infty} a_m^2\Big(\sum_{\substack{jd\in K\\ j|m}} 1\Big) \le \sum_{m=1}^{\infty} a_m^2 d(m).\end{aligned}$$

Since it is true for any $d \in F(K)$, we deduce that $B \le \sum_{m=1}^{\infty} a_m^2 d(m)$, and so Theorem 2.1 follows from Proposition 4.1.

Proof of Proposition 4.1. Let δ be the arithmetical function defined by

$$\delta(n) = \begin{cases} 1 & \text{if } n = 1,\\ 0 & \text{if } n \neq 1.\end{cases} \tag{4.2}$$

Let μ denote the Möbius function and recall that

$$\sum_{d|n} \mu(d) = \delta(n). \tag{4.3}$$

We have

$$\Big\| \sum_{k\in K} c_k f_k\Big\|_2^2 = \sum_{k,\ell\in K} c_k c_\ell \sum_{\nu=1}^{\infty} a_{\frac{\nu k}{(k,\ell)}} a_{\frac{\nu \ell}{(k,\ell)}}.$$

We decompose the right-hand side according to the values taken by (k, ℓ),

$$\sum_{k,\ell\in K} c_k c_\ell \sum_{\nu=1}^{\infty} a_{\frac{\nu k}{(k,\ell)}} a_{\frac{\nu \ell}{(k,\ell)}} = \sum_{d\in F(K)} S_d, \tag{4.4}$$

where

$$S_d = \sum_{\substack{k,\ell \in K \\ (k,\ell)=d}} c_k c_\ell \sum_{\nu=1}^{\infty} a_{\frac{\nu k}{d}} a_{\frac{\nu \ell}{d}}. \tag{4.5}$$

We claim that

$$|S_1| \le \Big(\sum_{k\in K} |c_k|^2 \psi(k)\Big)\Big(\sum_{k\in K} \frac{A_k}{\psi(k)}\theta(k)\Big). \tag{4.6}$$

Indeed, by (4.2), (4.3),

$$\begin{aligned} S_1 &= \sum_{k,\ell\in K} c_k c_\ell \sum_{\nu=1}^{\infty} a_{\nu k} a_{\nu \ell} \delta((k,\ell)) = \sum_{k,\ell\in K} c_k c_\ell \sum_{\nu=1}^{\infty} a_{\nu k} a_{\nu\ell} \sum_{d|(k,\ell)} \mu(d) \\ &= \sum_{\nu=1}^{\infty} \sum_{d\in F(K)} \mu(d) \sum_{\substack{k,\ell\in K \\ d|k, d|\ell}} c_k c_\ell a_{\nu k} a_{\nu \ell} = \sum_{\nu=1}^{\infty} \sum_{d\in F(K)} \mu(d) \Big(\sum_{\substack{k\in K \\ d|k}} c_k a_{\nu k}\Big)^2. \end{aligned}$$

This thus factorizes, and now we can apply Cauchy–Schwarz's inequality to get

$$\begin{aligned} |S_1| &\le \sum_{\nu=1}^{\infty} \sum_{d\in SF(K)} \Big(\sum_{\substack{k\in K \\ d|k}} |c_k| \sqrt{\psi(k)}. \frac{a_{\nu k}}{\sqrt{\psi(k)}}\Big)^2 \\ &\le \Big(\sum_{k\in K} |c_k|^2 \psi(k)\Big) \sum_{\nu=1}^{\infty} \sum_{d\in SF(K)} \sum_{\substack{k\in K \\ d|k}} \frac{a_{\nu k}^2}{\psi(k)} \\ &= \Big(\sum_{k\in K} |c_k|^2 \psi(k)\Big)\Big(\sum_{k\in K} \frac{A_k}{\psi(k)} \sum_{\substack{d\in SF(K) \\ d|k}} 1\Big) \\ &= \Big(\sum_{k\in K} |c_k|^2 \psi(k)\Big)\Big(\sum_{k\in K} \frac{A_k}{\psi(k)} \theta(k)\Big), \end{aligned}$$

Now let $K_d = \frac{1}{d}(d\mathbb{N} \cap K)$. For the sum S_d defined in (4.5), we have

$$S_d = \sum_{\substack{k,\ell\in K \\ (k,\ell)=d}} c_k c_\ell \sum_{\nu=1}^{\infty} a_{\frac{\nu k}{d}} a_{\frac{\nu\ell}{d}} = \sum_{\substack{k',\ell'\in K_d \\ (k',\ell')=1}} c_{k'd} c_{\ell' d} \sum_{\nu=1}^{\infty} a_{\nu k'} a_{\nu k'}.$$

Thus S_d has just the same form as the sum S_1 studied before, with $K_d, k', c_{k'd}, a_{k'd}$ in place of K, k, c_k, a_k. We deduce from (4.6) that

$$
\begin{aligned}
S_d &\le \Big(\sum_{k'\in K_d} |c_{k'd}|^2\psi(k')\Big)\Big(\sum_{k'\in K_d} \frac{A_{k'}}{\psi(k')}\theta(k')\Big)\\
&= \Big(\sum_{\substack{k\in K\\ d|k}} |c_k|^2\psi(\frac{k}{d})\Big)\Big(\sum_{\substack{k\in K\\ d|k}} \frac{A_{\frac{k}{d}}}{\psi(\frac{k}{d})}\theta(\frac{k}{d})\Big). \qquad (4.7)
\end{aligned}
$$

Using (4.4), we get

$$
\begin{aligned}
\|\sum_{k\in K} c_k f_k\|_2^2 &= \sum_{d\in F(K)} S_d \le \sum_{d\in F(K)} \Big(\sum_{\substack{k\in K\\ d|k}} |c_k|^2\psi(\frac{k}{d})\Big)\Big(\sum_{\substack{k\in K\\ d|k}} \frac{A_{\frac{k}{d}}}{\psi(\frac{k}{d})}\theta(\frac{k}{d})\Big)\\
&\le \Big(\sup_{d\in F(K)} \sum_{\substack{k\in K\\ d|k}} \frac{A_{\frac{k}{d}}}{\psi(\frac{k}{d})}\theta(\frac{k}{d})\Big) \sum_{k\in K} |c_k|^2 \sum_{\substack{d\in F(K)\\ d|k}} \psi(\frac{k}{d})\\
&= B\sum_{k\in K} |c_k|^2 \psi * \zeta_0(k). \qquad \square \qquad (4.8)
\end{aligned}
$$

5. Proof of Theorem 2.5

Let $(c_k)_k$ be a sequence of coefficients supported by K ($c_k = 0$ if $k \notin K$). Let $\varepsilon > 0$. We recall that the generalized Euler totient function J_ε is the multiplicative arithmetical function defined by

$$J_\varepsilon(n) = \zeta_\varepsilon * \mu(n) = \sum_{d|n} d^\varepsilon \mu(\frac{n}{d}).$$

By Möbius inversion theorem,

$$n^\varepsilon = \sum_{d|n} J_\varepsilon(d). \qquad (5.1)$$

Step (1) is as in [4], except that we introduce an arithmetic function ψ. It is necessary to display it here. Step (ii) uses basic properties of Dirichlet convolutions.

(1) Noticing that if $d|k$ and $k \in K$, then $d \in F(K)$, we have by (5.1)

$$(k,\ell)^\varepsilon = \sum_{d\in F(K)} J_\varepsilon(d)\mathbf{1}_{d|k}\mathbf{1}_{d|\ell}.$$

Thus

$$L := \sum_{k,\ell=1}^{n} c_k c_\ell \frac{(k,\ell)^{2s}}{k^s\ell^s} = \sum_{k,\ell\in K} \frac{c_k c_\ell}{k^s\ell^s}\Big\{\sum_{d\in F(K)} J_{2s}(d)\mathbf{1}_{d|k}\mathbf{1}_{d|\ell}\Big\}. \qquad (5.2)$$

Writing $k = ud$, $\ell = vd$ and noting that $u, v \in F(K)$, we have

$$L \le \sum_{u,v\in F(K)} \frac{1}{u^s v^s}\Big(\sum_{d\in F(K)} \frac{J_{2s}(d)}{d^{2s}} c_{ud}c_{vd}\Big).$$

By the Cauchy–Schwarz inequality,

$$\sum_{d\in F(K)} \frac{J_{2s}(d)}{d^{2s}} c_{ud}c_{vd} \le \Big(\sum_{d\in F(K)} \frac{J_{2s}(d)}{d^{2s}} c_{ud}^2\Big)^{1/2}\Big(\sum_{d\in F(K)} \frac{J_{2s}(d)}{d^{2s}} c_{vd}^2\Big)^{1/2}.$$

Hence,

$$L \le \Big[\sum_{u\in F(K)} \frac{1}{u^s}\Big(\sum_{d\in F(K)} \frac{J_{2s}(d)}{d^{2s}} c_{ud}^2\Big)^{1/2}\Big]^2.$$

Let ψ be a positive arithmetic function. Writing $\frac{1}{u^s} = \frac{1}{u^{s/2}\psi(u)^{1/2}}\frac{\psi(u)^{1/2}}{u^{s/2}}$ and applying Cauchy–Schwarz's inequality again gives

$$L \le \Big(\sum_{u\in F(K)} \frac{1}{u^s\psi(u)}\Big)\Big(\sum_{u\in F(K)} \frac{\psi(u)}{u^s}\sum_{d\in F(K)} \frac{J_{2s}(d)}{d^{2s}} c_{ud}^2\Big).$$

Let $F^2(K) = \{ud : u, d \in F(K)\}$. Then

$$\begin{aligned}\sum_{u\in F(K)} \frac{\psi(u)}{u^s}\sum_{d\in F(K)} \frac{J_{2s}(d)}{d^{2s}} c_{ud}^2 &\le \sum_{\nu\in F^2(K)} c_\nu^2 \sum_{\substack{u\in F(K)\\ u|\nu}} \frac{\psi(u)}{u^s}\frac{J_{2s}(\frac{\nu}{u})}{(\frac{\nu}{u})^{2s}} \\ &= \sum_{\nu\in K} \frac{c_\nu^2}{\nu^{2s}} \sum_{\substack{u\in F(K)\\ u|\nu}} J_{2s}\big(\frac{\nu}{u}\big)u^s\psi(u) \qquad (5.3)\end{aligned}$$

since $c_\nu = 0$ if $\nu \notin K$. Hence we get

$$L \le \Big(\sum_{u\in F(K)} \frac{1}{u^s\psi(u)}\Big)\Big(\sum_{\nu\in K} \frac{c_\nu^2}{\nu^{2s}} \sum_{\substack{u\in F(K)\\ u|\nu}} J_{2s}\big(\frac{\nu}{u}\big)u^s\psi(u)\Big). \qquad (5.4)$$

(2) Choose $\psi(u) = u^{-s}\psi_1(u)\sigma_\tau(u)$ (recalling that $\psi_1(u) > 0$ is non-decreasing). Then,

$$\begin{aligned}L &\le \Big(\sum_{u\in F(K)} \frac{1}{\psi_1(u)\sigma_\tau(u)}\Big)\Big(\sum_{\nu\in K} \frac{c_\nu^2}{\nu^{2s}} \sum_{\substack{u\in F(K)\\ u|\nu}} J_{2s}\big(\frac{\nu}{u}\big)\psi_1(u)\sigma_\tau(u)\Big) \\ &\le \Big(\sum_{u\in F(K)} \frac{1}{\psi_1(u)\sigma_\tau(u)}\Big)\Big(\sum_{\nu\in K} \frac{c_\nu^2\psi_1(\nu)}{\nu^{2s}} \sum_{\substack{u\in F(K)\\ u|\nu}} J_{2s}\big(\frac{\nu}{u}\big)\sigma_\tau(u)\Big).\end{aligned}$$

As $\nu \in K$,

$$\sum_{\substack{u\in F(K)\\ u|\nu}} J_{2s}\big(\frac{\nu}{u}\big)\sigma_\tau(u) = J_{2s} * \sigma_\tau(\nu).$$

By commutativity and associativity of the Dirichlet convolution,

$$J_{2s} * \sigma_\tau = (\zeta_{2s} * \mu) * (\zeta_\tau * \zeta_0) = (\zeta_{2s} * \zeta_\tau) * (\zeta_0 * \mu) = (\zeta_{2s} * \zeta_\tau) * \delta,$$

since by (4.3), $\zeta_0 * \mu = \delta$. Further,

$$\zeta_{2s} * \zeta_\tau(n) = \sum_{d|n} d^{2s}\big(\frac{n}{d}\big)^\tau = n^\tau \sum_{d|n} d^{2s-\tau} = n^\tau \sigma_{2s-\tau}(n).$$

Consequently,

$$J_{2s} * \sigma_\tau(\nu) = \sum_{n|\nu} n^\tau \sigma_{2s-\tau}(n)\delta\big(\frac{\nu}{n}\big) = \nu^\tau \sigma_{2s-\tau}(\nu).$$

By reporting

$$\begin{aligned} L &\le \Big(\sum_{u\in F(K)} \frac{1}{\psi_1(u)\sigma_\tau(u)}\Big)\Big(\sum_{\nu\in K} \frac{c_\nu^2\psi_1(\nu)}{\nu^{2s}}\nu^\tau\sigma_{2s-\tau}(\nu)\Big) \\ &= \Big(\sum_{u\in F(K)} \frac{1}{\psi_1(u)\sigma_\tau(u)}\Big)\Big(\sum_{\nu\in K} c_\nu^2\psi_1(\nu)\sigma_{\tau-2s}(\nu)\Big), \end{aligned} \tag{5.5}$$

as claimed. Taking $\psi_1(u) \equiv 1$ gives

$$L \le \Big(\sum_{u\in F(K)} \frac{1}{\sigma_\tau(u)}\Big)\Big(\sum_{\nu\in K} c_\nu^2\sigma_{\tau-2s}(\nu)\Big),$$

which is (i). Finally let $s = 1 = \tau$ and $\psi_1(u) = \log\log u$. Then, by (5.5) again,

$$\begin{aligned} L &\le \Big(\sum_{u\in F(K)} \frac{1}{\sigma(u)\log\log u}\Big)\Big(\sum_{\nu\in K} c_\nu^2\sigma_{-1}(\nu)\log\log\nu\Big) \\ &\le \frac{\pi^2}{6}\Big(\sum_{u\in F(K)} \frac{\varphi(u)}{u^2\log\log u}\Big)\Big(\sum_{\nu\in K} c_\nu^2\sigma_{-1}(\nu)\log\log\nu\Big), \end{aligned}$$

since [12, p. 10] $\sigma(n)\varphi(n) > 6n^2/\pi^2$. This is (ii) and the proof is now complete.

Remark 5.1. Quite similarly, one can also prove that

$$L \le \Big(\sum_{u\in F(K)} \frac{1}{\sum_{d|\nu} d\log\log d}\Big)\Big(\sum_{\nu\in K} c_\nu^2 \sum_{d|\nu}\frac{\log\log d}{d}\Big).$$

6. Proof of Theorems 2.14, 2.15

We begin with a lemma and a corollary which are slightly improving Proposition 3.3 in Hilberdink [19]. We displayed with care the necessary calculations.

Lemma 6.1. *Let $\sigma > 1/2$. There exists a constant C_σ depending on σ only such that for any positive integers k, ℓ and any real T,*

$$\int_1^T |\zeta(\sigma+it)|^2 \Big(\frac{k}{\ell}\Big)^{it} \mathrm{d}t = \begin{cases} T\zeta(2\sigma)\frac{(k,\ell)^{2\sigma}}{(k\ell)^\sigma} + H_1 & \text{if } T \ge \frac{k\vee\ell}{(k,\ell)} \\ H_2 & \text{if } T < \frac{k\vee\ell}{(k,\ell)}, \end{cases}$$

where

$$|H_1| \le C_\sigma\Big\{T\frac{(k,\ell)^{2\sigma}}{(k\ell)^\sigma}\Big(\frac{T}{\frac{k\vee\ell}{(k,\ell)}}\Big)^{1-2\sigma} + T^{2(1-\sigma)}\big\{1 + k^{1-\sigma}\log(kT) + \ell^{1-\sigma}\log(\ell T)\big\}\Big\}$$

$$|H_2| \le C_\sigma T^{2(1-\sigma)}\big\{1 + k^{1-\sigma}\log(kT) + \ell^{1-\sigma}\log(\ell T)\big\}.$$

Proof. Recall the basic approximation result of the Riemann Zeta function [34, Theorem 3.5]. Let $\sigma_0 > 0$, $0 < \delta < 1$. Then, uniformly for $\sigma \ge \sigma_0$, $x \ge 1$, $0 < |t| \le (1-\delta)2\pi x$,

$$\zeta(s) = \sum_{n\le x}\frac{1}{n^s} - \frac{x^{1-s}}{1-s} + \mathcal{O}(x^{-\sigma}). \tag{6.1}$$

By taking δ such that $2\pi(1-\delta) = 1$, $|t| = x$, and observing that $\big|\frac{t^{1-\sigma-it}}{1-\sigma-it}\big| \le |t|^{-\sigma}$, we deduce

$$\sup_{|t|\ge 1, \sigma\ge\sigma_0} |t|^\sigma \Big|\zeta(\sigma+it) - \sum_{k=1}^{|t|}\frac{1}{k^{\sigma+it}}\Big| < \infty.$$

And further,

$$\sup_{|t|\ge 1, \sigma\ge\sigma_0} |t|^{2\sigma-1}\Big||\zeta(\sigma+it)|^2 - \Big|\sum_{k=1}^{|t|}\frac{1}{k^{\sigma+it}}\Big|^2\Big| < \infty.$$

We choose throughout $\sigma = \sigma_0 > 1/2$. We thus have

$$\int_1^T |\zeta(\sigma+it)|^2\Big(\frac{k}{\ell}\Big)^{it}\mathrm{d}t = \int_1^T \Big|\sum_{m=1}^{|t|}\frac{1}{m^{\sigma+it}}\Big|^2\Big(\frac{k}{\ell}\Big)^{it}\mathrm{d}t + H, \tag{6.2}$$

where $|H| \le C_\sigma T^{2(1-\sigma)}$. And

$$\int_1^T \Big|\sum_{1\le n\le t} \frac{1}{n^{\sigma+it}}\Big|^2 \Big(\frac{k}{\ell}\Big)^{it} \mathrm{d}t = \sum_{1\le n,m\le T} \int_{n\vee m}^T \frac{1}{(nm)^\sigma}\Big(\frac{km}{\ell n}\Big)^{it} \mathrm{d}t.$$

• First assume that $(k,\ell) = 1$. The solutions of the equation $km = \ell n$ are $m = r\ell$, $n = rk$, and the condition $n \vee m \le T$ requires that $1 \le r \le T/(k \vee \ell)$. Hence,

— If k,ℓ are such that $(k \vee \ell) > T$, there is no solution and thus,

$$\int_1^T \Big|\sum_{1\le n\le t} \frac{1}{n^{\sigma+it}}\Big|^2 \Big(\frac{k}{\ell}\Big)^{it} \mathrm{d}t = \sum_{\substack{1\le n,m\le T\\ km\neq \ell n}} \int_{n\vee m}^T \frac{1}{(nm)^\sigma}\Big(\frac{km}{\ell n}\Big)^{it} \mathrm{d}t.$$

— If k,ℓ are such that $(k \vee \ell) \le T$, then

$$\sum_{\substack{1\le n,m\le T\\ km=\ell n}} \int_{n\vee m}^T \frac{1}{(nm)^\sigma}\Big(\frac{km}{\ell n}\Big)^{it} \mathrm{d}t = \frac{1}{(k\ell)^\sigma} \sum_{1\le r\le \frac{T}{k\vee \ell}} \frac{T - r(k\vee\ell)}{r^{2\sigma}} = \frac{T\zeta(2\sigma)}{(k\ell)^\sigma} + H,$$

where

$$|H| \le C_\sigma \frac{1}{(k\ell)^\sigma} \frac{T^{2(1-\sigma)}}{(k\vee\ell)^{1-2\sigma}}.$$

• Now if $d := (k,\ell) > 1$, writing $k = k'd$, $\ell = \ell'd$, where $(k',\ell') = 1$, our initial integral reduces to

$$\int_{n\vee m}^T \frac{1}{(nm)^\sigma}\Big(\frac{k'm}{\ell' n}\Big)^{it} \mathrm{d}t.$$

And by what precedes,

(i) If k,ℓ are such that $\frac{k\vee\ell}{(k,\ell)} > T$, then

$$\int_1^T \Big|\sum_{1\le n\le t} \frac{1}{n^{\sigma+it}}\Big|^2 \Big(\frac{k}{\ell}\Big)^{it} \mathrm{d}t = \sum_{\substack{1\le n,m\le T\\ km\neq \ell n}} \int_{n\vee m}^T \frac{1}{(nm)^\sigma}\Big(\frac{km}{\ell n}\Big)^{it} \mathrm{d}t.$$

(ii) If k,ℓ are such that $\frac{k\vee\ell}{(k,\ell)} \le T$, then

$$\begin{aligned}\sum_{\substack{1\le n,m\le T\\ km=\ell n}} \int_{n\vee m}^T \frac{1}{(nm)^\sigma}\Big(\frac{km}{\ell n}\Big)^{it} \mathrm{d}t &= \sum_{\substack{1\le n,m\le T\\ k'm=\ell' n}} \int_{n\vee m}^T \frac{1}{(nm)^\sigma}\Big(\frac{k'm}{\ell' n}\Big)^{it} \mathrm{d}t \\ &= \zeta(2\sigma)T\frac{(k,\ell)^{2\sigma}}{(k\ell)^\sigma} + H,\end{aligned}$$

where

$$|H| \le C_\sigma \frac{1}{(k'\ell')^\sigma} \frac{T^{2(1-\sigma)}}{(k' \vee \ell')^{1-2\sigma}} = C_\sigma T \frac{(k,\ell)^{2\sigma}}{(k\ell)^\sigma} \Big(\frac{T}{\frac{k\vee\ell}{(k,\ell)}} \Big)^{1-2\sigma}.$$

Now consider the contribution provided by the indices n, m such that $km \neq \ell n$,

$$\Big| \sum_{\substack{1\le n,m\le T \\ km\neq \ell n}} \int_{n\vee m}^{T} \frac{1}{(nm)^\sigma} \Big(\frac{km}{\ell n}\Big)^{it} \mathrm{d}t \Big| = \Big| \sum_{\substack{1\le n,m\le T \\ km\neq \ell n}} \frac{1}{(nm)^\sigma} \frac{e^{iT\log\frac{km}{\ell n}} - e^{i(n\vee m)\log\frac{km}{\ell n}}}{i\log\frac{km}{\ell n}} \Big|$$
$$\le \sum_{\substack{1\le n,m\le T \\ km\neq \ell n}} \frac{1}{(nm)^\sigma} \frac{1}{|\log\frac{km}{\ell n}|}.$$

We operate as in the proof of Lemma 7.2 in [35]. We have

$$\sum_{\substack{1\le n,m\le T \\ \ell n< km}} \frac{1}{(nm)^\sigma} \frac{1}{|\log\frac{km}{\ell n}|} = \Big(\sum_{\substack{1\le n,m\le T \\ \ell n<\frac12 km}} + \sum_{\substack{1\le n,m\le T \\ \frac12 km\le \ell n< km}} \Big) \frac{1}{(nm)^\sigma} \frac{1}{|\log\frac{km}{\ell n}|}$$
$$:= S_1 + S_2.$$

We have

$$S_1 \le \frac{1}{\log 2} \Big(\sum_{1\le n\le T} \frac{1}{n^\sigma} \Big)^2 \le C_\sigma T^{2(1-\sigma)}.$$

Now if $\frac12 km \le \ell n < km$, we can write $\ell n = km - r$ where $1 \le r < \frac12 km$ and we note that $\log\frac{km}{\ell n} = -\log\big(1 - \frac{r}{km}\big) > \frac{r}{km}$. Then

$$S_2 \le \sum_{m\le T} \sum_{1\le r<\frac12 km} \frac{1}{(km-r)^\sigma m^\sigma (\frac{r}{km})} \le 2^\sigma k^{1-\sigma} \sum_{m\le T} m^{1-2\sigma} \sum_{1\le r<\frac12 km} \frac{1}{r}$$
$$\le C_\sigma k^{1-\sigma} \sum_{m\le T} m^{1-2\sigma} \log(km) = k^{1-\sigma}(\log k) \sum_{m\le T} m^{1-2\sigma} + k^{1-\sigma} \sum_{m\le T} m^{1-2\sigma}\log m$$
$$\le C_\sigma k^{1-\sigma} T^{2(1-\sigma)} \log(kT).$$

Putting both estimates together and operating similarly with the sum corresponding to indices n, m such that $\ell n > km$ gives

$$\sum_{\substack{1\le n,m\le T \\ km\neq \ell n}} \frac{1}{(nm)^\sigma} \frac{1}{|\log\frac{km}{\ell n}|} \le C_\sigma T^{2(1-\sigma)} \big\{ 1 + k^{1-\sigma}\log(kT) + \ell^{1-\sigma}\log(\ell T) \big\}. \quad (6.3)$$

Therefore if $T \ge \frac{k\vee\ell}{(k,\ell)}$),

$$\int_1^T \Big|\sum_{1\le n\le t} \frac{1}{n^{\sigma+it}}\Big|^2 \Big(\frac{k}{\ell}\Big)^{it} \mathrm{d}t = T\zeta(2\sigma)\frac{(k,\ell)^{2\sigma}}{(k\ell)^{\sigma}} + H$$

with

$$|H| \le C_\sigma\Big\{T\frac{(k,\ell)^{2\sigma}}{(k\ell)^{\sigma}}\Big(\frac{T}{\frac{k\vee\ell}{(k,\ell)}}\Big)^{1-2\sigma} + T^{2(1-\sigma)}\big\{1+k^{1-\sigma}\log(kT)+\ell^{1-\sigma}\log(\ell T)\big\}\Big\}.$$

And by (6.2),

$$\int_1^T |\zeta(\sigma+it)|^2 \Big(\frac{k}{\ell}\Big)^{it} \mathrm{d}t = T\zeta(2\sigma)\frac{(k,\ell)^{2\sigma}}{(k\ell)^{\sigma}} + H$$

with

$$|H| \le C_\sigma\Big\{T\frac{(k,\ell)^{2\sigma}}{(k\ell)^{\sigma}}\Big(\frac{T}{\frac{k\vee\ell}{(k,\ell)}}\Big)^{1-2\sigma} + T^{2(1-\sigma)}\big\{1+k^{1-\sigma}\log(kT)+\ell^{1-\sigma}\log(\ell T)\big\}\Big\}.$$

Finally if $T < \frac{k\vee\ell}{(k,\ell)}$), by (6.3)

$$\begin{aligned}\Big|\int_1^T \Big|\sum_{1\le n\le t} \frac{1}{n^{\sigma+it}}\Big|^2 \Big(\frac{k}{\ell}\Big)^{it} \mathrm{d}t\Big| &= \Big|\sum_{\substack{1\le n,m\le T\\ km\neq \ell n}} \int_{n\vee m}^T \frac{1}{(nm)^{\sigma}}\Big(\frac{km}{\ell n}\Big)^{it}\mathrm{d}t\Big| \\ &\le 2\sum_{\substack{1\le n,m\le T\\ km\neq \ell n}} \frac{1}{(nm)^{\sigma}}\frac{1}{|\log\frac{km}{\ell n}|} \\ &\le C_\sigma T^{2(1-\sigma)}\big\{1+k^{1-\sigma}\log(kT)+\ell^{1-\sigma}\log(\ell T)\big\}. \qquad \square\end{aligned}$$

Corollary 6.2. *Let $\sigma > 1/2$. There exists a constant C_σ depending on σ only such that for any complex numbers $\{c_k, k\in K\}$ and any real $T\ge 2$, we have*

$$\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^2\Big|\sum_{k\in K} c_k k^{it}\Big|^2 \mathrm{d}t = \zeta(2\sigma)\sum_{\substack{k,\ell\in K\\ \frac{k\vee\ell}{(k,\ell)}\le T}} c_k\overline{c}_\ell\frac{(k,\ell)^{2\sigma}}{(k\ell)^{\sigma}} + H$$

with

$$\begin{aligned}|H| \le C_\sigma\Big\{&\sum_{\substack{k,\ell\in K\\ \frac{k\vee\ell}{(k,\ell)}\le T}} |c_k||c_\ell|\frac{(k,\ell)^{2\sigma}}{(k\ell)^{\sigma}}\Big(\frac{T}{\frac{k\vee\ell}{(k,\ell)}}\Big)^{1-2\sigma} \\ &+ T^{1-2\sigma}\Big(\sum_{\ell\in K}|c_\ell|\Big)\Big(\sum_{k\in K}|c_k|k^{1-\sigma}\log(kT)\Big\}.\end{aligned}$$

Proof. It follows from Lemma 6.1 that

$$\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^2 \Big|\sum_{k=1}^N c_k k^{it}\Big|^2 \mathrm{d}t = \zeta(2\sigma) \sum_{\substack{k,\ell\in K\\ \frac{k\vee\ell}{(k,\ell)}\le T}} c_k\overline{c}_\ell \frac{(k,\ell)^{2\sigma}}{(k\ell)^\sigma} + H$$

with

$$|H| \le C_\sigma\Big\{ \sum_{\substack{k,\ell\in K\\ \frac{k\vee\ell}{(k,\ell)}\le T}} |c_k||c_\ell| \frac{(k,\ell)^{2\sigma}}{(k\ell)^\sigma}\Big(\frac{T}{\frac{k\vee\ell}{(k,\ell)}}\Big)^{1-2\sigma} + T^{1-2\sigma}\Big(\sum_{\ell\in K}|c_\ell|\Big)\Big(\sum_{k\in K}|c_k|k^{1-\sigma}\log(kT)\Big\}.$$

This allows us to conclude easily. □

The following corollary is now straightforward.

Corollary 6.3. *Let $\sigma > 1/2$. There exists a constant C_σ depending on σ only such that for any complex numbers $\{c_k, k\in K\}$ and any real T such that $T \ge K_+ = \max\{k, k\in K\}$, we have*

$$(1 + C\frac{K_+}{T})\Big(\sum_{k\in K}|c_k|^2\Big) \max_{1\le t\le T} |\zeta(\sigma+it)|^2 \ge \Big(\zeta(2\sigma) - \frac{C_\sigma}{\varpi(K)^{2\sigma-1}}\Big)\sum_{k,\ell=1}^N c_k\overline{c}_\ell\frac{(k,\ell)^{2\sigma}}{(k\ell)^\sigma} - T^{1-2\sigma}\Big(\sum_{k\in K}|c_k|k^{1-\sigma}\log(kT)\Big)\sum_{\ell\in K}|c_\ell|,$$

where we have noted

$$\varpi(K) = \max_{k,\ell\in K}\frac{k\vee\ell}{(k,\ell)}.$$

Proof. A standard consequence of Montgomery and Vaughan version of Hilbert's inequality is

$$\Big|\int_0^T \Big|\sum_{k\in K} c_k k^{it}\Big|^2 \mathrm{d}t - T\sum_{k\in K}|c_k|^2\Big| \le \frac{4\pi}{\delta}\sum_{k\in K}|c_k|^2$$

where

$$\delta = \min\{\log\frac{\ell}{k}, \ell > k, \ell, k\in K\}.$$

Plainly $\delta \ge 1/CK_+$ where C is an absolute constant. Thus

$$\frac{1}{T}\int_0^T \Big|\sum_{k\in K} c_k k^{it}\Big|^2 \mathrm{d}t \le \sum_{k\in K} |c_k|^2(1 + C\frac{K_+}{T}).$$

This along with Lemma 6.2 implies that

$$\begin{aligned}&(1 + C\frac{K_+}{T})\Big(\sum_{k\in K}|c_k|^2\Big)\max_{1\le t\le T}|\zeta(\sigma+it)|^2\\ &\ge \Big(\zeta(2\sigma) - \frac{C_\sigma}{\varpi(K)^{2\sigma-1}}\Big)\sum_{k,\ell=1}^{N} c_k\overline{c}_\ell\frac{(k,\ell)^{2\sigma}}{(k\ell)^\sigma}\\ &\quad - T^{1-2\sigma}\Big(\sum_{\ell\in K}|c_\ell|\Big)\Big(\sum_{k\in K}|c_k|k^{1-\sigma}\log(kT)\Big\}. \qquad \square\end{aligned}$$

Proof of Theorem 2.14. Let K be an FC set and $c_d = d^{-\varepsilon}$, $d \in K$. We deduce from (2.12), that

$$\sum_{n\in K}\frac{1}{n^{2\varepsilon}}\sigma_{-\sigma+\varepsilon}(n)^2 \le \zeta(2\sigma)\sum_{d,\delta\in K}\frac{(d,\delta)^{2\sigma}}{(d\delta)^\sigma}(d\delta)^{-\varepsilon}$$

Consequently, if $K_+ \le T$ and $\varpi(K) \ge c_\sigma := (\frac{2C_\sigma}{\zeta(2\sigma)})^{1/(2\sigma-1)}$,

$$\begin{aligned}&(1 + C)\Big(\sum_{n\in K}\frac{1}{n^{2\varepsilon}}\Big)\max_{1\le t\le T}|\zeta(\sigma+it)|^2\\ &\ge \frac{\zeta(2\sigma)}{2}\sum_{n\in K}\frac{1}{n^{2\varepsilon}}\sigma_{-\sigma+\varepsilon}(n)^2\\ &\quad - \frac{1}{T^{2\sigma-1}}\Big(\sum_{\ell\in K}\frac{1}{\ell^\varepsilon}\Big)\Big(\sum_{k\in K}k^{1-\sigma-\varepsilon}\log(kT)\Big\}. \qquad \square\end{aligned}$$

Proof of Theorem 2.15. Let ν be some positive integer. Choose K to be the set of all divisors of ν, which is obviously an FC set. Then

$$\sum_{n\in K}\frac{1}{n^{2\varepsilon}}\sigma_{-\sigma+\varepsilon}(n)^2 = \sum_{n|\nu}\frac{\sigma_{-\sigma+\varepsilon}(n)^2}{n^{2\varepsilon}}, \qquad \sum_{n\in K}\frac{1}{n^{2\varepsilon}} = \sigma_{-2\varepsilon}(\nu).$$

Thus for ν and T such that $\nu \le T$ and $\max_{[k,\ell]|\nu}\frac{(k\vee\ell)}{(k,\ell)} \ge c_\sigma$

$$\begin{aligned}&(1 + C)\max_{1\le t\le T}|\zeta(\sigma+it)|^2\\ &\ge \frac{\zeta(2\sigma)}{2}\frac{1}{\sigma_{-2\varepsilon}(\nu)}\sum_{n|\nu}\frac{\sigma_{-\sigma+\varepsilon}(n)^2}{n^{2\varepsilon}} - \frac{\sigma_{-\varepsilon}(\nu)\sigma_{1-\sigma-\varepsilon}(\nu)\log(\nu T)}{\sigma_{-2\varepsilon}(\nu)T^{(2\sigma-1)}}.\end{aligned}$$

If ν and T are such that

$$\frac{\sigma_{-\varepsilon}(\nu)\sigma_{1-\sigma-\varepsilon}(\nu)\log(\nu T)}{\sum_{n|\nu}\frac{\sigma_{-\sigma+\varepsilon}(n)^2}{n^{2\varepsilon}}}\le\frac{\zeta(2\sigma)}{4}T^{(2\sigma-1)},$$

we deduce that

$$\max_{1\le t\le T}|\zeta(\sigma+it)|\ge c\Big(\frac{1}{\sigma_{-2\varepsilon}(\nu)}\sum_{n|\nu}\frac{\sigma_{-\sigma+\varepsilon}(n)^2}{n^{2\varepsilon}}\Big)^{1/2},$$

with $c^2=\frac{\zeta(2\sigma)}{4(1+C)}$ as claimed. $\square$

7. Proof of Theorem 3.1

Basically, the principle of the proof consists in showing that the studied case belongs to the "domain of attraction" of Carleson's theorem. First, recall for reader's convenience Lemma 8.3.4 from [37].

Lemma 7.1. *Let $\gamma>1$, $0<\beta\le 1$ and consider a finite collection of random variables $E=\big(X_1,\dots,X_N\big)\subset L^\gamma(\mathbb{P})$, and reals $0\le t_1\le t_2\le\dots\le t_N\le 1$ such that*

$$\|X_j-X_i\|_\gamma\le(t_j-t_i)^\beta\qquad(\forall 1\le i\le j\le N).$$

Then, there exists a constant $K_{\beta,\gamma}$ depending on β,γ only, such that

$$\Big\|\sup_{1\le i,j\le N}|X_i-X_j|\Big\|_\gamma\le\begin{cases}K_{\beta,\gamma} & \text{if } \beta\gamma>1,\\ K_{\beta,\gamma}\log N & \text{if } \beta\gamma=1,\\ K_{\beta,\gamma}N^{\frac{1}{\gamma}-\beta} & \text{if } \beta\gamma<1.\end{cases}$$

This standard lemma will be used repeatedly. Let $\{N_j,j\ge 1\}$ be an increasing sequence of integers to be specified later on. Let $S_n=\sum_{k=1}^n c_kf_k$, $n\ge 1$. Put

$$R^J=\sum_{m=1}^J a_me_m,\qquad r^J=\sum_{m=J+1}^\infty a_me_m.$$

We decompose S_n as follows: if $N_j\le n<N_{J+1}$, then for some $J=J(j)$ depending on j, the value of which being specified in the course of the proof, we write that

$$S_n=\sum_{k=1}^n c_kf_k=\sum_{k=1}^n c_kR_k^J+\sum_{k=1}^n c_kr_k^J.$$

This way to proceed is not new; we refer for instance to Theorem 2.6 in [5] where it is used already in the proof. By Carleson–Hunt's inequality,

$$\begin{aligned}\Big\| \sup_{N_j\le u\le v\le N_{j+1}} \Big| \sum_{u\le k\le v} c_k R_k^J\Big|\Big\|_2 &\le \sum_{m=1}^J |a_m| \Big\| \sup_{N_j\le u\le v\le N_{j+1}} \Big| \sum_{u\le k\le v} c_k e_{km}\Big|\Big\|_2 \\ &\le C\Big(\sum_{m=1}^J |a_m|\Big)\Big(\sum_{N_j\le k\le N_{j+1}} c_k^2\Big)^{1/2}.\end{aligned}$$

We will show in (7.2), (7.7) by using Abel summation that the series $\sum_{m\ge1}^\infty a_m^2 d(m)$ is convergent in each of the considered cases (i)–(iii), and we will estimate the tail $\sum_{m>L}^\infty a_m^2 d(m)$. By Theorem 2.1,

$$\Big\| \sum_{u\le k\le v} c_k r_k^J\Big\|_2^2 \le \Big(\sum_{m=J+1}^\infty a_m^2 d(m)\Big) \sum_{u\le k\le v} c_k^2 d(k^2).$$

By Lemma 7.1 we deduce

$$\begin{aligned}&\Big\| \sup_{N_j\le u\le v\le N_{j+1}} \Big| \sum_{u\le k\le v} c_k r_k^J\Big|\Big\|_2^2 \\ &\le C(\log N_{j+1})^2\Big(\sum_{m=J+1}^\infty a_m^2 d(m)\Big) \sum_{N_j\le k\le N_{j+1}} c_k^2 d(k^2).\end{aligned}$$

By combining both estimates we arrive to

$$\begin{aligned}\Big\| \sup_{N_j\le u\le v\le N_{j+1}} \Big| \sum_{u\le k\le v} c_k f_k\Big|\Big\|_2^2 &\le C\Big(\sum_{m=1}^J |a_m|\Big)^2\Big(\sum_{N_j\le k\le N_{j+1}} c_k^2\Big) \\ &\quad + C(\log N_{j+1})^2\Big(\sum_{m=J+1}^\infty a_m^2 d(m)\Big) \sum_{N_j\le k\le N_{j+1}} c_k^2 d(k^2). \qquad (7.1)\end{aligned}$$

Notice that if $a_m = o(m^{-\alpha})$, $\alpha > 1/2$, we have by applying Abel summation and using the well-known estimate $\sum_{m\le \ell} d(m) \le C\ell\log\ell$,

$$\begin{aligned}\sum_{m=L+1}^\infty a_m^2 d(m) \le \sum_{m=L+1}^\infty \frac{d(m)}{m^{2\alpha}} &\le C\Big\{\sum_{m=L+1}^\infty \frac{\log m}{m^{2\alpha}} + \sup_{m>L} m^{1-2\alpha}\log m\Big\} \\ &\le C_\alpha L^{1-2\alpha}(\log L). \qquad (7.2)\end{aligned}$$

Now we give the proof of assertion (i). Let $1/2 < \alpha < 1$. We then deduce from (7.1), (7.2)

$$
\begin{aligned}
&\Big\| \sup_{N_j \le u \le v \le N_{j+1}} \big| \sum_{u \le k \le v} c_k f_k \big| \Big\|_2^2 \\
&\le C\Big(\sum_{N_j \le k \le N_{j+1}} c_k^2 d(k^2) \Big) \Big\{ \Big(\sum_{m=1}^{J} |a_m| \Big)^2 + C(\log N_{j+1})^2 \Big(\sum_{m=J+1}^{\infty} a_m^2 d(m) \Big) \Big\} \\
&\le C_\alpha \Big(\sum_{N_j \le k \le N_{j+1}} c_k^2 d(k^2) \Big) \Big\{ J^{2(1-\alpha)} + (\log N_{j+1})^2 J^{1-2\alpha} \log J \Big\}.
\end{aligned} \tag{7.3}
$$

We choose N_j so that

$$
(\log N_j)^{4(1-\alpha)} (\log\log N_j)^{2(1-\alpha)} \sim j^4.
$$

Next choose J so that

$$
\frac{J}{\log J} \sim (\log N_{j+1})^2.
$$

Then $J^{2(1-\alpha)} \sim (\log N_{j+1})^{4(1-\alpha)} (\log\log N_{j+1})^{2(1-\alpha)}$. And we obtain

$$
\begin{aligned}
&\Big\| \sup_{N_j \le u \le v \le N_{j+1}} \big| \sum_{u \le k \le v} c_k f_k \big| \Big\|_2^2 \\
&\le C_\alpha \Big(\sum_{N_j \le k \le N_{j+1}} c_k^2 d(k^2) \Big) (\log N_{j+1})^{4(1-\alpha)} (\log\log N_{j+1})^{2(1-\alpha)} \\
&\le C_\alpha \Big(\sum_{N_j \le k \le N_{j+1}} c_k^2 (\log k)^{4(1-\alpha)} (\log\log k)^{2(1-\alpha)} d(k^2) \Big).
\end{aligned}
$$

In view of the assumption made, we deduce

$$
\sum_j \Big\| \sup_{N_j \le u \le v \le N_{j+1}} \big| \sum_{u \le k \le v} c_k f_k \big| \Big\|_2^2 < \infty. \tag{7.4}
$$

By Tchebycheff's inequality and by using Theorem 2.1 and (7.2) with $L = 2$,

$$
\begin{aligned}
&\lambda\Big\{ \big| \sum_{N_j < k \le N_{j+1}} c_k f_k \big| > j^{-2} \Big\} \le C_\alpha j^4 \sum_{N_j \le k \le N_{j+1}} c_k^2 d(k^2) \\
&\le C_\alpha \frac{j^4}{(\log N_j)^{4(1-\alpha)} (\log\log N_j)^{2(1-\alpha)}} \sum_{N_j \le k \le N_{j+1}} c_k^2 d(k^2) (\log N_j)^{4(1-\alpha)} (\log\log N_j)^{2(1-\alpha)} \\
&\le C_\alpha \sum_{N_j \le k \le N_{j+1}} c_k^2 d(k^2) (\log k)^{4(1-\alpha)} (\log\log k)^{2(1-\alpha)}.
\end{aligned}
$$

The assumption made implies that

$$\sum_j \lambda\Big\{\big|\sum_{N_j<k\le N_{j+1}} c_k f_k\big| > j^{-2}\Big\} < \infty.$$

By the Borel–Cantelli lemma, the series $\sum_j |\sum_{N_j<u\le N_{j+1}} c_k f_k|$ converges almost everywhere. As by (7.4), the oscillation of partial sums around this subsequence is almost surely asymptotically tending to 0, this allows us to conclude.

We continue with the proof of assertion (ii). If $\alpha = 1$, (7.3) is slightly modified as follows:

$$\begin{aligned}&\Big\|\sup_{N_j\le u\le v\le N_{j+1}} \big|\sum_{u\le k\le v} c_k f_k\big|\Big\|_2^2\\ &\le C\Big(\sum_{N_j\le k\le N_{j+1}} c_k^2 d(k^2)\Big)\Big\{(\log J)^2 + (\log N_{j+1})^2\frac{\log J}{J}\Big\}.\end{aligned}\tag{7.5}$$

We choose N_j so that

$$\log\log N_j \sim j^2,$$

and J so that

$$\frac{J}{\log J} \sim (\log N_{j+1})^2.$$

We deduce

$$\begin{aligned}\Big\|\sup_{N_j\le u\le v\le N_{j+1}} \big|\sum_{u\le k\le v} c_k f_k\big|\Big\|_2^2 &\le C\Big(\sum_{N_j\le k\le N_{j+1}} c_k^2 d(k^2)\Big)(\log\log N_{j+1})^2\\ &\le C\sum_{N_j\le k\le N_{j+1}} c_k^2 d(k^2)(\log\log k)^2.\end{aligned}$$

According to the assumption made,

$$\sum_j \Big\|\sup_{N_j\le u\le v\le N_{j+1}} \big|\sum_{u\le k\le v} c_k f_k\big|\Big\|_2^2 < \infty.\tag{7.6}$$

By Tchebycheff's inequality and by using Theorem 2.1,

$$\begin{aligned}\lambda\Big\{\big|\sum_{N_j<k\le N_{j+1}} c_k f_k\big| > j^{-2}\Big\} &\le Cj^4 \sum_{N_j\le k\le N_{J+1}} c_k^2 d(k^2)\\ &\le C\frac{j^4}{(\log\log N_{j+1})^2}\sum_{N_j\le k\le N_{j+1}} c_k^2 d(k^2)(\log\log k)^2\\ &\le C\sum_{N_j\le k\le N_{j+1}} c_k^2 d(k^2)(\log\log k)^2.\end{aligned}$$

By the Borel–Cantelli lemma, the series $\sum_j |\sum_{N_j<u\le N_{j+1}} c_k f_k|$ converges almost everywhere. This along with (7.6) allows us to conclude.

Finally we prove assertion (iii). By using Abel summation again,

$$\sum_{m>L} a_m^2 d(m) \le C\Big\{ \sum_{m>L} \frac{1}{m(\log m)^h} + \sup_{m>L} \frac{1}{(\log m)^h}\Big\} \le \frac{C_h}{(\log L)^{h-1}}. \tag{7.7}$$

And $\sum_{m=1}^J |a_m| \le C_h\sqrt{J/(\log J)^{1+h}}$. Then by (7.1),

$$\begin{aligned}
&\Big\| \sup_{N_j\le u\le v\le N_{j+1}} \big|\sum_{u\le k\le v} c_k f_k\big|\Big\|_2^2 \\
&\le C_h\Big(\sum_{N_j\le k\le N_{j+1}} c_k^2 d(k^2)\Big)\Big\{\frac{J}{(\log J)^{1+h}} + C_h\frac{(\log N_{j+1})^2}{(\log J)^{h-1}}\Big\}.
\end{aligned} \tag{7.8}$$

We choose N_j so that

$$\frac{(\log N_{j+1})^2}{(\log\log N_{j+1})^{h-1}} \sim j^4.$$

This time we choose J so that $\frac{J}{(\log J)^{1+h}} \sim \frac{(\log N_{j+1})^2}{(\log J)^{h-1}}$, namely

$$\frac{J}{(\log J)^2} \sim (\log N_{j+1})^2.$$

Thus $\log J \sim \log\log N_{j+1}$ and

$$\frac{J}{(\log J)^{1+h}} \sim \frac{(\log N_{j+1})^2}{(\log J)^{h-1}} \sim \frac{(\log N_{j+1})^2}{(\log\log N_{j+1})^{h-1}}.$$

We deduce

$$\begin{aligned}
&\Big\| \sup_{N_j\le u\le v\le N_{j+1}} \big|\sum_{u\le k\le v} c_k f_k\big|\Big\|_2^2 \\
&\le \Big(\sum_{N_j\le k\le N_{j+1}} c_k^2 d(k^2)\Big)\Big\{C_h\frac{J}{(\log J)^{1+h}} + C_h'\frac{(\log N_j)^2}{(\log J)^{h-1}}\Big\} \\
&\le C_h\Big(\sum_{N_j\le k\le N_{j+1}} c_k^2 d(k^2)\Big)\frac{(\log N_{j+1})^2}{(\log\log N_{j+1})^{h-1}} \\
&\le C_h \sum_{N_j\le k\le N_{j+1}} c_k^2 d(k^2)\frac{(\log k)^2}{(\log\log k)^{h-1}}.
\end{aligned}$$

Using the assumption made, it follows that

$$\sum_{j\ge 1}\Big\|\sup_{N_j\le u\le v\le N_{j+1}}\Big|\sum_{u\le k\le v}c_kf_k\Big|\Big\|_2^2<\infty.$$

We conclude by proceeding exactly as before. Noticing first from estimate (7.7) that $\sum_{m\ge 1}a_m^2 d(m)\le C_h$, by Tchebycheff's inequality and Theorem 2.1, we get

$$\begin{aligned}\lambda\Big\{\Big|\sum_{N_j<k\le N_{j+1}}c_kf_k\Big|>\frac{1}{j^2}\Big\}&\le Cj^4\sum_{N_j\le k\le N_{j+1}}c_k^2 d(k^2)\\ &\le C_h j^4\frac{(\log\log N_{j+1})^{h-1}}{(\log N_{j+1})^2}\sum_{N_j\le k\le N_{j+1}}c_k^2 d(k^2)\frac{(\log k)^2}{(\log\log k)^{h-1}}\\ &\le C_h\sum_{N_j\le k\le N_{j+1}}c_k^2 d(k^2)\frac{(\log k)^2}{(\log\log k)^{h-1}}.\end{aligned}$$

By the Borel–Cantelli lemma, the series $\sum_j|\sum_{N_j<u\le N_{j+1}}c_kf_k|$ converges almost everywhere. We conclude as before.

8. Proof of Theorem 3.2

Let $\{N_j, j\ge 1\}$ be an increasing unbounded sequence of positive reals. We write

$$\sum_{N_j\le k<N_{j+1}}c_kf_k=\sum_{N_j\le k<N_{j+1}}c_kR_k^J+\sum_{N_j\le k<N_{j+1}}c_kr_k^J$$

where

$$R^J(x)=\sum_{1\le \ell<J}\frac{\sin 2\pi\ell x}{\ell},\qquad r^J(x)=\sum_{\ell>J}\frac{\sin 2\pi\ell x}{j},$$

and J is a real number greater than 1 and defined later on with respect to j. Let $\beta>1$. Choose $N_j=e^{e^{j^B}}$ with $B=2\beta/\delta$; δ is a (small) positive real. As $f\in\mathrm{BV}(\mathbb{T})$, $a_j=\mathcal{O}(j^{-1})$, and so

$$\sup_{N_j\le u\le v\le N_{j+1}}\Big|\sum_{u\le k\le v}c_kR_k^J(x)\Big|\le C\sum_{\ell=1}^J\frac{1}{\ell}\sup_{N_j\le u\le v\le N_{j+1}}\Big|\sum_{u\le k\le v}c_k\sin 2\pi k\ell x\Big|.$$

By using Carleson–Hunt's maximal inequality

$$\begin{aligned}\Big\|\sup_{N_j\le u\le v\le N_{j+1}}\Big|\sum_{u\le k\le v}c_kR_k^J\Big|\Big\|_2&\le\sum_{\ell=1}^J\frac{1}{\ell}\Big\|\sup_{N_j\le u\le v\le N_{j+1}}\Big|\sum_{u\le k\le v}c_k\sin 2\pi k\ell x\Big|\Big\|_2\\ &\le C(\log J)\Big(\sum_{N_j\le u\le N_{j+1}}c_k^2\Big)^{1/2}.\end{aligned}\tag{8.1}$$

We now combine our Theorem 2.5 with the $(\varepsilon, 1-\varepsilon)$ argument introduced in [1]. Let $0 < \varepsilon < 1/2$. From the bound

$$\sum_{\substack{i,j>J\\ jk=i\ell}} \frac{1}{ij} \le C \min\Big(\frac{(k,\ell)}{(k\vee \ell)J}, \frac{(k,\ell)^2}{k\ell}\Big) \le C\Big(\frac{(k,\ell)}{(k\vee \ell)J}\Big)^{\varepsilon}\Big(\frac{(k,\ell)^2}{k\ell}\Big)^{1-\varepsilon}$$
$$\le \frac{C}{J^{\varepsilon}}\Big(\frac{(k,\ell)^2}{k\ell}\Big)^{1-\varepsilon/2} = \frac{C}{J^{\varepsilon}}\langle f_k^{1-\varepsilon/2}, f_\ell^{1-\varepsilon/2}\rangle,$$

we get by applying Theorem 2.5-(i),

$$\Big\| \sum_{u\le k\le v} c_k r_k^J\Big\|_2^2 \le \frac{C}{J^{\varepsilon}}\Big\| \sum_{u\le k\le v} |c_k| f_k^{1-\varepsilon/2}\Big\|_2^2$$
$$\le \frac{C}{J^{\varepsilon}}\Big(\sum_{w\in F([u,v])}\frac{1}{\sigma_\tau(w)}\Big)\Big(\sum_{u\le k\le v} c_k^2\sigma_{\tau-2+\varepsilon}(k)\Big). \tag{8.2}$$

By taking $\tau = 1+\varepsilon$ and using Corollary 2.6, this becomes

$$\Big\| \sum_{u\le k\le v} c_k r_k^J\Big\|_2^2 \le \frac{C}{\varepsilon J^{\varepsilon}}\sum_{u\le k\le v} c_k^2\sigma_{-1+2\varepsilon}(k).$$

From estimate (2.16), it follows that

$$\sigma_{-1+2\varepsilon}(k) \le \exp\Big\{\frac{\varrho}{2\varepsilon}\frac{(\log k)^{2\varepsilon}}{\log\log k}\Big\}.$$

where ϱ is some positive number. Thus,

$$\Big\| \sum_{u\le k\le v} c_k r_k^J\Big\|_2^2 \le \frac{C}{J^{\varepsilon}}\exp\Big\{\frac{\varrho}{2\varepsilon}\frac{(\log N_{j+1})^{2\varepsilon}}{\log\log N_{j+1}}\Big\}\Big(\sum_{u\le k\le v} c_k^2\Big).$$

By using Lemma 7.1, we obtain

$$\Big\| \sup_{N_j\le u\le v\le N_{j+1}}\Big| \sum_{u\le k\le v} c_k r_k^J\Big|\Big\|_2^2$$
$$\le \frac{C}{\varepsilon J^{\varepsilon}}(\log N_{j+1})^2\exp\Big\{\frac{\varrho}{2\varepsilon}\frac{(\log N_{j+1})^{2\varepsilon}}{\log\log N_{j+1}}\Big\}\Big(\sum_{N_j\le k\le N_{j+1}} c_k^2\Big).$$

Choose ε, J so that

$$\varepsilon J^{\varepsilon} = (\log N_{j+1})^2\exp\Big\{\frac{\varrho}{\varepsilon}\frac{(\log N_{j+1})^{2\varepsilon}}{\log\log N_{j+1}}\Big\}, \qquad \varepsilon = \frac{\log\log\log N_{j+1}}{2\log\log N_{j+1}}.$$

We get

$$\Big\| \sup_{N_j\le u\le v\le N_{j+1}} \Big| \sum_{u\le k\le v} c_k r_k^J \Big|\Big\|_2^2 \le \sum_{N_j\le k\le N_{j+1}} c_k^2 .$$

We have

$$\log J = \frac{1}{\varepsilon}\log\frac{1}{\varepsilon} + (\frac{2}{\varepsilon})\log\log N_{j+1} + \frac{\varrho}{\varepsilon^2}\frac{(\log N_{j+1})^{2\varepsilon}}{\log\log N_{j+1}}$$

Further,

$$\frac{1}{\varepsilon}\log\frac{1}{\varepsilon} = \frac{2\log\log N_{j+1}}{\log\log\log N_{j+1}}\log\Big(\frac{2\log\log N_{j+1}}{\log\log\log N_{j+1}}\Big) \sim 2\log\log N_{j+1},$$

and

$$(\log N_{j+1})^{2\varepsilon} = e^{(\log\log N_{j+1})(\log\log\log N_{j+1})/(\log\log N_{j+1})} = \log\log N_{j+1}.$$

Thus

$$\begin{aligned}
\log J &\sim 2(\log\log N_{j+1}) + \frac{4(\log\log N_{j+1})^2}{(\log\log\log N_{j+1})} + \frac{\varrho}{\varepsilon^2}\frac{(\log N_{j+1})^{2\varepsilon}}{\log\log N_{j+1}} \\
&= 2(\log\log N_{j+1}) + \frac{4(\log\log N_{j+1})^2}{(\log\log\log N_{j+1})} + 4\varrho\frac{(\log\log N_{j+1})^2}{(\log\log\log N_{j+1})^2} \\
&\le C\frac{(\log\log N_{j+1})^2}{(\log\log\log N_{j+1})},
\end{aligned}$$

Now by (8.1),

$$\begin{aligned}
\Big\| \sup_{N_j\le u\le v\le N_{j+1}} \Big| \sum_{u\le k\le v} c_k R_k^J \Big|\Big\|_2^2 &\le C(\log J)^2\Big(\sum_{N_j\le u\le N_{j+1}} c_k^2\Big) \\
&\le C\frac{(\log\log N_{j+1})^4}{(\log\log\log N_{j+1})^2}\Big(\sum_{N_j\le u\le N_{j+1}} c_k^2\Big) \\
&\le C\sum_{N_j\le k\le N_{j+1}} c_k^2\frac{(\log\log k)^4}{(\log\log\log k)^2}.
\end{aligned}$$

By combining

$$\Big\| \sup_{N_j\le u\le v\le N_{j+1}} \Big| \sum_{u\le k\le v} c_k f_k \Big|\Big\|_2^2 \le C\sum_{N_j\le u\le N_{j+1}} c_k^2\frac{(\log\log k)^4}{(\log\log\log k)^2}. \tag{8.3}$$

By the assumption made, this immediately implies that the series

$$\sum_j \sup_{N_j\le u\le v\le N_{j+1}} \Big|\sum_{u\le k\le v} c_k f_k\Big|^2$$

converges almost everywhere. And so the oscillation of the partial sum sequence $\{\sum_{k=1}^{N} c_k f_k, N \geq 1\}$ around the subsequence $\{\sum_{k=1}^{N_j} c_k f_k, j \geq 1\}$ tends to zero almost everywhere.

By Tchebycheff's inequality,

$$\begin{aligned}\lambda\Big\{\sup_{N_j\le u\le v\le N_{j+1}} \Big|\sum_{u\le k\le v} c_k r_k^J\Big| > j^{-\beta}\Big\} &\le Cj^{2\beta}\sum_{N_j\le k\le N_{j+1}} c_k^2 \\ &\le C\frac{j^{2\beta}}{(\log\log N_{j+1})}\sum_{N_j\le u\le N_{j+1}} c_k^2 \log\log k \\ &\le C\sum_{N_j\le u\le N_{j+1}} c_k^2(\log\log k).\end{aligned}$$

By the Borel–Cantelli lemma, the series

$$\sum_j \Big|\sum_{N_j<u\le N_{j+1}} c_k r_k^J\Big|$$

converges almost everywhere.

We shall prove that the series

$$\sum_j \Big|\sum_{N_j<u\le N_{j+1}} c_k R_k^J\Big|$$

also converges almost everywhere. This part is more tricky. We begin with a remark concerning the sum related to the component R^J. Recall that

$$\langle R_k^J, R_\ell^J\rangle = \sum_{\substack{i,h=1\\ hk=i\ell}}^{J} \frac{1}{ih} = \Big(\sum_{1\le u\le J(\frac{(k,\ell)}{k}\wedge\frac{(k,\ell))}{\ell}}^{J} \frac{1}{u^2}\Big)\frac{(k,\ell)^2}{k\ell}.$$

The existence of a solution in h and i (automatically of the form $i = u\frac{k}{(k,\ell)}, h = u\frac{\ell}{(k,\ell)}$, $u \ge 1$) imposes constraints on the integers k, ℓ. These must satisfy the following conditions:

$$\ell \le u\ell = (k,\ell)h \le J(k,\ell), \qquad k \le uk = (k,\ell)i \le J(k,\ell),$$

that is $(k,\ell) \ge \frac{(k\vee\ell)}{J}$. Consequently, as obviously $(k,\ell) \le (k\wedge\ell)$, it is necessary to have

$$\frac{1}{J}(k\vee\ell) \le (k\wedge\ell).$$

In this case $0 \le \langle R_k^J, R_\ell^J\rangle \le \zeta(2)\frac{(k,\ell)^2}{k\ell}$. Observe before continuing that in our situation $J \ll N_{j+1}$ while $N_j < k, \ell \le N_{j+1}$.

Let h and H be such that $J^h < N_j \le J^{h+1} \le \ldots \le J^{h+H-1} \le N_{j+1} < J^{h+H}$. It follows from the previously made remark and estimate (2.20) that

$$
\begin{aligned}
\Big\| \sum_{N_j<k\le N_{j+1}} c_k R_k^J \Big\|_2^2 &\le \zeta(2) \sum_{\substack{N_j<k,\ell\le N_{j+1}\\ (k\vee\ell)\le J(k\wedge\ell)}} |c_k||c_\ell| \frac{(k,\ell)^2}{k\ell} \\
&= \zeta(2) \sum_{\mu=h}^{H} \sum_{\substack{k\in]J^\mu, J^{\mu+1}]\\ \ell:(k\vee\ell)\le J(k\wedge\ell)}} |c_k||c_\ell| \frac{(k,\ell)^2}{k\ell} \\
&\le \zeta(2) \sum_{\mu=h}^{H} \sum_{\substack{J^\mu<k\le J^{\mu+1}\\ \frac{1}{J}.J^\mu\le\ell\le J.J^{\mu+1}}} |c_k||c_\ell| \frac{(k,\ell)^2}{k\ell} \\
&\le \zeta(2) \sum_{\mu=h}^{H} \sum_{J^{\mu-1}\le k,\ell\le J^{\mu+2}} |c_k||c_\ell| \frac{(k,\ell)^2}{k\ell} \\
&\le \big(4\zeta(2)\log J\big) \sum_{\mu=h}^{H} \sum_{J^{\mu-1}\le k\le J^{\mu+2}} c_k^2 \sigma_{-1}(k) \\
&\le C \sum_{\mu=h}^{H} \sum_{J^{\mu-1}\le k\le J^{\mu+2}} c_k^2 \frac{(\log\log k)^2}{\log\log\log k} \sigma_{-1}(k)
\end{aligned}
$$

since $N_j < k \le N_{j+1}$ and

$$
\log J \sim 4(1+o(1)) \frac{(\log\log N_{j+1})^2}{(\log\log\log N_{j+1})} \sim 4(1+o(1)) \frac{(\log\log k)^2}{\log\log\log k}
$$

Therefore

$$
\Big\| \sum_{N_j<k\le N_{j+1}} c_k R_k^J \Big\|_2^2 \le C \sum_{J^{-1}N_j<k\le N_{j+1}J^2} c_k^2 \frac{(\log\log k)^2}{\log\log\log k} \sigma_{-1}(k). \tag{8.4}
$$

By Tchebycheff's inequality,

$$
\begin{aligned}
&\lambda\Big\{ \Big| \sum_{N_j<k\le N_{j+1}} c_k R_k^J \Big| > j^{-\beta} \Big\} \\
&\le C j^{2\beta} \sum_{J^{-1}N_j<k\le N_{j+1}J^2} c_k^2 \frac{(\log\log k)^2}{\log\log\log k} \sigma_{-1}(k) \\
&\le C j^{2\beta - B\delta} (\log\log N_{j+1})^\delta \sum_{J^{-1}N_j<k\le N_{j+1}J^2} c_k^2 \frac{(\log\log k)^2}{\log\log\log k} \sigma_{-1}(k) \\
&\le C \sum_{J^{-1}N_j<k\le N_{j+1}J^2} c_k^2 \frac{(\log\log k)^{2+\delta}}{\log\log\log k} \sigma_{-1}(k)
\end{aligned}
$$

Recall that $J = J(j)$ is associated with the interval $]N_j, N_{j+1}]$. Now observe that

$$\frac{N_{j+2}}{J(j+2)} > N_{j+1}J(j)^2.$$

Indeed,

$$J(j)^2 J(j+2) \sim e^{\frac{8+o(1)}{B}\frac{j^B}{\log j}+\frac{8+o(1)}{B}\frac{(j+2)^B}{\log(j+2)}}.$$

Thus $J(j)^2J(j+2) \le e^{C_B j^B}$, whereas for j large

$$\frac{N_{j+2}}{N_{j+1}} = e^{e^{(j+2)^B}-e^{(j+1)^B}} \ge e^{e^{(j+2)^B}/2} \gg J(j)^2J(j+2).$$

This means that the intervals $]J(j)^{-1}N_j, N_{j+1}J(j)^2]$, $j = 2, 4, 6\ldots$, are disjoint. The same holds for the sequence of intervals with odd indices. Consequently,

$$\sum_j \lambda\Big\{\big|\sum_{N_j<k\le N_{j+1}} c_k R_k^J\big| > j^{-\beta}\Big\} \le C\sum_k c_k^2 \frac{(\log\log k)^{2+\delta}}{\log\log\log k}\sigma_{-1}(k) < \infty,$$

by assumption. Hence by the Borel–Cantelli lemma, the series

$$\sum_j \big|\sum_{N_j<k\le N_{j+1}} c_k R_k^J\big|$$

converges almost everywhere. This allows us to conclude.

Remark 8.1. It is interesting to notice that from estimates (8.1) and (8.4) and Gronwall's estimate (2.16),

$$\Big\|\sup_{N_j\le u\le v\le N_{j+1}} \big|\sum_{u\le k\le v} c_k R_k^J\big|\Big\|_2^2 \le C \sum_{N_j\le k\le N_{j+1}} c_k^2 \frac{(\log\log k)^4}{(\log\log\log k)^2}.$$

while

$$\Big\|\sum_{N_j<k\le N_{j+1}} c_k R_k^J\Big\|_2^2 \le C \sum_{J(j)^{-1}N_j<k\le N_{j+1}J(j)^2} c_k^2 \frac{(\log\log k)^3}{\log\log\log k}.$$

Remark 8.2. The following set $B = \big\{k,\ell; N < k,\ell \le M : J(k,\ell) \ge (k\vee\ell)\big\}$ appeared in the last step of the proof. Concerning the size of such sets, one can show the general bound

$$\#(B) \le CJM(\log JM)^3,$$

where C is absolute.

Remark 8.3. By suitably modifying the above proof, we can also obtain the following result where multipliers have arithmetical factors, and which is developing the recent works [5, Theorem 2.8], [38, Theorem 1.1].

Theorem 8.4. *Let $f \in \mathrm{BV}(\mathbb{T})$, $\langle f, 1\rangle = 0$. Assume that for some real $b > 0$,*

$$\sum_{k\ge 3} c_k^2 (\log\log k)^{2+b}\sigma_{-1+\frac{1}{(\log\log k)^{b/3}}}(k) < \infty. \tag{8.5}$$

Then the series $\sum_k c_k f_k$ converges almost everywhere.

In contrast with Theorem 3.2, condition (8.5) directly depends on the arithmetical structure of the support of the coefficient sequence $(c_k)_k$, call it $\mathcal{K}$, thereby making both results hardly comparable. Condition (8.5) can be much weaker than condition (3.1) on examples, typically when $\mathcal{K}$ is formed with large integers having only very small prime divisors. Suppose for instance that $c_k \neq 0$ if k is such that $\log P^+(k) \le \rho(\log\log k)^{b/3}$, for some small positive ρ, and write $e^\rho = 1 + \delta$. From the standard bound $\sigma_{-1+\varepsilon}(k) \le \prod_{p|k}(\frac{1}{1-1/p^{1-\varepsilon}})$ systematically applied in [17], letting $\varepsilon = (\log\log k)^{b/3}$ and using the fact that $\sum_{p\le x} 1/p = \log\log x + \mathcal{O}(1)$, we get

$$\sigma_{-1+\varepsilon}(k) \le \prod_{p\le P^+(k)} (\frac{1}{1-\frac{1+\delta}{p}}) \le C_\delta\big(\log P^+(k)\big)^{\frac{2(1+\delta)}{1-\delta}} \le C_\delta(\log\log k)^{\frac{2b(1+\delta)}{3(1-\delta)}}.$$

Hence condition (8.5) reduces to $\sum_{k\in\mathcal{K}} c_k^2(\log\log k)^{2+b\frac{5-\delta}{3(1-\delta)}} < \infty$, which is for b small clearly weaker than condition (3.1). For $\mathcal{K}$ arbitrary, it seems however quite difficult to efficiently estimate $\sigma_{-1+\frac{1}{(\log\log k)^{b/3}}}(k)$, notably from the above standard bound, thus providing some restriction.

Sketch of proof. Let $b > 0$. We choose N_j so that $\log\log N_j = j^{\beta/b}$ for some $\beta > 2$. We start with (8.1). Similarly as before, by using Corollary 2.6 and Lemma 7.1, we obtain

$$\Big\| \sup_{N_j\le u\le v\le N_{j+1}} \Big|\sum_{u\le k\le v} c_k r_k^J\Big|\Big\|_2^2 \le \frac{C(\log N_{j+1})^2}{\varepsilon J^\varepsilon}\Big(\sum_{N_j\le k\le N_{j+1}} c_k^2\sigma_{-1+2\varepsilon}(k)\Big). \tag{8.6}$$

By combining, we get

$$\begin{aligned}&\Big\| \sup_{N_j\le u\le v\le N_{j+1}} \Big|\sum_{u\le k\le v} c_k f_k\Big|\Big\|_2^2 \\ &\le C\Big(\sum_{N_j\le u\le N_{j+1}} c_k^2\sigma_{-1+2\varepsilon}(k)\Big)\Big((\log J)^2 + \frac{C(\log N_{j+1})^2}{\varepsilon J^\varepsilon}\Big).\end{aligned}$$

Choose ε, J as follows:

$$\log J = (\log\log N_{j+1})^{1+b}, \qquad \varepsilon = \frac{2}{(\log\log N_{j+1})^b}.$$

Then $J^\varepsilon = \exp\{\varepsilon \log J\} = \exp\{2\log\log N_{j+1}\} = (\log N_{j+1})^2$. So that

$$\frac{(\log N_{j+1})^2}{\varepsilon J^\varepsilon} = \frac{(\log\log N_{j+1})^b}{2}$$

We deduce that

$$\begin{aligned}\Big\| \sup_{N_j\le u\le v\le N_{j+1}} \Big|\sum_{u\le k\le v} c_k f_k\Big|\Big\|_2^2 &\le C(\log\log N_{j+1})^{2(1+b)} \sum_{N_j\le k\le N_{j+1}} c_k^2 \sigma_{-1+2\varepsilon}(k)\\ &\le C_b \sum_{N_j\le k\le N_{j+1}} c_k^2(\log\log k)^{2(1+b)}\sigma_{-1+\frac{4}{(\log\log k)^b}}(k).\end{aligned}$$

Similarly, by using Tchebycheff's inequality,

$$\begin{aligned}&\lambda\Big\{ \sup_{N_j\le u\le v\le N_{j+1}} \Big|\sum_{u\le k\le v} c_k f_k\Big| > j^{-\beta/2}\Big\}\\ &\le C_b j^\beta \sum_{N_j\le k\le N_{j+1}} c_k^2(\log\log k)^{2(1+b)}\sigma_{-1+\frac{4}{(\log\log k)^b}}(k)\\ &\le C_b \sum_{N_j\le k\le N_{j+1}} c_k^2\sigma_{-1+\frac{4}{(\log\log k)^b}}(k)(\log\log k)^{2+3b}.\end{aligned}$$

By the Borel–Cantelli lemma and the assumption made, the series

$$\sum_j \sup_{N_j\le u\le v\le N_{j+1}} \Big|\sum_{u\le k\le v} c_k f_k\Big|$$

converges almost everywhere. This allows us to conclude. □

9. Proof of Theorem 3.6

Changing f for f/c if necessary, we may assume for our purpose that $c = 1$ in (3.5). Let $G_n = (\gamma_{k,\ell})$, where $\gamma_{k,\ell} = \int_E f_k f_\ell \,\mathrm{d}x$ denotes the Gram matrix of the system $f_1, \dots, f_n$. As $H_n = I - G_n$ is nonnegative definite, there exist in $\mathbb{R}^n$ vectors $u_1, \dots, u_n$ with Gram matrix H_n, for instance the rows of $H_n^{1/2}$. Given any bounded interval Y, it follows that there exist in $L^2(Y)$ (in fact in any separable Hilbert space) vectors $v_1, \dots, v_n$ with Gram matrix $I - G_n$. By induction (using isometry), it is plain that if $v_1, \dots, v_n$ are already chosen with Gram matrix H_n, a vector v_{n+1} can be added so that the new system $v_1, \dots, v_{n+1}$ will have Gram matrix H_{n+1}. Consequently there exist (g_k) supported on Y such that $(f_k + g_k)$ is an orthonormal system on $\mathbb{T} \times Y$. Thus for any $(c_k)_k$ universal, the series $\sum_k c_k(f_k + g_k)$ converges a.e. on $\mathbb{T} \times Y$, and thereby converges a.e. on $\mathbb{T}$. Since $g_k \equiv 0$ on $\mathbb{T}$, it follows that $\sum_k c_k f_k$ converges a.e.

Remark 9.1. The construction of (g_k) is exactly the same as in the proof of Schur's Lemma [30, p. 56].

Final note. In a very recent work, Lewko and Radziwill (arXiv:1408.2334v1) proposed a new, simple (with no combinatorial argument) and very informative approach to Gál's theorem and the recent extensions obtained in [1]. They further applied their results to the convergence almost everywhere of series of dilates of functions in BV($\mathbb{T}$), and were able to reduce the condition $\gamma > 4$ in (3.2) to $\gamma > 2$. This naturally includes our Theorem 3.2, but not our Theorem 3.1 with arithmetical multipliers, as well as Theorem 8.4. Further, the new argument we introduced in the proof of Theorem 3.2 suggests a possibility to improve Lewko and Radziwill's convergence condition by requiring only that $\sum_k c_k^2 \frac{(\log\log k)^2}{(\log\log\log k)^2} < \infty$. This will be investigated elsewhere.

We also mention Aistleitner's recent result (arXiv:1409.6035v1) deriving from Hilberdink's approach, a lower bound for the maximum of $|\zeta(\sigma + it)|$, $0 \le t \le T$, T large, of the same kind as Montgomery well-known result, with slightly better constant.

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